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Research article

Some weighted estimates for the commutators of p-adic Hardy operator on two weighted p-adic Herz-type spaces

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Abstract: In the present artice we discuss the weighted p-adic central bounded mean oscillations (CMO) and p-adic Lipschtiz estimates for the commutators of p-adic Hardy operator on two weighted p-adic Herz-type spaces.

Keywords: p-adic Hardy operator; commutators; two weighted p-adic Morrey-Herz spaces; central bounded mean oscillations; weighted p-adic Lipschitz spaces
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1. Introduction

In mathematical analysis Hardy operator is considered an important averaging operator as it plays a vital role in many branches of mathematics, such as complex analysis, partial differential equations and harmonic analysis (for example, see [2,7,8,10,29]). In [6], Hardy introduced the one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \tag{1.1}$$

for a measurable function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$. The operator in (1.1) satisfies the below inequality

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty, \tag{1.2}$$

where the constant $q/(q-1)$ is sharp. An extension of the operator H on higher dimensional space $\mathbb{R}^n$ was defined in [3] by Faris as

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{t}|\leq|\mathbf{x}|} f(\mathbf{t})d\mathbf{t}, \tag{1.3}$$

where $|\mathbf{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ for $\mathbf{x} = (x_1, \ldots, x_n)$. Furthermore, Christ and Grafakos [1] acquired the exact value of the norm of operator H defined by (1.3). Recently, Hardy operator has gained a tremendous amount of consideration, see for example [16, 20, 24, 30, 33, 34] and the references therein.

In the past few decades there has been a relentless attention in p-adic models appearing in various branches of science. The applications of p-adic analysis are found mainly in the field of mathematical physics (see, for example, [15, 26, 27]). Importantly, many current researchers are paying a valiant effort to harmonic analysis on p-adic field [9–11, 13, 17, 21, 22, 25].

Let $\mathbb{Q}$ be a field of rational numbers and p a prime number. We introduce a so called p-adic norm $|x|_p$ on $\mathbb{Q}$ by a rule $|x|_p = \{0\} \cup \{p^{-\gamma} : \gamma \in \mathbb{Z}\}$, where $\gamma = \gamma(x)$ is defined from the following representation

$$x = p^{\gamma} s/t,$$

integers s and t are coprime to p. $|\cdot|_p$ fulfills all the axioms of a real norm along with the following non-Archimedean property:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \tag{1.4}$$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Any nonzero p-adic number can be written in canonical form (see [28]) as:

$$x = p^{\gamma} \sum_{j=0}^{\infty} \alpha_j p^j, \tag{1.5}$$

where $\alpha_j, \gamma \in \mathbb{Z}, \alpha_j \in \frac{\mathbb{Z}}{p\mathbb{Z}_p}, \alpha_0 \neq 0$. Interestingly, the series in (1.5) is convergent with respect to $|\cdot|_p$ because $|p^{\gamma}\alpha_k p^j|_p = p^{-\gamma-j}$.

The higher dimensional p-adic vector space $\mathbb{Q}_p^n$ consists of points $\mathbf{x} = (x_1, x_2, ..., x_n)$, where $x_i \in \mathbb{Q}_p, i = 1, 2, ..., n$, with the following norm

$$|\mathbf{x}|_p = \max_{1 \leq i \leq n} |x_i|_p. \tag{1.6}$$

Let

$$B_{\gamma}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^{\gamma}\},\ S_{\gamma}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^{\gamma}\}$$

be the ball and sphere respectively with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius p^{γ}. If $\mathbf{a} = \mathbf{0}$, we may write $B_{\gamma}(\mathbf{0}) = B_{\gamma},\ S_{\gamma}(\mathbf{0}) = S_{\gamma}$.

It is well known that the space $\mathbb{Q}_p^n$ is locally compact commutative group under addition, then there exists a translation invariant Haar measure $d\mathbf{x}$ which is normalized such that

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1,$$

where $|A|_H$ represents the Haar measure of a measurable subset A of $\mathbb{Q}_p^n$. Moreover, one can easily show that $|B_{\gamma}(\mathbf{a})|_H = p^{n\gamma}$ and $|S_{\gamma}(\mathbf{a})|_H = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

In what follows the p-adic Hardy operator

$$H^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^n} \int_{|\mathbf{t}|_p \leq |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t}$$

and its commutator

$$H_b^p f(\mathbf{x}) = bH^p(f) - H^p(bf)$$

were defined and studied for $f, b \in L_1^{loc}(\mathbb{Q}_p^n)$ in [4]. In the same paper, Fu et al. acquired the boundedness of p-adic Hardy operator and its commutator on Lebesgue spaces and Herz spaces. On the Morrey-Herz spaces, the p-adic Hardy type operators and their commutators are reported in [5]. For complete comprehension of p-Hardy operator and its commutator, we refer the publications [12, 18, 31, 32].

The purpose of the current article is to discuss the weighted central bounded mean oscillations and weighted p-adic Lipschitz estimates of H_b^p on two weighted p-adic Herz spaces and p-adic Morrey-Herz spaces. Throughout this article a letter C denotes a constant whose value may change at its different places. It is mandatory to recall the definitions of relevant p-adic function spaces before moving to our results.

Suppose $w(\mathbf{x})$ is a nonnegative function on $\mathbb{Q}_p^n$. The weighted measure of A is denoted and defined as $w(A) = \int_A w(\mathbf{x})d\mathbf{x}$. The weighted p-adic Lebesgue space $L^q(w, \mathbb{Q}_p^n), (0 < q < \infty)$ is defined to be the space of all measurable functions f on $\mathbb{Q}_p^n$ such that:

$$\|f\|_{L^q(w,\mathbb{Q}_p^n)} = \Big(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\Big)^{\frac{1}{q}} < \infty.$$

The theory of A_q weights on $\mathbb{R}^n$ was introduced by Benjamin Muckenhoupt in [19]. Let us recall the definition of A_q weights in p-adic setting.

Definition 1.1. [23] A weight function $w \in A_q(1 \leq q < \infty)$ if there exists a constant C free from choice of $B \subset \mathbb{Q}_p^n$ such that

$$\Big(\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x}\Big)\Big(\frac{1}{|B|}\int_B w(\mathbf{x})^{-\frac{1}{q-1}}d\mathbf{x}\Big)^{1/q} \leq C.$$

For the case $q = 1, w \in A_1$, we have

$$\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x} \leq C ess \inf_{\mathbf{x}\in B} w(\mathbf{x}),$$

for every $B \subset \mathbb{Q}_p^n$.

Remark 1.2. A weight function $w \in A_\infty$ if it undergoes the stipulation of $A_q(1 \leq q < \infty)$ weights.

Definition 1.3. Suppose w is a weight function and $1 \leq q < \infty$. The p-adic space $CMO^q(w, \mathbb{Q}_p^n)$ is defined by

$$\|f\|_{CMO^q(w,\mathbb{Q}_p^n)} = \sup_{\gamma\in Z}\Big(\frac{1}{w(B_\gamma)}\int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q w(\mathbf{x})^{1-q}d\mathbf{x}\Big)^{1/q},$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|}\int_{B_\gamma} f(\mathbf{x})d\mathbf{x}. \tag{1.7}$$

Definition 1.4. [22] Suppose w_1 and w_2 are weight functions, $0 < r, q < \infty$ and $\alpha \in \mathbb{R}$. Then the two weighted p-adic Herz space $K_q^{\alpha,r}(w_1, w_2)$ is defined as

$$K_q^{\alpha,r}(w_1, w_2) = \{f \in L^q_{\text{loc}}(w_2, \mathbb{Q}_p^n\backslash\{0\}) : \|f\|_{K_q^{\alpha,r}(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,r}(w_1,w_2)} = \Big(\sum_{k=-\infty}^{\infty} w_1(B_k)^{\alpha r/n} \|f\chi_k\|^r_{L^q(w_2,\mathbb{Q}_p^n)} \Big)^{1/r} \tag{1.8}$$

and χ_k is the characteristic function of the sphere $S_k = B_k \setminus B_{k-1}$.

Remark 1.5. Obviously $K_q^{0,q}(w_1, w_2) = L^q(w_2, \mathbb{Q}_p^n)$.

Definition 1.6. [22] Suppose w_1 and w_2 are weight functions, $0 < r, q < \infty$, $\alpha \in \mathbb{R}$ and $\lambda \geq 0$. Then the two weighted p-adic Morrey-Herz space $MK_{r,q}^{\alpha,\lambda}(w_1, w_2)$ is defined as follows

$$MK_{r,q}^{\alpha,\lambda}(w_1, w_2) = \{f \in L^q_{loc}(w_2, \mathbb{Q}_p^n \backslash \{0\}) : \|f\|_{MK_{r,q}^{\alpha,\lambda}(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{MK_{r,q}^{\alpha,\lambda}(w_1,w_2)} = \sup_{k_0\in\mathbb{Z}} w_1(B_{k_0})^{-\lambda/n} \Big(\sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha r/n} \|f\chi_k\|^r_{L^q(w_2,\mathbb{Q}_p^n)} \Big)^{1/r}. \tag{1.9}$$

Remark 1.7. It is evident that $MK_{r,q}^{\alpha,0}(w_1, w_2) = K_q^{\alpha,r}(w_1, w_2)$.

Definition 1.8. [23] Suppose $1 \leq q < \infty$, $0 < \beta < 1$ and w is a weight function. The p-adic space $Lip_\beta(w, \mathbb{Q}_p^n)$ is defined as

$$\|f\|_{Lip_\beta(w,\mathbb{Q}_p^n)} = \sup_{B\subset\mathbb{Q}_p^n} \frac{1}{w(B)^{\beta/n}} \Big(\frac{1}{w(B)} \int_B |f(\mathbf{x}) - f_B|^q w(\mathbf{x})^{1-q} d\mathbf{x} \Big)^{1/q},$$

where

$$f_B = \frac{1}{|B|} \int_B f(\mathbf{x}) d\mathbf{x}. \tag{1.10}$$

2. Weighted CMO estimates of H_b^p on two weighted p-adic Herz-type spaces

The following section discusses the weighted CMO estimates of H_b^p on two weighted p-adic Herz-type spaces. We open up the section with few lemmas which are useful in proving key results.

Lemma 2.1. *[14] Suppose $w \in A_1$, then there exist constants C_1, C_2 and $0 < \mu < 1$ such that*

$$C_1 \frac{|A|}{|B|} \leq \frac{w(A)}{w(B)} \leq C_2 \Big(\frac{|A|}{|B|} \Big)^{\mu},$$

for any measurable subset A of a ball B.

Remark 2.2. If $w \in A_1$, then it follows from lemma (2.1) that there exist constants C and μ $(0 < \mu < 1)$ such that $\frac{w(B_k)}{w(B_i)} \leq Cp^{(k-i)n}$ as $i < k$ and $\frac{w(B_k)}{w(B_i)} \leq Cp^{(k-i)n\mu}$ as $i \geq k$.

Lemma 2.3. *[23] Suppose $w \in A_1$ and $b \in CMO^q(w, \mathbb{Q}_p^n)$, then there is a constant C such that for i, $k \in \mathbb{Z}$,*

$$|b_{B_i} - b_{B_k}| \leq C(i-k)\|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \frac{w(B_k)}{|B_k|}.$$

Lemma 2.4. *[23] Suppose $w \in A_1$, then for $1 < q < \infty$,*

$$\int_B w(x)^{1-q'} dx \leq C|B|^{q'} w(B)^{1-q'},$$

where $1/q + 1/q' = 1$.

Now we state the result about the boundedness of H_b^p on two weighted p-adic Herz-type spaces.

Theorem 2.5. *Let $0 < r_1 \leq r_2 < \infty$, $1 \leq r, q < \infty$ and let $w \in A_1$. If $\alpha < \frac{n\mu}{q'}$, then the inequality*

$$\|H_b^p f\|_{\dot{K}_q^{\alpha,r_2}(w,w^{1-q})} \leq C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|_{\dot{K}_q^{\alpha,r_1}(w,w)}$$

holds for all $b \in CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

If $\alpha = 0, r_1 = r_2 = q$, then we have the following result.

Corollary 2.6. *Let $1 \leq r, q < \infty$ and $w \in A_1$, then*

$$\|H_b^p f\|_{L^q(w^{1-q},\mathbb{Q}_p^n)} \leq C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|_{L^q(w,\mathbb{Q}_p^n)}$$

holds for all $b \in CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Theorem 2.7. *Let $0 < r_1 \leq r_2 < \infty$, $1 \leq r, q < \infty$ and let also $w \in A_1$ and $\lambda > 0$. If $\alpha < \frac{n\mu}{q'} + \lambda$, then*

$$\|H_b^p f\|_{M\dot{K}_{r_2,q}^{\alpha,\lambda}(w,w^{1-q})} \leq C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}_{r_1,q}^{\alpha,\lambda}(w,w)}$$

holds for all $b \in CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.5: First, by the definition we have

$$\begin{aligned}
&\|(H_b^p f)\chi_k\|_{L^q(w^{1-q},\mathbb{Q}_p^n)}^q \\
&= \int_{S_k} |\mathbf{x}|_p^{-qn} \left| \int_{|\mathbf{t}|_p \leq |\mathbf{x}|_p} f(\mathbf{t})(b(\mathbf{x}) - b(\mathbf{t})) d\mathbf{t} \right|^q w(\mathbf{x})^{1-q} d\mathbf{x} \\
&\leq Cp^{-kqn} \int_{S_k} \left(\int_{|\mathbf{t}|_p \leq p^k} |f(\mathbf{t})(b(\mathbf{x}) - b(\mathbf{t}))| d\mathbf{t} \right)^q w(\mathbf{x})^{1-q} d\mathbf{x} \\
&= Cp^{-kqn} \int_{S_k} \left(\sum_{i=-\infty}^{k} \int_{S_i} |f(\mathbf{t})(b(\mathbf{x}) - b(\mathbf{t}))| d\mathbf{t} \right)^q w(\mathbf{x})^{1-q} d\mathbf{x} \\
&\leq Cp^{-kqn} \int_{S_k} \left(\sum_{i=-\infty}^{k} \int_{S_i} |f(\mathbf{t})(b(\mathbf{x}) - b_{B_k})| d\mathbf{t} \right)^q w(\mathbf{x})^{1-q} d\mathbf{x} \\
&\quad + Cp^{-kqn} \int_{S_k} \left(\sum_{i=-\infty}^{k} \int_{S_i} |f(\mathbf{t})(b(\mathbf{t}) - b_{B_k})| d\mathbf{t} \right)^q w(\mathbf{x})^{1-q} d\mathbf{x} \\
&= I + II.
\end{aligned} \tag{2.1}$$

Since $w \in A_1 \subset A_q$, making use of Hölder's inequality along with lemma 2.4, we have

$$\int_{S_i} f(\mathbf{t})d\mathbf{t} \leq \Big(\int_{S_i} |f(\mathbf{t})|^q w(\mathbf{t})d\mathbf{t}\Big)^{1/q}\Big(\int_{S_i} w(\mathbf{t})^{-q'/q}d\mathbf{t}\Big)^{1/q'}$$
$$\leq C\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}|B_i|w(B_i)^{-1/q}. \tag{2.2}$$

To estimate I, by the application of Hölder's inequality, Remark 2.2 along with inequality (2.2), we are down to

$$I \leq Cp^{-kqn}\|b\|^q_{CMO^q(w,\mathbb{Q}_p^n)} w(B_k)\Big(\sum_{i=-\infty}^{k} \|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}|B_i|w(B_i)^{-1/q}\Big)^q$$
$$\leq Cp^{-knq}\|b\|^q_{CMO^q(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k} \|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}|B_i|\Big(\frac{w(B_k)}{w(B_i)}\Big)^{1/q}\Big)^q$$
$$\leq C\|b\|^q_{CMO^q(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k} p^{(i-k)n/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q. \tag{2.3}$$

Now, we estimate II as follows

$$II \leq Cp^{-kqn}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b(\mathbf{t})-b_{B_i})|d\mathbf{t}\Big)^q w(\mathbf{x})^{1-q}d\mathbf{x}$$
$$+ Cp^{-kqn}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b_{B_k}-b_{B_i})|d\mathbf{t}\Big)^q w(\mathbf{x})^{1-q}d\mathbf{x}$$
$$= II_1 + II_2. \tag{2.4}$$

Next, applying Hölder's inequality to deduce

$$\int_{S_i}|f(\mathbf{t})(b(\mathbf{t})-b_{B_i})|d\mathbf{t}$$
$$\leq\Big(\int_{S_i}|f(\mathbf{t})|^q w(\mathbf{t})d\mathbf{t}\Big)^{1/q}\Big(\int_{S_i}|b(\mathbf{t})-b_{B_i}|^{q'}w(\mathbf{t})^{-q'/q}d\mathbf{t}\Big)^{1/q'}$$
$$\leq w(B_i)^{-1/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\|b\|_{CMO^{q'}(w,\mathbb{Q}_p^n)}. \tag{2.5}$$

By the application of Hölder's inequality, inequality (2.5), lemma 2.4 and Remark 2.2, we are in a position to estimate II_1.

$$II_1 \leq Cp^{-kqn}\int_{S_k} w(\mathbf{x})^{1-q}d\mathbf{x}\Big(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}w(B_i)^{1/q'}\Big)^q$$
$$\leq Cp^{-kqn}|B_k|^q w(B_k)^{1-q}\|b\|^q_{CMO^{q'}(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}w(B_i)^{1/q'}\Big)^q$$
$$\leq C\|b\|^q_{CMO^{q'}(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k}\Big(\frac{w(B_i)}{w(B_k)}\Big)^{1-1/q}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q$$
$$\leq C\|b\|^q_{CMO^{q'}(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k} p^{(i-k)n\mu/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q. \tag{2.6}$$

Next task is to estimate II_2. For this, we use Hölder's inequality, lemmas 2.3 and 2.4, Remark 2.2 and inequality (2.2)

$$\begin{aligned} II_2 &\le Cp^{-kqn} \\ &\quad \times \int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{y})(i-k)||b||_{CMO^r(w,\mathbb{Q}_p^n)}\frac{w(B_i)}{|B_i|}|d\mathbf{y}\Big)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\ &\le Cp^{-kqn}\|b\|^q_{CMO^r(w,\mathbb{Q}_p^n)}|B_k|^q w(B_k)^{1-q} \\ &\quad \times\Big(\sum_{i=-\infty}^{k}(k-i)\frac{w(B_i)^{1-1/q}}{|B_i|}|B_i|\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q \\ &\le C\|b\|^q_{CMO^r(w,\mathbb{Q}_p^n)} \\ &\quad \times\Big(\sum_{i=-\infty}^{k}(k-i)\Big(\frac{w(B_i)}{w(B_k)}\Big)^{1-1/q}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q \\ &\le C\|b\|^q_{CMO^r(w,\mathbb{Q}_p^n)} \\ &\quad \times\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^q. \end{aligned} \tag{2.7}$$

From (2.3), (2.6) and (2.7) together with Jensen's Inequality, we have

$$\begin{aligned} &\|H_b^p f\|_{\dot K_q^{\alpha,r_2}(w,w^{1-q})} \\ &= \Big(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_2/n}\|(H_b^p f)\chi_k\|^{r_2}_{L^q(w^{1-q},\mathbb{Q}_p^n)}\Big)^{1/r_2} \\ &\le \Big(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n}\|(H_b^p f)\chi_k\|^{r_1}_{L^q(w^{1-q},\mathbb{Q}_p^n)}\Big)^{1/r_1} \\ &\le C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\Big(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n}\Big(\sum_{i=-\infty}^{k}p^{(i-k)n/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^{r_1}\Big)^{1/r_1} \\ &\quad + C\|b\|_{CMO^{q'}(w,\mathbb{Q}_p^n)}\Big(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n}\Big(\sum_{i=-\infty}^{k}p^{(i-k)n\mu/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^{r_1}\Big)^{1/r_1} \\ &\quad + C\|b\|_{CMO^r(w,\mathbb{Q}_p^n)}\Big(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^{r_1}\Big)^{1/r_1} \\ &= J. \end{aligned}$$

Therefore,

$$\begin{aligned} J^{r_1} &\le C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)} \\ &\quad \times\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q'}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^{r_1} \\ &\le C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)} \end{aligned}$$

$$\times \sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q'-\alpha}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}\Big)^{r_1}.$$

In what follows we consider two cases, $0<r_1\leq 1$ and $r_1>1$.
Case 1: When $0<r_1\leq 1$ and $\alpha<n\mu/q'$, we have

$$\begin{aligned}
J^{r_1} &\leq C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\\
&\quad\times\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k}(k-i)^{r_1}w(B_i)^{\alpha r_1/n}p^{(i-k)(n\mu/q'-\alpha)r_1}\|f\chi_i\|^{r_1}_{L^q(w,\mathbb{Q}_p^n)}\\
&= C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\\
&\quad\times\sum_{k=-\infty}^{\infty}w(B_i)^{\alpha r_1/n}\|f\chi_i\|^{r_1}_{L^q(w,\mathbb{Q}_p^n)}\sum_{k=i}^{\infty}(k-i)^{r_1}p^{(i-k)(n\mu/q'-\alpha)r_1}\\
&= C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|^{r_1}_{\dot{K}_q^{\alpha,r_1}(w,w)}.
\end{aligned}$$

Case 2: Whenever $r_1>1$, an application of Hölder's inequality with $\alpha<n\mu/q'$, we get

$$\begin{aligned}
J^{r_1} &\leq C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k}w(B_i)^{\alpha r_1/n}\|f\chi_i\|^{r_1}_{L^q(w,\mathbb{Q}_p^n)}p^{(i-k)(n\mu/q'-\alpha)r_1/2}\\
&\quad\times\Big(\sum_{i=-\infty}^{k}(k-i)^{r_1'}p^{(i-k)(n\mu/q'-\alpha)r_1'/2}\Big)^{r_1/r_1'}\\
&= C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\infty}w(B_i)^{\alpha r_1/n}\|f\chi_i\|^{r_1}_{L^q(w,\mathbb{Q}_p^n)}\sum_{k=i}^{\infty}p^{(i-k)(n\mu/q'-\alpha)r_1/2}\\
&= C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|^{r_1}_{\dot{K}_{q_1}^{\alpha,r_1}(w,w)}.
\end{aligned}$$

Hence, the proof of theorem is completed.

Proof of Theorem 2.7: From theorem 2.5, we have

$$\|(H_b^p)f\chi_k\|_{L^q(w^{1-q},\mathbb{Q}_p^n)}\leq C\|b\|^{r_1}_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\sum_{i=-\infty}^{k}(k-i)p^{(i-k)(n\mu/q')}\|f\chi_i\|_{L^q(w,\mathbb{Q}_p^n)}.$$

By definition of weighted p-adic Morrey-Herz spaces and Jensen's Inequality along with $\alpha<n\mu/q'+\lambda$,

$\lambda > 0$ and $1 < r_1 < \infty$, we reach at

$$
\begin{aligned}
&\|H_b^p f\|_{M\dot{K}^{\alpha,\lambda}_{r_2,q}(w,w^{1-q})}\\
&= \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\Big(\sum_{k=-\infty}^{k_0} w(B_k)^{\alpha r_2/n}\|(H_b^p f)\chi_k\|^{r_2}_{L^q(w^{1-q},\mathbb{Q}_p^n)}\Big)^{1/r_2}\\
&\le \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\Big(\sum_{k=-\infty}^{k_0} w(B_k)^{\alpha r_1/n}\|(H_b^p f)\chi_k\|^{r_1}_{L^q(w^{1-q},\mathbb{Q}_p^n)}\Big)^{1/r_1}\\
&\le C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\\
&\quad\times\Big(\sum_{k=-\infty}^{k_0} w(B_k)^{\lambda r_1/n}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q'}\Big(\frac{w(B_k)}{w(B_i)}\Big)^{\lambda r_1/n}\\
&\quad\times w(B_i)^{-\lambda/n}\Big(\sum_{l=-\infty}^{i} w(B_l)^{\alpha r_1/n}\|f\chi_i\|^{r_1}_{L^q(w,\mathbb{Q}_p^n)}\Big)^{1/r_1}\Big)^{r_1}\Big)^{1/r_1}\\
&\le C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\\
&\quad\times\Big(\sum_{k=-\infty}^{k_0} w(B_k)^{\lambda r_1/n}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)(n\mu/q'-\alpha+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}\Big)^{r_1}\Big)^{1/r_1}\\
&\le C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\\
&\quad\times\Big(\sum_{k=-\infty}^{k_0} w(B_k)^{\lambda r_1/n}\Big)^{1/r_1}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}\\
&\le C\|b\|_{CMO^{r\max\{q,q'\}}(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}.
\end{aligned}
$$

3. Weighted p-adic Lipschitz estimates of H_b^p on two weighted p-adic Herz-type Spaces

The current section deals the weighted p-adic Lipschitz estimates of H_b^p on two weighted p-adic Herz-type spaces. The outset of a section is with a lemma which is helpful in proving main results.

Lemma 3.1. *[23] Suppose $w \in A_1$ and $b \in Lip_\beta(w,\mathbb{Q}_p^n)$, then there is a constant C such that for i, $k \in \mathbb{Z}$,*

$$|b_{B_i} - b_{B_k}| \le C(i-k)\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)} w(B_i)^{\beta/n}\frac{w(B_k)}{|B_k|}.$$

Now, we state the result about the boundedness of commutator of p-adic Hardy operator on two weighted p-adic Herz-type spaces.

Theorem 3.2. *Let $0 < r_1 \le r_2 < \infty$, $1 \le q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and let $w \in A_1$. If $\alpha < \frac{n\mu}{q_1'}$, then the inequality*

$$\|H_b^p f\|_{\dot{K}^{\alpha,r_2}_{q_2}(w,w^{1-q_2})} \le C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{\dot{K}^{\alpha,r_1}_{q_1}(w,w)}$$

holds for all $b \in Lip_\beta(w,\mathbb{Q}_p^n)$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

If $\alpha = 0, r_1 = q_1 = p$ and $r_2 = q_2 = q$, then we have the following corollary.

Corollary 3.3. *Let $1 \le q < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and $w \in A_1$, then*

$$\|H_b^p f\|_{L^q(w^{1-q},\mathbb{Q}_p^n)} \le C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{L^q(w,\mathbb{Q}_p^n)}$$

holds for all $b \in Lip_\beta(w,\mathbb{Q}_p^n)$ and $f \in L_{\text{loc}}(\mathbb{Q}_p^n)$.

Theorem 3.4. *Let $0 < r_1 \le r_2 < \infty$, $1 \le q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and let $w \in A_1$. If $\alpha < \frac{n\mu}{q_1'} + \lambda$, then*

$$\|H_b^p f\|_{M\dot{K}^{\alpha,\lambda}_{r_2,q_2}(w,w^{1-q_2})} \le C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q_1}(w,w)}$$

holds for all $b \in Lip_\beta(w,\mathbb{Q}_p^n)$ and $f \in L_{\text{loc}}(\mathbb{Q}_p^n)$.

Proof of Theorem 3.2: In a similar fashion as that of theorem 2.5, we get

$$\begin{aligned}
&\|(H_b^p f)\chi_k\|^{q_2}_{L^{q_2}(w^{1-q_2},\mathbb{Q}_p^n)}\\
&\le Cp^{-kq_2n}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b(\mathbf{x})-b_{B_k})|d\mathbf{t}\Big)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x}\\
&\quad + Cp^{-kq_2n}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b(\mathbf{t})-b_{B_k})|d\mathbf{t}\Big)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x}\\
&= L + LL. \qquad (3.1)
\end{aligned}$$

For the evaluation of L, we apply Hölder's inequality, Remark 2.2, $\beta/n = 1/q_1 - 1/q_2$, $w \in A_1 \subset A_{q_1}$, and inequality (2.2) to get

$$\begin{aligned}
L &\le Cp^{-kq_2n}\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)} w(B_k)^{1+\beta q_2/n}\Big\{\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}|B_i|w(B_i)^{-1/q_1}\Big\}^{q_2}\\
&\le Cp^{-knq_2}\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big\{\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}|B_i|\Big(\frac{w(B_k)}{w(B_i)}\Big)^{1/q_1}\Big\}^{q_2}\\
&\le C\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big(p^{(k-i)n/q_1'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}. \qquad (3.2)
\end{aligned}$$

In order to evaluate LL, we proceed as follows

$$\begin{aligned}
LL &\le Cp^{-kq_2n}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b(\mathbf{t})-b_{B_i})|d\mathbf{t}\Big)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x}\\
&\quad + Cp^{-kq_2n}\int_{S_k}\Big(\sum_{i=-\infty}^{k}\int_{S_i}|f(\mathbf{t})(b_{B_k}-b_{B_i})|d\mathbf{t}\Big)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x}\\
&= LL_1 + LL_2. \qquad (3.3)
\end{aligned}$$

The following preparation will do world of good to estimate LL_1. Using Hölder's inequality, we have

$$\begin{aligned}
&\int_{S_i}|f(\mathbf{t})(b(\mathbf{t})-b_{B_i})|d\mathbf{t}\\
&\qquad\le\Big(\int_{S_i}|f(\mathbf{t})|^{q_1}w(\mathbf{t})d\mathbf{t}\Big)^{1/q_1}\Big(\int_{S_i}|b(\mathbf{t})-b_{B_i}|^{q_1'}w(\mathbf{t})^{-q_1'/q_1}d\mathbf{t}\Big)^{1/q_1'}\\
&\qquad\le w(B_i)^{-1/q_1'+\beta/n}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}. \qquad (3.4)
\end{aligned}$$

To evaluate LL_1, we apply Hölder's inequality, inequality (3.4), lemma 2.4 and Remark 2.2.

$$\begin{aligned} LL_1 &\le Cp^{-kq_2n}\int_{S_k} w(\mathbf{x})^{1-q_2}d\mathbf{x}\Big(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}w(B_i)^{1/q_1'+\beta/n}\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big)^{q_2}\\ &\le Cp^{-kq_2n}|B_k|^{q_2}w(B_k)^{1-q_2}\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}w(B_i)^{1/q_1'+\beta/n}\Big)^{q_2}\\ &\le C\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k}\Big(\frac{w(B_i)}{w(B_k)}\Big)^{1-1/q_2}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}\\ &\le C\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\Big(\sum_{i=-\infty}^{k}p^{(i-k)n\mu/q_2'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}. \end{aligned} \tag{3.5}$$

Next step is to evaluate LL_2. For this we use Hölder's inequality, lemmas 3.1 and 2.4, inequality (2.2), and Remark 2.2 to get

$$\begin{aligned} LL_2 &\le Cp^{-kq_2n}\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}|B_k|^{q_2}w(B_k)^{1-q_2}\\ &\quad\times\Big(\sum_{i=-\infty}^{k}(k-i)w(B_k)^{\beta/n}\frac{w(B_i)}{|B_i|}|B_i|w(B_i)^{-1/q_1}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}\\ &= C\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\\ &\quad\times\Big(\sum_{i=-\infty}^{k}(k-i)\Big(\frac{w(B_i)}{w(B_k)}\Big)^{1-1/q_1}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}\\ &\le C\|b\|^{q_2}_{Lip_\beta(w,\mathbb{Q}_p^n)}\\ &\quad\times\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q_1'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{q_2}. \end{aligned} \tag{3.6}$$

Remaining proof is more or less same to the proof of theorem 2.5. Thus, we conclude the theorem.

Proof of Theorem 3.4: Let $\alpha < n\mu/q_1' + \lambda$. By the definition of weighted p-adic Morrey-Herz spaces together with inequalities (3.2), (3.5), and (3.6), we have

$$\begin{aligned} &\|H_b^p f\|_{M\dot{K}^{\alpha,\lambda}_{r_2,q_2}(w,w^{1-q_2})}\\ &\le C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\Big(\sum_{k=-\infty}^{\infty}w(B_k)^{\beta r_2/n}\Big(\sum_{i=-\infty}^{k}p^{(i-k)n/q_1'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{r_2}\Big)^{1/r_2}\\ &\quad+C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\Big(\sum_{k=-\infty}^{\infty}w(B_k)^{\beta r_2/n}\Big(\sum_{i=-\infty}^{k}p^{(i-k)n\mu/q_2'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{r_2}\Big)^{1/r_2}\\ &\quad+C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/n}\Big(\sum_{k=-\infty}^{\infty}w(B_k)^{\beta r_2/n}\Big(\sum_{i=-\infty}^{k}(k-i)p^{(i-k)n\mu/q_1'}\|f\chi_i\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Big)^{r_2}\Big)^{1/r_2}\\ &= S_1+S_2+S_3. \end{aligned}$$

Next by applying the similar arguments as in theorem 2.7, we get

$$S_1 \leq C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}, \quad \alpha < n/q_1' + \lambda,$$

$$S_2 \leq C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}, \quad \alpha < n\mu/q_2' + \lambda,$$

$$S_3 \leq C\|b\|_{Lip_\beta(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{r_1,q}(w,w)}, \quad \alpha < n\mu/q_1' + \lambda.$$

So, the proof of the theorem is finished.

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Conflict of interest

The authors declare that they have no conflict of interest.

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