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Research article

Soliton solutions for a class of generalized quasilinear Schrödinger equations

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Abstract: In this paper, critical point theory is used to show the existence of nontrivial solutions for a class of generalized quasilinear Schrödinger equations

$$-\Delta_p u - |u|^{\sigma-2}uh'(|u|^\sigma)\Delta_p h(|u|^\sigma) = f(x,u)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions. Our result covers some typical physical models.

Keywords: soliton solution; quasilinear Schrödinger equation; critical point theory; change of variables
Mathematics Subject Classification: 35B38, 35J20

1. Introduction

In this article, we study the generalized quasilinear Schrödinger equations

$$\begin{cases} -\Delta_p u - |u|^{\sigma-2}uh'(|u|^\sigma)\Delta_p h(|u|^\sigma) = f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $h(t) \in C^2(\mathbb{R}^+, \mathbb{R})$, $N \geq 3$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian with $1 < p < N$ and the parameter $\sigma > 1$.

When $p = \sigma = 2$, Eq (1.1) is a special case for some physical phenomena, see [1–3]. In fact, solutions for the Eq (1.1) for $p = \sigma = 2$ are the existence of standing wave solutions for the following quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + Wz - f(|z|^2)z - \kappa z h'(|z|^2)\Delta h(|z|^2), \tag{1.2}$$

where $W(x)$, $x \in \mathbb{R}^N$ is a given potential, κ is a real constant and f, h are real functions of essentially pure power forms. The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years. Quasilinear Schrödinger equation of the form (1.2) appears more naturally in mathematical physics and has been derived as a model of several physical phenomena corresponding to various types of h. For instance, the case of $h(t) = t$ was used for the superfluid film equation in plasma physics by Kurihara in [4]. In the case $h(t) = (1+t)^{1/2}$, Eq (1.2) models the self-channeling of a high-power ultrashort lasers in matter, see [5–7] and the references in [8]. Eq (1.2) also appears in plasma physics and fluid mechanics [9–11], in the theory of Heisenberg ferromagnets and magnons [12–14], in dissipative quantum mechanics [15], and in condensed matter theory [16].

In recent years, problem (1.1) was studied primarily in the case $p = \sigma = 2$ and $h(t) = t$. Recently, some works dealing with problem (1.1) for $p \neq 2$, $\sigma = 2$ and $h(t) = t$ appeared in [17–19]; for $p = 2, \sigma \neq 2$ and $h(t) = t$ appeared in [20–22]; for $p = 2, \sigma = 2$ and $h(t) \neq t$ appeared in [23, 24]. But, to our best knowledge, so far there is not any result on the existence of solutions for problem (1.1) for $p \neq 2, \sigma \neq 2$ and $h(t) \neq t$.

We consider the existence of weak solutions for a more general form of (1.2) of the following quasilinear Schrödinger equation

$$i\partial_t z = -\Delta_p z + Wz - |z|^{\sigma-2} z f(|z|^{\sigma}) - \kappa |z|^{\sigma-2} z h'(|z|^{\sigma}) \Delta_p h(|z|^{\sigma})$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with the Dirichlet boundary condition, in which $\kappa = 1$ and $f = f(t)$ is a Caratheodory function under some power growth with respect to t. At the same time we assume $W(x) \equiv W$ (a constant) to indicate that the solution stays at a constant potential level. Putting $z(x,t) = \exp(-iWt)u(x)$ we obtain the corresponding Eq (1.1) of elliptic type with a formal variational structure, see in Section 2.

For a deep insight into this problem one can find that a major difficulty of the problem (1.1) is that the functional corresponding to the equation is not well defined for all $u \in W_0^{1,p}(\Omega)$ if $p < N$. We generalized the method of a change of variables developed in [25] to overcome this difficulty, and make a slight different definition of weak solution. Then by a standard argument by critical point theory, we develop the existence of nontrivial solutions to our problem.

This article is organized as follows. In Section 2, we developed the properties of changing of variables and give the definition of weak solution for our problem; in Section 3, we give existence theorems of solutions; and in Section 4, we prove the main theorems.

2. Variational structure

We assume the following conditions on f:

(F_1) There exist constants $C > 0$, $\alpha > 1$ and $p < q < p^* := \frac{Np}{N-p}$, such that for any $x \in \Omega$ and $t \in \mathbb{R}$,

$$|f(x,t)| \leq C(1 + |t|^{\alpha q - 1}).$$

We assume that $h(t)$ satisfies the following conditions:

(h_0) There exists a constant $\beta > 0$, such that for $t \in (0, +\infty)$ and $\alpha > 1$ (the constant appeared in the assumption (F_1)),

$$0 \leq \frac{\sigma}{p} t H'(t) \leq (\alpha - 1) H(t),$$

where

$$H(t) = 1 + \sigma^{p-1} t^{\frac{p(\sigma-1)}{\sigma}} |h'(t)|^p$$

with h satisfying the following

$$\lim_{t\to+\infty} t^{1-\alpha/\sigma}h'(t) = \beta,$$
$$\lim_{t\to 0+} t^{1-1/\sigma}h'(t) = 0.$$

By a direct, but a bit of complex computation, we observe that (1.1) is the Euler-Lagrange equation associated to the energy functional

$$J(u) = \frac{1}{p}\int_\Omega \left(1 + \sigma^{p-1}|u|^{p(\sigma-1)}|h'(|u|^\sigma)|^p\right)|\nabla u|^p dx - \int_\Omega F(x,u)dx, \tag{2.1}$$

where $F(x,t) = \int_0^t f(x,t)dt$. But this functional J may be not well defined in $u \in W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\|^p = \int_\Omega |\nabla u|^p dx.$$

To overcome this difficulty, we generalize the changing of variables developed in [25]. That is $v = g^{-1}(u)$, where g is defined by the following ODE

$$\begin{cases} g'(t) = \left[1 + \sigma^{p-1}|g(t)|^{p(\sigma-1)}|h'(|g(t)|^\sigma)|^p\right]^{-1/p}, \ t \in [0,+\infty) \\ g(t) = -g(-t), \ t \in (-\infty, 0]. \end{cases}$$

It follows from the theory of ODE that g is uniquely defined in $\mathbb{R}$. We summarize the properties of g as follows.

Lemma 2.1. *The function g defined above satisfies the following properties:*

(1) $g(0) = 0$;
(2) *g is uniquely defined in $\mathbb{R}$, C^2 and invertible;*
(3) $0 < g'(t) \leq 1$, *for any $t \in \mathbb{R}$;*
(4) $g(t) \leq \alpha t g'(t) \leq \alpha g(t)$, *for any $t > 0$;*
(5) $g(t)/t \nearrow 1$, *as $t \to 0^+$;*
(6) $|g(t)| \leq |t|$, *for any $t \in \mathbb{R}$;*
(7) $g(t)/t^{1/\alpha} \nearrow K_0 = \left(\frac{\alpha}{\sigma^{1-1/p}\beta}\right)^{1/\alpha}$, *as $t \to +\infty$;*
(8) $|g(t)| \leq K_0|t|^{1/\alpha}$, *for any $t \in \mathbb{R}$;*
(9) $g^2(t) - g(t)g'(t)t \geq 0$, *for any $t \in \mathbb{R}$;*
(10) *There exists a positive constant L_0 such that*

$$|g(t)| \geq \begin{cases} L_0|t|^{1/\alpha}, & |t| \geq 1; \\ L_0|t|, & |t| \leq 1; \end{cases}$$

(11) $|g^{\alpha-1}(t)g'(t)| < {K_0}^\alpha$;
(12) $g''(t) < 0$ *for $t > 0$ and* $g''(t) > 0$ *for $t < 0$.*

Proof. The conclusions (1)–(3) are trivial. To establish the left hand side of the inequality (4), we need to show that, for any $t > 0$,

$$\left[1 + \sigma^{p-1}|g(t)|^{p(\sigma-1)}|h'(|g(t)|^\sigma)|^p\right]^{1/p} g(t) \leq \alpha t.$$

To prove this we study the function $l(t) : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$l(t) := \alpha t - \left[1 + \sigma^{p-1}|g(t)|^{p(\sigma-1)}|h'(|g(t)|^{\sigma})|^p\right]^{1/p} g(t).$$

It is clear that $l(0) = 0$, and from (h_0), we get

$$l'(t) = \alpha - 1 - \frac{\sigma^{p-1}|g|^{p(\sigma-1)}|h'(|g|^{\sigma})|^{p-2}\left[(\sigma-1)(h'(|g|^{\sigma}))^2 + \sigma h'(|g|^{\sigma})h''(|g|^{\sigma})|g|^{\sigma}\right]}{1 + \sigma^{p-1}|g|^{p(\sigma-1)}|h'(|g|^{\sigma})|^p} \geq 0.$$

Hence the left hand side of the inequality (4) is proved. The right hand side of the inequality (4) can be proved in a similar way.

It is easy to get (5) and (6) from (4). We give the proof of (7) by (h_0) and the principle of L'Hospital. In fact, since $g(t) \to +\infty$ as $t \to +\infty$, we get

$$\begin{aligned}
\lim_{t\to+\infty} \frac{g(t)}{t^{1/\alpha}} &= \lim_{t\to+\infty} \left(\frac{g^{\alpha}(t)}{t}\right)^{1/\alpha} = \lim_{t\to+\infty} \left(\frac{\alpha g^{\alpha-1}(t)g'(t)}{1}\right)^{1/\alpha} \\
&= \lim_{t\to+\infty} \left(\frac{\alpha^p g^{p(\alpha-1)}(t)}{1 + \sigma^{p-1}|g(t)|^{p(\sigma-1)}|h'(|g(t)|^{\sigma})|^p}\right)^{1/\alpha p} \\
&= \lim_{y\to+\infty} \left(\frac{\alpha^p y^{\frac{p(\alpha-1)}{\sigma}}}{1 + \sigma^{p-1}y^{\frac{p(\sigma-1)}{\sigma}}|h'(y)|^p}\right)^{1/\alpha p} \\
&= \left(\frac{\alpha}{\sigma^{1-1/p}\beta}\right)^{1/\alpha} = K_0.
\end{aligned}$$

Then (7) is proved by (4).

It is easy to get (8) by (7) and (9) by (4). The inequalities in (10) are trivial and (11) is from (4) and (8).

For (12), it is easy to see that

$$g''(t) = \frac{\sigma^{p-1}|g|^{p(\sigma-1)-2}g|h'(|g|^{\sigma})|^{p-2}\left[(\sigma-1)(h'(|g|^{\sigma}))^2 + \sigma h'(|g|^{\sigma})h''(|g|^{\sigma})|g|^{\sigma}\right]}{-[1 + \sigma^{p-1}|g|^{p(\sigma-1)}|h'(|g|^{\sigma})|^p]^{1+2/p}}.$$

So the conclusion of (12) is true. $\square$

After the changing of variables by $u = g(v)$, we obtain the following functional

$$\Phi(v) := J(g(v)) = \frac{1}{p}\int_{\Omega} |\nabla v|^p dx - \int_{\Omega} F(x, g(v))dx \tag{2.2}$$

which is well defined on the space $W_0^{1,p}(\Omega)$. It belongs to $C^1(W_0^{1,p}(\Omega); \mathbb{R})$ by the assumption (F_1) and Lemma 2.1. Then for all $w \in W_0^{1,p}(\Omega)$, we get

$$\langle \Phi'(v), w\rangle = \int_{\Omega} |\nabla v|^{p-2}\nabla v \nabla w dx - \int_{\Omega} f(x, g(v))g'(v)w dx.$$

Thus the critical point of Φ is the weak solution of the problem

$$\begin{cases} -\Delta_p v = f(x, g(v))g'(v), \text{ in } \Omega, \\ \quad v = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.3}$$

By setting $v = g^{-1}(u)$, it is easy to see that Eq (2.3) is equivalent to our problem (1.1), which takes $u = g(v)$ as its solution.

Motivated by the above, we give the following definition of the weak solution for problem (1.1).

Definition 2.1. *We say u is a weak solution of problem (1.1), if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a critical point of the following functional corresponding to problem (2.3):*

$$\Phi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v))dx.$$

3. Main theorems

For the rest of this paper, we make use of the following notations: X denotes the Sobolev space $W_0^{1,p}(\Omega)$; X^* denotes the conjugate space of X; $\langle \cdot, \cdot \rangle$ is the dual pairing on the space X^* and X; by $\to$ (resp. $\rightharpoonup$) we mean strong (resp. weak) convergence; $|\Omega|$ denotes the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$; $L^p(\Omega)$ denotes Lebesgue space with the norm $|\cdot|_p$; C, C_1, C_2,...denote (possibly different) positive constants.

It is well known (see [26]) that the p-homogeneous boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega \end{cases}$$

has the first eigenvalue $\lambda_1 > 0$, which is simple and has an associated eigenfunction which is positive in Ω. It is also known that λ_1 is an isolated point of $\sigma(-\Delta_p)$, the spectrum of $-\Delta_p$, which contains at least an increasing eigenvalue sequence obtained by Lusternik-Schnirelman theory.

Let $V = \text{span}\{\phi_1\}$ be the one-dimensional eigenspace associated to λ_1, where $\phi_1 > 0$ in Ω and $\|\phi_1\| := (\int_\Omega |\nabla\phi_1|^p dx)^{1/p} = 1$. Taking the subspace $Y \subset X$ completing V such that $X = V \oplus Y$, there exists $\overline{\lambda} > \lambda_1$ such that

$$\int_\Omega |\nabla u|^p dx \geq \overline{\lambda} \int_\Omega |u|^p dx, \ u \in Y.$$

When $p = 2$, one can take $\overline{\lambda} = \lambda_2$, the second eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Let us recall the following useful notion from nonlinear operator theory. If X is a Banach space and $A : X \to X^*$ is an operator, we say that A is of type (S_+), if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \rightharpoonup x$ weakly in X, and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, we have that $x_n \to x$ in X.

Let us consider the map $A : X \to X^*$, corresponding to $-\Delta_p u$ with Dirichlet boundary data, defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \ \forall u, v \in X. \tag{3.1}$$

Then we have the following result:

Lemma 3.1. *[27] The map $A : X \to X^*$ defined by (3.1) is continuous and of type (S_+).*

We shall use Palais-Smale compactness condition:

Definition 3.1. *Let X be a Banach space. Let $\Phi \in C^1(X;\mathbb{R})$, we say Φ satisfies (PS) (resp. $(PS)_c$) condition if any sequence $\{u_n\} \subset X$ for which $\Phi(u_n)$ is bounded (resp. $\Phi(u_n) \to c$) and $\Phi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.*

Lemma 3.2. *[28] (Mountain Pass Theorem) Let X be a Banach space, and let $f \in C^1(X, \mathbb{R})$ satisfy $f(0) = 0$ and (PS) condition. Assume*
(1) There exists a $\delta > 0$ such that $f|_{\|u\|=\delta} > 0$;
(2) There is a $v \in X$ satisfying $\|v\| > \delta$ and $\Phi(v) < 0$.
Then f has a critical value c characterized by $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$,
where $\Gamma = \{\gamma \in C([0, 1], X)|\gamma(0) = 0,\ \gamma(1) = v\}$.

We shall also assume the following condition f:
(F_2) There exist $p < \theta$, $0 < \eta < \frac{\alpha\theta\lambda_1(1/p-1/\theta)}{K_0{}^{\alpha p}}$ and $\mu \in [0, p]$ such that

$$\liminf_{|t|\to\infty} \frac{f(x,t)t - \alpha\theta F(x,t)}{|t|^{\alpha\mu}} > -\eta \text{ uniformly in } x \in \Omega;$$

(F_3) There exists a constant $M > 0$ such that $f(x,t)t > 0$ for $|t| \geq M$.

Remark 3.1. *The Ambrosetti-Rabinowitz type growth condition "There exists $M > 0$, such that $\alpha\theta F(x,t) \leq f(x,t)t$, $|t| \geq M$, $x \in \Omega$" implies that $|t| \geq M$, $x \in \Omega$,*

$$f(x,t)t - \alpha\theta F(x,t) \geq 0 > -\eta|t|^{\alpha\mu}.$$

Then

$$\liminf_{|t|\to\infty} \frac{f(x,t)t - \alpha\theta F(x,t)}{|t|^{\alpha\mu}} > -\eta,$$

uniformly in $x \in \Omega$. Hence (F_2) is weaker than Ambrosetti-Rabinowitz type growth condition.

Our main results are the following.

Theorem 3.1. *Assume (F_1)–(F_3) and the following*
(F_4) $\limsup_{|t|\to 0} \frac{pK_0^{\alpha p}F(x,t)}{|t|^{\alpha p}} < \lambda_1$ uniformly in $x \in \Omega$;
(F_5) $\liminf_{|t|\to\infty} \frac{pL_0^{\alpha p}F(x,t)}{|t|^{\alpha p}} > \lambda_1$ uniformly in $x \in \Omega$,
hold, where K_0 and L_0 are constants appeared in Lemma 2.1. Then (1.1) has at least one nontrivial weak solution in the sense of Definition 2.1.

Theorem 3.2. *Assume (F_1)–(F_3), (F_5) and the following*
(F_6) $f(x, -t) = -f(x,t)$, $x \in \Omega$, $|t| \leq r$,
hold. Then (1.1) has a sequence of weak solutions $\{\pm u_k\}_{k=1}^{\infty}$ such that $\Phi(\pm u_k) \to +\infty$ as $k \to +\infty$ in the sense of Definition 2.1.

Theorem 3.3. *Assume (F_1), and the following*
(F_7) There exist $r > 0$, $\hat{\lambda}_1, \hat{\lambda}_2 \in (\lambda_1, \overline{\lambda})$ such that $\hat{\lambda}_1 < \hat{\lambda}_2$ and $|t| \leq r$ implies $\hat{\lambda}_1|t|^p \leq pF(x,t) \leq \hat{\lambda}_2|t|^p$, $x \in \Omega$, $t \in \mathbb{R}$;

(F_8) $\limsup_{|t|\to\infty} \frac{pK_0^{\alpha p} F(x,t)}{|t|^{\alpha p}} < \lambda_1$ *uniformly in* $x \in \Omega$,
hold, where K_0 *is the constant appeared in Lemma 2.1. Then (1.1) has at least two nontrivial weak solutions in the sense of Definition 2.1.*

Theorem 3.4. *Assume* (F_1), (F_7), *and the following*
(F_9) $\lim_{|t|\to\infty} \frac{pK_0^{\alpha p} F(x,t)}{|t|^{\alpha p}} = \lambda_1$ *uniformly in* $x \in \Omega$;
(F_{10}) $\lim_{|t|\to\infty}(f(x,t)t - \alpha pF(x,t)) = +\infty$ *uniformly in* $x \in \Omega$,
hold, where K_0 *is the constant appeared in Lemma 2.1. Then (1.1) has at least two nontrivial weak solutions in the sense of Definition 2.1.*

4. The proof of the theorems

We decompose the proof of Theorem 3.1 into the following three lemmas.

Lemma 4.1. *Under condition* (F_1), *any bounded sequence* $\{v_n\} \subset X$ *such that* $\Phi'(v_n) \to 0$ *in* X^*, *as* $n \to \infty$, *has a convergent subsequence.*

Proof. Since $\{v_n\}$ is bounded, by the self-reflexive property of X, there exists a subsequence of $\{v_n\}$ (we may also denote it by $\{v_n\}$) and $v \in X$, such that $v_n \rightharpoonup v$. From (F_1), Lemma 2.1, Hölder inequality and the compact Sobolev embedding, we can see that as $n \to \infty$

$$\begin{aligned}
&\left|\int_\Omega f(x, g(v_n))g'(v_n)(v_n - v)dx\right| \\
&\leq \int_\Omega \left(1 + |g(v_n)|^{\alpha q-1} g'(v_n)\right)|v_n - v|dx \\
&\leq \int_\Omega C_1(1 + |v_n|^{q-1})|v_n - v|dx \\
&\leq C_2\left(\int_\Omega (1 + |v_n|^{q-1})^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |v_n - v|^q dx\right)^{\frac{1}{q}} \to 0.
\end{aligned} \tag{4.1}$$

By (4.1) and the following

$$|\langle \Phi'(v_n), v_n - v\rangle| \leq C\|\Phi'(v_n)\|_{X^*} \to 0,$$

we get

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)dx \to 0.$$

Then we conclude that $v_n \to v$ by the property of (S_+) in Lemma 3.1. □

Lemma 4.2. *Under assumptions* (F_2) *and* (F_3), *any sequence* $\{v_n\} \subset X$ *such that* $|\Phi(v_n)| \leq B$, *and* $\Phi'(v_n) \to 0$ *in* X^*, *as* $n \to \infty$, *is bounded in* X.

Proof. Suppose that $\{v_n\} \subset X$, $|\Phi(v_n)| \leq B$, and $\Phi'(v_n) \to 0$ in X^* as $n \to \infty$. By (F_2), there exists $C_1 > 0$ such that

$$f(x,t)t - \alpha\theta F(x,t) > -\eta|t|^{\alpha\mu},\ |t| > C_1,\ x \in \Omega.$$

Let $C := \sup_n \Phi(v_n)$. From Lemma 2.1 (4), (9) and (10), (F_2) and (F_3) we have

$$\begin{aligned}
&C + 1 + \|v_n\| \\
&\geq \Phi(v_n) - \frac{1}{\theta}\langle \Phi'(v_n), v_n\rangle \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|v_n\|^p + \int_\Omega \frac{1}{\theta} f(x, g(v_n))g'(v_n)v_n - F(x, g(v_n))dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|v_n\|^p + \frac{1}{\alpha\theta}\int_\Omega f(x, g(v_n))g(v_n) - \alpha\theta F(x, g(v_n))dx - C_2 \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|v_n\|^p - \frac{\eta}{\alpha\theta}\int_\Omega |g(v_n)|^{\alpha\mu}dx - C_3 \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|v_n\|^p - \frac{\eta K_0^{\alpha\mu}}{\alpha\theta}\int_\Omega |v_n|^{\mu}dx - C_4 \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|v_n\|^p - \frac{\eta K_0^{\alpha\mu}}{\alpha\theta}\int_\Omega |v_n|^{p}dx - C_5 \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta} - \frac{\eta K_0^{\alpha\mu}}{\alpha\theta\lambda_1}\right)\|v_n\|^p - C_6.
\end{aligned}$$

Noticing that $\left(\frac{1}{p} - \frac{1}{\theta} - \frac{\eta K_0^{\alpha\mu}}{\alpha\theta\lambda_1}\right) > 0$, we obtain the boundedness of $\{v_n\}$ in X. □

Lemma 4.3. *Assume that $(F_1), (F_4)$ and (F_5) hold. Then the functional Φ satisfies:*
(1) There exists a $\delta > 0$ such that $\Phi|_{\|v\|=\delta} > 0$;
(2) There is an $\overline{e} \in V$ satisfying $\|\overline{e}\| > \delta$ and $\Phi(\overline{e}) < 0$.

Proof. We obtain from the assumptions (F_1) and (F_4) that for some small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$F(x, t) \leq \frac{\lambda_1 - \varepsilon}{pK_0^{\alpha p}}|t|^{\alpha p} + C_\varepsilon |t|^{\alpha q},\ \forall x \in \Omega,\ t \in \mathbb{R}.$$

Taking $v \in X$, using the inequality $\int_\Omega |\nabla v|^p dx \geq \lambda_1 \int_\Omega |v|^p dx$, the Sobolev inequality $|v|_q^q \leq \tau\|v\|^q$ and Lemma 2.1, we get

$$\begin{aligned}
\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v))dx \\
&\geq \frac{1}{p}\|v\|^p - \int_\Omega \frac{\lambda_1 - \varepsilon}{pK_0^{\alpha p}}|g(v)|^{\alpha p} + C_\varepsilon |g(v)|^{\alpha q}dx \\
&\geq \frac{1}{p}\|v\|^p - \int_\Omega \frac{\lambda_1 - \varepsilon}{p}|v|^p + C_1|v|^q dx \\
&\geq \frac{1}{p}\|v\|^p - \frac{\lambda_1 - \varepsilon}{p\lambda_1}\|v\|^p - C_2\|v\|^q \\
&= \frac{\varepsilon}{p\lambda_1}\|v\|^p - C_2\|v\|^q.
\end{aligned}$$

Then there exists a $\delta > 0$ such that $\Phi|_{\|v\|=\delta} > 0$.

By (F_5), there exist $\varepsilon > 0$ and $T > 0$ such that

$$F(x,t) \geq \frac{\lambda_1 + \varepsilon}{pL_0^{\alpha p}}|t|^{\alpha p}, \ |t| > T, \ x \in \Omega.$$

For any $v \in V$, we can denote $v = t\phi_1$, $t \in \mathbb{R}$. Then

$$\begin{aligned}
\Phi(t\phi_1) &= \frac{1}{p}\int_\Omega |\nabla t\phi_1|^p dx - \int_\Omega F(x, g(t\phi_1))dx \\
&\leq \frac{1}{p}|t|^p - \int_\Omega \frac{\lambda_1 + \varepsilon}{pL_0^{\alpha p}}|g(t\phi_1)|^{\alpha p} dx + C_3|\Omega| \\
&\leq \frac{1}{p}|t|^p - \int_\Omega \frac{\lambda_1 + \varepsilon}{p}|t\phi_1|^p dx + C_4 \\
&= \frac{1}{p}|t|^p - \frac{\lambda_1 + \varepsilon}{p\lambda_1}|t|^p + C_4 \\
&= -\frac{\varepsilon}{p\lambda_1}|t|^p + C_4 \to -\infty \text{ as } t \to \infty.
\end{aligned}$$

Hence there exists $\bar{e} = \bar{t}\phi_1 \in X$, $\|\bar{e}\| > \delta$ such that $\Phi(\bar{e}) < 0$. □

Proof of Theorem 3.1. Obviously we have $\Phi(0) = 0$. By Lemmas 4.1 and 4.2, we know that the functional Φ satisfies the (PS) condition. Sum up the above fact, Theorem 3.1 follows from Lemmas 4.1–4.3 by Lemma 3.2. □

We will use the Fountain Theorem to prove Theorem 3.2. Since X is a reflexive and separable Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \ X^* = \overline{span\{e_j^* : j = 1, 2, ...\}}$$

in which

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We will write $X_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$.

Lemma 4.4. *[28] (Fountain Theorem) Assume*
(A_1) *X is a Banach space, $\Phi \in C^1(X, \mathbb{R})$ is an even functional;*
(A_2) *Φ satisfies $(PS)_c$ condition for every $c > 0$,*
and for each $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that
(A_3) *$a_k := \max_{v \in Y_k, \|v\| = \rho_k} \Phi(v) \leq 0$;*
(A_4) *$b_k := \inf_{v \in Z_k, \|v\| = r_k} \Phi(v) \to +\infty$ as $k \to +\infty$.*
Then Φ admits a sequence of critical values tending to $+\infty$.

Lemma 4.5. *[28] Denote $\beta_k = \sup\{|v|_q : \|v\| = 1, \ v \in Z_k\}$. Then $\lim_{k \to +\infty} \beta_k = 0$.*

Proof of Theorem 3.2. Obviously Φ is even by (F_6). Further more, by Lemmas 4.1 and 4.2, Φ satisfies the $(PS)_c$ condition. We need only to prove that there exist $\rho_k > r_k > 0$ such that condition (A_3) and (A_4) in Lemma 4.4 hold.

(A_3) From assumption (F_5) and Lemma 2.1, there exist $\varepsilon > 0$ and $T > 0$ large enough such that

$$F(x, g(t)) \geq \frac{\lambda_1 + \varepsilon}{pL_0^{\alpha p}} |g(t)|^{\alpha p} \geq \frac{\lambda_1 + \varepsilon}{p} |t|^p, \ |t| > T, \ x \in \Omega.$$

For any $w \in Y_k$ with $\|w\| = 1$ and $\rho_k = t > 1$, we have

$$\begin{aligned} \Phi(tw) &= \frac{1}{p} \int_\Omega |\nabla tw|^p dx - \int_\Omega F(x, g(tw)) dx \\ &\leq \frac{1}{p} t^p - \frac{\lambda_1 + \varepsilon}{p} \int_\Omega |tw|^p dx + C_1 \\ &= \frac{1}{p} t^p - \frac{\lambda_1 + \varepsilon}{p\lambda_1} t^p + C_1 \\ &= -\frac{\varepsilon}{p\lambda_1} t^p + C_1 \to -\infty \ as \ t \to \infty. \end{aligned}$$

(A_4) After integrating, we obtain from (F_1) that the existence of $C_2 > 0$ such that

$$|F(x, t)| \leq C_2 (1 + |t|^{\alpha q}). \tag{4.2}$$

Let $v \in Z_k$, $\|v\| = r_k := (C_2 q K_0^{\alpha q} \beta_k^q)^{1/p-q}$, in which K_0 is the constant appeared in Lemma 2.1. By (4.2), Lemmas 2.1 and (4.5), we get

$$\begin{aligned} \Phi(v) &= \frac{1}{p} \int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v)) dx \\ &\geq \frac{1}{p} \|v\|^p - C_2 \int_\Omega |g(v)|^{\alpha q} dx - C_2 |\Omega| \\ &\geq \frac{1}{p} \|v\|^p - C_2 K_0^{\alpha q} \int_\Omega |v|^q dx - C_2 |\Omega| \\ &\geq \frac{1}{p} \|v\|^p - C_2 K_0^{\alpha q} \beta_k^q \|v\|^q - C_2 |\Omega| \\ &= \left(\frac{1}{p} - \frac{1}{q} \right) (C_2 q K_0^{\alpha q} \beta_k^q)^{\frac{p}{p-q}} - C_2 |\Omega| \to +\infty, \ as \ k \to +\infty. \end{aligned}$$

Then the conclusion of Theorem 3.2 is obtained by Lemma 4.4. □

Remark 4.1. *We can obtain the existence of a sequence solutions by symmetric mountain pass theorem under similar odd condition* (F_6).

For the proof of Theorems 3.3 and 3.4, we need the following lemma from [29].

Lemma 4.6. *Let X be a Banach space with a direct sum decomposition* $X = X_1 \oplus X_2$*, with* $k = dimX_2 < \infty$*, let f be a* C^1 *functional on X with* $f(0) = 0$*, satisfying* (PS) *condition. Assume that, for some* $\rho > 0$*,*
(1) $f(u) \geq 0$*, for* $u \in X_1$, $\|u\|_X \leq \rho$*;*
(2) $f(u) \leq 0$*, for* $u \in X_2$, $\|u\|_X \leq \rho$*.*
Assume also that f is bounded below and $\inf_X f < 0$*. Then f has at least two nonzero critical points.*

Lemma 4.7. *Under assumptions* (F_1) *and* (F_8) *(or substitute* (F_9) *and* (F_{10}) *for* (F_8)*), the functional* Φ *is coercive in X, that is,* $\Phi(v) \to +\infty$ *as* $\|v\| \to \infty$.

Proof. (1) Let (F_8) holds. From (F_1) and (F_8) we can see for some $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$F(x,t) \le \frac{\lambda_1 - \varepsilon}{pK_0^{\alpha p}}|t|^{\alpha p} + C_\varepsilon,\ t \in \mathbb{R},\ x \in \Omega.$$

So by Sobolev inequality, we get for $v \in X$,

$$\begin{aligned}\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v))dx \\ &\ge \frac{1}{p}\|v\|^p - \frac{\lambda_1 - \varepsilon}{pK_0^{\alpha p}}\int_\Omega |g(v)|^{\alpha p} dx - C_\varepsilon|\Omega| \\ &\ge \frac{1}{p}\|v\|^p - \frac{\lambda_1 - \varepsilon}{p}\int_\Omega |v|^p dx - C_\varepsilon|\Omega| \\ &\ge \frac{\varepsilon}{p\lambda_1}\|v\|^p - C_\varepsilon|\Omega| \to +\infty,\ \text{as } \|v\| \to +\infty.\end{aligned}$$

(2) Let (F_9) and (F_{10}) hold. Write $F(x,t) = \frac{\lambda_1}{pK_0^{\alpha p}}|t|^{\alpha p} + H(x,t)$ and $f(x,t) = \frac{\alpha\lambda_1}{K_0^{\alpha p}}|t|^{\alpha p-2}t + h(x,t)$. Then

$$\lim_{|t|\to\infty} \frac{pK_0^{\alpha p}H(x,t)}{|t|^{\alpha p}} = 0$$

and

$$\lim_{|t|\to\infty} (h(x,t)t - \alpha p H(x,t)) = +\infty \text{ uniformly in } x \in \Omega.$$

It follows that for any $M > 0$, there is a $T_M > 0$ such that

$$h(x,t)t - \alpha p H(x,t) \ge M,\ |t| \ge T_M,\ x \in \Omega.$$

Integrating the equality

$$\frac{d}{dt}\left(\frac{H(x,t)}{|t|^{\alpha p}}\right) = \frac{h(x,t)t - \alpha p H(x,t)}{|t|^{\alpha p} t}$$

over the interval $[t_1, t_2] \subset [T_M, +\infty)$, we have

$$\frac{H(x,t_2)}{t_2^{\alpha p}} - \frac{H(x,t_1)}{t_1^{\alpha p}} \ge \frac{M}{\alpha p}\left(\frac{1}{t_1^{\alpha p}} - \frac{1}{t_2^{\alpha p}}\right).$$

Letting $t_2 \to +\infty$, we have $H(x,t) \le -\frac{M}{\alpha p}$ for $t \ge T_M$, $x \in \Omega$. In a similar way, we have $H(x,t) \le -\frac{M}{\alpha p}$ for $t \le -T_M$, $x \in \Omega$. So we can see

$$\lim_{|t|\to\infty} H(x,t) \to -\infty \text{ uniformly in } x \in \Omega. \tag{4.3}$$

We suppose on the contrary, there exists a sequence $\{v_n\} \subset X$ such that $\|v\| \to \infty$ as $n \to \infty$, but $\Phi(v_n) \le C$ for some constant $C \in \mathbb{R}$. Set $w_n = \frac{v_n}{\|v_n\|}$, then up to a subsequence, we assume there is some $w_0 \in X$ such that $w_n \rightharpoonup w_0$ in X, $w_n \to w_0$ in $L^p(\Omega)$, and $w_n(x) \to w_0(x)$ for a.e. $x \in \Omega$. Moreover, we have the following:

$$\begin{aligned}\frac{C}{\|v_n\|^p} \geq \frac{\Phi(v_n)}{\|v_n\|^p} &= \frac{1}{p\|v_n\|^p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{\|v_n\|^p}\int_\Omega F(x,g(v_n))dx \\ &\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)dx - \frac{1}{\|v_n\|^p}\int_\Omega H(x,g(v_n))dx \\ &\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)dx + \frac{M|\Omega|}{\alpha p\|v_n\|^p} - \frac{1}{\|v_n\|^p}\int_{|g(v_n)|\leq T_M} H(x,g(v_n))dx \\ &\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)dx - \frac{C_1}{\|v_n\|^p},\end{aligned}$$

which implies

$$\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^p dx \leq \lambda_1 \int_\Omega |w_0|^p dx. \tag{4.4}$$

By the weakly semicontinuous property of the norm and the Sobolev inequality again, we have the converse inequality of (4.4),

$$\lambda_1 \int_\Omega |w_0|^p dx \leq \int_\Omega |\nabla w_0|^p dx \leq \liminf_{n\to\infty}\int_\Omega |\nabla w_n|^p dx \leq \limsup_{n\to\infty}\int_\Omega |\nabla w_n|^p dx. \tag{4.5}$$

By (4.4) and (4.5), $\int_\Omega |\nabla w_0|^p dx = \lambda_1 \int_\Omega |w_0|^p dx$ and $w_n \to w_0$ in X with $\|w_0\| = 1$. Hence $w_0 = \pm\phi_1$. Take $w_0 = \phi_1$. Then $v_n \to +\infty$ *a.e.* $x \in \Omega$, which implies $H(x,g(v_n)) \to -\infty$ by (4.3). So we have

$$\begin{aligned}C \geq \Phi(v_n) &= \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \int_\Omega F(x,g(v_n))dx \\ &\geq \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \int_\Omega H(x,g(v_n))dx \\ &\geq -\int_\Omega H(x,g(v_n))dx \to +\infty, \text{ as } n\to\infty,\end{aligned}$$

which is a contradiction. So we have Φ is coercive in X. □

Lemma 4.8. *Under assumptions* (F_1) *and* (F_7)*, for the decomposition of the space* $X = V \oplus Y$*, there is a small ball* B_ρ *with the center at 0 and small radius* $\rho > 0$ *such that*
(1) $\Phi(v) \leq 0$*, for* $v \in V$*,* $v \in B_\rho$*;*
(2) $\Phi(v) \geq 0$*, for* $v \in Y$*,* $v \in B_\rho$*.*

Proof. (1) Take $v \in V$, we can see that $\|v\| \leq \rho$ implies $|g(v)| \leq r$, $\forall x \in \Omega$ for $\rho > 0$ small enough. So by (F_7), for $\|v\| \leq \rho$,

$$\begin{aligned}\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x,g(v))dx \\ &= \frac{\lambda_1}{p}\int_\Omega |v|^p dx - \int_\Omega F(x,g(v))dx \\ &= \int_{|g(v)|\leq r}\left(\frac{\lambda_1}{p}|v|^p - F(x,g(v))\right)dx \\ &< \int_{|g(v)|\leq r}\left(\frac{\hat{\lambda}_1}{p}|g(v)|^p - F(x,g(v))\right)dx \\ &\leq 0.\end{aligned}$$

(2) Take $v \in Y$. From Lemma 2.1, assumptions (F_1) and (F_7), Sobolev embedding and the definition of $\bar{\lambda}$, we have the following

$$\begin{aligned}
\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v))dx \\
&= \frac{1}{p}\int_\Omega (|\nabla v|^p - \hat{\lambda}_2 |v|^p)dx - \int_\Omega \left(F(x, g(v)) - \frac{\hat{\lambda}_2}{p}|v|^p\right)dx \\
&\geq \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\bar{\lambda}}\right)\|v\|^p - \int_{|g(v)|>r}\left(F(x, g(v)) - \frac{\hat{\lambda}_2}{p}|v|^p\right)dx \\
&\geq \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\bar{\lambda}}\right)\|v\|^p - C_1\int_{|g(v)|>r} |g(v)|^{\alpha q}dx \\
&\geq \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\bar{\lambda}}\right)\|v\|^p - C_2\int_{|g(v)|>r} |v|^{q}dx \\
&\geq \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\bar{\lambda}}\right)\|v\|^p - C_3\|v\|^q.
\end{aligned}$$

So we can derive, when $v \in Y$ and $\|v\| \leq \rho$ for $\rho > 0$ small, that $\Phi(v) \geq 0$, which completes the proof. □

Proof of Theorems 3.3 and 3.4. Obviously we have $\Phi(0) = 0$. Further more, by Lemmas 4.1 and 4.7, Φ is coercive and satisfies the (PS) condition. Hence Φ is bounded below. From Lemma 4.8 (1) we have $\inf_X \Phi < 0$. By summing up the above fact the conclusion follows from Lemmas 4.7 and 4.8 by Lemma 4.6. □

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Conflict of interest

The author declares no conflict of interest.

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