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Research article

Integral inequalities of Hermite-Hadamard type for GA-F-convex functions

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Dedicated to Dr. Professor Wing-Sum Cheung at the University of Hong Kong

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Abstract: In the paper, the authors define a notion of geometric-arithmetic-F-convex functions and, via an integral identity and other analytic techniques, establish several integral inequalities of the Hermite-Hadamard type for geometric-arithmetic-F-convex functions.

Keywords: integral inequality; Hermite-Hadamard type; GA-convex function; s-GA-convex function; GA-F-convex function
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1. Motivations

In this paper, we let I denote an interval on $\mathbb{R}$ and I° be the interior of I.

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex function if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$. If for any $x, y \in I$ and $t \in [0, 1]$, the inequality

$$f(x^t y^{1-t}) \le tf(x) + (1-t)f(y)$$

is valid, then we call f a geometric-arithmetic convex function on I. For simplicity, we call it a GA-convex function.

In the paper [11], the notion of s-GA-convex functions was defined.

Definition 1.3 ([11, Definition 2.1]). Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ and $s \in (0, 1]$. If the inequality

$$f(x^t y^{1-t}) \le t^s f(x) + (1-t)^s f(y)$$

is valid for any $x, y \in I$ and $t \in [0, 1]$, then we call f an s-geometric-arithmetic convex function on I. For shortness, we call it an s-GA-convex function.

Now we recall several Hermite-Hadamard type inequalities for GA-convex functions in the form of theorems.

Theorem 1.1 ([6, Theorem 1]). *Let $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be GA-convex. Then*

$$f\left(\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \left(\frac{1}{\ln b - \ln a} - \frac{a}{b-a}\right)f(a) + \left(\frac{b}{b-a} - \frac{1}{\ln b - \ln a}\right)f(b).$$

Theorem 1.2 ([17, Theorem 3.1]). *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a, b])$. If $|f'|^q$ is GA-convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{bf(b) - af(a)}{b-a} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{[A(a,b)]^{1-1/q}}{[2(b-a)]^{1/q}}\{[L(a^2,b^2) - a^2]|f'(a)|^q + [b^2 - L(a^2,b^2)]|f'(b)|^q\}^{1/q},$$

where $A(u, v) = \frac{u+v}{2}$ and

$$L(u,v) = \begin{cases} \dfrac{u-v}{\ln u - \ln v}, & u \ne v \\ u, & u = v \end{cases} \tag{1.1}$$

for $u, v > 0$ are the arithmetic mean and logarithmic mean respectively.

Theorem 1.3 ([11, Theorem 3.3]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable and decreasing on I°, $a, b \in I^\circ$ with $a < b$, and $f \in L_1([a, b])$. If $|f'|^p$ is s-GA-convex on $[a, b]$ for $s \in (0, 1]$ and $p > 1$, then*

$$\left|\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}\,u\right| \le \left(\frac{p-1}{2p-1}\right)^{1-1/p}\left[\frac{(x-a)^2}{b-a}\left(\frac{|f'(a)|^p + |f'(x)|^p}{s+1}\right)^{1/p} + \frac{(b-x)^2}{b-a}\left(\frac{|f'(x)|^p + |f'(b)|^p}{s+1}\right)^{1/p}\right].$$

For recent developments related to integral inequalities of the Hermite-Hadamard type for convex functions, please refer to [3–5, 7–10, 12–14, 16] and closely related references therein.

In this paper, we will define a new notion of GA-F-convex functions and establish several integral inequalities of the Hermite-Hadamard type for GA-F-convex functions.

2. A new notion and two lemmas

In the papers [1, 2], as a generalization of strongly convex functions, the notion of F-convex functions was defined as follows.

Definition 2.1 ([2, p. 868, Definition 1])**.** Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $f : I \to \mathbb{R}$ is called to be F-convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - t(1-t)F(x-y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Motivated by Definition 2.1, we now define a new notion of GA-F-convex functions as follows.

Definition 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is called GA-F-convex if

$$f(x^t y^{1-t}) \le tf(x) + (1-t)f(y) - t(1-t)F(x-y) \tag{2.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Remark 1. It is easy to see that, if $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is a decreasing and GA-F-convex function, then f is an F-convex function on I.

For establishing integral inequalities of the Hermite-Hadamard type for GA-F-convex functions, we need the following lemmas.

Lemma 2.1 ([15])**.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then*

$$\begin{aligned} f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}\,u \\ = \frac{(\ln x - \ln a)^2}{\ln b - \ln a}\int_0^1 t a^{1-t}x^t f'(a^{1-t}x^t)\,\mathrm{d}\,t - \frac{(\ln b - \ln x)^2}{\ln b - \ln a}\int_0^1 t x^t b^{1-t} f'(x^t b^{1-t})\,\mathrm{d}\,t. \end{aligned} \tag{2.2}$$

Lemma 2.2. *Let $u, v \in \mathbb{R}_+$. Then*

$$\int_0^1 u^{1-t}v^t\,\mathrm{d}\,t = L(u, v);$$

$$G_1(u, v) = \int_0^1 t u^{1-t}v^t\,\mathrm{d}\,t = \begin{cases} \dfrac{v - L(u, v)}{\ln v - \ln u}, & u \ne v; \\ \dfrac{1}{2}u, & u = v, \end{cases}$$

$$G_2(u, v) = \int_0^1 t^2 u^{1-t}v^t\,\mathrm{d}\,t = \begin{cases} \dfrac{v(\ln v - \ln u) - 2v + 2L(u, v)}{(\ln v - \ln u)^2}, & u \ne v; \\ \dfrac{1}{3}u, & u = v, \end{cases}$$

$$G_3(u, v) = \int_0^1 t^3 u^{1-t}v^t\,\mathrm{d}\,t = \begin{cases} \dfrac{v[(\ln v - \ln u)(\ln v - \ln u - 3) + 6] - 6L(u, v)}{(\ln v - \ln u)^3}, & u \ne v; \\ \dfrac{1}{4}u, & u = v, \end{cases}$$

where $L(u, v)$ is defined by (1.1).

Proof. The proof is straightforward. □

3. Some new integral inequalities of Hermite-Hadamard type

In this section, we establish several integral inequalities of the Hermite-Hadamard type for GA-F-convex functions.

Theorem 3.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a given function. Suppose that $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is differentiable, that $a, b \in I$ with $a < b$, and that $x \in [a, b]$. If $|f'|^q$ is GA-F-convex on I for $q \geq 1$, then*

$$\begin{aligned}\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}\,u\right| \leq{}& \frac{(\ln x - \ln a)^2}{\ln b - \ln a}[G_1(a,x)]^{1-1/q}\{[G_1(a,x) - G_2(a,x)]|f'(a)|^q \\ &+ G_2(a,x)|f'(x)|^q - [G_2(a,x) - G_3(a,x)]F(x-a)\}^{1/q} \\ &+ \frac{(\ln b - \ln x)^2}{\ln b - \ln a}[G_1(b,x)]^{1-1/q}\{[G_1(b,x) - G_2(b,x)]|f'(b)|^q \\ &+ G_2(b,x)|f'(x)|^q - [G_2(b,x) - G_3(b,x)]F(x-b)\}^{1/q},\end{aligned} \tag{3.1}$$

where $G_1(u, v)$, $G_2(u, v)$, and $G_3(u, v)$ are defined as in Lemma 2.2.

Proof. By Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}\,u\right| \leq{}& \frac{(\ln x - \ln a)^2}{\ln b - \ln a}\int_0^1 ta^{1-t}x^t\left|f'(a^{1-t}x^t)\right|\mathrm{d}\,t \\ &+ \frac{(\ln b - \ln x)^2}{\ln b - \ln a}\int_0^1 tx^tb^{1-t}\left|f'(x^tb^{1-t})\right|\mathrm{d}\,t \\ \leq{}& \frac{(\ln x - \ln a)^2}{\ln b - \ln a}\left(\int_0^1 ta^{1-t}x^t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 ta^{1-t}x^t\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t\right]^{1/q} \\ &+ \frac{(\ln b - \ln x)^2}{\ln b - \ln a}\left(\int_0^1 tx^tb^{1-t}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 tx^tb^{1-t}\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t\right]^{1/q}.\end{aligned} \tag{3.2}$$

Since $|f'|^q$ is a GA-F-convex function on $[a, b]$, using Lemma 2.2, we obtain

$$\begin{aligned}&\int_0^1 ta^{1-t}x^t\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t \leq \int_0^1 ta^{1-t}x^t[(1-t)|f'(a)|^q + t|f'(x)|^q - t(1-t)F(x-a)]\,\mathrm{d}\,t \\ &= [G_1(a,x) - G_2(a,x)]|f'(a)|^q + G_2(a,x)|f'(x)|^q - [G_2(a,x) - G_3(a,x)]F(x-a),\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\int_0^1 tx^tb^{1-t}\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t \leq \int_0^1 tx^tb^{1-t}[(1-t)|f'(b)|^q + t|f'(x)|^q - t(1-t)F(x-b)]\,\mathrm{d}\,t \\ &= [G_1(b,x) - G_2(b,x)]|f'(b)|^q + G_2(b,x)|f'(x)|^q - [G_2(b,x) - G_3(b,x)]F(x-b),\end{aligned} \tag{3.4}$$

and

$$\int_0^1 ta^{1-t}x^t\,\mathrm{d}\,t = G_1(a,x), \qquad \int_0^1 tx^tb^{1-t}\,\mathrm{d}\,t = G_1(b,x). \tag{3.5}$$

Substituting inequalities (3.3) –(3.5) into the inequality (3.2) leads to the inequality (3.1). The proof of Theorem 3.1 is complete. □

Corollary 3.1.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a given function. Suppose that $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ is differentiable, $a,b\in I$ with $a<b$, and $x\in[a,b]$. If $|f'|$ is GA-F-convex on I, then*

$$\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}\,u\right|\le\frac{(\ln x-\ln a)^2}{\ln b-\ln a}\{[G_1(a,x)-G_2(a,x)]|f'(a)|$$
$$+G_2(a,x)|f'(x)|-[G_2(a,x)-G_3(a,x)]F(x-a)\}$$
$$+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\{[G_1(b,x)-G_2(b,x)]|f'(b)|+G_2(b,x)|f'(x)|-[G_2(b,x)-G_3(b,x)]F(x-b)\},$$

where $G_1(u,v)$, $G_2(u,v)$, and $G_3(u,v)$ are defined as in Lemma 2.2.

Theorem 3.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be a given function. Suppose that $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ is differentiable, $a,b\in I$ with $a<b$, and $x\in[a,b]$. If $|f'|^q$ is GA-F-convex on I for $q>1$, then*

$$\begin{aligned}\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}\,u\right|\le\left(\frac{q-1}{2q-1}\right)^{1-1/q}\Bigg[&\frac{(\ln x-\ln a)^2}{\ln b-\ln a}\{[L(a^q,x^q)-G_1(a^q,x^q)]|f'(a)|^q\\&+G_1(a^q,x^q)|f'(x)|^q-[G_1(a^q,x^q)-G_2(a^q,x^q)]F(x-a)\}^{1/q}\\&+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\{[L(b^q,x^q)-G_1(b^q,x^q)]|f'(b)|^q\\&+G_1(b^q,x^q)|f'(x)|^q-[G_1(b^q,x^q)-G_2(b^q,x^q)]F(x-b)\}^{1/q}\Bigg],\end{aligned}\tag{3.6}$$

where $L(u,v)$, $G_1(u,v)$, and $G_2(u,v)$ are defined as in Lemma 2.2.

Proof. By Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}\,u\right|&\le\frac{(\ln x-\ln a)^2}{\ln b-\ln a}\int_0^1 ta^{1-t}x^t\left|f'(a^{1-t}x^t)\right|\mathrm{d}\,t\\&+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\int_0^1 tx^tb^{1-t}\left|f'(x^tb^{1-t})\right|\mathrm{d}\,t\\&\le\frac{(\ln x-\ln a)^2}{\ln b-\ln a}\left(\int_0^1 t^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 a^{q(1-t)}x^{qt}\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t\right]^{1/q}\\&+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\left(\int_0^1 t^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 b^{q(1-t)}x^{qt}\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t\right]^{1/q}.\end{aligned}\tag{3.7}$$

Using the GA-F-convexity of $|f'|^q$ and Lemma 2.2, we obtain

$$\begin{aligned}&\int_0^1 a^{q(1-t)}x^{qt}\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t\le\int_0^1 a^{q(1-t)}x^{qt}[(1-t)|f'(a)|^q+t|f'(x)|^q-t(1-t)F(x-a)]\,\mathrm{d}\,t\\&=[L(a^q,x^q)-G_1(a^q,x^q)]|f'(a)|^q+G_1(a^q,x^q)|f'(x)|^q-[G_1(a^q,x^q)-G_2(a^q,x^q)]F(x-a),\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\int_0^1 x^{qt}b^{q(1-t)}\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t\le\int_0^1 x^{qt}b^{q(1-t)}[(1-t)|f'(b)|^q+t|f'(x)|^q-t(1-t)F(x-b)]\,\mathrm{d}\,t\\&=[L(b^q,x^q)-G_1(b^q,x^q)]|f'(b)|^q+G_1(b^q,x^q)|f'(x)|^q-[G_1(b^q,x^q)-G_2(b^q,x^q)]F(x-b),\end{aligned}\tag{3.9}$$

and

$$\int_0^1 t^{q/(q-1)}\,\mathrm{d}\,t=\frac{q-1}{2q-1}.\tag{3.10}$$

Substituting inequalities (3.8)–(3.10) into the inequality (3.7) results in the inequality (3.6). The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a given function. Suppose that $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ is differentiable, $a,b\in I$ with $a<b$, and $x\in[a,b]$. If $|f'|^q$ is GA-F-convex on I for $q>1$, then*

$$\begin{aligned}\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}\,u\right| \le{}& \frac{(\ln x-\ln a)^2}{\ln b-\ln a}\left[L\left(a^{q/(q-1)},x^{q/(q-1)}\right)\right]^{1-1/q}\\ &\times\left[\frac{|f'(a)|^q}{(q+1)(q+2)}+\frac{|f'(x)|^q}{q+2}-\frac{F(x-a)}{(q+2)(q+3)}\right]^{1/q}\\ &+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\left[L\left(b^{q/(q-1)},x^{q/(q-1)}\right)\right]^{1-1/q}\\ &\times\left[\frac{|f'(b)|^q}{(q+1)(q+2)}+\frac{|f'(x)|^q}{q+2}-\frac{F(x-b)}{(q+2)(q+3)}\right]^{1/q},\end{aligned}\tag{3.11}$$

where $L(u,v)$ is defined as in Lemma 2.2.

Proof. By Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}&\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}\,u\right|\\ &\quad\le\frac{(\ln x-\ln a)^2}{\ln b-\ln a}\left[\int_0^1 a^{q(1-t)/(q-1)}x^{qt/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^q\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t\right]^{1/q}\\ &\qquad+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\left[\int_0^1 x^{qt/(q-1)}b^{q(1-t)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^q\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t\right]^{1/q}.\end{aligned}\tag{3.12}$$

Since $|f'|^q$ is GA-F-convex on $[a,b]$, using Lemma 2.2, we obtain

$$\begin{aligned}\int_0^1 t^q\left|f'(a^{1-t}x^t)\right|^q\mathrm{d}\,t&\le\int_0^1 t^q[(1-t)|f'(a)|^q+t|f'(x)|^q-t(1-t)F(x-a)]\,\mathrm{d}\,t\\ &=\frac{|f'(a)|^q}{(q+1)(q+2)}+\frac{|f'(x)|^q}{q+2}-\frac{F(x-a)}{(q+2)(q+3)},\end{aligned}\tag{3.13}$$

$$\begin{aligned}\int_0^1 t^q\left|f'(x^tb^{1-t})\right|^q\mathrm{d}\,t&\le\int_0^1 t^q[(1-t)|f'(b)|^q+t|f'(x)|^q-t(1-t)F(x-b)]\,\mathrm{d}\,t\\ &=\frac{|f'(b)|^q}{(q+1)(q+2)}+\frac{|f'(x)|^q}{q+2}-\frac{F(x-b)}{(q+2)(q+3)},\end{aligned}\tag{3.14}$$

$$\int_0^1 a^{q(1-t)/(q-1)}x^{qt/(q-1)}\,\mathrm{d}\,t=L\left(a^{q/(q-1)},x^{q/(q-1)}\right),\tag{3.15}$$

$$\int_0^1 x^{qt/(q-1)}b^{q(1-t)/(q-1)}\,\mathrm{d}\,t=L\left(b^{q/(q-1)},x^{q/(q-1)}\right).\tag{3.16}$$

Substituting (3.13)–(3.16) into the inequality (3.12) arrives at the inequality (3.11). The proof of Theorem 3.3 is thus complete. □

4. Conclusions

In this paper, we defined the notion of geometric-arithmetic-F-convex functions in Definition 2.2 and, via the integral identity (2.2) and other analytic techniques such as those in Lemma 2.2, establish those integral inequalities (3.1) in Theorem 3.1, (3.6) in Theorem 3.2, and (3.11) in Theorem 3.3 of the Hermite-Hadamard type for geometric-arithmetic-F-convex functions defined by the inequality (2.1) in Definition 2.2.

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Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

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