Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9647–9659.
DOI: 10.3934/math.2021562
Received: 03 January 2021
Accepted: 03 June 2021
Published: 28 June 2021

Research article

Perturbed uncertain differential equations and perturbed reflected canonical process

Ma Yuanbin and Li Zhi*

School of Information and Mathematics, Yangtze University, Jingzhou 434023, China

* **Correspondence:** Email: lizhi_csu@126.com.

Abstract: In this paper, we consider a class of perturbed uncertain differential equations, which is a type of differential equations driven by canonical process. By the reflection principle and a successive approximation method, we obtain the existence and uniqueness of the solution to the considered equations. As an application, we establish the existence and uniqueness of some perturbed reflected canonical process.

Keywords: perturbed uncertain differential equations; canonical process; reflected diffusions; existence and uniqueness
Mathematics Subject Classification: 26A33, 60G15, 60H15

1. Introduction

There now exists a considerable body of literature devoted to the study of 'perturbed' versions of familiar stochastic and deterministic equations. An example is Carmona et al. [1] and Norris et al. [2] investigated the following doubly perturbed Brownian motion

$$x(t) = B(t) + \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s). \tag{1.1}$$

We study 'perturbed canonical process', that can be, loosely speaking, described as follows: they behave exactly as a canonical process has stationary and independent increments except when they hit their past maximum or/and minimum where they get an extra 'push'. Many researchers have devoted themselves to studying the perturbed process (see [3–6]). Following them, Doney and Zhang [7] studied the following singly perturbed Skorohod equations

$$x(t) = x_0 + \int_0^t g(s, x(s))dB(s) + \int_0^t f(s, x(s))ds + \alpha \max_{0\le s\le t} x(s). \tag{1.2}$$

The authors proved the existence and uniqueness of the solution for (1.2) when the coefficients b, σ is the global Lipschitz.

On the other hand, to describe the evolution of the uncertain phenomenon, Liu [8] proposed uncertain process and designed a Liu process [9]. Meanwhile, Liu [9] introduced uncertain calculus to handle the integral and differential with respect to an uncertain process. Uncertain differential equations driven by a Liu process, which were first proposed by Liu [8], have attracted the increasing attentions due to the wide applications in in many fields such as finance ([10]), optimal control ([11, 12]), differential game ([13]), population growth ([14]), heat conduction ([15]), string vibration ([16]), spring vibration ([17]), and epidemic spread ([18]).

As far as we known, there is no result on the perturbed uncertain differential equations. Motivated by the need of the applications and in connection with the above discussions, it is worthwhile to develop some techniques and methods to explore the perturbed uncertain differential equations. To this end, in this paper, we will investigate the following perturbed uncertain differential equations,

$$X_t = X_0 + \int_0^t \sigma(X_s)dC_s + \int_0^t b(X_s)ds + \alpha \max_{0\le s\le t} X_s, \tag{1.3}$$

where C is a canonical process starting from 0, $\alpha < 1$ is a real constant, $\sigma(x)$, $b(x)$ be Lipschitz continuous function on R. By the reflection principle and a successive approximation method, we obtain the existence and uniqueness of the solution to the considered equations.

Our other main aim is to deal with the analogous question for a general diffusion. Specifically we study the equation

$$X_t = x + \int_0^t \sigma(X_s)dC_s + \alpha \max_{0\le s\le t} X_s + L_t, \tag{1.4}$$

where σ is a Lipschitz continuous function on R, $\alpha < 1$ is a real constant, $x \ge 0$, and L_t denotes a local time at zero of X. Since the cases $x = 0$ and $x > 0$ are quite different, we will treat them separately. Finally we exploit our result on the uncertain differential equation together with Picard iteration to establish existence and uniqueness of a solution to (1.4).

The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are recalled in Section 2. The method to solve perturbed uncertain differential equations is presented in Section 3. An existence and uniqueness theorem for perturbed reflected canonical process is proved in Section 4. At last, a brief summary is given in Section 5.

2. Preliminaries

In this section, we will introduce some foundational concepts and properties of uncertainty theory, which will be used throughout this paper.

Theorem 2.1. *([19]) An uncertain process C_t is said to be a canonical process if*

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;

(ii) C_t has stationary and independent increments;

(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and vartiance t^2.

It is clear that a canonical process C_t is a stationary independent increment process with normal uncertainary distribution

$$\Phi_t(x) = \left(1 + exp(-\frac{\pi x}{\sqrt{3}t})\right)^{-1} \tag{2.1}$$

and inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{2.2}$$

Theorem 2.2. *([20]) Let C_t be a canonical process. Then its expected value is*

$$E[C_t] = 0 \tag{2.3}$$

and variance is

$$V[C_t] = t^2. \tag{2.4}$$

In other words, Liu process C_t is a normal uncertain process with expected value 0 and variance t^2, i.e., $C_t \sim \mathcal{N}(0, t)$.

Theorem 2.3. *([21]) Set $W_0 = \{f \in C([0,\infty) \to R); f(0) = 0\}$ and $W^+ = \{f \in C([0,\infty) \to R); f(t) \geq 0 \ for\ all\ t \geq 0\}$. Given $f \in W_0$ and $0 \leq \alpha < 1$, there exist unique $g \in W^+$ and $h \in W^+$ such that*

(i) $g(t) = f(t) + \alpha \max_{0\leq s\leq t} g(s) + h(t)$;

(ii) $h(0) = 0$ *and* $t \to h(t))$ *is non-decreasing;*

(iii) $\int_0^t \chi\{g(s) = 0\} dh(s) = h(t)$.

(g, h) is called a solution to the perturbed Skorohod equation for the function f.

Lemma 2.1. *([22]) Suppose that C_t is a canonical process, and X_t is an integrable uncertain process on $[a, b]$ with respect to t. Then the inequality*

$$\left| \int_a^b X_t(\gamma) dC_t(\gamma) \right| \leq K(\gamma) \int_a^b | X_t(\gamma) | dt$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

3. Perturbed diffusion processes

In this section, we assume that $\sigma(x)$, $b(x)$ be Lipschitz continuous function on R, i.e., there exists a constant c such that

$$|\sigma(x) - \sigma(y)| \leq c|x - y| \tag{3.1}$$

$$|b(x) - b(y)| \leq c|x - y| \tag{3.2}$$

and linear growth condition

$$|\sigma(x)| + |b(x)| \leq c(1 + |x|). \tag{3.3}$$

For $\alpha < 1$, consider the following uncertain differential equation:

$$X_t = X_0 + \int_0^t \sigma(X_s) dC_s + \int_0^t b(X_s) ds + \alpha \max_{0\leq s\leq t} X_s. \tag{3.4}$$

Theorem 3.1. *Assume that the random variable X_0 is independent of L. There exists a unique, continuous, $\mathscr{F}$-adpted solution X_t, $t \geq 0$ to the uncertain differential (3.4) for any $T > 0$ if the coefficients $\sigma(X_t)$ and $b(X_t)$ satisfy the assumption (3.1)-(3.3) for some constants $c > 0$.*

We construct the solution by iteration. Let

$$X_t^0 = \frac{X_0}{1-\alpha}, \ 0 \leq t < \infty. \tag{3.5}$$

For $n \geq 0$ define X_t^{n+1} to be the unique, continuous, adapted solution to the following equation:

$$X_t^{n+1} = X_0 + \int_0^t \sigma(X_s^n) dC_s + \int_0^t b(X_s^n) ds + \alpha \max_{0\leq s\leq t} X_s^{n+1}. \tag{3.6}$$

Such a solution exists and can be expressed explicitly as

$$X_t^{n+1} = \frac{X_0}{1-\alpha} + \int_0^t \sigma(X_s^n) dC_s + \int_0^t b(X_s^n) ds + \frac{\alpha}{1-\alpha} \max_{0\leq s\leq t} \Big(\int_0^s \sigma(X_u^n) dC_u + \int_0^s b(X_u^n) du \Big). \tag{3.7}$$

This is a consequence of the reflection principle. We will show that X^n converges uniformly on compact intervals almost surely. It following from (3.7) that

$$\begin{aligned}
|X_s^{n+1} - X_s^n| \leq & \Big| \int_0^s \sigma(X_u^n) dC_u - \int_0^s \sigma(X_u^{n-1}) dC_u \Big| + \Big| \int_0^s b(X_u^n) du - \int_0^s b(X_u^{n-1}) du \Big| \\
& + \frac{|\alpha|}{1-\alpha} \max_{0\leq v\leq s} \Big(\int_0^v \sigma(X_u^n) dC_u + \int_0^v b(X_u^n) du \Big) \\
& - \max_{0\leq v\leq s} \Big(\int_0^v \sigma(X_u^{n-1}) dC_u + \int_0^v b(X_u^{n-1}) du \Big) \\
\leq & \Big| \int_0^s \sigma(X_u^n) dC_u - \int_0^s \sigma(X_u^{n-1}) dC_u \Big| + \int_0^s |b(X_u^n) du - b(X_u^{n-1})| du \\
& + \frac{|\alpha|}{1-\alpha} \max_{0\leq v\leq s} \Big| \int_0^v \Big(\sigma(X_u^n) - \sigma(X_u^{n-1}) \Big) dC_u \Big| \\
& + \frac{|\alpha|}{1-\alpha} \max_{0\leq v\leq s} \Big| \int_0^v \Big(b(X_u^n) - b(X_u^{n-1}) \Big) du \Big|,
\end{aligned} \tag{3.8}$$

where we used the fact that $|\max_{0\leq v\leq s} f(v) - \max_{0\leq v\leq s} g(v)| \leq \max_{0\leq v\leq s} |f(v) - g(v)|$ holds for any two continuous functions f and g. Thus,

$$\max_{0\leq s\leq t} |X_s^{n+1} - X_s^n| \leq \Big(1 + \frac{|\alpha|}{1-\alpha}\Big) \Big[\max_{0\leq s\leq t} \Big| \int_0^s \sigma(X_u^n) dC_u - \sigma(X_u^{n-1}) dC_u \Big| + \int_0^s |b(X_u^n) - b(X_u^{n-1})| du \Big]. \tag{3.9}$$

For any sample γ, we define

$$D_t^n = \max_{0\leq s\leq t} |X_s^{n+1}(\gamma) - X_s^n(\gamma)|, \ n = 1, 2, ... \tag{3.10}$$

We claim that

$$\begin{gathered}
D_t^n \leq \Big(1 + |\frac{x_0}{1-\alpha}|\Big) \frac{c^{n+1}(1 + \frac{|\alpha|}{1-\alpha})^{n+1}(1 + k(\gamma))^{n+1}}{(n+1)!} t^{n+1}, \\
n = 0, 1, 2, ..., 0 \leq t \leq T,
\end{gathered} \tag{3.11}$$

where T is a constant. Indeed for $n = 0$, it follows from Lemma 2.1 that

$$
\begin{aligned}
D_t^0(\gamma) &= \max_{0\le s\le t} |X_s^1 - X_s^0| \\
&= \max_{0\le s\le t} \Big| X_0 + \int_0^s \sigma(X_s^0)dC_s + \int_0^s b(X_s^0)ds + \alpha \max_{0\le u\le s} X_u^1 - X_s^0 \Big| \\
&= \max_{0\le s\le t} \Big| X_0 + \int_0^s \sigma(X_s^0)dC_s + \int_0^s b(X_s^0)ds + \alpha \max_{0\le u\le s} X_u^1 - \frac{X_0}{1-\alpha} \Big| \\
&= \max_{0\le s\le t} \Big| \int_0^s \sigma(X_s^0)dC_s + \int_0^s b(X_s^0)ds + \alpha \max_{0\le u\le s} X_u^1 - \frac{\alpha}{1-\alpha} X_0) \Big| \\
&= \max_{0\le s\le t} \Big| \int_0^s \sigma(X_s^0)dC_s + \int_0^s b(X_s^0)ds + \alpha (\max_{0\le u\le s} X_u^1 - \frac{X_0}{1-\alpha}) \Big| \\
&\le \max_{0\le s\le t} \Big| \int_0^s \sigma(X_u^0)dC_u + \int_0^s b(X_u^0)du + \alpha (\max_{0\le u\le s} X_u^1 - \max_{0\le u\le s} X_u^0) \Big| \\
&\le \max_{0\le s\le t} \Big| \int_0^s \sigma(X_u^0)dC_u + \int_0^s b(X_u^0)du + \alpha \max_{0\le u\le s} (X_u^1 - X_u^0) \Big| \\
&\le \max_{0\le s\le t} \Big| \int_0^s \sigma(X_u^0)dC_u + \int_0^s b(X_u^0)du \Big| + \alpha D_t^0(\gamma).
\end{aligned} \tag{3.12}
$$

Then

$$
\begin{aligned}
D_t^0(\gamma) &\le \frac{1}{1-\alpha} \max_{0\le s\le t} \Big| \int_0^s \sigma(X_u^0)dC_u + \int_0^s b((X_u^0))du \Big| \\
&\le \Big(1 + \frac{|\alpha|}{1-\alpha}\Big) \max_{0\le s\le t} \Big| \int_0^s \sigma(X_u^0)dC_u + \int_0^s b(X_u^0)du \Big| \\
&\le \Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big(K(\gamma) \max_{0\le s\le t} \int_0^s |\sigma(X_u^0)|du + \max_{0\le s\le t} \int_0^s |b(X_u^0)|du\Big) \\
&\le \Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big(K(\gamma) \int_0^t |\sigma(X_u^0)|du + \int_0^t |b(X_u^0)|du\Big) \\
&\le \Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big(1 + |\frac{X_0}{1-\alpha}|\Big)\Big(1 + K(\gamma)\Big)t \ (\textit{by the linear growth condition}).
\end{aligned} \tag{3.13}
$$

This confirms the claim for $n = 0$. Next we assume the claim is true for $n - 1$. Then

$$\begin{aligned}
D_t^n &= \max_{0\le s\le t} |X_s^{n+1} - X_s^n| \\
&\le \Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big\{\max_{0\le s\le t}\Big[\Big|\int_0^s \sigma(X_u^n)dC_u - \sigma(X_u^{n-1})dC_u\Big| + \Big|\int_0^s b(X_u^n) - b(X_u^{n-1})du\Big|\Big]\Big\} \\
&\le c\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\max_{0\le s\le t}\Big[\int_0^s |X_u^n - X_u^{n-1}|dC_u + \int_0^s |X_u^n - X_u^{n-1}|du\Big] \\
&\le c\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\max_{0\le s\le t}\Big[\Big(1 + K(\gamma)\Big)\int_0^s |X_u^n - X_u^{n-1}|du\Big] \\
&\le c\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big(1 + K(\gamma)\Big)\int_0^t |X_u^n - X_u^{n-1}|du \\
&\le c\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)\Big(1 + K(\gamma)\Big)\int_0^t \Big(1 + |\frac{x_0}{1-\alpha}|\Big)\frac{c^n\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)^n\Big(1 + K(\gamma)\Big)^n u^n}{(n+1)!}du \\
&\le \Big(1 + \Big|\frac{x_0}{1-\alpha}\Big|\Big)\frac{c^{n+1}\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)^{n+1}\Big(1 + K(\gamma)\Big)^{n+1}}{(n+1)!}t^{n+1}.
\end{aligned} \tag{3.14}$$

Note that (3.13) and (3.14) are induced form Lemma 2.1 and the inductive assumption, respectively. This proves the claim. Therefore,

$$\begin{aligned}
D_t^n &= \max_{0\le s\le t} |X^{n+1}(\gamma) - X_s^n(\gamma)| \\
&\le \Big(1 + |\frac{x_0}{1-\alpha}|\Big)\frac{c^{n+1}\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)^{n+1}\Big(1 + K(\gamma)\Big)^{n+1}}{(n+1)!}t^{n+1},
\end{aligned}$$

holds for all $n \ge 0$. It follows from Weierstrass' criterion that, for each sample γ,

$$\begin{aligned}
&\sum_{n=0}^{+\infty}\Big(1 + |\frac{x_0}{1-\alpha}|\Big)\frac{c^{n+1}\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)^{n+1}\Big(1 + K(\gamma)\Big)^{n+1}}{(n+1)!}t^{n+1} \\
&\qquad \le \sum_{n=0}^{+\infty}\Big(1 + |\frac{x_0}{1-\alpha}|\Big)\frac{c^{n+1}\Big(1 + \frac{|\alpha|}{1-\alpha}\Big)^{n+1}\Big(1 + K(\gamma)\Big)^{n+1}}{(n+1)!}T^{n+1} \le +\infty.
\end{aligned}$$

Thus $X_t^k(\gamma)$ converges uniformly in $t \in [0, T]$. We denote the limit by

$$X_t(\gamma) = \lim_{k\to\infty} X_t^k(\gamma),\ \gamma \in \Gamma,\ t \in [0, T].$$

Then

$$X_t = X_0 + \int_0^t \sigma(X_s)dC_s + \int_0^t b(X_s)ds + \alpha \max_{0\le s\le t} X_s.$$

Therefore X_t is the solution of (3.4) for all $t \ge 0$ since T is arbitrary.

Next, we will prove that the solution of uncertain differential (3.4) is unique. Assume that both of X_t and X_t^* are solutions of (3.4) with the same initial value X_0. Then

$$\begin{aligned}
X_t &= \frac{X_0}{1-\alpha} + \int_0^t \sigma(X_s)dC_s + \frac{\alpha}{1-\alpha}\max_{0\le s\le t}\Big(\int_0^s \sigma(X_u)dC_u + \int_0^s b(X_u)du\Big), \\
X_t^* &= \frac{X_0}{1-\alpha} + \int_0^t \sigma(X_s^*)dC_s + \frac{\alpha}{1-\alpha}\max_{0\le s\le t}\Big(\int_0^s \sigma(X_u^*)dC_u + \int_0^s b(X_u^*)du\Big).
\end{aligned} \tag{3.15}$$

Arguing as above, there is a constant C such that

$$|X_t - X_t^*| \leq c \max_{0\leq s\leq t}\left| \int_0^s \sigma(X_u^n)dC_u - \sigma(X_u^{n-1})dC_u\right| + c\int_0^s |b(X_u) - b(X_u^*)|du. \tag{3.16}$$

Then for each $\gamma \in \Gamma$, we have

$$\begin{aligned}
|X_{t(\gamma)} - X_t^*(\gamma)| \leq & C\left|\int_0^t \Big(\sigma(X_v(\gamma)) - \sigma(X_v^*(\gamma))\Big)dC_v\right| \\
& + C\int_0^t \Big|b(X_v(\gamma)) - b(X_v^*(\gamma))\Big|dv \\
\leq & C\cdot K(\gamma)\left|\int_0^t \Big(\sigma(X_v(\gamma)) - \sigma(X_v^*(\gamma))\Big)dv\right| \\
& + C\int_0^t \Big|b(X_v(\gamma)) - b(X_v^*(\gamma))\Big|dv \ (by\ Lemma\ 2.1) \\
\leq & C\cdot L\cdot K(\gamma)\int_0^t |X_v(\gamma) - X_v^*(\gamma)|dv \\
& + C\cdot L\int_0^t |X_v(\gamma) - X_v^*(\gamma)|dv \ (by\ Lipschitz\ condition) \\
\leq & C\cdot L\cdot\Big(1 + K(\gamma)\Big)\int_0^t |X_v(\gamma) - X_v^*(\gamma)|dv.
\end{aligned}$$

It follows from Gronwall inequality that

$$|X_{t(\gamma)} - X_t^*(\gamma)| \leq 0\cdot exp(C\cdot L\cdot(1 + K(\gamma))t) = 0$$

for any γ. Hence $X_t = X_t^*$, the solution is unique. The theorem is proved.

4. Perturbed reflected diffusions

Let σ be as in Section 2. For $x \geq 0$, consider the uncertain differential equation:

$$X_t = x + \int_0^t \sigma(X_s)dC_s + \alpha\max_{0\leq s\leq t} X_s + L_t. \tag{4.1}$$

Definition 4.1 We say that $(X_t, L_t, t \geq 0)$ is a solution to (4.1) if

(i) $X_0 = x$, $X_t \geq 0$ for $t \geq 0$;

(ii) X_t,L_t are adapted to the filtration of C;

(iii) L_t is non-decreasing with $L_0 = 0$ and

$$\int_0^t \chi\{X_s = 0\}dL_s = L_t;$$

(iv) $(X_t, L_t, t \geq 0)$ satisfies (4.1) almost surely for every $t > 0$.

The cases $x = 0$ and $x > 0$ are quite different. We will treat them separately.

Theorem 4.1. *Assume $\alpha < 1$ and σ is Lipschitz. If $x > 0$, there exists a unique solution $(X_t, L_t, t \geq 0)$ to (4.1).*

Proof. We construct the solution iteratively in a similar way to (3.11). Define Y_t^0 to be the unique solution to the equation:

$$Y_t^0 = x + \int_0^t \sigma(Y_s^0) dC_s + \alpha \max_{0 \leq s \leq t} Y_s^0. \tag{4.2}$$

It is known from Section 2 that such a solution exists. Set $T_1 = \inf\{t \geq 0; Y_t^0 = 0\}$. Then $T_1 > 0$ a.s. as $x > 0$. Define

$$X_t = Y_t^0, L_t = 0 \; for \; 0 \leq t \leq T_1. \tag{4.3}$$

Put $C_t^1 = C_{t+T_1} - C_{T_1}$ for $t \geq 0$. It is well known that C_t^1, $t \geq 0$ is a normal uncertain variable with expected value 0 and variance t^2. Consider the uncertain differential equation with reflecting boundary:

$$\begin{aligned} &Z_t^1 = \int_0^t \sigma(Z_s^1) dC_s^1 + L_t^1, \\ &Z_t^1 \geq 0, Z_0^1 = 0, \\ &L_0^1 = 0, \int_0^t \chi Z_s^1 = 0 dL_s^1 = L_t^1. \end{aligned} \tag{4.4}$$

The definition of a solution to this equation is the same as Definition 4.1 with $x = 0$ and $\alpha = 0$. It is known that a unique solution (Z_t^1, L_t^1) to the (4.4) exists, see e.g. [11] or [15]. In general, suppose that (X_t, L_t) has been defined for $0 \leq t \leq T_{2n-1}$. We can construct (X_t, L_t) for $T_{2n-1} \leq t \leq T_{2n+1}$ as follows. Let Z_t^{2n-1} be the solution to the equation:

$$\begin{aligned} &Z_t^{2n-1} = \int_0^t \sigma(Z_s^{2n-1}) dC_s^{2n-1} + L_t^{2n-1}, \\ &Z_t^{2n-1} \geq 0, Z_0^{2n-1} = 0, \\ &L_0^{2n-1} = 0, \int_0^t \chi\{Z_s^{2n-1} = 0\} dL_s^{2n-1} = L_t^{2n-1}, \end{aligned} \tag{4.5}$$

where $C_t^{2n-1} = C_{t+T_{2n-1}}$. Put $T_{2n} = \inf\{t > T_{2n-1}; Z_{t-T_{2n-1}}^{2n-1} = \max_{0 \leq s \leq T_{2n-1}} X_s\}$ and set

$$X_t = Z_{t-T_{2n-1}}^{2n-1}, L_t = L_{T_{2n-1}} + L_{t-T_{2n-1}}^{2n-1} \; for \; T_{2n-1} \leq t \leq T_{2n}. \tag{4.6}$$

Let Y_t^{2n} denote the solution to equation:

$$Y_t^{2n} = (1 - \alpha) X_{T_{2n}} + \int_0^t \sigma(Y_s^{2n}) dC_s^{2n} + \alpha \max_{0 \leq s \leq t} Y_s^{2n}, \tag{4.7}$$

where $C_t^{2n} = C_{t+T_{2n}} - C_{T_{2n}}$. Set $T_{2n+1} = \inf\{t > T_{2n}, Y_{t-T_{2n}=0}\}$ and

$$X_t = Y_{t-T_{2n}}^{2n}, L_t = L_{T_{2n}} z for T_{2n} \leq t \leq T_{2n+1}. \tag{4.8}$$

By this procedure, we obtain a sequence of increasing stopping times T_n, $n \geq 0$. Set $T = \lim_{n \to \infty T_n}$. Then T is again a stopping time, and (X_t, L_t) is a well defined continuous process for all $0 \leq t < T$. We will show that $(X_t, L_t, t < T)$ satisfies (4.1) in the sense of Definition 4.1. To achieve this, it is sufficient to prove that (X_t, L_t) satisfies (4.1) for $T_{2n} \leq t \leq T_{2n+1}$ and $n = 0, 1....$ We will do this by induction. It is obvious that (X_t, L_t) is a solution to (4.1) for $0 \leq t < T_1$. If $T_1 \leq t \leq T_2$, it follows that

$$\begin{aligned}
X_t &= Z^1_{t-T_1} \\
&= \int_0^{t-T_1} \sigma(Z^1_s)dC^1_s + L^1_{t-T_1} \\
&= \int_0^{t-T_1} \sigma(Z^1_s)dC_{s+T_1} + L_t \\
&= \int_{T_1}^t \sigma(X_u)dC_u + L_t \\
&= X_{T_1} + \int_{T_1}^t \sigma(X_u)dC_u + L_t \\
&= x + \int_0^{T_1} \sigma(X_s)dC_s + \alpha \max_{0\leq s\leq T_1} X_s + \int_{T_1}^t \sigma(X_u)dC_u + L_t \\
&= x + \int_0^t \sigma(X_s)dC_s + \alpha \max_{0\leq s\leq T_1} X_s + L_t,
\end{aligned} \tag{4.9}$$

since $max_{0\leq s\leq T_1} X_s = max_{0\leq s\leq t} X_s$ for $T_1 \leq t \leq T_2$, and $X_{T_1} = 0$.

Furthermore, if $T_1 \leq t \leq T_2$, we see that

$$\int_0^t \chi\{X_{s=0}\}dL_s = \int_{T_1}^t \chi\{X_s = 0\}dL^1_{s-T_1} = \int_0^{t-T_1} \chi\{Z^1_s = 0\}dL^1_s = L^1_{t-T_1} = L_t. \tag{4.10}$$

Thus we have showed that (X_t, L_t) is a solution to (4.1) for $0 \leq t \leq T_2$. Suppose that (X_t, L_t) satisfies (4.1) for $0 \leq t \leq T_{2n}$. If $T_{2n} \leq t \leq T_{2n+1}$, it follows that

$$\begin{aligned}
X_t &= Y^{2n}_{t-T_{2n}} \\
&= (1-\alpha)X_{T_{2n}} + \int_0^{t-T_{2n}} \sigma(Y^{2n}_s)dC^{2n}_s + \alpha \max_{0\leq s\leq t-T_{2n}} Y^{2n}_s \\
&= x + \int_0^{T_{2n}} \sigma(X_s)dC_s + \alpha \max_{0\leq s\leq T_{2n}} X_s + L_{T_{2n}} - \alpha X_{T_{2n}} \\
&\quad + \int_0^{t-T_{2n}} \sigma(Y^{2n}_s)dC_{s+T_{2n}} + \alpha \max_{0\leq s\leq t-T_{2n}} Y^{2n}_s \\
&= x + \int_0^t \sigma(X_s)dC_s + \alpha \max_{T_{2n}\leq s\leq t} X_s + L_t \\
&= x + \int_0^t \sigma(X_s)dC_s + \alpha \max_{0\leq s\leq t} X_s + L_t,
\end{aligned} \tag{4.11}$$

where we have used the fact that $X_{T_{2n}} = \max_{0\leq s\leq T_{2n}} X_s$ and $Y^{2n}_0 = X_{T_{2n}}$ from their definitions. Since $X_t \neq 0$ for $T_{2n} \leq t < T_{2n+1}$, we also have

$$\int_0^t \chi\{X_s = 0\}dL_s = \int_0^{T_{2n}} \chi\{X_s = 0\}dL_s = L_{T_{2n}} = L_t. \tag{4.12}$$

So (X_t, L_t) satisfies (4.1) also for $T_{2n} \le t < T_{2n+1}$. Repeating similar arguments as for (4.10), we also can show that (X_t, L_t) satisfies (4.1) for $T_{2n+1} \le t < T_{2n+2}$.

Finally we show that $T = \infty$ a.s. By the construction of X, we can have that

$$0 = X_{T_{2n+1}} = \max_{0\le s\le T_{2n}} X_s + \int_{T_{2n}}^{T_{2n+1}} \sigma(X_s)dC_s + \alpha\Big(\max_{0\le s\le T_{2n+1}} X_s - \max_{0\le s\le T_{2n}} X_s\Big) + L_{T_{2n+1}} - L_{T_{2n}}. \tag{4.13}$$

Suppose $T < \infty$ with positive probability. Letting $n \to \infty$ in (4.13), we get $0 = \max_{0\le s\le T} X_s$ which contradicts the fact that $X_0 = (1-\alpha)^{-1}x > 0$. The proof of existence is complete.

On the other hand, it is easily seen that the solution is unique since it is unique on each interval $[T_n, T_{n+1}]$. □

Theorem 4.2. *Assume $x = 0$. If $0 \le \alpha < \frac{1}{2}$, then there exists a unique solution $(X_t, T_t, t \ge 0)$ to (4.1).*

Proof. We will use the Picard iteration method. Define $X_t^0 \equiv 0$ and (X_t^{n+1}, L_t^{n+1}) to be the unique solution to the equation:

$$X_t^{n+1} = \int_0^t \sigma(X_s^n)dC_s + \alpha \max_{0\le s\le t} X_s^{n+1} + L_t^{n+1}. \tag{4.14}$$

The existence and uniqueness of this solution follow from Section 3. Observe that by the reflection principle,

$$L_t^{n+1} = -\inf\{\Big(\int_0^t \sigma(X_u^n)dC_u + \alpha \max_{0\le u\le s} X_u^{n+1}\Big) \wedge 0\}. \tag{4.15}$$

Now (4.14) and (4.15) imply that

$$\begin{aligned}|X_t^{n+1} - X_t^n| \le &\Big|\int_o^t \Big(\sigma(X_s^n) - \sigma(X_s^{n-1})\Big)dC_s\Big| + \sup_{s\le t}\Big|\int_0^s \Big(\sigma(X_u^n) - \sigma(X_u^{n-1})\Big)dC_u\Big| \\ &+ 2\alpha \sup_{s\le t}|X_s^{n+1} - X_s^n|.\end{aligned} \tag{4.16}$$

Consequently,

$$\sup_{s\le t}|X_s^{n+1} - X_s^n| \le \frac{2}{1-2\alpha}\sup_{s\le t}\Big|\int_0^t \Big(\sigma(X_u^n) - \sigma(X_u^{n-1})\Big)dC_u\Big|. \tag{4.17}$$

Let $\beta = \sup|X_s^1 - X_s^0|$. Then

$$\begin{aligned}\sup_{s\le t}|X_s^{n+1} - X_s^n| &\le \frac{2}{1-2\alpha}\sup_{s\le t}\Big|\int_0^s \Big(\sigma(X_u^n) - \sigma(X_u^{n-1})\Big)dC_u\Big| \\ &\le \frac{2}{1-2\alpha}K(\gamma)\sup_{s\le t}\Big|\int_0^s \Big(\sigma(X_u^n) - \sigma(X_u^{n-1})\Big)du\Big| \\ &\le \frac{2}{1-2\alpha}L \cdot K(\gamma) \cdot \sup_{s\le t}\Big|\int_0^s (X_u^n - X_u^{n-1})du\Big| \\ &\le \frac{(\frac{2}{1-2\alpha} \cdot L \cdot K(\gamma) \cdot \beta \cdot t)^n}{n!},\end{aligned} \tag{4.18}$$

holds for all $n \ge 1$. It follows from Weierstrass' criterion that, for each sample γ,

$$\sum_{n=1}^{+\infty} \frac{(\frac{2}{1-2\alpha} \cdot L \cdot K(\gamma) \cdot \beta \cdot t)^n}{n!} \le \frac{(\frac{2}{1-2\alpha} \cdot L \cdot K(\gamma) \cdot \beta \cdot T)^n}{n!}.$$

Thus X_s^n converges uniformly to a continuous, adapted process X on $[0,T]$ almost surely. It is also seen that $M_n(t) := \int_0^t \sigma(X_s^n)dC_s$ converges uniformly on $[0,T]$ to $M_n(t) := \int_0^t \sigma(X_s^n)dC_s$ almost surly. Thus, by (4.14), we see that L_t^n converges uniformly to a continuous non-decreasing process L on $[0,T]$ almost surly. Letting $n \to \infty$ in (4.14) gives

$$X_t(\gamma) = \lim_{k\to\infty} X_t^k(\gamma),\ \gamma \in \Gamma,\ t \in [0,T].$$

Then

$$X_t = X_0 + \int_0^t \sigma(X_s)dC_s + \int_0^t b(X_s)ds + \alpha \max_{0\le s\le t} X_s.$$

Therefore, X_t is the solution of (3.4) for all $t \geq 0$ since T is arbitrary.

$$X_t = \int_0^t \sigma(X_s)dC_s + \alpha \max_{0\le s\le t} X_s + L_t. \tag{4.19}$$

To show that (X_t, L_t) is a solution to (4.1), we need to prove

$$\int_0^t \chi\{X_{s=0}\}dL_s = L_t. \tag{4.20}$$

This will follow if we can show that for any $f \in C_0(0,\infty)$

$$\int_0^t f(X_s)dL_s = 0. \tag{4.21}$$

Indeed,

$$\int_0^t f(X_s)dL_s = \lim_{n\to\infty} \int_0^t f(X_s^n)dL_s^n = 0. \tag{4.22}$$

Next we show the uniqueness. Let (X_t^1, L_t^1), (X_t^2, L_t^2) be two solutions to (4.1). Using the similar arguments as above, it can be shown that

$$|X_t^1 - X_t^2| \le C_\alpha \int_0^t |X_s^1 - X_s^2| ds.$$

By Gronwall's inequality, it follows that $X^1 = X^2$, and hence $L^1 = L^2$. □

5. Conclusions

In this paper, a new type of differential equations within the framework of uncertainty theory was discussed for the first time. First of all, we was first to provide an existence and uniqueness theorem under Lipschitz condition and linear growth condition. And then, as an application, we establish the existence and uniqueness of some perturbed reflected canonical process. In the future work, we will try to explore the stability for this type of perturbed uncertain differential equations.

Acknowledgement

This research is partially supported by the NNSF of China (No.11901058).

Conflict and interest

The authors declare that they have no conflict of interest.

References

1. P. Carmona, F. Petit, M. Yor, Beta variables as times spent in $[0, \infty)$ by certain perturbed Brownian motions, *J. London Math. Soc.,* **58** (1998), 239–256.
2. J. Norris, L. Rogers, D. Williams, Self-avoiding random walk: A Brownian motion model with local time drift, *Probab. Theory Rel.,* **74** (1987), 271–287.
3. B. Davis, Brownian motion and random walk perturbed at extrema, *Prob. Theory Rel.,* **113** (1999), 501–518.
4. P. Carmona, F. Petit, M. Yor, Some extentions of the arcsine law as partial consequences of the scaling property of Brownian motion, *Probab. Theory. Rel.,* **100** (1994), 1–29.
5. M. Perman, W. Werner, Perturbed Brownian motions, *Probab. Theory Rel.,* **108** (1997), 357–383.
6. L. Chaumont, R. A. Doney, Some calculations for doubly perturbed Brownian motion, *Stoch. Proc. Appl.,* **85** (2000), 61–74.
7. R. A. Doney, T. Zhang, Perturbed Skorohod equations and perturbed reflected diffusion processes, *Ann. Inst. H. Poincaré Probab. Statist,* **41** (2005), 107–121.
8. B. Liu, Fuzzy process, hybrid process and uncertain process, *J. Uncertain Syst.,* **2** (2008), 3–16.
9. B. Liu, Some research problems in uncertainty theory, *J. Uncertain Syst.,* **3** (2009), 3–10.
10. B. Liu, Toward uncertain finance theory, *J. Uncertainty Anal. Appl.,* **1** (2013), 1–15.
11. Y. Zhu, Uncertain optimal control with application to a portfolio selection model, *Cybernetics Syst.,* **41** (2010), 535–547.
12. Y. Zhu, *Uncertain Optimal Control*, Singapore: Springer, 2019.
13. X. Yang, J. Gao, Uncertain differential games with applications to capitallism, *J. Uncertainty Anal. Appl.,* **1** (2013), 1–11.
14. Z. Zhang, X. Yang, Uncertain population model, *Soft Comput.,* **24** (2020), 2417–2423.
15. X. Yang, K. Yao, Uncertain partial differential equation with application to heat conduction, *Fuzzy Optim. Decis. Ma.,* **16** (2017), 379–403.
16. X. Yang, J. Gao, Y. Ni, Resolution principle in uncertain random environment, *IEEE Trans. Fuzzy Syst.,* **26** (2017), 1578–1588.
17. L. Jia, W. Dai, Uncertain forced libration equation of spring mass system, Technical Report, 2018.
18. Z. Li, Y. Sheng, Z. Teng, H. Miao, An uncertian differential equation for SIS epidemic model, *J. Int. Fuzzy Sys.,* **33** (2017), 2317–2327.
19. B. Liu, Some research problems in uncertainy theory, *J. Uncertain Syst.,* **3** (2009), 3–10.
20. B. Liu, *Uncertainty Theory*, Berlin: Springer, 2007.

21. J. C. Pedjeu, G. S. Ladde, Stochastic fractional differential equations: Modeling, method and analysis, *Chaos Soliton Fract.,* **45** (2012), 279–293.
22. X. Chen, B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optim. Decis. Ma.,* **9** (2010), 69–81.
23. J. Luo, Doubly perturbed jump-diffusion processes, *J. Math. Anal. Appl.,* **351** (2009), 147–151.
24. W. Mao, L. Hu, X. Mao, Approximate solutions for a class of doubly perturbed stochastic differential equations, *Adv. Differ. Equ-Ny,* **2018** (2018), 1–17.
25. R. Khas'minskii, A limit theorem for the solutions of differential equations with random right-hand sides, *Theor. Prob. Appl.,* **11** (1966), 390–406.
26. B. Liu, Theory and practice of uncertain programming, Berlin: Springer, 2009.
27. L. Hu, Y. Ren, Doubly perturbed neutral stochastic functional equations, *J. Comput. Appl. Mat.,* **231** (2009), 319–326.
28. R. A Doney, T. Zhang, Perturbed Skorohod equation and perturbed reflected diffusion processes, *Ann. Inst. H. Poincaré Probab. Statist,* **41** (2005), 107–121.