



Research article

Almost primes in Piatetski-Shapiro sequences

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Abstract: The Piatetski-Shapiro sequences are sequences of the form $(\lfloor n^c \rfloor)_{n=1}^\infty$ for $c > 1$ and $c \notin \mathbb{N}$. It is conjectured that there are infinitely many primes in Piatetski-Shapiro sequences for $c \in (1, 2)$. For every $R \geq 1$, we say that a natural number is an R -almost prime if it has at most R prime factors, counted with multiplicity. In this paper, we prove that there are infinitely many R -almost primes in Piatetski-Shapiro sequences if $c \in (1, c_R)$ and c_R is an explicit constant depending on R .

Keywords: almost primes; exponent pair; Piatetski-Shapiro sequences

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1. Introduction

The Piatetski-Shapiro sequences are sequences of the form

$$\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^\infty \quad (c > 1, c \notin \mathbb{N}).$$

Such sequences have been named in honor of Piatetski-Shapiro [13], who published the first paper in this problem. He showed that the counting function

$$\pi^{(c)}(x) := |\{\text{prime } p \leq x : p \in \mathcal{N}^{(c)}\}|$$

satisfies the asymptotic relation

$$\pi^{(c)}(x) \sim \frac{x^{1/c}}{\log x} \quad \text{as } x \rightarrow \infty,$$

if

$$1 < c < \frac{12}{11} = 1.0909\dots$$

The range for c of the asymptotic formula of $\pi^{(c)}(x)$ has been improved by several mathematicians over the years. Kolesnik [7] improved this result to

$$1 < c < \frac{10}{9} = 1.1111\dots$$

Graham and Leitmann [10] using the method of exponent pairs independently improved the range to

$$1 < c < \frac{69}{62} = 1.1129\dots$$

Graham did not publish his paper. Heath-Brown [4] applied the Weyl's shift and the exponent pair method, together with his decomposition of the Von Mangoldt function, extended the range to

$$1 < c < \frac{755}{662} = 1.1404\dots$$

Kolesnik [8] using the method of multiple exponential sums improved the range to

$$1 < c < \frac{39}{34} = 1.1470\dots$$

Liu and Rivat [12] applied the double large sieve to Type I sums and extended the range to

$$1 < c < \frac{15}{13} = 1.1538\dots$$

Rivat [14] improved the range in his PhD thesis to

$$1 < c < \frac{6121}{5302} = 1.1544\dots$$

Rivat and Sargos [15] held the best record to be

$$1 < c < \frac{2817}{2426} = 1.1611\dots$$

We mention that Leitmann and Wolke [11] proved that the asymptotic formula holds for almost all $c \in (1, 2)$ in the sense of Lebesgue measure.

Rivat also considered to prove that there are infinitely many Piatetski-Shapiro primes by giving a lower bound of $\pi^{(c)}(x)$. He used a sieve method and showed that

$$\pi^{(c)}(x) \gg \frac{x^{1/c}}{\log x}$$

if

$$1 < c < \frac{7}{6} = 1.1666\dots$$

Later, Baker, Harman and Rivat [1] and Jia [6] improved this range to

$$1 < c < \frac{20}{17} = 1.1764\dots$$

Jia [5] extended the range again to

$$1 < c < \frac{13}{11} = 1.1818\dots$$

Kumchev [9] improved the range to

$$1 < c < \frac{45}{38} = 1.1842\dots$$

Eventually, Rivat and Wu [16] gave the best range up to now, which is

$$1 < c < \frac{243}{205} = 1.1853\dots$$

We remark that if $c \in (0, 1)$ then $\mathcal{N}^{(c)}$ contains all natural numbers, hence contains all primes. The estimation of Piatetski-Shapiro primes is an approximation of the well-known conjecture that there exists infinitely many primes of the form $n^2 + 1$.

It is conjectured that there are infinitely many Piatetski-Shapiro primes for $c \in (1, 2)$. However, the best known bound for c is still far from 2 and the range for c has not been improved for almost 20 years. We approach this problem in a different direction. For every $R \geq 1$, we say that a natural number is an R -almost prime if it has at most R prime factors, counted with multiplicity. The study of almost primes is an intermediate step to the investigation of primes. In this paper, we prove there are infinitely many almost primes in Piatetski-Shapiro sequences.

THEOREM 1.1. *For any fixed $c \in (1, c_R)$ we have*

$$|\{n \leq x : [n^c] \text{ is an } R\text{-almost prime}\}| \gg \frac{x}{\log x}$$

holds for all sufficiently large x . In particular, we have

$$c_3 := \frac{889}{741} = 1.1997\dots, \quad c_4 := \frac{25882}{16071} = 1.6104\dots,$$

and

$$c_R := 3 - \frac{128}{3(8R - 1)} \quad (R \geq 5).$$

Recall that the best known range that there are infinitely many Piatetski-Shapiro primes is $(1, 1.1853)$ and $c_3 = 1.1997 > 1.1853$. Hence our theorem for 3-almost primes provides a bigger range of c than that of prime numbers. Moreover, when $R = 6$ we have that

$$c_6 = \frac{295}{141} = 2.0921,$$

which is greater than 2.

2. Preliminaries

2.1. Notation

We denote by $[t]$ and $\{t\}$ the integer part and the fractional part of t , respectively. As is customary, we put $\mathbf{e}(t) := e^{2\pi it}$. We make considerable use of the sawtooth function defined by

$$\psi(t) := t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2} \quad (t \in \mathbb{R}).$$

The letter p always denotes a prime. For the Piatetski-Shapiro sequence $([n^c])_{n=1}^{\infty}$, we denote $\gamma := c^{-1}$. We use notation of the form $m \sim M$ as an abbreviation for $M < m \leq 2M$.

Throughout the paper, implied constants in symbols O , \ll and \gg may depend (where obvious) on the parameters c, ε but are absolute otherwise. For given functions F and G , the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq C|G|$ holds with some constant $C > 0$. $F \asymp G$ means that $F \ll G \ll F$.

2.2. The weighted sieve

As we have mentioned the following notion plays a crucial role in our arguments. We specify it to the form that is suited to our applications; it is based on a result of Greaves [3] that relates level of distribution to R -almost primality. More precisely, we say that an N -element set of integers \mathcal{A} has a level of distribution D if for a given multiplicative function $f(d)$ we have

$$\sum_{d \leq D} \max_{\gcd(s,d)=1} \left| \left| \{a \in \mathcal{A}, a \equiv s \pmod{d}\} \right| - \frac{f(d)}{d} N \right| \leq \frac{N}{\log^2 N}.$$

As in [3, pp. 174–175] we define the constants

$$\delta_2 := 0.044560, \quad \delta_3 := 0.074267, \quad \delta_4 := 0.103974$$

and

$$\delta_R := 0.124820, \quad R \geq 5.$$

We have the following result, which is [3, Chapter 5, Proposition 1].

LEMMA 2.1. *Suppose \mathcal{A} is an N -element set of positive integers with a level of distribution D and degree ρ in the sense that*

$$a < D^\rho \quad (a \in \mathcal{A})$$

holds with some real number $\rho < R - \delta_R$. Then

$$\left| \{a \in \mathcal{A} : a \text{ is an } R\text{-almost prime}\} \right| \gg_\rho \frac{N}{\log N}.$$

2.3. Technical lemmas

LEMMA 2.2. *Let $M \geq 1$ and λ be positive real numbers and let H be a positive integer. If $f : [1, M] \rightarrow \mathbb{R}$ is a real valued function with three continuous derivatives, which satisfies*

$$\lambda \leq |f^{(3)}(x)| \ll \lambda \quad \text{for } 1 \leq x \leq M,$$

then for the sum

$$S = \frac{1}{H} \sum_{h=H+1}^{2H} \left| \sum_{m=1}^{M_h} \mathbf{e}\left(\frac{h}{H}f(m)\right) \right|,$$

where the integer M_h satisfies $1 \leq M_h \leq M$ for each $h \in [H+1, 2H]$, we have

$$S \ll M^\varepsilon \left(M\lambda^{1/6}H^{-1/9} + M\lambda^{1/5} + M^{3/4} \right) + \lambda^{-1/3}.$$

Proof. See [17, Theorem 1]. □

LEMMA 2.3. For any $H \geq 1$ there are numbers a_h, b_h such that

$$\left| \psi(t) - \sum_{0 < |h| \leq H} a_h \mathbf{e}(th) \right| \leq \sum_{|h| \leq H} b_h \mathbf{e}(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.$$

Proof. See [18]. □

We also need the method of exponent pair. A detailed definition of exponent pair can be found in [2, Page 31]. For an exponent pair (k, l) , we denote

$$A(k, l) := \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2} \right)$$

and

$$B(k, l) := \left(l - \frac{1}{2}, k + \frac{1}{2} \right)$$

the A-process and B-process of the exponent pair, respectively.

3. Proof of Theorem 1.1

3.1. Initial approach

The set we sieve is

$$\mathcal{A} := \{m \leq x^c : m = \lfloor n^c \rfloor \text{ for integer } n\}.$$

For any $d \leq D$, where D is a fixed power of x , we estimate

$$\mathcal{A}_d := \{m \in \mathcal{A} : d \mid m\}.$$

We know that $rd \in \mathcal{A}$ if and only if

$$rd \leq n^c < rd + 1 \quad \text{and} \quad rd \leq x.$$

Within $O(1)$ the cardinality of \mathcal{A}_d is equal to the number of integers $n \leq x$ for which the interval $((n^c - 1)d^{-1}, n^c d^{-1}]$ contains a natural number. Hence

$$|\mathcal{A}_d| = \sum_{n \leq x} \left(\lfloor n^c d^{-1} \rfloor - \lfloor (n^c - 1)d^{-1} \rfloor \right) + O(1)$$

$$= \mathcal{X}d^{-1} + \sum_{n \leq x} (\psi((n^c - 1)d^{-1}) - \psi(n^c d^{-1})) + O(1),$$

where

$$\mathcal{X} := \sum_{n \leq x} 1 = x.$$

By Lemma 2.1 we need to show that for any sufficiently small $\varepsilon > 0$

$$\sum_{d \leq D} |\mathcal{A}_d - \mathcal{X}d^{-1}| \leq \mathcal{X}x^{-\varepsilon/3} = x^{1-\varepsilon/3}$$

for sufficiently large x . Splitting the range of d into dyadic subintervals, it is sufficient to prove that

$$\sum_{d \sim D_1} \left| \sum_{N < n \leq N_1} (\psi((n^c - 1)d^{-1}) - \psi(n^c d^{-1})) \right| \ll x^{1-\varepsilon/2} \quad (3.1)$$

holds uniformly for $D_1 \leq D, N \leq x, N_1 \sim N$. Our aim is to establish (3.1) with D as large as possible. We define

$$S := \sum_{N < n \leq N_1} (\psi((n^c - 1)d^{-1}) - \psi(n^c d^{-1})). \quad (3.2)$$

3.2. Estimation of S by the method of exponent pair

By Lemma 2.3 and taking that

$$H = Dx^\varepsilon,$$

we have

$$S = S_1 + O(S_2),$$

where

$$S_1 := \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} a_h (\mathbf{e}(h(n^c - 1)d^{-1}) - \mathbf{e}(hn^c d^{-1}))$$

and

$$S_2 := \sum_{N < n \leq N_1} \sum_{|h| \leq H} b_h (\mathbf{e}(h(n^c - 1)d^{-1}) + \mathbf{e}(hn^c d^{-1})).$$

We consider S_1 . Writing that

$$\phi_h := \mathbf{e}(hd^{-1}) - 1 \ll 1,$$

we obtain that

$$\begin{aligned} S_1 &= \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} a_h \phi_h \mathbf{e}(hn^c d^{-1}) \\ &\ll \sum_{0 < |h| \leq H} h^{-1} \sum_{N < n \leq N_1} \mathbf{e}(hn^c d^{-1}). \end{aligned} \quad (3.3)$$

Using the exponent pair (k, l) , we have

$$\sum_{N < n \leq N_1} \mathbf{e}(hn^c d^{-1}) \ll (hd^{-1}N^{c-1})^k N^l + (hd^{-1})^{-1} N^{1-c}. \quad (3.4)$$

Substituting (3.4) to S_1 and (3.1), it becomes that

$$\begin{aligned} \sum_{d \sim D_1} |S_1| &\ll \sum_{d \sim D_1} \left| \sum_{0 < h \leq H} h^{-1} (h^k d^{-k} N^{kc-k+l} + h^{-1} d N^{1-c}) \right| \\ &\ll \sum_{d \sim D_1} \left| H^k d^{-k} N^{kc-k+l} + H^{-1} d N^{1-c} \right| \\ &\ll H^k D^{1-k} N^{kc-k+l} + H^{-1} D^2 N^{1-c} \\ &\ll D x^{kc-k+l+k\varepsilon} + D^2 x^{1-c}. \end{aligned}$$

Now we consider S_2 . The contribution of S_2 from $h = 0$ is

$$\sum_{N < n \leq N_1} b_h \ll NH^{-1}. \quad (3.5)$$

Substituting (3.5) to (3.1), we have

$$\sum_{d \sim D_1} NH^{-1} \ll DNH^{-1} \ll x^{1-\varepsilon/2},$$

The contribution of S_2 from $h \neq 0$ is

$$\begin{aligned} &= \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} b_h \left(\mathbf{e}(h(m^c - 1)d^{-1}) + \mathbf{e}(hm^c d^{-1}) \right) \\ &\ll \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} b_h \phi_h \mathbf{e}(hn^c d^{-1}) \\ &\ll \sum_{0 < h \leq H} H^{-1} \sum_{N < n \leq N_1} \mathbf{e}(hn^c d^{-1}), \end{aligned} \quad (3.6)$$

which can be estimated by the same method of S_1 . By (3.4), we write (3.6) to be

$$\begin{aligned} &\ll \sum_{d \sim D_1} \left| \sum_{0 < h \leq H} H^{-1} (h^k d^{-k} N^{kc-k+l} + h^{-1} d N^{1-c}) \right| \\ &\ll H^k D^{1-k} N^{kc-k+l} + H^{-1} D^2 N^{1-c} \log H \\ &\ll D x^{kc-k+l+k\varepsilon} + D^2 x^{1-c}. \end{aligned}$$

Therefore, to make (3.1) to be true, we need that

$$D x^{kc-k+l+k\varepsilon} \ll x^{1-\varepsilon/2}, \quad (3.7)$$

and

$$D^2 x^{1-c} \ll x^{1-\varepsilon/2}. \quad (3.8)$$

Combining (3.7) and (3.8), we obtain that

$$D \ll \min \left(x^{c/2-\varepsilon/4}, x^{1-kc+k-l-\varepsilon} \right). \quad (3.9)$$

3.3. Exponent pair estimation for $R = 3$

We apply the weighted sieve with the choice

$$R = 3, \quad \delta_3 = 0.074267$$

and choose

$$\Lambda_R = 3 - \frac{3}{40} = \frac{117}{40} < R - \delta_R.$$

By (3.9) we require that

$$1 - kc + k - l > \frac{40}{117} \quad \text{and} \quad \frac{c}{2} > \frac{40}{117}, \quad (3.10)$$

then by Lemma 2.1, \mathcal{A} contains $\gg x/\log x$ R -almost primes. To achieve (3.10), we need that

$$c < \frac{70 - 117l}{117k} + 1.$$

Taking the exponent pair

$$BAAAAAB(0, 1) = \left(\frac{19}{42}, \frac{32}{63} \right),$$

we have that

$$c < \frac{889}{741} = 1.1997 \dots$$

3.4. Exponent pair estimation for $R = 4$

We apply the weighted sieve with the choice

$$R = 4, \quad \delta_4 = 0.103974$$

and choose

$$\Lambda_R = 4 - \frac{13}{125} = \frac{487}{125} < R - \delta_R.$$

Similarly to (3.10), we need that

$$1 - kc + k - l > \frac{125}{487} \quad \text{and} \quad \frac{c}{2} > \frac{125}{487}, \quad (3.11)$$

which requires that

$$c < \frac{362 - 487l}{487k} + 1 = \frac{25882}{16071} = 1.6104 \dots,$$

by taking the exponent pair

$$BABABAABAAB(0, 1) = \left(\frac{33}{128}, \frac{75}{128} \right).$$

3.5. A refinement for $R \geq 5$

For $R \geq 5$, we estimate (3.2) by Lemma 2.2. By (3.3) we have that

$$S_1 \ll \log H \max_{1 \leq T \leq H} S(T, N),$$

where

$$S(T, N) := \frac{1}{T} \sum_{h \sim T} \sum_{n \sim N} \mathbf{e}(hd^{-1}n^c).$$

By Lemma 2.2 with $f(n) = Td^{-1}(n + N)^c$ and

$$\lambda = c(c-1)(c-2)Td^{-1}N^{c-3},$$

it follows that

$$\begin{aligned} S(T, N) &\ll N^{1+\varepsilon}(Td^{-1}N^{c-3})^{1/6}T^{-1/9} + N^{1+\varepsilon}(Td^{-1}N^{c-3})^{1/5} \\ &\quad + N^{3/4+\varepsilon} + (Td^{-1}N^{c-3})^{-1/3} \\ &\ll T^{1/18}d^{-1/6}N^{c/6+1/2+\varepsilon} + T^{1/5}d^{-1/5}N^{c/5+2/5+\varepsilon} \\ &\quad + N^{3/4+\varepsilon} + T^{-1/3}d^{1/3}N^{1-c/3}. \end{aligned}$$

Hence

$$\begin{aligned} S_1 &\ll H^{1/18}d^{-1/6}N^{c/6+1/2+\varepsilon} + H^{1/5}d^{-1/5}N^{c/5+2/5+\varepsilon} \\ &\quad + N^{3/4+\varepsilon} + d^{1/3}N^{1-c/3}. \end{aligned}$$

The contribution of S_2 from $h \neq 0$ can be estimated by the same method and achieve the same upper bound, which means that (3.6) is

$$\begin{aligned} &\ll H^{1/18}d^{-1/6}N^{c/6+1/2+\varepsilon} + H^{1/5}d^{-1/5}N^{c/5+2/5+\varepsilon} \\ &\quad + N^{3/4+\varepsilon} + d^{1/3}N^{1-c/3}. \end{aligned}$$

Together with the contribution of S_2 from $h = 0$, by (3.5) we obtain that the left-hand side of (3.1) is

$$\begin{aligned} \sum_{d \sim D_1} |S| &\ll \sum_{d \sim D_1} \left| H^{1/18}d^{-1/6}N^{c/6+1/2+\varepsilon} + H^{1/5}d^{-1/5}N^{c/5+2/5+\varepsilon} \right. \\ &\quad \left. + N^{3/4} \log H + d^{1/3}N^{1-c/3} \right| \\ &\ll H^{1/18}D^{5/6}N^{c/6+1/2+\varepsilon} + H^{1/5}D^{4/5}N^{c/5+2/5+\varepsilon} \\ &\quad + DN^{3/4} \log H + D^{4/3}N^{1-c/3} \\ &\ll D^{8/9}x^{c/6+1/2+19\varepsilon/18} + Dx^{c/5+2/5+6\varepsilon/5} + Dx^{3/4+\varepsilon} + D^{4/3}x^{1-c/3}. \end{aligned}$$

To ensure the left-hand side of (3.1) is $\ll x^{1-\varepsilon/2}$, we require that

$$D \ll \min \left(x^{9/16-3c/16-\varepsilon}, x^{3/5-c/5-\varepsilon}, x^{1/4-\varepsilon}, x^{c/4-\varepsilon} \right). \quad (3.12)$$

3.6. The bound of c for $R \geq 5$

We apply the weighted sieve with the choice

$$\delta_R = 0.124820 \quad (R \geq 5)$$

and choose

$$\Lambda_R = R - \frac{1}{8} < R - \delta_R.$$

To apply Lemma 2.1, by (3.12) we need that

$$\min\left(\frac{9}{16} - \frac{3}{16}c, \frac{3}{5} - \frac{c}{5}, \frac{1}{4}, \frac{c}{4}\right) > \frac{1}{R - \frac{1}{8}},$$

which gives that

$$c < 3 - \frac{128}{3(8R - 1)}.$$

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Conflict of interest

The author declares no conflicts of interest in this paper.

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