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Research article

Bifurcation results of positive solutions for an elliptic equation with nonlocal terms

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Abstract: In this paper, we investigate the local and global nature for the connected components of positive solutions set of an elliptic equation with nonlocal terms. The local bifurcation results of positive solutions are obtained by using the local bifurcation theory, Lyapunov-Schmidt reduction technique, etc. Under suitable conditions, we show two proofs of priori estimates by using blow-up technique, upper and lower solution method, etc. Finally, the global bifurcation results of positive solutions are obtained by using priori bounds, global bifurcation theory.

Keywords: nonlocal elliptic equation; bifurcation; a priori estimate
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1. Introduction

In this paper, we consider the elliptic problem with nonlocal terms

$$\begin{cases} -\Delta u = \lambda m(x)u + h(x)u^p + \int_\Omega u^\beta, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary, $N \geq 2$; n is the outward unit normal to $\partial\Omega$; $m(x), h(x) \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$ and $m(x), h(x)$ may change sign in Ω; $p, \beta > 1$ and $p < \frac{N+2}{N-2}$ for $N \geq 3$, $\lambda \in \mathbb{R}$ is a parameter.

Many physical phenomena were formulated into nonlocal mathematical models [1–3, 11, 12] and studied by many authors. For example, J. Bebernes and A. Bressan [11] studied an ignition model similar to (1.1) for a compressible reactive gas which is a nonlocal reaction-diffusion equation. In [11], u is the temperature perturbation of the gas and nonlocal term is due to the compressibility of the gas.

Subsequently, some researchers [2, 3, 12] discussed the parabolic problems related to the equation

$$\begin{cases} -\Delta u = f(x,u) + g\left(\int_\Omega u^\beta\right), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

This type of problem is frequently encountered in nuclear reaction process, where it is known that the reaction is very strong, say like $f(x,u) = \lambda m(x)u + h(x)u^p$ with $p > 1$ and constant functions $m(x)$ and $h(x)$, but the rate with respect to this power is unknown, say like $g\left(\int_\Omega u^\beta\right) = \int_\Omega u^\beta$. The above mathematical problem can also be used to population dynamics and biological science where the total mass is often conserved or known, but the growth of a certain cell is known to be of some form (see [12]). Thus, the problem (1.1) is worthy to be considered.

Mathematically, the problem (1.1) combines local and nonlocal terms. It is well known [2, 3] that the authors discussed that the case $m(x) = 0$, $h(x) \equiv h_0 < 0$, $\beta > 1$ and $p \geq 1$. In two articles, the parabolic problem related to the equation was studied and the authors showed that the value $p = \beta$ represents a critical blow-up exponent. They proved that if $\beta > p$ or $\beta = p$ and $h_0 > -|\Omega|$, the blow-up phenomenon can occur in finite time. If $\beta < p$ or $\beta = p$ and $h_0 \leq -|\Omega|$, all the solutions are global and bounded. The authors also proved the existence of positive solution for h_0 small in the particular case $h_0 < 0$, $p > \beta > 1$. In [1], F. Corrêa and A. Suárez made a further study for the problem and proved the existence, uniqueness, stability and asymptotic properties of positive solutions for some values of $p \geq 1$ and $\beta > 0$.

We want to further consider the global bifurcation structure of the positive solutions set of the problem (1.1) when the function $m(x)$ and $h(x)$ are nonconstant functions because for general function $h(x)$, especially sign-changing function, comparing with the local elliptic equation, we see that many methods that prove the boundedness of positive solutions cannot be used in nonlocal elliptic equation, such as the extremum principle, parameter control. Finally, we can obtain the existence, multiplicity and nonexistence of the positive solution for the problem (1.1) when a bounded connected branch of the positive solutions set is established by the global bifurcation theory.

In an early paper, K. J. Brown [4] studied the local and global bifurcation of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda m(x)u + b(x)u^\gamma, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < \gamma < \frac{N+2}{N-2}$, $m(x)$, $b(x)$ may change sign in Ω. The cases where $\int_\Omega m(x)dx \neq 0$ and $\int_\Omega m(x)dx = 0$ were discussed respectively and the author concluded that there are continua of positive solutions of (1.2) connecting $\lambda = 0$ to the other principal eigenvalue for $\int_\Omega m(x)dx \neq 0$ when $m(x)$ and $b(x)$ are under suitable conditions. It was also showed that the closed loops of positive solutions occur naturally and properties of these loops are investigated.

In this paper, we are interested in the problem (1.1), namely, the problem (1.2) added a nonlocal term. We want to investigate whether the local and global structures of positive solutions set for the problem (1.1) have similar properties to the problem (1.2). We also investigate sufficient conditions for a bounded continuum of positive solutions. In Theorem 2.2, we see that the direction of bifurcation curve is related to β and p. In Theorem 3.1, we get a priori bound of positive solution when $m(x)$ is under suitable conditions and $\beta > \max\{p, N(p-1)/2\}$ by using upper and lower solution method, blow-up technique and boot-strapping method. Moreover, another way of proving boundedness shows that the priori bound still exists when $\beta > \max\{p, N(p-1)/2\}$ vanishes.

Before proceeding to the study of local and global nature of positive solutions, we need to introduce some notations. If $u > 0$ in Ω, we say u is a positive solution of the problem (1.1). (λ, u) is called a nonnegative solution of the problem (1.1) if u is a nonnegative solution of the problem (1.1) with λ. Obviously, $(\lambda, 0)$ is a nonnegative solution of the problem (1.1), we say it is a trivial solution.

To investigate the bifurcation of problem (1.1) at the trivial solution $(\lambda, 0)$, we discuss the linear eigenvalue problem

$$\begin{cases} -\Delta w = \lambda m(x) w, & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $m(x)$ changes sign in Ω. According to [5], we have the following results.

(i) If $\int_\Omega m < 0$, the problem (1.3) has the principal eigenvalues $\lambda_+ > 0$ and $\lambda_0 = 0$.

(ii) If $\int_\Omega m > 0$, the problem (1.3) has the principal eigenvalues $\lambda_- < 0$ and $\lambda_0 = 0$.

(iii) If $\int_\Omega m = 0$, the problem (1.3) has the unique principal eigenvalue $\lambda_0 = 0$.

The usual norms of the space $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\bar{\Omega})$ are, respectively,

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}, \quad \|u\|_{C(\bar{\Omega})} = \max_{\bar{\Omega}} |u(x)| .$$

Let

$$\Omega_+^h = \{x \in \Omega : h(x) > 0\}, \quad \Omega_-^h = \{x \in \Omega : h(x) < 0\}.$$

We use the following hypothesis.

(H_1) $\Lambda = \left\{ u \in H_0^1(\Omega_+^h) : \int_{\Omega_+^h} m u^2 > 0 \right\} \neq \emptyset$.

If $m(x)$ changes sign in Ω_+^h and (H_1) holds, then the equation

$$\begin{cases} -\Delta u = \lambda m(x) u, & x \in \Omega_+^h, \\ u = 0, & x \in \partial\Omega_+^h \end{cases}$$

has unique positive principal eigenvalue [5, 6].

We have the following main global results of positive solutions in two cases where $\int_\Omega m < 0$ and $\int_\Omega m > 0$ by using priori bounds, global bifurcation theory.

Theorem 1.1. *Assume* $\int_\Omega m < 0$ *and*

$$\beta > p, \quad \int_\Omega h\varphi_1^{p+1} < 0,$$

where φ_1 *is the positive eigenfunction of* λ_+. *Then there exists a continuous curve* $(\lambda(s), u(s))$ *of* $s \in (0, \varepsilon)$ *such that* $(\lambda(0), u(0)) = (\lambda_+, 0)$, $\lambda(s) > \lambda_+$, $(\lambda(s), u(s))$ *are positive solutions of the problem (1.1), and for any positive solution* (λ, u) *of the problem (1.1) in a neighborhood of* $(\lambda_+, 0)$, *we have* $(\lambda, u) = (\lambda(s), u(s))$.

Moreover, if $m(x)$ *changes sign in* Ω_+^h *and* (H_1) *holds, then the connected branch* C^+ *of positive solutions set containing* $(\lambda(s), u(s))$ *satisfies the following conclusions.*

(i) The projection of C^+ *on* λ*-axis is bounded, that is,*

$$\lambda^* = \sup\{|\lambda| > 0 : (\lambda, u) \in C^+\} < +\infty.$$

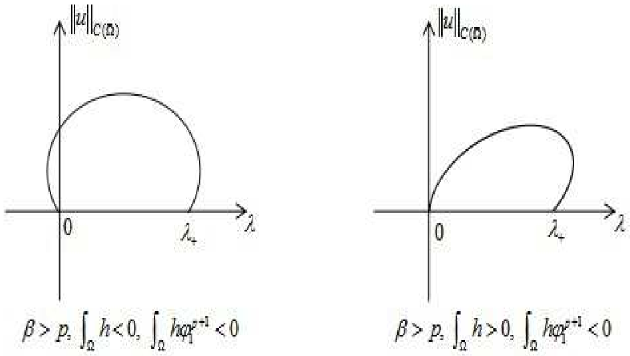

Figure 1. Global bifurcation in case $\int_\Omega m < 0$.

More generally, there is no positive solution of the problem (1.1) for any sufficiently large $|\lambda|$.

(ii) If $(\lambda, u) \in C^+$*, then* u *is bounded in* $C(\bar\Omega)$.

(iii) The closure $\overline{C^+}$ *of* C^+ *in* $\mathbb{R} \times C(\bar\Omega)$ *satisfies*

$$\overline{C^+} \cap \{(\lambda, 0) : \lambda \in \mathbb{R}, \lambda \neq \lambda_+, 0\} = \emptyset.$$

The assertions of Theorem 1.1 may be illustrated by the bifurcation diagram shown in Figure 1.

Remark 1.1. *(i) Theorem 1.1 shows the conditions that the connected branch is supercritical at* $(\lambda_+, 0)$*. In fact, according to Theorem 2.2, we obtain the conditions that the connected branch is supercritical or subcritical at* $(0, 0)$*. In Figure 1, it is clear that if* $\beta > p$, $\int_\Omega h < 0$ *(resp.* $\beta > p$, $\int_\Omega h > 0$ *), the connected branch at* $(0, 0)$ *is subcritical (resp. supercritical). Furthermore, by using Rabinowitz global bifurcation theory, we have that* C^+ *bifurcates from* $(\lambda_+, 0)$ *and backs to* $(0, 0)$.

(ii) Comparing with the results of [4], we see that directions of bifurcation curve depend on not only the sign of $\int_\Omega m$ *and* $h\varphi_1^{p+1}$ *but also the relationship of* β *and* p*. In Theorem 1.1, we only show the case of* $\beta > p$*, while the cases of* $\beta < p$ *and* $\beta = p$ *can also be listed.*

(iii) According to Theorem 1.1, we can obtain the existence, multiplicity and nonexistence of positive solutions for the problem (1.1). For example, the conditions of Theorem 1.1 hold and $\int_\Omega h < 0$*, then there exist constants* $\sigma_1 < 0 < \lambda_+ < \sigma_2$*, such that*

- *the problem (1.1) has at least one positive solution for* $\lambda \in [0, \lambda_+]$;
- *the problem (1.1) has at least two positive solutions for* $\lambda \in (\sigma_1, 0) \cup (\lambda_+, \sigma_2)$.
- *the problem (1.1) has no positive solution for* $\lambda \in (-\infty, \sigma_1) \cup (\sigma_2, +\infty)$.

These results are clearly shown in Figure 1.

Similarly, we have the global bifurcation of the positive solutions set for $\int_\Omega m > 0$.

Theorem 1.2. *Assume* $\int_\Omega m > 0$ *and*

$$\beta > p, \int_\Omega h\varphi_1^{p+1} < 0,$$

where φ_1 *is the positive eigenfunction of* λ_-*. Then there exists a continuous curve* $(\lambda(s), u(s))$ *of* $s \in (0, \varepsilon)$ *such that* $(\lambda(0), u(0)) = (\lambda_-, 0)$, $\lambda(s) < \lambda_-$, $(\lambda(s), u(s))$ *are positive solutions of the problem*

(1.1), and for any positive solution (λ, u) *of the problem (1.1) in a neighborhood of* $(\lambda_-, 0)$*, we have* $(\lambda, u) = (\lambda(s), u(s))$*.*

Moreover, if $m(x)$ *changes sign in* Ω_+^h *and* (H_1) *holds, then the connected components* C^+ *of positive solutions set containing* $(\lambda(s), u(s))$ *satisfies the claims (i), (ii) of Theorem 1.1 and the closure* $\overline{C^+}$ *of* C^+ *in* $\mathbb{R} \times C(\bar{\Omega})$ *satisfies*

$$\overline{C^+} \cap \{(\lambda, 0) : \lambda \in \mathbb{R}, \lambda \neq \lambda_-, 0\} = \emptyset.$$

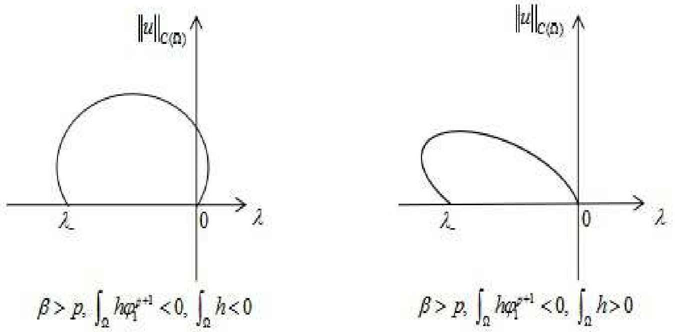

Figure 2. Global bifurcation in case $\int_\Omega m > 0$.

The assertions of Theorem 1.2 may be illustrated by the bifurcation diagram shown in Figure 2.

Next, we consider the case $\int_\Omega m = 0$. We shall then study the global bifurcation of positive solutions for the problem (1.1) by the approximation method. Let $m_\varepsilon(x) = m(x) - \varepsilon$ for $\varepsilon > 0$, we have the following conclusions.

Theorem 1.3. *Assume* $\int_\Omega m = 0$*,* $p = 2$ *or* 3*,* $m(x)$ *changes sign in* Ω_+^h *and* (H_1) *holds. If*

$$\tfrac{2N}{N-2} > \beta > p, \ \int_\Omega h < 0,$$

then there exists a connected components C^+ *of positive solutions set for the problem (1.1), which bifurcates from the origin and backs to the origin in* λ*-norm plane, namely, the closure* $\overline{C^+}$ *of* C^+ *in* $\mathbb{R} \times C(\bar{\Omega})$ *is a closed loop.*

Remark 1.2. *For the case* $\int_\Omega m = 0$*, the hypotheses for the Crandall and Rabinowitz theorem are no longer satisfied, and we use the Lyapunov-Schmidt technique to investigate how bifurcation occurs at* $(\lambda, u) = (0, 0)$*. Moreover, comparing with the global bifurcation results of [4], we see that the relationship of p and nonlocal term power* β *also has influence on the continuum.*

The assertions of Theorem 1.3 may be illustrated by the bifurcation diagram shown in Figure 3.

The rest of this article is organized as follows. In Section 2, we discuss the local properties of positive solutions set in the cases where $\int_\Omega m \neq 0$ and $\int_\Omega m = 0$ by using the local bifurcation theory, Liapunov-Schmidt reduction technique. In Section 3, we show that a priori estimate of positive solutions by using the blow-up technique, global bifurcation theory and upper and lower solution. In Section 4, we complete the proof of main results in two cases.

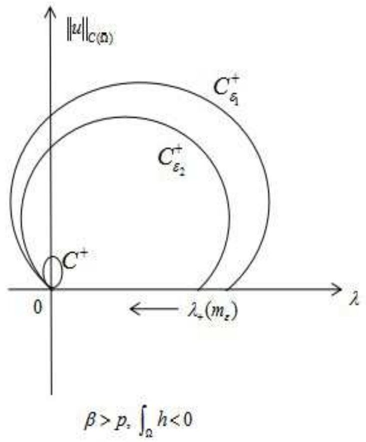

Figure 3. $\overline{C^+_\varepsilon}$ approaches a closed loop as $\varepsilon \to 0$.

2. Local bifurcation results

Let's investigate local bifurcation in the cases where $\int_\Omega m \neq 0$ and $\int_\Omega m = 0$, respectively.

2.1. *Local bifurcation when $\int_\Omega m \neq 0$*

If $\int_\Omega m < 0$, let $\lambda_1 = \lambda_+$ or λ_0; if $\int_\Omega m > 0$, let $\lambda_1 = \lambda_-$ or λ_0. Let φ_1 be the positive eigenfunction of λ_1. If $\lambda_1 = 0$, φ_1 is a constant and we take $\varphi_1 = 1$. Let $0 < \varepsilon \ll 1$ be a constant. We have the following result.

Theorem 2.1. *Assume $\int_\Omega m \neq 0$, then there exists a bifurcation curve $(\lambda(s), u(s))$ of positive solutions of the problem (1.1) at $(\lambda_1, 0)$ parameterized by $s \in (0, \varepsilon)$, which satisfies*

$$\lambda(s) = \lambda_1 + \gamma(s), \quad u(s) = s\left(\varphi_1 + z(s)\right),$$
$$z(\cdot) \in Z, \quad \mathrm{span}\{\varphi_1\} \oplus Z = C^{2+\theta}(\bar\Omega).$$

Here, $\gamma(0) = 0$, $z(0) = 0$. $\gamma(s)$ and $z(s)$ is analytic at $s = 0$.

Proof. We define the mapping

$$F : \mathbb{R} \times X \to C^\theta(\Omega),$$
$$F(\lambda, u) = \Delta u + \lambda m(x) u + h(x) u^p + \int_\Omega u^\beta, \tag{2.1}$$

where $X = \left\{u \in C^{2+\theta}(\Omega) : \frac{\partial u}{\partial n} = 0, x \in \partial\Omega\right\}$.

For any $w \in X$, we have

$$F_u(\lambda, u)[w] = \Delta w + \lambda m(x) w + p h(x) u^{p-1} w + \beta \int_\Omega u^{\beta-1} w,$$

then

$$F_u(\lambda, 0)[w] = \Delta w + \lambda m(x) w.$$

Thus, we have

$$\ker(F_u(\lambda_1, 0)) = span\{\varphi_1\}, \ \dim(\ker(F_u(\lambda_1, 0))) = 1.$$

The range of $F_u(\lambda_1, 0)$ is $R\left(F_u(\lambda_1, 0)\right) = \left\{u \in C^{\theta}(\Omega) : \int_\Omega u\varphi_1 = 0\right\}$, so

$$\operatorname{codim} R\left(F_u(\lambda_1, 0)\right) = 1.$$

Next, we prove

$$F_{\lambda u}(\lambda_1, 0)[\varphi_1] \notin R\left(F_u(\lambda_1, 0)\right). \tag{2.2}$$

Since $F_{\lambda u}(\lambda_1, 0)[\varphi_1] = m(x)\varphi_1$, we have

(i) if $\lambda_1 = 0$, then $\varphi_1 = 1$ and $m(x)\varphi_1 \notin R\left(F_u(\lambda_1, 0)\right)$;

(ii) if $\lambda_1 \neq 0$, then $\int_\Omega |\nabla\varphi_1|^2 = \lambda_1 \int_\Omega m\varphi_1^2 > 0$, namely $\int_\Omega m\varphi_1^2 \neq 0$. Thus, we have $m(x)\varphi_1 \notin R\left(F_u(\lambda_1, 0)\right)$.

Hence, we get (2.2). By virtue of the Crandall-Rabinowitz local bifurcation theory, we obtain Theorem 2.1.

□

Next, we discuss the direction of bifurcation.

Theorem 2.2. *Assume $\int_\Omega m \neq 0$. $(\lambda(s), u(s))$ for $s \in (0, \varepsilon)$ is a bifurcation curve of positive solutions obtained by Theorem 2.1, then we have the following conclusions.*

(1) $\lambda_1 = 0$.

(i) $\beta = p$, then

$$\lim_{s\to 0} \frac{\gamma(s)}{s^{p-1}} = -\frac{|\Omega|^2 + \int_\Omega h}{\int_\Omega m}.$$

If $|\Omega|^2 + \int_\Omega h$ and $\int_\Omega m$ have same (resp. opposite) sign, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

(ii) $\beta < p$, then

$$\lim_{s\to 0} \frac{\gamma(s)}{s^{\beta-1}} = -\frac{|\Omega|^2}{\int_\Omega m}.$$

If $\int_\Omega m > 0$ $(resp. < 0)$, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

(iii) $\beta > p$, then

$$\lim_{s\to 0} \frac{\gamma(s)}{s^{p-1}} = -\frac{\int_\Omega h}{\int_\Omega m}.$$

If $\int_\Omega h$ and $\int_\Omega m$ have same (resp. opposite) sign, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

(2) $\lambda_1 \neq 0$.

(i) $\beta = p$, if $\int_\Omega h\varphi_1^{p+1} + \int_\Omega \varphi_1 \int_\Omega \varphi_1^p$ and $\int_\Omega m$ have opposite (resp. same) sign, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

(ii) $\beta < p$, if $\int_\Omega \varphi_1 \int_\Omega \varphi_1^\beta$ and $\int_\Omega m$ have opposite (resp. same) sign, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

(iii) $\beta > p$, if $\int_\Omega h\varphi_1^{p+1}$ and $\int_\Omega m$ have opposite (resp. same) sign, then the bifurcation curve at $(\lambda_1, 0)$ is subcritical (resp. supercritical).

Proof. Since $(\lambda(s), u(s))$ is a positive solution of the problem (1.1), we have

$$\begin{cases} -\Delta\left[s\left(\varphi_1+z(s)\right)\right]=\lambda(s)ms\left(\varphi_1+z(s)\right) \\ \qquad\qquad\qquad + hs^p(\varphi_1+z(s))^p+\int_\Omega\left[s\left(\varphi_1+z(s)\right)\right]^\beta, & x\in\Omega \\ \frac{\partial(s(\varphi_1+z(s)))}{\partial n}=0, & x\in\partial\Omega, \end{cases}$$

then

$$\begin{cases} -\Delta z(s)=\gamma(s)m\varphi_1+\lambda(s)mz(s) \\ \qquad\qquad + hs^{p-1}(\varphi_1+z(s))^p+s^{\beta-1}\int_\Omega(\varphi_1+z(s))^\beta, & x\in\Omega, \\ \frac{\partial(z(s))}{\partial n}=0, & x\in\partial\Omega. \end{cases} \tag{2.3}$$

Multiplying the Eq (2.3) by φ_1, integrating in Ω, and using the Green formula, it follows that

$$0=\int_\Omega\gamma(s)m\varphi_1^2+\int_\Omega\gamma(s)mz\varphi_1+s^{p-1}\int_\Omega h(\varphi_1+z(s))^p\varphi_1 + s^{\beta-1}\int_\Omega\varphi_1\int_\Omega(\varphi_1+z(s))^\beta.$$

(i) If $\beta=p$, we get

$$\frac{\gamma(s)}{s^{p-1}}=-\frac{\int_\Omega h(x)(\varphi_1+z(s))^p\varphi_1+\int_\Omega\varphi_1\int_\Omega(\varphi_1+z(s))^p}{\int_\Omega m\varphi_1^2+\int_\Omega mz\varphi_1}.$$

So

$$\lim_{s\to0}\frac{\gamma(s)}{s^{p-1}}=-\frac{\int_\Omega h\varphi_1^{p+1}+\int_\Omega\varphi_1\int_\Omega\varphi_1^p}{\int_\Omega m\varphi_1^2}.$$

(ii) If $\beta<p$, we get

$$\frac{\gamma(s)}{s^{\beta-1}}=-\frac{s^{p-\beta}\int_\Omega h(\varphi_1+z(s))^p\varphi_1+\int_\Omega\varphi_1\int_\Omega(\varphi_1+z(s))^\beta}{\int_\Omega m\varphi_1^2+\int_\Omega mz\varphi_1}.$$

So

$$\lim_{s\to0}\frac{\gamma(s)}{s^{\beta-1}}=-\frac{\int_\Omega\varphi_1\int_\Omega\varphi_1^\beta}{\int_\Omega m\varphi_1^2}.$$

(iii) If $\beta>p$, we get

$$\frac{\gamma(s)}{s^{p-1}}=-\frac{\int_\Omega h(\varphi_1+z(s))^p\varphi_1+s^{\beta-p}\int_\Omega\varphi_1\int_\Omega(\varphi_1+z(s))^\beta}{\int_\Omega m\varphi_1^2+\int_\Omega mz\varphi_1}.$$

So

$$\lim_{s\to0}\frac{\gamma(s)}{s^{p-1}}=-\frac{\int_\Omega h\varphi_1^{p+1}}{\int_\Omega m\varphi_1^2}.$$

Since $\int_\Omega m\varphi_1^2$ and $\int_\Omega m$ have opposite sign, then we get Theorem 2.2.

□

2.2. Local bifurcation when $\int_\Omega m = 0$

If $\int_\Omega m = 0$, (2.2) doesn't work, then the hypotheses for the Crandall and Rabinowitz theorem are no longer satisfied. However we can use the Lyapunov-Schmidt technique to investigate how bifurcation occurs.

Assume $u \in X$ is the solution of the problem (1.1), let $u = s + w$, where s is a constant and $\int_\Omega w = 0$. Let Q be the projection of X onto W, where $W = \left\{w \in X, \int_\Omega w = 0\right\}$. Then $w = Q[u] = u - \frac{1}{|\Omega|}\int_\Omega u$, so $u = s + w$ is the solution of the problem (1.1) if and only if

$$-\Delta w = \lambda m(x)(s+w) + h(x)(s+w)^p + \int_\Omega (s+w)^\beta.$$

The condition

$$Q\left[-\Delta w\right] = Q\left[\lambda m(x)(s+w) + h(x)(s+w)^p + \int_\Omega (s+w)^\beta\right],$$

implies that

$$-\Delta w = Q\left[\lambda m(x)(s+w) + h(x)(s+w)^p\right]. \tag{2.4}$$

The condition

$$(I-Q)\left[-\Delta w\right] = (I-Q)\left[\lambda m(x)(s+w) + h(x)(s+w)^p + \int_\Omega (s+w)^\beta\right],$$

implies that

$$\int_\Omega \left(\lambda m w + h(s+w)^p + |\Omega|(s+w)^\beta\right) = 0. \tag{2.5}$$

We consider $F(\lambda, s, w) = 0$. Here $F : \mathbb{R}\times\mathbb{R}\times W \to W$,

$$F(\lambda, s, w) = -\Delta w - Q\left[\lambda m(x)(s+w) + h(x)(s+w)^p\right].$$

Note that $F(0,0,0) = 0$, $F_w(0,0,0)w = -\Delta w$: $W \to W$ is homeomorphism, by using implicit function theorem, there exists a unique solution $w = w(\lambda, s)$ of the equation $F(\lambda, s, w) = 0$ around $(\lambda, s, w) = (0,0,0)$, being analytic at $(0,0)$ and having the condition $w(0,0) = 0$.

Since W is complete, so w satisfies $\int_\Omega \frac{\partial^k w}{\partial s^{k-l}\partial\lambda^k}(0,0) = 0$, namely $\frac{\partial^k w}{\partial s^{k-l}\partial\lambda^k}(0,0) \in W$. We substitute $w = w(\lambda, s)$ for (2.5), then

$$\Phi(\lambda, s) := \int_\Omega \left(\lambda m w(\lambda, s) + h(s + w(\lambda, s))^p + |\Omega|(s + w(\lambda, s))^\beta\right) = 0,$$

where (λ, s) is in a neighborhood of $(0,0)$ and $\Phi(\lambda, s)$ is analytic at $(0,0)$.

Let w_m be the solution of the problem

$$\begin{cases} -\Delta w = m, & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ \int_\Omega w = 0. & \end{cases}$$

We have the following conclusions.

Theorem 2.3. *Assume $\int_\Omega m = 0$, $p = 2$ or 3. If $\beta = p$, let $|\Omega| + \int_\Omega h < 0$; if $\beta > p$, let $\int_\Omega h < 0$. Then there exists a continuous curve $(\lambda(s), u(s))$ of $s \in (0, \varepsilon)$ such that $(\lambda(0), u(0)) = (0, 0)$, $(\lambda(s), u(s))$ are positive solutions of the problem (1.1), and for any positive solution (λ, u) of the problem (1.1) in a neighborhood of $(0, 0)$, we have $(\lambda, u) = (\lambda(s), u(s))$. Moreover, if $\beta = p$, we have*

$$\lim_{s\to 0^+} \frac{u(s)}{|\lambda(s)|^{\frac{2}{p-1}}} = \left(\frac{\int_\Omega |\nabla w_m|^2}{-\left(|\Omega| + \int_\Omega h\right)}\right)^{\frac{1}{p-1}} > 0. \tag{2.6}$$

If $\beta > p$, we have

$$\lim_{s\to 0^+} \frac{u(s)}{|\lambda(s)|^{\frac{2}{p-1}}} = \left(\frac{\int_\Omega |\nabla w_m|^2}{-\int_\Omega h}\right)^{\frac{1}{p-1}} > 0. \tag{2.7}$$

Proof. We shall solve that $\Phi(\lambda, s) = 0$ by considering the Taylor expansion of Φ at $(\lambda, s) = (0, 0)$. Since $w(0, 0) = 0$, we have $\Phi(0, 0) = 0$. From $F(\lambda, s, w) = 0$, we can calculate the partial derivative of $w = w(\lambda, s)$ with respect to λ and s at $(0, 0)$ respectively.

Calculating derivative of $F(\lambda, s, w) = 0$ with respect to λ, when $\lambda = 0$ and $s = 0$, we have

$$\begin{cases} -\Delta w_\lambda(0, 0) = 0, & x \in \Omega, \\ \frac{\partial w_\lambda}{\partial n}(0, 0) = 0, & x \in \partial\Omega. \end{cases}$$

So $w_\lambda(0, 0)$ is a constant. By virtue of $\frac{\partial^k w}{\partial s^{k-l}\partial\lambda^k}(0, 0) \in W$, we get $w_\lambda(0, 0) = 0$. Similarly, we have

$$\frac{\partial^k w}{\partial \lambda^k}(0, 0) = 0, \quad k \geq 1.$$

Calculating derivative of $F(\lambda, s, w) = 0$ with respect to s, when $\lambda = 0$ and $s = 0$, we get

$$\frac{\partial^k w}{\partial s^k}(0, 0) = \begin{cases} 0, & 1 \leq k \leq p - 1, \\ w_p, & k = p, \end{cases}$$

where w_p is the solution of the equation

$$\begin{cases} -\Delta w = p! Q\left[h(x)\right], & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ \int_\Omega w = 0. \end{cases}$$

Moreover, we have

$$\frac{\partial^2 w}{\partial s \partial \lambda}(0, 0) = w_m.$$

Next, we calculate partial derivative of $\Phi(\lambda, s)$ with respect to λ and s at $(0, 0)$ respectively. By direct calculations, we have

$$\frac{\partial^k \Phi}{\partial \lambda^k}(0, 0) = 0, \ k \geq 1,$$

(i) $\beta = p$,

$$\frac{\partial^k \Phi}{\partial s^k}(0, 0) = \begin{cases} 0, & 1 \leq k \leq p - 1, \\ p!\left(|\Omega| + \int_\Omega h\right), & k = p, \end{cases}$$

(ii) $\beta > p$,

$$\frac{\partial^k \Phi}{\partial s^k}(0,0) = \begin{cases} 0, & 1 \le k \le p-1, \\ p! \int_\Omega h, & k = p, \end{cases}$$

(iii) $\beta < p$,

$$\frac{\partial^k \Phi}{\partial s^k}(0,0) = \begin{cases} 0, & 1 \le k \le \beta - 1, \\ p!|\Omega|, & k = \beta. \end{cases}$$

Moreover, we have

$$\frac{\partial^2 \Phi}{\partial s \partial \lambda}(0,0) = \int_\Omega m = 0,$$

$$\frac{\partial^3 \Phi}{\partial s^2 \partial \lambda}(0,0) = 0, \ p = 3, \ \beta \ge 3,$$

$$\frac{\partial^3 \Phi}{\partial s \partial \lambda^2}(0,0) = 2 \int_\Omega m w_m = 2 \int_\Omega |\nabla w_m|^2 > 0.$$

Therefore, the Taylor expansion of Φ at $(0,0)$ is $\Phi(\lambda, s) = s\psi(\lambda, s)$.

(1) $p = 2$.

(i) If $\beta = 2$, then

$$\psi(\lambda, s) = s\left(|\Omega| + \int_\Omega h\right) + \lambda^2 \int_\Omega |\nabla w_m|^2 + \textit{higher order terms}.$$

(ii) If $\beta > 2$, then

$$\psi(\lambda, s) = s \int_\Omega h + \lambda^2 \int_\Omega |\nabla w_m|^2 + \textit{higher order terms}.$$

(2) $p = 3$.

(i) If $\beta = 2$, then

$$\psi(\lambda, s) = s\,|\Omega| + \lambda^2 \int_\Omega |\nabla w_m|^2 + \textit{higher order terms}.$$

(ii) If $\beta = 3$, then

$$\psi(\lambda, s) = \lambda^2 \int_\Omega |\nabla w_m|^2 + s^2 \left(|\Omega| + \int_\Omega h\right) + \textit{higher order terms}.$$

(iii) If $\beta > 3$, then

$$\psi(\lambda, s) = \lambda^2 \int_\Omega |\nabla w_m|^2 + s^2 \int_\Omega h + \textit{higher order terms}.$$

For (1) (i), we note that $\psi_s(0,0) = |\Omega| + \int_\Omega h < 0$, and that by using implicit function theorem, there exists a unique solution $s = s(\lambda)$ of the equation $\psi(\lambda, s) = 0$ around $(0,0)$, which satisfies

$$s'(0) = -\frac{\frac{\partial \psi}{\partial \lambda}(0,0)}{\frac{\partial \psi}{\partial s}(0,0)} = 0, \quad s''(0) = -\frac{\frac{\partial^2 \psi}{\partial \lambda^2}(0,0)}{\frac{\partial \psi}{\partial s}(0,0)} = -\frac{2\int_\Omega |\nabla w_m|^2}{|\Omega| + \int_\Omega h}.$$

So
$$
\begin{aligned}
s(\lambda) &= s(0) + s'(0)\lambda + \frac{s''}{2}\lambda^2 + o(\lambda^2) \\
&= \lambda^2 \left(-\frac{\int_\Omega |\nabla w_m|^2}{|\Omega| + \int_\Omega h} + o(1) \right).
\end{aligned}
$$
Moreover, we have
$$
\begin{aligned}
u =& s + w(\lambda, s) \\
=& s + w(0,0) + (w_\lambda(0,0)\lambda + w_s(0,0)s) \\
&+ \frac{1}{2}\left(w_{\lambda\lambda}(0,0)\lambda^2 + 2w_{\lambda s}(0,0)\lambda s + w_{ss}(0,0)s^2 \right) + ... \\
=& s + w_m \lambda s + \frac{1}{2} w_p s^2 + ... \\
=& s\,(1 + o(1)).
\end{aligned}
$$
Therefore, combining the above two equations, we obtain (2.6).

For (1) (ii), $\psi_s(0,0) = \int_\Omega h < 0$, using a similar argument as that of (1) (i), we have
$$
s(\lambda) = \lambda^2 \left(-\frac{\int_\Omega |\nabla w_m|^2}{\int_\Omega h} + o(1) \right), \ u = s\,(1 + o(1)).
$$
For (2) (i), since $|\Omega| > 0$, $\int_\Omega |\nabla w_m|^2 > 0$, so the problem (1.1) is no positive solution in the neighborhood of $(0,0)$.

For (2) (ii), since
$$
\begin{aligned}
\det D^2\psi(0,0) &= \det \begin{pmatrix} 2\int_\Omega |\nabla w_m|^2 & 0 \\ 0 & 2\left(|\Omega| + \int_\Omega h\right) \end{pmatrix} \\
&= 4\left(|\Omega| + \int_\Omega h\right) \int_\Omega |\nabla w_m|^2 < 0,
\end{aligned}
$$
by using the Morse lemma, we see that for any $s > 0$, $\psi(\lambda, s) = 0$ has a unique solution $s = s(\lambda)$ in a neighborhood of $(0,0)$ and we have
$$
s(\lambda) = \left(\left(\frac{\int_\Omega |\nabla w_m|^2}{-\left(|\Omega| + \int_\Omega h\right)} \lambda^2 \right)^{\frac{1}{2}} + o(1) \right), \ u = s\,(1 + o(1)).
$$
So we get the conclusion.

For (2) (iii), since
$$
\begin{aligned}
\det D^2\psi(0,0) &= \det \begin{pmatrix} 2\int_\Omega |\nabla w_m|^2 & 0 \\ 0 & 2\int_\Omega h \end{pmatrix} \\
&= 4\int_\Omega h \int_\Omega |\nabla w_m|^2 < 0,
\end{aligned}
$$
using a similar argument as that of (2) (ii), we get
$$
s(\lambda) = \left(\left(\frac{\int_\Omega |\nabla w_m|^2}{-\int_\Omega h} \lambda^2 \right)^{\frac{1}{2}} + o(1) \right), \ u = s\,(1 + o(1)).
$$
□

3. A priori estimate

We first prove that under the suitable conditions, the problem (1.1) has no positive solution for any sufficiently large $|\lambda|$. More precisely, we have the following results.

Proposition 3.1. *Assume that $m(x)$ changes sign in Ω_+^h, (H_1) holds and (λ, u) is a positive solution of the problem (1.1). Then $\lambda \in (\lambda_1^-, \lambda_1^+)$, where $\lambda_1^- < 0$ and $\lambda_1^+ > 0$ are the principal eigenvalue of the equation*

$$\begin{cases} -\Delta u = \lambda m(x) u, & x \in \Omega_+^h, \\ u = 0, & x \in \partial\Omega_+^h. \end{cases}$$

Proof. If $(\tilde{\lambda}, \tilde{u})$ is a positive solution of the problem (1.1), then

$$\begin{cases} -\Delta \tilde{u} = \tilde{\lambda} m(x) \tilde{u} + h(x)\tilde{u}^p + \int_\Omega \tilde{u}^\beta, & x \in \Omega, \\ \frac{\partial \tilde{u}}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

thus, we have

$$\begin{cases} -\Delta \tilde{u} - \tilde{\lambda} m(x) \tilde{u} = h(x)\tilde{u}^p + \int_\Omega \tilde{u}^\beta > 0, & x \in \Omega_+^h, \\ \tilde{u} \geq 0, & x \in \partial\Omega_+^h. \end{cases}$$

It follows that the principal eigenvalue $\mu_1(\tilde{\lambda})$ of the eigenvalue problem

$$\begin{cases} -\Delta u - \tilde{\lambda} m(x) u = \mu(\tilde{\lambda}) u, & x \in \Omega_+^h, \\ u = 0, & x \in \partial\Omega_+^h \end{cases}$$

is positive. Then $\lambda_1^- < \tilde{\lambda} < \lambda_1^+$. □

Assume $m(x)$ changes sign in Ω_+^h, (H_1) holds. For any $\lambda \in \left(\lambda_1^-, \lambda_1^+\right)$, e_λ is the unique positive solution of the equation

$$\begin{cases} -\Delta u - \lambda m(x) u = 1, & x \in \Omega_+^h, \\ u = 0, & x \in \partial\Omega_+^h. \end{cases}$$

We have the following lemma.

Lemma 3.1. *Assume $m(x)$ changes sign in Ω_+^h, (H_1) holds and (λ, u) is a positive solution of the problem (1.1). Then*

$$u \geq e_\lambda \int_\Omega u^\beta, \ x \in \Omega_+^h.$$

Proof. If (λ, u) is a positive solution of the problem (1.1), then $\lambda \in \left(\lambda_1^-, \lambda_1^+\right)$, and we have

$$\begin{cases} -\Delta u - \lambda m(x) u = h(x) u^p + \int_\Omega u^\beta \geq \int_\Omega u^\beta, & x \in \Omega_+^h, \\ u \geq 0, & x \in \partial\Omega_+^h, \end{cases}$$

so

$$\begin{cases} -\Delta \left(\frac{u}{\int_\Omega u^\beta}\right) - \lambda m(x) \left(\frac{u}{\int_\Omega u^\beta}\right) \geq 1, & x \in \Omega_+^h, \\ \frac{u}{\int_\Omega u^\beta} \geq 0, & x \in \partial\Omega_+^h. \end{cases}$$

Thus, we have $\frac{u}{\int_\Omega u^\beta} \geq e_\lambda$, namely, $u \geq e_\lambda \int_\Omega u^\beta$ for $x \in \Omega_+^h$.

□

We will use the method of Gidas-Spruck [7] to discuss the priori estimate of positive solutions.

Lemma 3.2. *Assume $m(x)$ changes sign in Ω_+^h, (H_1) holds and $\{(\lambda_k, u_k)\}$ is a sequence of positive solutions of the problem (1.1) with $\|u_k\|_{C(\bar{\Omega}_+^h)} \to \infty$ as $k \to \infty$. Then there exists a constant $C > 0$ such that $\|u_k\|_{C(\bar{\Omega}_+^h)} \leq C t_k^{\frac{1}{p}}$ for sufficiently large k, where $t_k = \int_\Omega u_k^\beta$.*

Proof. If (λ_k, u_k) is a positive solution of the problem (1.1), then $\lambda_k \in \left(\lambda_1^-, \lambda_1^+\right)$. Choose $x_k \in \bar{\Omega}_+^h$ such that

$$u_k(x_k) = \max_{\bar{\Omega}_+^h} u_k, \ k = 1, 2...$$

and let $M_k = u_k(x_k)$. Assume $M_k t_k^{-\frac{1}{p}} \to \infty$ as $k \to +\infty$. Take a change of variables

$$u_k(x) = M_k v_k(y), \ \ y = \rho_k(x - x_k), \ \ x \in \Omega_+^h, \tag{3.1}$$

where $\rho_k = M_k^{\frac{p-1}{2}}$, $y \in \Omega_k := \rho_k(\Omega_+^h - \{x_k\})$. It is clear that $0 < v_k \leq 1$, $v_k(0) = 1$. Substituting (3.1) into the problem (1.1), by direct calculations, we have

$$-\Delta v_k(y) = \lambda_k \bar{m}_k(y) M_k^{1-p} v_k(y) + \bar{h}_k(y) v_k^p(y) + \left(M_k t_k^{-\frac{1}{p}}\right)^{-p}, \ \ y \in \Omega_k, \tag{3.2}$$

where $\bar{m}_k(y) = m(\rho_k^{-1} y + x_k)$, $\bar{h}_k(y) = h(\rho_k^{-1} y + x_k)$. Since $\bar{\Omega}_+^h$ is compact, then there exists a subsequence of $\{x_k\}$, still denoted by $\{x_k\}$, such that $x_k \to x_0 \in \bar{\Omega}_+^h$. Now, we distinguish two cases.

Case 1. $x_0 \in \Omega_+^h$. It is seen in this case that $\Omega_k \to \mathbb{R}^N$ as $k \to \infty$. Hence, for any compact subset $\mathbb{K}_1$, we have $\mathbb{K}_1 \subset \Omega_k$ for sufficiently large k. Since $0 < v_k \leq 1$, there exist a positive constant C_2, such that

$$\left|\lambda_k \bar{m}_k(y) M_k^{1-p} v_k(y) + \bar{h}_k(y) v_k^p(y) + \left(M_k t_k^{-\frac{1}{p}}\right)^{-p}\right| \leq C_2.$$

By using the regularity theory of the elliptic equation, we know that, there exists a subsequence of $\{v_k\}$, still denoted by itself, such that

$$v_k \to v \ in \ C^1(\mathbb{K}_1), \ \ k \to +\infty,$$

where $v \in C^1(\mathbb{K}_1)$. Since $\mathbb{K}_1 \subset\subset \Omega_k$ is arbitrarily given, by a diagonal process, we can choose a subsequence, still denoted by $\{v_k\}$, such that

$$v_k \to v \ in \ C_{loc}^1(\mathbb{R}^N), \ \ k \to +\infty.$$

Thus, we have

$$-\Delta v(y) = h(x_0) v^p(y), \ \ y \in \mathbb{R}^N. \tag{3.3}$$

Note that $v(0) = 1$, by (3.3) and a linear change of coordinates, we find that there exists a nontrivial non-negative function $w \in C^2(\mathbb{R}^N)$ satisfying $-\Delta w = w^p$, which contradicts [7].

Case 2. $x_0 \in \partial\Omega_+^h$. By an additional change of coordinates, we can assume that a neighborhood of x_0 in $\partial\Omega_+^h$ is a hyperplane $x^N = 0$ and $\bar{\Omega}_+^h \subset H = \{x \in \mathbb{R}^N, x^N > 0\}$. Hence, given $R > 0$, there exists k_R such that for $k \geq k_R$, v_k is well defined on

$$H_{R,k} := B(0, R) \cap \left\{y^N > -M_k^{\frac{p-1}{2}} x_k^N\right\}.$$

Now, we have the following three cases.

(i) $\left\{M_k^{\frac{p-1}{2}} x_k^N\right\}$ is not bounded from upper. Assume without loss of generality that $M_k^{\frac{p-1}{2}} x_k^N \to \infty$. Then, we have

$$H_{R,k} \to B(0,R),\ k \to \infty.$$

We may argue exactly as in Case 1.

(ii) $\left\{M_k^{\frac{p-1}{2}} x_k^N\right\}$ is not bounded from below. Assume without loss of generality that $M_k^{\frac{p-1}{2}} x_k^N \to 0$. Then, we have

$$H_{R,k} \to B(0,R) \cap H,\ k \to \infty.$$

Arguing as in Case 1, there exists $v \in C^2(\bar{H})$ such that $v \geq 0$, $v(0) = 1$, and v satisfies

$$-\Delta v = h(x_0)v^p,\ in\ \mathbb{R}^N_+.$$

This contradicts Corollary 2.1 of [7].

(iii) $\left\{M_k^{\frac{p-1}{2}} x_k^N\right\}$ is bounded from below. Assume without loss of generality that $M_k^{\frac{p-1}{2}} x_k^N \to s,\ s > 0$. Then, we have

$$H_{R,k} \to B(0,R) \cap \left\{y \in \mathbb{R}^N : y^N > -s\right\} = B_R \cap H_s.$$

We can proceed as in (ii) and there exists $v \in C^2_{loc}(H_s)$ such that $v \geq 0$, $v(0) = 1$, and v satisfies

$$-\Delta v = h(x_0)v^p,\ in\ H_s.$$

Taking a change of variable through $y^N = -s$, we have that $v \in C^2(\bar{H})$, $v \geq 0$, $v(0) = 1$, and v satisfies

$$-\Delta v = h(x_0)v^p,\ in\ \mathbb{R}^N_+,$$

a contradiction.

□

Lemma 3.3. *Assume $m(x)$ changes sign in Ω^h_+ and* (H$_1$) *holds, then there exists a constant $C > 0$ such that $\|u\|_{C(\bar{\Omega}^h_+)} \leq C$ for any positive solution (λ, u) of the problem (1.1).*

Proof. If (λ_k, u_k) is a positive solution of the problem (1.1), then $\lambda_k \in \left(\lambda_1^-, \lambda_1^+\right)$. Assume

$$\lambda_k \to \lambda_0 \in \left(\lambda_1^-, \lambda_1^+\right),\ \|u_k\|_{C(\bar{\Omega}^h_+)} \to \infty,\ k \to \infty.$$

By virtue of Lemma 3.2, we have

$$u_k \leq \|u_k\|_{C(\bar{\Omega}^h_+)} \leq C t_k^{\frac{1}{p}}, x \in \Omega^h_+,$$

where $t_k = \int_\Omega u_k^\beta$. Moreover, we have $t_k \to \infty$. But by using Lemma 3.1, we have $u_k \geq e_{\lambda_k} t_k$, a contradiction. Therefore, there exists a constant $C > 0$ such that $\|u\|_{C(\bar{\Omega}^h_+)} \leq C$.

□

Theorem 3.1. *Assume $m(x)$ changes sign in Ω^h_+,* (H$_1$) *holds and $\beta > \max\{p, N(p-1)/2\}$. Then there exists a constant $C > 0$ such that $\|u\|_{C(\bar{\Omega})} \leq C$ for any positive solution (λ, u) of the problem (1.1).*

Proof. Let $f(u) = \lambda m(x)u + h(x)u^p + \int_\Omega u^\beta$. By Lemma 3.3, there exists a constant $C_1 > 0$, such that $u \leq \|u\|_{C(\bar{\Omega}_+^h)} \leq C_1, x \in \Omega_+^h$. It follows that $\int_\Omega u^\beta$ is bounded by Lemma 3.1, so $f(u)$ is bounded in $L^{\beta/p}(\Omega)$. Thus, u is bounded in $W^{2,\beta/p}(\Omega)$. By using boot-strapping method [1], it follows that there exists a constant $C > 0$, such that $\|u\|_{C(\bar{\Omega})} \leq C$.

□

Next, we give another way of proving boundedness. We first show two results about eigenvalue problem. We consider the eigenvalue problem

$$\begin{cases} -\Delta u + a(x)u = \sigma u, & x \in \Omega, \\ B_\Omega u = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

where

$$B_\Omega u = \begin{cases} v, & x \in \Gamma_1(\Omega), \\ \frac{\partial u}{\partial n}, & x \in \Gamma_2(\Omega), \end{cases}$$

$\partial\Omega = \Gamma_1(\Omega) \cup \Gamma_2(\Omega)$, $\Gamma_1(\Omega)$ is nonempty. $\sigma_1(-\Delta + a(x), B_\Omega, \Omega)$ is the principal eigenvalue of problem (3.4). Generally, we denote B_Ω by B. We have the following results.

(1) If $a(x) \leq \tilde{a}(x)$, then $\sigma_1(-\Delta + a(x), B, \Omega) \leq \sigma_1(-\Delta + \tilde{a}(x), B, \Omega)$.

Proof. Let $\tilde{a}(x) = a(x) + b(x)$, then $b(x) \geq 0$. Let $F(u) = \int_\Omega |\nabla u|^2 + \int_\Omega au^2$, $\tilde{F}(u) = \int_\Omega |\nabla u|^2 + \int_\Omega \tilde{a}u^2$, then

$$\begin{aligned} \sigma_1(-\Delta + a(x), B, \Omega) &= \inf\left\{F(u) : u \in H^1(\Omega), \|u\|_{L^2(\Omega)} = 1, Bu = 0\right\} \\ &\leq \inf\left\{F(u) + \int_\Omega bu^2 : u \in H^1(\Omega), \|u\|_{L^2(\Omega)} = 1, Bu = 0\right\} \\ &= \inf\left\{\tilde{F}(u) : u \in H^1(\Omega), \|u\|_{L^2(\Omega)} = 1, Bu = 0\right\} \\ &\leq \sigma_1(-\Delta + \tilde{a}(x), B, \Omega). \end{aligned}$$

□

(2) If $\Omega \subset \Omega^*$ and $\mathrm{int}(\Omega^*) \cap \Gamma_2(\Omega) = \emptyset$, then

$$\sigma_1(-\Delta + a(x), B, \Omega) \geq \sigma_1(-\Delta + a(x), B, \Omega^*).$$

Proof. Let $F_\Omega(u) = \int_\Omega |\nabla u|^2 + \int_\Omega au^2$, $\tilde{u}$ denote that the function $u \in H^1(\Omega)$ extends to Ω^* and $\mathrm{int}(\Omega^*) \cap$

$\Gamma_2(\Omega) = \emptyset$. Then $\tilde{u}$ satisfies $\tilde{u}|_{\partial\Omega^*\setminus\partial\Omega} = 0,\ B_{\Omega^*}\tilde{u}|_{\partial\Omega^*\cap\partial\Omega} = 0$. Therefore,

$$
\begin{aligned}
&\sigma_1(-\Delta + a(x), B, \Omega)\\
&= \inf\left\{F_\Omega(u) : u \in H^1(\Omega), \|u\|_{L^2(\Omega)} = 1, B_\Omega u = 0\right\}\\
&= \inf\left\{F_{\Omega^*}(\tilde{u}) : \tilde{u} \in H^1(\Omega), \|\tilde{u}\|_{L^2(\Omega)} = 1,\ B_{\Omega^*}\tilde{u}|_{\partial\Omega^*\cap\partial\Omega} = 0,\ \tilde{u}|_{\partial\Omega^*\setminus\partial\Omega} = 0\right\}\\
&\geq \inf\left\{F_{\Omega^*}(u^*) : u^* \in H^1(\Omega), \|u^*\|_{L^2(\Omega)} = 1,\ B_{\Omega^*}u^*|_{\partial\Omega^*\cap\partial\Omega} = 0,\ u^*|_{\partial\Omega^*\setminus\partial\Omega} = 0\right\}\\
&= \inf\Big\{F_{\Omega^*}(u^*) : u^* \in H^1(\Omega), \|u^*\|_{L^2(\Omega)} = 1,\ B_{\Omega^*}u^*|_{\partial\Omega^*\cap\partial\Omega} = 0,\ u^*|_{\Gamma_1(\partial\Omega^*\setminus\partial\Omega)} = 0,\\
&\quad u^*|_{\Gamma_2(\partial\Omega^*\setminus\partial\Omega)} = 0\Big\}\\
&\geq \inf\Big\{F_{\Omega^*}(u^*) : u^* \in H^1(\Omega), \|u^*\|_{L^2(\Omega)} = 1,\ B_{\Omega^*}u^*|_{\partial\Omega^*\cap\partial\Omega} = 0,\ u^*|_{\Gamma_1(\partial\Omega^*\setminus\partial\Omega)} = 0,\\
&\quad \left.\frac{\partial u^*}{\partial n}\right|_{\Gamma_2(\partial\Omega^*\setminus\partial\Omega)} = 0\Big\}\\
&= \inf\left\{F_{\Omega^*}(u) : u \in H^1(\Omega^*), \|u\|_{L^2(\Omega^*)} = 1, B_{\Omega^*}u = 0\right\}\\
&= \sigma_1(-\Delta + a(x), B, \Omega^*).
\end{aligned}
$$

□

Let S is the positive solution set of the problem (1.1) and $\Lambda_S := \{\lambda \in R; (\lambda, u) \in S\}$ is bounded. we have

Theorem 3.2. *If* $\sup\limits_{(\lambda,u)\in S} \sup\limits_{\Omega_+^h} \mathrm{u} < \infty$*, then* $\sup\limits_{(\lambda,u)\in S} \sup\limits_{\Omega} u < \infty$.

Proof. If (λ, u) is a positive solution of the problem (1.1), then

$$
\begin{cases}
-\Delta u - \lambda m(x)u = h(x)u^p + \int_\Omega u^\beta > 0, & x \in \Omega\setminus\bar{\Omega}_-^h,\\
u \geq 0, & x \in \Gamma_1(\Omega\setminus\bar{\Omega}_-^h),\\
\frac{\partial u}{\partial n} = 0, & x \in \Gamma_2(\Omega\setminus\bar{\Omega}_-^h),
\end{cases}
$$

where $\Gamma_1(\Omega\setminus\bar{\Omega}_-^h) \subset \Omega$, $\Gamma_2(\Omega\setminus\bar{\Omega}_-^h) \subset \partial\Omega$, $\Gamma_1(\Omega)$ is nonempty. Then (λ, u) is a strict upper solution of the equation

$$
\begin{cases}
-\Delta v - \lambda m(x)v = 0, & x \in \Omega\setminus\bar{\Omega}_-^h,\\
v = 0, & x \in \Gamma_1(\Omega\setminus\bar{\Omega}_-^h),\\
\frac{\partial v}{\partial n} = 0, & x \in \Gamma_2(\Omega\setminus\bar{\Omega}_-^h),
\end{cases}
$$

so

$$\sigma_1(-\Delta - \lambda m(x), B, \Omega\setminus\bar{\Omega}_-^h) > 0.$$

By $\Omega_0^h \subset \Omega\setminus\bar{\Omega}_-^h$ and result (2), we have

$$\sigma_1(-\Delta - \lambda m(x), B, \Omega_0^h) \geq \sigma_1(-\Delta - \lambda m(x), B, \Omega\setminus\bar{\Omega}_-^h).$$

Let

$$\Omega_\delta = \Omega_0^h \cup \left\{x \in \Omega_-^h : d(x, \partial\Omega_-^h) < \delta\right\} \cup \left\{\partial\Omega_-^h \cap (\Omega\setminus\partial\Omega_+^h)\right\},$$

then $\Omega_\delta \to \Omega_0^h$ when $\delta \to 0$. So there exists a sufficiently small $\delta > 0$, such that $\sigma_1(-\Delta - \lambda m(x), B, \Omega_\delta) > 0$, so that $\sigma_1(-\Delta - \lambda m(x), B, \Omega_\delta) > 0$ satisfies the strong maximum principle.

Let $M = \sup_{(\lambda,u)\in S} \sup_{\Omega_+^h} u$ and ψ be the unique solution of

$$\begin{cases} -\Delta v - \lambda m(x)v = 1, & x \in \Omega_\delta, \\ v = M, & x \in \Gamma_1(\Omega_\delta), \\ \frac{\partial v}{\partial n} = 0, & x \in \Gamma_2(\Omega_\delta). \end{cases}$$

Since $\sigma_1(-\Delta - \lambda m(x), B, \Omega_\delta) > 0$, we have $\psi > 0, x \in \Omega_\delta$ by the strong maximum principle. Denote by w an extension of $\psi|_{\Omega_{\delta/2}}$ with $\min_{\bar{\Omega}} w > 0$, $\frac{\partial w}{\partial n}\Big|_{\Gamma_2(\Omega)} = 0$. Then $\bar{u} = kw$ is for sufficiently large $k > 0$ a positive strict upper solution of

$$\begin{cases} -\Delta v = \lambda m(x)v + h(x)v^p + k_2, & x \in \Omega\backslash\bar{\Omega}_+^h, \\ v = M, & x \in \Gamma_1(\Omega\backslash\bar{\Omega}_+^h), \\ \frac{\partial v}{\partial n} = 0, & x \in \Gamma_2(\Omega\backslash\bar{\Omega}_+^h), \end{cases}$$

where $k_2 > \int_\Omega u^\beta$. Indeed, in $\Omega_{\delta/2}$ we have

$$\begin{aligned} -\Delta kw &= -k\Delta w \\ &= k\lambda m(x)w + k \\ &\geq k\lambda m(x)w + h(x)(kw)^p + k \\ &\geq \lambda m(x)(kw) + h(x)(kw)^p + k_2. \end{aligned}$$

In $\Sigma_\delta = \left\{x \in \Omega_-^h : d(x, \partial\Omega_-^h) \geq \frac{\delta}{2}\right\}$, since $w(x)$ and $-h(x)$ are positive and bounded away from zero, then $h(x)k^{p-1}w^p \to -\infty$ as $k \to \infty$, namely

$$-\Delta w \geq \lambda m(x)w + h(x)k^{p-1}w^p + k_2/k.$$

On $\Gamma_1(\Omega\backslash\bar{\Omega}_+^h)$, since $\min_{\bar{\Omega}} w > 0$, we know $kw \geq M$; on $\Gamma_2(\Omega\backslash\bar{\Omega}_+^h)$, we have $\frac{\partial kw}{\partial n} = 0$. Thus $\bar{u} = kw$ is a positive strict upper solution. Moreover, by the relationship between the strict upper solution and the principal eigenvalues, we have

$$\sigma_1(-\Delta - \lambda m(x) - \bar{u}^{p-1}h(x), B, \Omega\backslash\bar{\Omega}_+^h) > 0.$$

If $(\lambda, u) \in S$ then it follows that $v = \bar{u} - u$ satisfies

$$\begin{cases} -\Delta v - \lambda m(x)v - (\bar{u}^{p-1} + \bar{u}^{p-2}u + ... + u^{p-1})h(x)v > 0, & x \in \Omega\backslash\bar{\Omega}_+^h, \\ v \geq 0, & x \in \Gamma_1(\Omega\backslash\bar{\Omega}_+^h), \\ \frac{\partial v}{\partial n} = 0, & x \in \Gamma_2(\Omega\backslash\bar{\Omega}_+^h). \end{cases}$$

Indeed, in $\Omega\backslash\bar{\Omega}_+^h$, since

$$-\Delta\bar{u} \geq \lambda m(x)\bar{u} + h(x)\bar{u}^p + k_2,$$
$$-\Delta u = \lambda m(x)u + h(x)u^p + \int_\Omega u^\beta,$$

so

$$\begin{aligned} -\Delta v &> \lambda m(x)v + h(x)(\bar{u}^p - u^p) + (k_2 - \int_\Omega u^\beta) \\ &> \lambda m(x)v + (\bar{u}^{p-1} + \bar{u}^{p-2}u + ... + u^{p-1})h(x)v. \end{aligned}$$

Thus by result (1) we have

$$\begin{aligned}&\sigma_1(-\Delta-\lambda m(x)-(\bar{u}^{p-1}+\bar{u}^{p-2}u+...+u^{p-1})h(x),B,\Omega\backslash\bar{\Omega}^h_+)\\&\quad\geq\sigma_1(-\Delta-\lambda m(x)-\bar{u}^{p-1}h(x),B,\Omega\backslash\bar{\Omega}^h_+)\\&\quad>0.\end{aligned}$$

By the relationship between the principal eigenvalues and the strong maximum principle, we have $u\leq\bar{u}$. The proof is completed. □

4. Proof of main results

proof of Theorem 1.1. If $\int_\Omega m<0$, by virtue of Theorem 2.1, we obtain the bifurcation curve of positive solutions of the problem (1.1), and the direction of bifurcation is given by Theorem 2.2. When $m(x)$ changes sign in Ω^h_+ and (H_1) holds, we have (i) by Proposition 3.1. Finally, we have (ii) by the priori estimate of positive solutions of Theorem 3.2.

Next, we prove (iii). We assume by contradiction that (λ_k,u_k) are positive solutions of the problem (1.1), and $(\lambda_k,u_k)\to(\gamma,0)$ in $\mathbb{R}\times C(\bar{\Omega})$, where $\gamma\neq\lambda_+$ and 0. Let $v_k=\frac{u_k}{\|u_k\|_{C(\bar{\Omega})}}$, by the problem (1.1), we have

$$\begin{cases}-\Delta v_k=\lambda_k m(x)v_k+h(x)v_k^p\|u_k\|^{p-1}_{C(\bar{\Omega})}+\|u_k\|^{\beta-1}_{C(\bar{\Omega})}\int_\Omega v_k^\beta, & x\in\Omega,\\ \frac{\partial v_k}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

By virtue of the regularity theory of the elliptic equation, we know that there is a subsequence, still denoted by $\{v_k\}$, such that $v_k\to v_0$ in $C^2(\bar{\Omega})$, v_0 is a solution of the equation

$$\begin{cases}-\Delta v_0=\gamma m(x)v_0, & x\in\Omega,\\ \frac{\partial v_0}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

So $\gamma=\lambda_+$ or 0, a contradiction. Moreover, by using Rabinowitz global bifurcation theory, we see that C^+ bifurcates from $(\lambda_+,0)$ and backs to $(0,0)$. □

Similarly, we obtain the global bifurcation results of the problem (1.1) for $\int_\Omega m>0$.

proof of Theorem 1.3. Since $m(x)$ changes sign in Ω^h_+ and $m_\varepsilon(x)=m(x)-\varepsilon$ for $\varepsilon>0$, so m_ε changes sign and $\int_\Omega m_\varepsilon<0$ for sufficiently small ε. Thus, we know that

$$\begin{cases}-\Delta u=\lambda m_\varepsilon(x)u, & x\in\Omega,\\ \frac{\partial u}{\partial n}=0, & x\in\partial\Omega\end{cases}$$

has the principal eigenvalue 0 and $\lambda_+(m_\varepsilon)>0$. Substituting $m_\varepsilon(x)$ for $m(x)$, we can see that u is bounded in $C(\bar{\Omega})$ with $m_\varepsilon(x)$ through the proof of a priori estimate of positive solutions. By Theorem 1.1, we see that the problem (1.1) with m_ε has a connected branch C^+_ε of positive solutions set in $\mathbb{R}\times C(\bar{\Omega})$ such that its closure $\overline{C^+_\varepsilon}$ contains $(0,0)$ and $(\lambda_+(m_\varepsilon),0)$. Suppose that $\varphi_\varepsilon>0$ is the principal eigenfunction corresponding to $\lambda_+(m_\varepsilon)$ normalized so that $\|\varphi_\varepsilon\|_{W^{1,2}(\Omega)}=1$. According to [4], we have $\lim_{\varepsilon\to0}\lambda_+(m_\varepsilon)=0$ and $\varphi_\varepsilon\to C$ in $W^{1,2}(\Omega)$, where C is a positive constant.

If $\beta>p$ and $\int_\Omega h<0$, then C^+_ε is subcritical at $(0,0)$ by Theorem 2.2. Since $\lambda_+(m_\varepsilon)\to0$ and $\varphi_\varepsilon\to C$, so for sufficiently small ε, we have $\int_\Omega h\varphi_1^{p+1}$ and $\int_\Omega h$ have same sign for $\beta>p$. So C^+_ε is supercritical at $(\lambda_+(m_\varepsilon), 0)$ for $\beta > p$ and $\int_\Omega h < 0$. Therefore, $\overline{C_\varepsilon^+}$ is likely to approach a closed loop as $\varepsilon \to 0$, which bifurcates from the origin and backs to the origin, as Figure 3.

We now investigate $\overline{C_\varepsilon^+}$ as $\varepsilon \to 0$. Although it seems likely in Figure 3 that $\overline{C_\varepsilon^+}$ approaches a closed loop joining the origin to itself as $\varepsilon \to 0$, this seems difficult to establish. We can, however, prove that $\overline{C_\varepsilon^+}$ does not shrink to a point. For sets E_n, $n \in \mathbb{N}$, we define

$$\liminf_{n\to\infty} E_n = \{x : \text{there exists } N_0 \in \mathbb{N} \text{ such that any neighborhood of } x \text{ intersects } E_n \text{ for all } n \geq N_0 \},$$
$$\limsup_{n\to\infty} E_n = \{x : \text{any neighborhood of } x \text{ intersects } E_n \text{ for infinitely many } n \}.$$

According to [8], if $\cup_{n\geq 1} E_n$ is precompact in M and $\underline{\lim}_{n\to\infty} E_n \neq \emptyset$, then $\overline{\lim}_{n\to\infty} E_n$ is non-empty, closed and connected. Here, $\{E_n\}$ is a sequence of connected sets in a complete metric space M.

Obviously, $(0, 0) \in \overline{C_\varepsilon^+}$, so $(0, 0) \in \underline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$. It follows from the results of the priori bounds that $\cup_{\varepsilon>0} C_\varepsilon^+$ is precompact in $C^2(\bar{\Omega})$. Then $\overline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$ is non-empty, closed and connected. We note that $(0, 0) \in \overline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$, and also, from the definition, that $\overline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$ consists of nonnegative solutions of the problem (1.1).

Since $\overline{C_\varepsilon^+}$ joining $(0, 0)$ and $(\lambda_+(m_\varepsilon), 0)$ is subcritical at $(0, 0)$ and supercritical at $(\lambda_+(m_\varepsilon), 0)$, then $\overline{C_\varepsilon^+}$ must join $(0, u_\varepsilon)$, u_ε is a positive solution of the equation

$$\begin{cases} -\Delta u = h(x)u^p + \int_\Omega u^\beta, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

By a priori estimate of positive solutions, we see that u_ε is bounded. By virtue of the regularity theory of the elliptic equation, it follows that $\{u_\varepsilon\}$ must have a convergent subsequence in $C^2(\bar{\Omega})$ converging to u, where u is a solution of the Eq (4.1). Moreover, we have $(0, u) \in \overline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$.

Next, we prove $u \not\equiv 0$. Otherwise, we have $u_\varepsilon \to 0$. Let $v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|}$, by the Eq (4.1), we see that

$$\int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega h(x)u_\varepsilon^{p+1} + \int_\Omega u_\varepsilon \int_\Omega u_\varepsilon^\beta,$$

then

$$\int_\Omega |\nabla v_\varepsilon|^2 = \|u_\varepsilon\|^{p-1} \int_\Omega h(x)v_\varepsilon^{p+1} + \|u_\varepsilon\|^{\beta-1} \int_\Omega v_\varepsilon \int_\Omega v_\varepsilon^\beta. \tag{4.2}$$

Thus, we have $\int_\Omega |\nabla v_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$. Since v_ε is bounded, so we may assume that $v_\varepsilon \rightharpoonup v_0$ in $W^{1,2}(\Omega)$, $v_\varepsilon \to v_0$ in $L^{p+1}(\Omega)$ and $L^\beta(\Omega)$, hence, we claim that $v_\varepsilon \to v_0$ in $W^{1,2}(\Omega)$. Otherwise, we have

$$\int_\Omega |\nabla v_0|^2 < \underline{\lim}_{\varepsilon\to 0} \int_\Omega |\nabla v_\varepsilon|^2 \leq 0,$$

a contradiction, so v_0 is a positive constant c, then $v_\varepsilon \to c$ in $L^{p+1}(\Omega)$ and $L^\beta(\Omega)$. Thus, when ε is sufficiently small, we have $\int_\Omega v_\varepsilon \int_\Omega v_\varepsilon^\beta < 0$ for $\beta > p$, but this is impossible because of the equality in (4.2).

Therefore, under the conditions of the Theorem 1.3, there exists a connected components C^+ of positive solutions set such that its closure $\overline{C^+}$ includes $\overline{\lim}_{\varepsilon\to 0} C_\varepsilon^+$, which bifurcates from the origin and backs to the origin, namely, $\overline{C^+}$ is a closed loop.

□

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Conflict of interest

The authors declare no conflict of interest.

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