AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9619–9632.
DOI: 10.3934/math.2021560
Received: 26 February 2021
Accepted: 21 June 2021
Published: 25 June 2021

Research article

θ-type generalized fractional integral and its commutator on some non-homogeneous variable exponent spaces

Guanghui Lu*and Li Rui

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: lghwmm1989@126.com; Tel: +8615214042802.

Abstract: Let (X, d, μ) be a non-homogeneous space satisfying certain growth conditions. In this paper, the authors obtain the boundedness of θ-type generalized fractional integral T_α on variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ and variable exponent Morrey spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$. Furthermore, by establishing the sharp maximal function for commutator $[b, T_\alpha]$ generated by $b \in \mathrm{RBMO}(\mu)$ and T_α, the authors prove that the $[b, T_\alpha]$ is bounded from spaces $L^{p(\cdot)}(X)$ into spaces $L^{q(\cdot)}(X)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \alpha$ and $\alpha \in (0, 1)$, and bounded from spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$, where $\frac{t(\cdot)}{s(\cdot)} = \frac{q(\cdot)}{p(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} - \alpha$ and $\bar{a} > 0$ is a constant.

Keywords: non-homogeneous variable exponent space; θ-type generalized fractional integral; commutator; space RBMO(μ)
Mathematics Subject Classification: 42B20, 42B35

1. Introduction

In 1931, Orlicz first obtained the definition of Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ (see [15]), i.e., for any measurable functions f and sets $\Omega \subset \mathbb{R}^n$, if there exists a positive constant η such that,

$$\int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty,$$

where p is a function on Ω satisfying $1 < p(x) < \infty$. Respectively, the norm of Luxemburg-Nakano is defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

Since then, many papers focus on the variable exponent spaces and their applications. For example, Kováčik and Rákosník [9] systematically researched variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$. In [16], Radulescu and Repovs studied the Lebesgue and Morrey spaces with variable exponent on $\mathbb{R}^n$, and also obtained some applications in partial differential equations. In [17], Ragusa and Tachikawa established the $C^{1,\gamma}_{\text{loc}}(\Omega)$-regularity result for $W^{1,1}$-local minimizers μ of the double phase functional with x-dependent exponents. In 2021, with the nonstandard growth conditions, Mingione and Rădulescu provide an overview of recent results concerning elliptic variational problems (see [12]). The more development and research on the variable exponents, we refer readers to see [3, 4, 8, 11, 13, 21–23] and reference therein.

On the other hand, fractional integrals, which regard as an important class of operators in harmonic analysis, have played a key role in the fields of harmonic analysis, applied probability and physics communities. For example, Sawano and Tanaka in [18] proved that fractional integral is bounded on Morrey space over non-doubling measures. Based on this work, the boundedness of fractional integral on Morrey space over non-homogeneous metric measure space is obtained by Cao and Zhou in [1]. Shen *et.al* used the generalization of a parameterized inexact Uzawa method to solve such a kind of saddle point problem for fractional diffusion equations (see [19]). However, in this paper, we will mainly consider the boundedness of θ-type generalized fractional integrals, which are slightly modified in [5], on Lebesgue and Morrey spaces with variable exponents over non-homogeneous spaces. What's more, the results of this paper extend the contents of fractional integral on variable exponent spaces over $\mathbb{R}^n$ and non-homogeneous spaces.

Let $X := (X, d, \mu)$ be a *quasimetric measure space*, if μ is a complete measure, and there exists a non-negative real-valued function d on $X \times X$ satisfying the following conditions:

(1) $d(x, x) = 0$ for all x in X; (2) $d(x, y) > 0$ for all $x \neq y$, x, $y \in X$;

(3) for all x, y, $z \in X$, there exists a constant $a_1 > 0$, such that $d(x, y) \leq a_1(d(x, z) + d(y, z))$;

(4) there exists a constant $a_0 > 0$, such that $d(x, y) \leq a_0 d(y, x)$ for all $x, y \in X$.

Moreover, we always assume that balls $B(x, r) := \{y \in X : d(x, y) < r\}$ are measurable, $0 \leq \mu(B(x, r)) < \infty$, $\mu(X) < \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$ and $r > 0$ in this paper.

A measure μ on X is said to satisfy the following growth condition, if there exists a constant $C > 0$ such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x, r)) \leq Cr. \tag{1.1}$$

Then the space (X, d, μ) with measure μ satisfying (1.1) is called a *non-homogeneous space*. In this setting, Kokilashvili and Meskhi obtained the boundedness of Maximal function and Riesez potential on variable Morrey spaces(see [7]). In [10], Lu proved that parameter Marcinkiewicz integral and its commutator are bounded on Morrey spaces with variable exponent and so on.

In this paper, we set that p is a μ-measurable function on X, and respectively define

$$p_-(E) := \inf_E p(x), \qquad p_+(E) := \inf_E p(x),$$

where $E \subset X$ is a μ-measurable. Moreover, we also denote $p_- = p_-(X)$ and $p_+ = p_+(X)$.

We now recall the following definitions introduced in [7].

Definition 1.1. Let $N \geq 1$ be a constant. Suppose that p is a function on X such that $0 < p_- < p_+ < \infty$. We say that $p \in \mathcal{P}(N)$ if there exists a constant $C > 0$ such that,

$$[\mu(B(x, Nr))]^{p_-(B(x,r)) - p_+(B(x,r))} \leq C, \tag{1.2}$$

for all $x \in X$ and $r > 0$.

Definition 1.2. Let $0 < p_- \leq p_+ < \infty$. We say that a function p on X satisfies the Log-Hölder continuity condition $p \in \mathrm{LH}(X)$ if

$$|p(x) - p(y)| \leq \frac{A}{-\log(d(x,y))}, \quad d(x,y) \leq \frac{1}{2}, \tag{1.3}$$

where constant $A > 0$ does not depend on $x,\ y \in X$.

For any ball B, we respectively denote its center and radius by c_B and r_B (or $r(B)$). Let $\eta > 1$ and $\beta > \eta$, a ball B is said to be an (η,β)-doubling ball if $\mu(\eta B) \leq \beta\mu(B)$, where ηB denotes the ball with the same center as B and $r(\eta B) = \eta r(B)$. Especially, for any given ball B, we denote by $\widetilde{B}$ the smallest doubling ball which contains B and has the same center as B. Given two balls $B \subset S$ in X, set

$$K_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(2^k B)}{r(2^k B)}, \tag{1.4}$$

where $N_{B,S}$ is denoted by the smallest integer k such that $r(2^k B) \geq r(S)$.

The following notion of regular bounded mean oscillation (RBMO) space is from [20].

Definition 1.3. Let $\tau > 1$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{RBMO}(\mu)$ if there exists a constant $C > 0$ such that for any ball B centered at some point of $\mathrm{supp}(\mu)$,

$$\frac{1}{\mu(\tau B)} \int_B |f(y) - m_{\widetilde{B}}(f)| \mathrm{d}\mu(y) \leq C \tag{1.5}$$

and

$$|m_B(f) - m_S(f)| \leq C K_{B,S} \tag{1.6}$$

for any two doubling balls $B \subset S$, where $m_B(f)$ represents the mean value of function f over ball B, that is,

$$m_B(f) = \frac{1}{\mu(B)} \int_B f(x) \mathrm{d}\mu(x).$$

Moreover, the minimal constant C satisfying (1.5) and (1.6) is defined to be the norm of f in the space $\mathrm{RBMO}(\mu)$ and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

Now we state the definition of θ-type generalized fractional integral kernel as follows.

Definition 1.4. Let $\alpha \in (0,1)$, and θ be a non-negative and non-decreasing function on $(0,\infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} |\log t| \mathrm{d}t < \infty. \tag{1.7}$$

A function $K_\alpha \in L^1_{\mathrm{loc}}(X \times X \setminus \{(x,x):\ x \in X\})$ is called an θ-type generalized fractional integral kernel if there exists a positive constant C_{K_α} depending on K_α, such that

(1) for all $x,\ y \in X$ with $x \neq y$,

$$|K_\alpha(x,y)| \leq C_{K_\alpha} \frac{1}{[d(x,y)]^{1-\alpha}}, \tag{1.8}$$

(2) there exists a constant $c_{K_\alpha} \in (0,\infty)$ such that, for all $x, \tilde{x}, y \in X$ with $d(x,y) \geq c_{K_\alpha} d(x,\tilde{x})$,

$$|K_\alpha(x,y) - K_\alpha(\tilde{x},y)| + |K_\alpha(y,x) - K_\alpha(y,\tilde{x})| \leq C_{K_\alpha} \theta\Big(\frac{d(x,\tilde{x})}{d(x,y)}\Big) \frac{1}{[d(x,y)]^{1-\alpha}}. \tag{1.9}$$

Remark 1.1. If we take the function $\theta(t) \equiv t^{\delta}$ with $\delta \in (0, 1]$, then the θ-type generalized fractional integral kernel K_α is just the fractional kernel of order 1 (see [7]).

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A linear T_α is called an θ-type generalized fractional integral with K_α satisfying (1.8) and (1.9) if, for all $f \in L^\infty_b(\mu)$ and $x \notin \operatorname{supp}(f)$,

$$T_\alpha(f)(x) = \int_X K_\alpha(x, y)f(y)\mathrm{d}\mu(y). \tag{1.10}$$

Given a function $b \in \mathrm{RBMO}(\mu)$, the commutator $[b, T_\alpha]$ which is generated by T_α and b is defined by

$$[b, T_\alpha](f)(x) = b(x)T_\alpha f(x) - T_\alpha(bf)(x), \quad \text{for any } x \in X. \tag{1.11}$$

The following definition of variable exponent Morrey space $M^{p(\cdot)}_{q(\cdot)}(X)_N$ is from [7].

Definition 1.5. Let $N \geq 1$ be a constant and $1 < q_- \leq q(x) \leq p(x) \leq p_+ < \infty$. Then, the variable exponent Morrey space $M^{p(\cdot)}_{q(\cdot)}(X)_N$ is defined by

$$M^{p(\cdot)}_{q(\cdot)}(X)_N := \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mu) : \ \|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N} < \infty \right\},$$

where

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N} := \sup_{x\in X, r>0} [\mu(B(x, Nr))]^{\frac{1}{p(x)} - \frac{1}{q(x)}} \|f\|_{L^{q(\cdot)}(B(x,r))}. \tag{1.12}$$

Remark 1.2. If we take $p(x) = q(x)$ in (1.12), then, the variable exponent Morrey space $M^{p(\cdot)}_{q(\cdot)}(X)_N$ is just variable exponent Lebesgue space $L^{p(\cdot)}(X)$ (see [7]), namely, for any μ-measurable subset $E \subset X$ and $1 \leq p_-(E) \leq p_+(E) < \infty$, then variable exponent Lebesgue space $L^{p(\cdot)}(E)$ is defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\lambda > 0 : \ S_p(f/\lambda) \leq 1\}, \tag{1.13}$$

where

$$S_p(f) := \int_E |f(x)|^{p(x)}\mathrm{d}\mu(x) < \infty.$$

The organization of this paper is stated as follows. In section 2, via some known results, we prove that θ-type fractional integral T_α is bounded from variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ into spaces $L^{q(\cdot)}(X)$ for $\alpha \in (0, 1)$ and $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \alpha$, and bounded from variable exponent Morrey spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$, where $\frac{t(\cdot)}{s(\cdot)} = \frac{q(\cdot)}{p(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} - \alpha$, $\bar{a} := a_1(a_1(a_0 + 1) + 1)$ and N is a constant with $N \geq 1$. By establishing the sharp maximal function for commutator $[b, T_\alpha]$ generated by T_α and $b \in \mathrm{RBMO}(\mu)$, the boundedness of the $[b, T_\alpha]$ on spaces $L^{p(\cdot)}(X)$ and on spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ is also obtained in sections 3 and 4.

Finally, we make some conventions on notation. Throughout the whole paper, C represents a positive constant being independent of the main parameters. For any subset E of X, we use χ_E to denote its characteristic function.

2. Estimate for T_α on variable exponent spaces

In this section, by applying some known results, the boundedness of θ-type generalized fractional integral T_α on variable Lebsgue spaces $L^{p(\cdot)}(X)$ and on variable exponent Morrey spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ is obtained. Now we state the main theorems as follows.

Theorem 2.1. Let $N \geq 1$ be a constant, K_α satisfy (1.8) and (1.9), $1 < p_- \leq p(x) \leq q(x) \leq q_+ < \infty$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \alpha$ and $0 < p_+ < \frac{1}{\alpha}$. Suppose that $p \in \mathcal{P}(N)$, $q \in \mathcal{P}(1)$ and μ satisfies (1.1). Then T_α defined as in (1.10) is bounded from variable Lebesgue spaces $L^{p(\cdot)}(X)$ into spaces $L^{q(\cdot)}(X)$.

Theorem 2.2. Let $N \geq 1$ be a constant, K_α satisfy (1.8) and (1.9), $1 < t_- \leq t(x) \leq s(x) \leq s_+ < \infty$, $\frac{s(x)}{t(x)} = \frac{p(x)}{q(x)}$ and $\frac{1}{s(x)} = \frac{1}{p(x)} - \alpha$ with $0 < p_+ < \frac{1}{\alpha}$. Suppose that μ satisfies (1.1), $p \in \mathcal{P}(N)$ and $q \in \mathcal{P}(1)$. Then T_α defined as in (1.10) is bounded from variable Morrey spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$.

Remark 2.1. By Remark 1.7, once Theorem 2.2 is proved, it is easy to see that Theorem 2.1 holds. Thus, we only prove Theorem 2.2 in this section.

Proof of Theorem 2.2. For any $x \in X$, by (1.9), we can deduce that

$$\begin{aligned}|T_\alpha f(x)| &\leq \int_X |K_\alpha(x,y)||f(y)|\mathrm{d}\mu(\mathrm{y}) \\ &\leq C\int_X \frac{|f(y)|}{[d(x,y)]^{1-\alpha}}\mathrm{d}\mu(y) \leq CI_\alpha(|f|)(x),\end{aligned}$$

where I_α represents the homogeneous fractional integral operator (see [7]), namely, for any $x \in X$, set

$$I_\alpha f(x) := \int_X \frac{f(y)}{[d(x,y)]^{1-\alpha}}\mathrm{d}\mu(y), \quad \text{for } 0 < \alpha < 1.$$

Further, by applying the $(M^{p(\cdot)}_{q(\cdot)}(X)_N, M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}})$-boundedness of I_α in [7], we have

$$\|T_\alpha f\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}} \leq C\|I_\alpha(|f|)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}} \leq C\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N}.$$

3. Estimate for $[b, T_\alpha]$ on $L^{p(\cdot)}(X)$

In this section, by establishing the sharp maximal function for commutator $[b, T_\alpha]$, which is generated by T_α and $b \in \mathrm{RBMO}(\mu)$, we prove that the $[b, T_\alpha]$ is bounded from space $L^{p(\cdot)}(X)$ into space $L^{q(\cdot)}(X)$. The main theorem of this section is as follows.

Theorem 3.1. Let $N \geq 1$ be a constant, $b \in \mathrm{RBMO}(\mu)$, K_α satisfy (1.8) and (1.9), $1 < p_- \leq p(x) < p_+ < \frac{1}{\alpha}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \alpha$ with $0 < \alpha < 1$. Suppose that μ satisfies condition (1.1). Then $[b, T_\alpha]$ defined as in (1.11) is bounded from $L^{p(\cdot)}(X)$ into $L^{q(\cdot)}(X)$.

To prove the above theorem, we need to recall and establish the following corollary and lemmas, see [6, 7], respectively.

Corollary 3.1. If $f \in \mathrm{RBMO}(\mu)$, then there exists a constant $C > 0$ such that, for any balls B, $\rho \in (1, \infty)$ and $r \in [1, \infty)$,

$$\left(\frac{1}{\mu(\rho B)}\int_B |f(y) - m_{\widetilde{B}}(f)|^r \mathrm{d}\mu(x)\right)^{\frac{1}{r}} \leq C\|f\|_{\mathrm{RBMO}(\mu)}. \tag{3.1}$$

Lemma 3.1. Let $\mu(X) < \infty$, $N \geq 1$ be a constant, $1 < p_- \leq p(x) \leq p_+ < \infty$ and $s \in (1, p_-)$. If there exists a positive constant C such that for all $x \in X$ and $r > 0$, the following inequality

$$[\mu(B(x, Nr))]^{p_-(B(x,r)) - p(x)} \leq C$$

holds, then $M_{s,N}$ is bounded on $L^{p(\cdot)}(X)$, where maximal operator $M_{s,N}$ is defined by, for any $f \in L^1_{\rm loc}(X)$,

$$M_{s,N}f(x) = \sup_{B \ni x}\left(\frac{1}{\mu(B(x,Nr))}\int_B |f(y)|^s {\rm d}\mu(y)\right)^{\frac{1}{s}}. \tag{3.2}$$

Moreover, if $s = 1$ in (3.2), we simply denote $M_N := M_{1,N}$.

Lemma 3.2. Let $\mu(B) < \infty$, $N \ge 1$ be a constant, $\tau \in (0,1)$, $s \in (1, \frac{1}{\tau})$, $s < p_- \le p(\cdot) < \frac{1}{\tau}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \tau$. Then there exists a constant $C > 0$ such that, for all $f \in L^{p(\cdot)}(X)$,

$$\|M^{(\tau)}_{s,N}f\|_{L^{q(\cdot)}(X)} \le C\|f\|_{L^{p(\cdot)}}(X),$$

where

$$M^{(\tau)}_{s,N}f(x) = \sup_{B \ni x}[\mu(B(x,Nr))]^{\tau-\frac{1}{s}}\left(\int_B |f(y)|^s {\rm d}\mu(y)\right)^{\frac{1}{s}}, \tag{3.3}$$

and the supremum is taken over all balls $B \ni x$.

Remark 3.1. With a way similar to that used in the proof of Theorem 1.3 in [2], it is easy to show that Lemma 3.4 hold on (X, d, μ).

Also, by applying Theorem 1.13 in [5], we have the following result on (X, d, μ).

Lemma 3.3. Let K_α satisfying (1.8) and (1.9), $\alpha \in (0,1)$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. Suppose that T_α defined as in (1.10) is bounded on $L^2(\mu)$. Then T_α is bounded from $L^p(\mu)$ into $L^q(\mu)$.

From [6], the sharp maximal function $\widetilde{M}^{\sharp,\alpha}$ is defined by, for all $x \in X$, $\alpha \in [0,1)$ and $f \in L^1_{\rm loc}(\mu)$,

$$\widetilde{M}^{\sharp,\alpha}f(x) = \sup_{B \ni x}\frac{1}{\mu(\frac{3}{2}B)}\int_B |f(y) - m_{\widetilde{B}}f|{\rm d}\mu(y) + \sup_{(B,S)\in\Delta_x}\frac{|m_B f - m_S f|}{\widetilde{K}^{(\alpha)}_{B,S}}, \tag{3.4}$$

where $\Delta_x = \{x \in B \subset S \text{ and } B,\ S \text{ are doubling balls}\}$ and coefficient $\widetilde{K}^{(\alpha)}_{B,S}$ is defined by

$$\widetilde{K}^{(\alpha)}_{B,S} := 1 + \sum_{k=1}^{N_{B,S}}\left[\frac{\mu(2^kB)}{r(2^kB)}\right]^{1-\alpha}.$$

For $0 < r < \infty$ and $x \in X$, set $M^{\sharp,\alpha}_r f(x) = [M^{\sharp,\alpha}(|f|^r)(x)]^{\frac{1}{r}}$. A simple computation shows that if $0 < r < 1$, we have

$$M^{\sharp,\alpha}_r f(x) \le C_r M^{\sharp,\alpha}f(x), \quad x \in X. \tag{3.5}$$

Lemma 3.4. Let $\tau \in (0,1)$, $g \in L^1_{\rm loc}(X)$ and μ-measurable function f satisfy the following condition

$$\mu(\{x \in X : |f(x)| > t\}) < \infty, \quad \text{for all } t > 0,$$

then

$$\int_X |f(x)g(x)|{\rm d}\mu(x) \le \int_X M^{\sharp,\alpha}_\tau(f)(x)M_N(g)(x){\rm d}\mu(x). \tag{3.6}$$

Lemma 3.5. Let K_α satisfy the conditions (1.8) and (1.9), $s \in (1,\infty)$ and $p_0 \in (1,\infty)$. If T_α is bounded on $L^2(\mu)$, then there exists a positive constant C such that, for all $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$,

$$M^{\sharp,\alpha}([b,T_\alpha]f)(x) \le C\|b\|_{{\rm RBMO}(\mu)}\left\{M^{(\alpha)}_{s,\frac{3}{2}}f(x) + M_{s,\frac{3}{2}}(T_\alpha f)(x) + T_\alpha(|f|)(x)\right\}. \tag{3.7}$$

Proof. By applying the definition of sharp maximal function $M^{\sharp,\alpha}$ defined as in (3.4), for any ball B, it suffices to show that, for all x and B with $B \ni x$,

$$\frac{1}{\mu(\frac{3}{2}B)}\int_B |[b,T_\alpha]f(y)-h_B|\mathrm{d}\mu(y)\le C\|b\|_{\mathrm{RBMO}(\mu)}\Big\{M^{(\alpha)}_{s,\frac{3}{2}}f(x)+M_{s,\frac{3}{2}}(T_\alpha f)(x)\Big\}. \tag{3.8}$$

and, for all balls $B,\ S$ with $B\subset S$ and $B\ni x$,

$$|h_B-h_R|\le C\|b\|_{\mathrm{RBMO}(\mu)}\Big\{M^{(\alpha)}_{s,\frac{3}{2}}f(x)+T_\alpha(|f|)(x)\Big\}K_{B,S}\widetilde{K}^{(\alpha)}_{B,S}, \tag{3.9}$$

where

$$h_B=m_B(T_\alpha([b-b_B]f\chi_{X\setminus(\frac{3}{2}B)}))\quad h_S=m_S(T_\alpha([b-b_S]f\chi_{X\setminus(\frac{3}{2}S)}))$$

To prove (3.8), decompose f as $f:=f_1+f_2:=f\chi_{\frac{3}{2}B}+f\chi_{X\setminus(\frac{3}{2}B)}$, then write,

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}B)}\int_B |[b,T_\alpha]f(y)-h_B|\mathrm{d}\mu(y)\\
&=\frac{1}{\mu(\frac{3}{2}B)}\int_B |(b(y)-b_B)T_\alpha(f)(y)+T_\alpha([b-b_B]f)(y)-h_B|\mathrm{d}\mu(y)\\
&\le\frac{1}{\mu(\frac{3}{2}B)}\int_B |(b(y)-b_B)T_\alpha(f)(y)|\mathrm{d}\mu(y)+\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha([b-b_B]f_1)(y)|\mathrm{d}\mu(y)\\
&\quad+\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha([b-b_B]f_2)(y)-h_B|\mathrm{d}\mu(y)\\
&=\mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3.
\end{aligned}$$

From Hölder inequality, Corollary 3.2 and (3.2), it follows that

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}B)}\int_B |(b(y)-b_B)T_\alpha(f)(y)|\mathrm{d}\mu(y)\\
&\le\Big(\frac{1}{\mu(\frac{3}{2}B)}\int_B |b(y)-b_B|^{s'}\mathrm{d}\mu(y)\Big)^{\frac{1}{s'}}\Big(\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha(f)(y)|^{s}\mathrm{d}\mu(y)\Big)^{\frac{1}{s}}\\
&\le C\|b\|_{\mathrm{RBMO}(\mu)}M_{s,\frac{3}{2}}(T_\alpha f)(x).
\end{aligned}$$

To estimate D_2, take $t=\sqrt{s}$ and $\frac{1}{r}=\frac{1}{t}-\alpha$. By applying Hölder inequality, Corollary 3.2 and Lemma 3.6, we obtain that

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha([b-b_B]f_1)(y)|\mathrm{d}\mu(y)\\
&\le\frac{1}{\mu(\frac{3}{2}B)}\Big(\int_B |T_\alpha([b-b_B]f_1)(y)|^{r}\mathrm{d}\mu(y)\Big)^{\frac{1}{r}}[\mu(B)]^{1-\frac{1}{r}}\le C\frac{[\mu(B)]^{1-\frac{1}{r}}}{\mu(\frac{3}{2}B)}\|(b-b_B)f_1\|_{L^t(\mu)}\\
&\le C\frac{[\mu(\frac{3}{2}B)]^{\frac{1}{tt'}+\frac{1}{tt}-\alpha}}{[\mu(\frac{3}{2}B)]^{\frac{1}{r}}}\Big(\frac{1}{\mu(\frac{3}{2})}\int_B |b(y)-b_B|^{tt'}\mathrm{d}\mu(y)\Big)^{\frac{1}{tt'}}\Big\{\frac{1}{[\mu(\frac{3}{2})]^{1-\alpha s}}\int_B |f(y)|^{s}\mathrm{d}\mu(y)\Big\}^{\frac{1}{s}}\\
&\le C\|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{s,\frac{3}{2}}(f)(x).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha([b-b_B]f_2)(y)-h_B|\mathrm{d}\mu(y)\\
&=\frac{1}{\mu(\frac{3}{2}B)}\int_B \left|T_\alpha([b-b_B]f_2)(y)-\frac{1}{\mu(B)}\int_B T_\alpha([b-b_B]f_2)(z)\mathrm{d}\mu(z)\right|\mathrm{d}\mu(y)\\
&\le \frac{1}{\mu(\frac{3}{2}B)}\frac{1}{\mu(B)}\int_B\int_B |T_\alpha([b-b_B]f_2)(y)-T_\alpha([b-b_B]f_2)(z)|\mathrm{d}\mu(y)\mathrm{d}\mu(z),
\end{aligned}$$

thus, we only estimate the difference $|T_\alpha([b-b_B]f_2)(y)-T_\alpha([b-b_B]f_2)(z)|$. For all $y,\ z\in B$, by applying (1.7), (1.9), Corollary 3.2 and Hölder inequality, we have

$$\begin{aligned}
&|T_\alpha([b-b_B]f_2)(y)-T_\alpha([b-b_B]f_2)(z)|\\
&\le \int_{X\setminus(\frac{3}{2}B)}|K_\alpha(y,w)-K_\alpha(z,w)||b(w)-b_B||f(w)|\mathrm{d}\mu(w)\\
&\le C\int_{X\setminus(\frac{3}{2}B)}\theta\Big(\frac{d(y,z)}{d(y,w)}\Big)\frac{|b(w)-b_B|}{[d(y,w)]^{1-\alpha}}|f(w)|\mathrm{d}\mu(w)\\
&\le C\sum_{k=1}^{\infty}\int_{2^k\times(\frac{3}{2}B)\setminus 2^{k-1}\times(\frac{3}{2}B)}\theta\Big(\frac{d(y,z)}{d(y,w)}\Big)\frac{|b(w)-b_B|}{[d(y,w)]^{1-\alpha}}|f(w)|\mathrm{d}\mu(w)\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[r(2^{k-1}\times\frac{3}{2}B)]^{1-\alpha}}\theta\Big(2^{1-k}\times\frac{2}{3}\Big)\int_{2^k\times(\frac{3}{2}B)}|b(w)-b_B||f(w)|\mathrm{d}\mu(w)\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[r(2^{k-1}\times\frac{3}{2}B)]^{1-\alpha}}\theta\Big(2^{1-k}\times\frac{2}{3}\Big)\Big(\int_{2^k\times(\frac{3}{2}B)}|b(w)-b_{2^k\times(\frac{3}{2}B)}||f(w)|\mathrm{d}\mu(w)\\
&\qquad+|b_B-b_{2^k\times(\frac{3}{2}B)}|\int_{2^k\times(\frac{3}{2}B)}|f(w)|\mathrm{d}\mu(w)\Big)\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[r(2^{k-1}\times\frac{3}{2}B)]^{1-\alpha}}\theta\Big(2^{1-k}\times\frac{2}{3}\Big)\Big\{\Big(\int_{2^k\times(\frac{3}{2}B)}|f(w)|^s\mathrm{d}\mu(w)\Big)^{\frac{1}{s}}\\
&\quad\times\Big(\int_{2^k\times(\frac{3}{2}B)}|b(w)-b_{2^k\times(\frac{3}{2}B)}|^{s'}\mathrm{d}\mu(w)\Big)^{\frac{1}{s'}}\\
&\qquad+|b_B-b_{2^k\times(\frac{3}{2}B)}|[\mu(2^k\times(\frac{3}{2}B))]^{1-\frac{1}{s}}\Big(\int_{2^k\times(\frac{3}{2}B)}|f(w)|^s\mathrm{d}\mu(w)\Big)^{\frac{1}{s}}\Big\}\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[r(2^{k-1}\times\frac{3}{2}B)]^{1-\alpha}}\theta\Big(2^{1-k}\times\frac{2}{3}\Big)\Big\{\Big(\frac{1}{[\mu(2^k\times(\frac{9}{4}B))]^{1-\alpha s}}\int_{2^k\times(\frac{3}{2}B)}|f(w)|^s\mathrm{d}\mu(w)\Big)^{\frac{1}{s}}\\
&\quad\times\Big(\frac{1}{\mu(2^k\times(\frac{9}{4}B))}\int_{2^k\times(\frac{3}{2}B)}|b(w)-b_{2^k\times(\frac{3}{2}B)}|^{s'}\mathrm{d}\mu(w)\Big)^{\frac{1}{s'}}[\mu(2^k\times(\frac{9}{4}B))]^{1-\alpha}\\
&\qquad+k\|b\|_{\mathrm{RBMO}(\mu)}[\mu(2^k\times(\frac{9}{4}B))]^{1-\alpha}\\
&\qquad\quad\times\Big(\frac{1}{[\mu(2^k\times(\frac{9}{4}B))]^{1-\alpha s}}\int_{2^k\times(\frac{3}{2}B)}|f(w)|^s\mathrm{d}\mu(w)\Big)^{\frac{1}{s}}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x) \sum_{k=1}^{\infty}(k+1)\theta\Big(2^{1-k}\times\frac{2}{3}\Big)\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x) \sum_{k=1}^{\infty}k\theta(2^{-k})\int_{2^{-k}}^{2^{-k+1}}\frac{1}{t}\mathrm{d}t\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x) \sum_{k=1}^{\infty}\theta(2^{-k})\int_{2^{-k}}^{2^{-k+1}}|\log t|\frac{1}{t}\mathrm{d}t\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x) \sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}|\log t|\frac{\theta(t)}{t}\mathrm{d}t\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x),
\end{aligned}
$$

where we have used the following fact that

$$
|b_B - b_{2^k\times(\frac{3}{2}B)}| \leq Ck\|b\|_{\mathrm{RBMO}(\mu)}. \tag{3.10}
$$

Thus,

$$
\frac{1}{\mu(\frac{3}{2}B)}\int_B |T_\alpha([b-b_B]f_2)(y)-h_B|\mathrm{d}\mu(y) \leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x),
$$

which, together with D_1 and D_2, implies (3.8).

Now let us estimate (3.9). Consider two balls $B \subset S$ with $x \in B$ and let $N := N_{B,S}+1$. Write

$$
\begin{aligned}
&|h_B - h_S|\\
&= \Big|m_B(T_\alpha([b-b_S]f\chi_{X\setminus\frac{3}{2}B})) + m_B(T_\alpha([b_S-b_B]f\chi_{X\setminus\frac{3}{2}B})) - m_S(T_\alpha([b-b_S]f\chi_{X\setminus\frac{3}{2}S}))\Big|\\
&= \Big|m_B(T_\alpha([b-b_S]f\chi_{X\setminus 2^N B})) + m_B(T_\alpha([b-b_S]f\chi_{2^N B\setminus\frac{3}{2}B})) + m_B(T_\alpha([b_S-b_B]f\chi_{X\setminus\frac{3}{2}B}))\\
&\quad - m_S(T_\alpha([b-b_S]f\chi_{X\setminus 2^N B})) - m_S(T_\alpha([b-b_S]f\chi_{2^N B\setminus\frac{3}{2}S}))\Big|\\
&\leq \Big|m_B(T_\alpha([b-b_S]f\chi_{X\setminus 2^N B})) - m_S(T_\alpha([b-b_S]f\chi_{X\setminus 2^N B}))\Big|\\
&\quad + |m_B(T_\alpha([b_S-b_B]f\chi_{X\setminus\frac{3}{2}B}))| + |m_B(T_\alpha([b-b_S]f\chi_{2^N B\setminus\frac{3}{2}B}))| + |m_S(T_\alpha([b-b_S]f\chi_{2^N B\setminus\frac{3}{2}S}))|\\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned}
$$

With arguments similar to that used in the estimate of D_3 and Theorem 1 in [22], it is not difficult to obtain that

$$
E_1 \leq C\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{s,\frac{3}{2}}(f)(x)
$$

and

$$
E_2 \leq CK_{B,S}\|b\|_{\mathrm{RBMO}(\mu)}[T_\alpha(|f|)(x) + M^{(\alpha)}_{s,\frac{3}{2}}(f)(x)].
$$

For any $y \in B$, by applying Hölder inequality, Corollary 3.2 and (3.2), we obtain that

$$
|T_\alpha([b-b_S]f\chi_{2^N B\setminus\frac{3}{2}B})(y)|
$$

$$
\begin{aligned}
&\leq \int_{2^N B\setminus\frac{3}{2}B} |K_\alpha(x,y)||b(w)-b_S||f(w)|\mathrm{d}\mu(w)\\
&\leq C\sum_{k=1}^{N-1}\int_{(2^{k+1}\times\frac{3}{2}B)\setminus(2^k\times\frac{3}{2}B)} \frac{|b(w)-b_S||f(w)|}{[d(y,w)]^{1-\alpha}}\mathrm{d}\mu(w)\\
&\leq C\sum_{k=1}^{N-1}\frac{1}{[r(2^k\times\frac{3}{2}B)]^{1-\alpha}}\bigg(|b_{2^{k+1}\times\frac{3}{2}B}-b_S|\int_{2^{k+1}\times\frac{3}{2}B}|f(w)|\mathrm{d}\mu(w)\\
&\qquad+\int_{2^{k+1}\times\frac{3}{2}B}|b(w)-b_{2^{k+1}\times\frac{3}{2}B}||f(w)|\mathrm{d}\mu(w)\bigg)\\
&\leq C\sum_{k=1}^{N-1}\frac{1}{[r(2^k\times\frac{3}{2}B)]^{1-\alpha}}\bigg\{|b_{2^{k+1}\times\frac{3}{2}B}-b_S|\bigg(\int_{2^{k+1}\times\frac{3}{2}B}|f(w)|^s\mathrm{d}\mu(w)\bigg)^{\frac{1}{s}}\\
&\qquad\times[\mu(2^{k+1}\times\frac{3}{2}B)]^{1-\frac{1}{s}}+\bigg(\int_{2^{k+1}\times\frac{3}{2}B}|f(w)|^s\mathrm{d}\mu(w)\bigg)^{\frac{1}{s}}\\
&\qquad\quad\times\bigg(\int_{2^{k+1}\times\frac{3}{2}B}|b(w)-b_{2^{k+1}\times\frac{3}{2}B}|^{s'}\mathrm{d}\mu(w)\bigg)^{\frac{1}{s'}}\bigg\}\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{s,\frac{3}{2}}(f)(x)\bigg\{\sum_{k=1}^{N-1}\frac{[\mu(2^{k+1}\times\frac{3}{2}B)]^{1-\alpha}}{[r(2^k\times\frac{3}{2}B)]^{1-\alpha}}\bigg\}\\
&\leq C\widetilde{K}_{B,S}\|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{s,\frac{3}{2}}(f)(x).
\end{aligned}
$$

Taking the mean over ball B, we get $\mathrm{E}_3 \leq C\widetilde{K}_{B,S}\|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{s,\frac{3}{2}}(f)(x)$. Similarly, we have

$$
\mathrm{E}_4 \leq C\widetilde{K}_{B,S}\|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{s,\frac{3}{2}}(f)(x).
$$

Which, combining the estimates E_1, E_2 and E_3, implies (3.9).

Proof of Theorem 3.1. By applying Lemmas 3.3 and 3.4 , Lemmas 3.6-3.8 and Hölder inequality, we can deduce that

$$
\begin{aligned}
&\|[b,T_\alpha]f\|_{L^{q(\cdot)}(X)}\\
&\quad=\sup_{\|g\|_{L^{q'(\cdot)}(X)}}\bigg|\int_X([b,T_\alpha]f)(x)g(x)\mathrm{d}\mu(x)\bigg|\\
&\quad\leq C\sup_{\|g\|_{L^{q'(\cdot)}(X)}}\bigg|\int_X M^{\sharp,\alpha}_{\tau}([b,T_\alpha]f)(x)M_N(g)(x)\mathrm{d}\mu(x)\bigg|\\
&\quad\leq C\sup_{\|g\|_{L^{q'(\cdot)}(X)}}\bigg|\int_X M^{\sharp,\alpha}([b,T_\alpha]f)(x)M_N(g)(x)\mathrm{d}\mu(x)\bigg|\\
&\quad\leq C\|b\|_{\mathrm{RBMO}(\mu)}\sup_{\|g\|_{L^{q'(\cdot)}(X)}}\bigg|\int_X\bigg\{M^{(\alpha)}_{s,\frac{3}{2}}f(x)M_N(g)(x)+M_{s,\frac{3}{2}}(T_\alpha f)(x)M_N(g)(x)\\
&\qquad\quad+T_\alpha(|f|)(x)M_N(g)(x)\bigg\}\mathrm{d}\mu(x)\bigg|\\
&\quad\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^{p(\cdot)}(X)}.
\end{aligned}
$$

4. Estimate for $[b, T_\alpha]$ on spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$

The main theorem of this section is stated as follows.

Theorem 4.1. Let $b \in \mathrm{RBMO}(\mu)$, K_α satisfy (1.8) and (1.9), $\mu(X) < \infty$, $N := a_1(1 + 2a_0)$, $1 < p_- \le p(x) \le q(x) \le q_+ < \infty$, $1 < t_- \le t(x) \le s(x) \le s_+ < \infty$, $\frac{t(x)}{s(x)} = \frac{q(x)}{p(x)}$, $\frac{1}{s(x)} = \frac{1}{p(x)} - \alpha$ and $0 < p(\cdot) < \frac{1}{\alpha}$ and $p \in \mathcal{P}(N)$. Suppose that μ satisfies (1.1). Then $[b, T_\alpha]$ defined as in (1.11) is bounded from spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$.

To prove the above theorem, we need to establish the following lemmas.

Lemma 4.1. Let $\mu(X) < \infty$, $1 < p_- \le p(x) \le q(x) \le q_+ < \infty$. Suppose that $N = a_1(1 + 2a_0)$ and $p \in \mathcal{P}(N)$, $q \in \mathcal{P}(1)$. Then $M_{s,N}$ defined as in (3.2) is bounded from spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}$.

Remark 4.1. With a slight modified argument similar to that use in the proof of Theorem 3.4 in [7], it is not difficult to prove that Lemma 4.2 also holds.

Lemma 4.2. Let N be a constant satisfying the condition $N \ge 1$ and $\tau \in (0, 1)$. Suppose that $1 \le s < q_- \le q(x) \le p(x) \le p_+ < \infty$, $s < \frac{1}{\tau}$ and $1 < p(x) < \frac{1}{\tau}$. Suppose that μ satisfies condition (1.1). Then

$$|M^{(\tau)}_{s,N} f(x)| \le C \|f\|^{\tau p(x)}_{M^{p(\cdot)}_{q(\cdot)}(X)_N} [M_{s,N} f(x)]^{1-\tau p(x)}. \tag{4.1}$$

Proof. For any $x \in X$, we set $\ell_x^{\frac{1}{p(x)}} = \frac{\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N}}{M_{s,N} f(x)}$. Then

$$\begin{aligned}
|M^{(\tau)}_{s,N} f(x)| \le & \sup_{x\in B, \mu(B(x,Nr)) \le \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} \Big(\int_B |f(y)|^s \mathrm{d}\mu(y) \Big)^{\frac{1}{s}} \\
& + \sup_{x\in B, \mu(B(x,Nr)) > \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} \Big(\int_B |f(y)|^s \mathrm{d}\mu(y) \Big)^{\frac{1}{s}} =: \mathrm{H}_1 + \mathrm{H}_2.
\end{aligned}$$

For H_1, we obtain that

$$\begin{aligned}
\mathrm{H}_1 &= \sup_{x\in B, \mu(B(x,Nr)) \le \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} \Big(\int_B |f(y)|^s \mathrm{d}\mu(y) \Big)^{\frac{1}{s}} \\
&= \sup_{x\in B, \mu(B(x,Nr)) \le \ell_x} [\mu(B(x, Nr))]^{\tau} \Big(\frac{1}{\mu(B(x, Nr))} \int_B |f(y)|^s \mathrm{d}\mu(y) \Big)^{\frac{1}{s}} \\
&\le \ell_x^\tau M_{s,N} f(x) = \|f\|^{\tau p(x)}_{M^{p(\cdot)}_{q(\cdot)}(X)_N} [M_{s,N} f(x)]^{1-\tau p(x)}.
\end{aligned}$$

If $\mu(B(x, Nr)) > \ell_x$, then there exists a $i \in \mathbb{N}$ such that $2^{i-1}\ell_x < \mu(B(x, Nr)) < 2^i \ell_x$. By applying Hölder inequality and Definition 1.6, we can deduce that

$$\begin{aligned}
\mathrm{H}_2 \le & \sup_{x\in B, \mu(B(x,Nr)) > \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} \|f\chi_B\|_{L^{q(x)}(X)} \|\chi_B\|_{L^{\frac{1}{s} - \frac{1}{q(x)}}(X)} \\
\le & \sup_{x\in B, \mu(B(x,Nr)) > \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} \|f\chi_B\|_{L^{q(x)}(X)} [\mu(B)]^{\frac{1}{s} - \frac{1}{q(x)}} \\
\le & \|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} \sup_{x\in B, \mu(B(x,Nr)) > \ell_x} [\mu(B(x, Nr))]^{\tau - \frac{1}{s}} [\mu(B(x, Nr))]^{\frac{1}{q(x)} - \frac{1}{p(x)}} [\mu(B)]^{\frac{1}{s} - \frac{1}{q(x)}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} \sup_{x\in B,\mu(B(x,Nr))>\ell_x} [\mu(B(x,Nr))]^{\tau-\frac{1}{q_+}}[\mu(B(x,Nr))]^{\frac{1}{q(x)}-\frac{1}{p(x)}}\\
&\leq C\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} \sup_{i\in\mathbb{N}}(2^i)^{\tau-\frac{1}{p(x)}}(\ell_x)^{\tau-\frac{1}{p(x)}}\\
&\leq C\|f\|^{\tau p(x)}_{M^{p(\cdot)}_{q(\cdot)}(X)_N}[M_{s,N}f(x)]^{1-\tau p(x)}.
\end{aligned}
$$

Which, together with estimate H_1, the proof of Lemma 4.4 is completed.

By applying Lemmas 4.2 and 4.4, it is easy to get the following result.

Lemma 4.3. Let $\mu(X) < \infty$, $1 < p_- \leq p(x) \leq q(x) \leq q_+ < \infty$, $1 < t_- \leq t(x) \leq s(x) \leq s_+ < \infty$, $\frac{t(x)}{s(x)} = \frac{q(x)}{p(x)}$ and $\frac{1}{s(x)} = \frac{1}{p(x)} - \tau$ satisfying $0 < \tau < \frac{1}{p_+}$. Suppose that $N = a_1(1 + 2a_0)$ and $p \in \mathcal{P}(N)$, $q \in \mathcal{P}(1)$. Then $M^{(\tau)}_{s,N}$ defined as in (3.3) is bounded from spaces $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into spaces $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$.

Proof of Theorem 4.1 From Theorem 2.2, Lemmas 3.8, 4.2 and 4.4, it follows that

$$
\begin{aligned}
&\|[b,T_\alpha](f)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}} \leq C\|M^{\sharp,(\alpha)}_{\tau}[b,T_\alpha](f)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}}\\
&\leq C\|M^{\sharp,(\alpha)}[b,T_\alpha](f)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}}\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)}\left\{\|M^{(\alpha)}_{s,\frac{3}{2}}f\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}} + \|M_{s,\frac{3}{2}}(T_\alpha f)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}} + \|T_\alpha(|f|)\|_{M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}}\right\}\\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N}.
\end{aligned}
$$

5. Conclusions

In this paper, we mainly obtain the boundedness of θ-type generalized fractional integral T_θ and its commutator $[b, T_\theta]$ generated by b and T_θ on variable Lebesgue space $L^{p(\cdot)}(X)$ and Morrey space $M^{p(\cdot)}_{q(\cdot)}(X)_N$.

Acknowledgments

The authors would like to express sincere thanks to the reviewers for their helpful comments and suggestions. This research is supported by the Innovation Fund Project for Higher Education of Gansu Province (2020A-010) and Young Teachers' Scientific Research Ability Promotion Project of Northwest Normal University (NWNU-LKQN2020-07).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Y. Cao, J. Zhou, Morrey spaces for nonhomogenerous metric measure spaces, *Abstr. Appl. Anal.,* **2013** (2013), 1–8.

2. C. Capone, D. Cruz-Uribe, A. Fiorenza, The fractional maximal operator and fractional integrals on variable L^p spaces, *Rev. Mat. Iberoamer.,* **23** (2007), 743–770.

3. L. Ephremidze, V. Kokilashvili, S. Samko, Fractional, maximal and singular operators in variable exponent Lorentz spaces, *Fract. Calc. Appl. Anal.,* **11** (2008), 407–420.

4. X. Fan, Variable exponent Morrey and Campanato spaces, *Nonlinear Anal.,* **72** (2010), 4148–4161.

5. X. Fu, D. Yang, W. Yuan, Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces, *Taiwanese J. Math.,* **18** (2014), 509–557.

6. G. Hu, H. Lin, D. Yang, Marcinkiewicz integrals with non-doubling measures, *Integral Equations Operator Theory,* **58** (2007), 205–238.

7. V. Kokilashvili, A. Meskhi, Maximal functions and potentials in variable exponent Morrey spaces with non-doubling measure, *Complex Var. Elliptic Equ.,* **55** (2010), 923–936.

8. V. Kokilashvili, A. Meskhi, Maximal and Calderón-Zygmund operators in weighted grand variable exponent Lebesgue space, *Trans. A. Razmadze Math. Inst.,* **173** (2019), 127–131.

9. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math.,* **41** (1991), 592–618.

10. G. Lu, Parameter Marcinkiewicz integral on non-homogeneous Morrey space with variable exponent, *Politehn. Univ. Bucharest Sci. Bull. Ser. A,* **83** (2021), 89–98.

11. G. Lu, Commutators of bilinear pseudo-differential operators on local Hardy spaces with variable exponents, *Bull. Braz. Math. Soc.,* **51** (2020), 975–1000.

12. G. Mingione, V. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, *J. Math. Anal. Appl.,* **501** (2021), 125197.

13. E. Nakai, Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, *J. Funct. Anal.,* **262** (2012), 3665–3748.

14. A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$, *Math. Inequal. Appl.,* **7** (2004), 255–265.

15. W. Orlicz, Über konjugierte exponentenfolgen, *Studia Math.*, **3** (1931), 200–212.

16. V. Radulescu, D. Repovs, *Partial differential equations with variable exponents: Variational methods and qualitative analysis*, Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.

17. M. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.,* **9** (2020), 710–728.

18. Y. Sawano, H. Tanaka, Morrey spaces for non-doubling measures, *Acta Math. Sin. (Engl. Ser.),* **21** (2005), 1535–1544.

19. H. Shen, Y. Li, X. Shao, A GPIU method for fractional diffusion equations, *Adv. Difference Equ.,* **2020** (2020), 1–17.

20. X. Tolsa, BMO, H^1, and Calderón-Zygmund operators for non-doubling measures, *Math. Ann.,* **319** (2001), 89–149.

21. L. Wang, L. Shu, Multilinear commutators of singular integral operators in variable exponent Herz-type spaces, *Bull. Malays. Math. Sci. Soc.,* **42** (2019), 1413–1432.

22. L. Wang, L. Shu, Higher order commutators of fractional integrals on Morrey type spaces with variable exponents, *Math. Nachr.,* **291** (2018), 1437–1449.

23. L. Wang, S. Tao, Parameterized Littlewood-Paley operators and their commutators on Herz spaces with variable exponent, *Turkish J. Math.,* **40** (2016), 122–145.

AIMS Press