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Research article

The third-power moment of the Riesz mean error term of symmetric square L-function

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Abstract: Let $f(z)$ be a holomorphic Hecke eigenform of weight k with respect to the full modular group $SL(2, Z)$, and let $\Delta_\rho(x; sym^2 f)$ be the error term of the Riesz mean of the symmetric square L-function $L(s, sym^2 f)$. In this paper, using a Voronoï type formula for $\Delta_\rho(x; sym^2 f)$, we consider the third-power moment of $\Delta_\rho(x; sym^2 f)$ and derive the asymptotic formula for

$$\int_1^T \Delta_\rho^3(x; sym^2 f)dx.$$

Keywords: symmetric square L-function; power moment; Riesz mean; Voronöi type formula
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1. Introduction

Let $\mathbb{H}$ be the upper half plane, *i.e.*, $\mathbb{H} = \{z = x + iy|\ x, y \in \mathbb{R}, \text{and } y > 0\}$. Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even weight $k \geq 2$ for the full modular group $\Gamma = SL(2, Z)$. Then for $f \in S_k(\Gamma)$ and $z \in \mathbb{H}$, we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz),$$

where $a_f(n)$ denotes the n-th Fourier coefficient, and $e(x) = e^{2\pi ix}$. Set $\lambda_f(n) = a_f(n)n^{-\frac{k-1}{2}}$, then $\lambda_f(n)$ is multiplicative. The analytic properties of $\lambda_f(n)$ were studied by many authors (see [5, 7–10, 12–14, 16, 20, 22–25]). The symmetric square L-function attached to f can be written as

$$L(s, sym^2 f) = \zeta(2s)\sum_{n=1}^{\infty} \lambda_f(n^2)n^{-s} = \sum_{n=1}^{\infty} c_n n^{-s}$$

for $\Re s > 1$. And it can be continued to an entire function on the whole complex plane. Then it is known that for any $\epsilon > 0$,

$$|c_n| \le d_3(n) \ll n^{\epsilon},$$

where $d_3(n)$ is the number of ways to write n as a product of three factors.

Hafner [3] considered the Riesz mean of the form

$$D_\rho(x; \mathrm{sym}^2 f) = \frac{1}{\Gamma(\rho+1)} \sum_{n \le x}{}' (x-n)^\rho c_n,$$

where $\rho \ge 0$ is a fixed number and $\sum'$ indicates that c_n is replaced by $c_n/2$ if $\rho = 0$. Also, the Riesz mean can be represented as a sum of "residue function" and "error term":

$$D_\rho(x; \mathrm{sym}^2 f) = \frac{L(0, \mathrm{sym}^2 f)}{\Gamma(\rho+1)} x^\rho + \Delta_\rho(x; \mathrm{sym}^2 f).$$

Then let $\Delta_\rho(x; sym^2 f)$ be the error term of the Riesz mean of the symmetric square L-function. Hafner [3] gave that for $\rho = 0$, one has

$$\sum_{n \le x} c_n = L(0, sym^2 f) + \Delta_0(x; sym^2 f).$$

The symmetric square L-function and the Riesz mean error term have been studied by many authors, for example, see [1, 4, 6, 11, 15, 18, 21, 25]. Fomenko [2] considered $\Delta_\rho(x; \mathrm{sym}^m f)$ when $m = 2$ and obtained a truncated Voronoï type formula, and we give it in Lemma 2.1. Let $N = x^{\frac{1}{2}}$ in Lemma 2.1, one obtain

$$\Delta_\rho(x; sym^2 f) \ll x^{\frac{1+\rho}{2}}.$$

Using the Voronöi formula, Fomenko got

$$\int_1^X \Delta_\rho^2(x; sym^2 f) dx = C X^{\frac{4\rho+5}{3}} + O(X^{\rho+\frac{5}{3}+\epsilon}),$$

where

$$C = 2^{-2\rho-1} \pi^{-2\rho-2} (4\rho+5)^{-1} \sum_{n=1}^{\infty} c_n^2 n^{-\frac{2\rho+4}{3}}.$$

Wang [19] generalized the truncated Voronöi type formula to the case $m \ge 3$ under the hypothesis Nice(m, f):

$$\begin{aligned}\Delta_\rho(x; sym^m f) =& \epsilon(sym^m f) 2^{-\rho} \pi^{-\rho-1} (m+1)^{-\frac{1}{2}} x^{\frac{m}{m+1}\rho + \frac{m}{2(m+1)}} \\ &\times \sum_{n \le N} c_n n^{-\frac{1}{m+1}\rho - \frac{m+2}{2(m+1)}} \cos\left(2\pi(m+1)(nx)^{\frac{1}{m+1}} - (A(m,k) + \frac{\rho}{2})\pi\right) \\ &+ O\left(x^{\frac{m}{m+1}\rho + \frac{m-1}{2(m+1)}} N^{-\frac{\rho}{m+1}\rho + \frac{m-1}{2(m+1)} + \epsilon}\right) + O\left(x^{\frac{m}{m+1}(\rho+1)+\epsilon} N^{-\frac{\rho+1}{m+1}}\right),\end{aligned}$$

where

$$A(m,k) = \begin{cases} \frac{m(m+2)(k-1)}{8} + \frac{1}{2}, & m \equiv 0 \pmod 4; \\ \frac{m(m+2)(k-1)}{8} + 1, & m \equiv 2 \pmod 4; \\ \frac{(m+1)^2(k-1)}{8} + \frac{3}{4}, & m \text{ is odd}, \end{cases}$$

$$\epsilon\left(\mathrm{sym}^m f\right)=\begin{cases}+1, & \text{if } m \text{ even};\\ \epsilon(k,m), & \text{otherwise},\end{cases}$$

with

$$\epsilon(k,m):=\mathrm{i}^{\left(\frac{m+1}{2}\right)^2(k-1)+\frac{m+1}{2}}=\begin{cases}\mathrm{i}^k, & \text{if } m\equiv 1 \pmod 8,\\ -1, & \text{if } m\equiv 3 \pmod 8,\\ -\mathrm{i}^k, & \text{if } m\equiv 5 \pmod 8,\\ +1, & \text{if } m\equiv 7 \pmod 8).\end{cases}$$

Liu and Wang [15] studied the higher-power moments of $\Delta_\rho(x; sym^2 f)$. If exists a real number $A_0 = A_0(\rho) > 3$ such that

$$\int_1^T \Delta_\rho^{A_0}(x; sym^2 f)dx \ll T^{1+\frac{2\rho+1}{3}A_0+\epsilon},$$

then for any integer $3 \le h < A_0$, they had the following asymptotic formula

$$\begin{aligned}\int_1^T \Delta_\rho^h(x; sym^2 f)dx =&\frac{6B_\rho(h,c)}{(3+(2\rho+1)h)\,(2\pi)^{(\rho+1)h}3^{\frac{h}{2}}}T^{1+\frac{2\rho+1}{3}h}\\ &+O\left(T^{1+\frac{2\rho+1}{3}h+\epsilon}(T^{-\delta_\rho(h,A_0)}+T^{-\frac{\rho}{3}})\right),\end{aligned}$$

where

$$B_\rho(h;f):=\sum_{l=1}^{h-1}\binom{h-1}{l}s_\rho(l,h;f)\cos\left(\frac{\pi\rho}{2}(h-2l)\right)$$

with

$$s_\rho(l,h;f):=\sum_{\sqrt[3]{n_1}+\cdots+\sqrt[3]{n_l}=\sqrt[3]{n_{l+1}}+\cdots+\sqrt[3]{n_h}}\frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(\rho+2)/3}}\quad (1\leqslant l<h),$$

and

$$\delta_\rho(h,A_0):=\sigma_\rho(h,A_0)\min\left(\frac{2\rho}{2\rho+1},\frac{1}{3b_\rho(H_0)}\right)$$

with

$$\sigma_\rho(h,A_0):=\frac{(2\rho+1)(A_0-h)}{3(A_0-2)},\quad 3\leqslant h<A_0,$$

$$b_\rho(h):=3^{h-2}-\frac{1}{3}+\frac{(1-\rho)h}{3}$$

and H_0 is the least even integer such that $n \geqslant A_0$. They also proved the result for $\rho=\frac{1}{2}$, $h = 3,4,5$,

$$\int_1^T \Delta_\rho^h(x; sym^2 f)dx=\frac{6B_{\frac{1}{2}}(h,c)}{(3+2h)(2\pi)^{\frac{3}{2}h}3^{\frac{h}{2}}}T^{1+\frac{2}{3}h}+O\left(T^{1+\frac{2}{3}h-\lambda_{1/2}(h,6)+\epsilon}\right),$$

where $\lambda_{1/2}(3,6)=\frac{1}{22}$, $\lambda_{1/2}(4,6)=\frac{1}{87}$, $\lambda_{1/2}(5,6)=\frac{1}{498}$. Zhang, Han and Zhang [26] studied the power moment of the Riesz mean error term in short intervals. For $k \ge 3$, $T^{\frac{2}{3}+\frac{2\epsilon d_{\rho,k}}{(2\rho+1)\delta}} \le H \le T$ and $\epsilon > 0$, they got

$$\int_{T-H}^{T+H}\Delta_\rho^k(x; sym^2 f)dx = B_k\int_{T-H}^{T+H}x^{\frac{2\rho+1}{3}k}dx+O\left(HT^{\frac{2\rho+1}{3}k+\epsilon}(HT^{-\frac{2}{3}})^{-\frac{(2\rho+1)\delta}{d_{\rho,k}}}\right),$$

where

$$B_k = 2^{-k\rho-k+1}\pi^{-k\rho-k}3^{-\frac{k}{2}}c_k,\ d_{\rho,k} = (3^{k-1} - 4 + k - \rho k)(k + \delta - 2) + (2\rho + 1)\delta.$$

Their results improved the results in [15] when $\rho = \frac{1}{2}$, $k = 3, 4, 5$ and $\delta > \frac{50}{31}$.

In this paper, we will use the Voronöi type formula for $\Delta_\rho(x; sym^2 f)$ to study the third-power moment estimates of $\Delta_\rho(x; sym^2 f)$ and obtain the following theorem.

Theorem 1. *For any $f \in S_k(\Gamma)^+$, $x > 1$ and $\frac{\sqrt{612}}{18} - 1 < \rho < \frac{2}{3}$, we have*

$$\int_1^T \Delta_\rho^3(x; sym^2 f)dx = \left(2^\rho \pi^{\rho+1}\sqrt{3}\right)^{-3} \frac{3}{8\rho+8} \cos\left(\frac{\pi\rho}{2}\right) A T^{2\rho+2} + O\left(T^{\frac{6\rho^2+21\rho+16}{8+3\rho}+\epsilon}\right),$$

where

$$A = \sum_{\alpha,\ \beta=1}^{\infty} (\alpha\beta(\alpha+\beta))^{-\rho-2} \sum_{\substack{h=1 \\ h\ is\ cube-free}}^{\infty} h^{-\rho-2} c_{\alpha^3 h} c_{\beta^3 h} c_{(\alpha+\beta)^3 h}.$$

Particularly, for $\rho = \frac{1}{2}$, we can get the following result.

Corollary 2. *Let $\rho = \frac{1}{2}$. For any $\epsilon > 0$, we have*

$$\int_1^T \Delta_\rho^3(x; sym^2 f)dx = \frac{\sqrt{3}\pi^{-\frac{9}{2}}}{144} B T^3 + O(T^{\frac{56}{19}+\epsilon}),$$

where

$$B = \sum_{\alpha,\ \beta=1}^{\infty} (\alpha\beta(\alpha+\beta))^{-\frac{3}{2}} \sum_{\substack{h=1 \\ h\ is\ cube-free}}^{\infty} h^{-\frac{3}{2}} c_{\alpha^3 h} c_{\beta^3 h} c_{(\alpha+\beta)^3 h}.$$

Remark. *Note that our results improve the results in [15].*

2. Some preliminary lemmas

To prove our theorem, we need the following lemmas.

Lemma 2.1. *Let $x > 1$, $N \geq 1$. Then for any fixed ρ, $0 \leq \rho \leq 1$, we have*

$$\begin{aligned} \Delta_\rho(x; sym^2 f) =& 2^{-\rho}\pi^{-\rho-1}3^{-\frac{1}{2}}\Sigma(x) + O\left(x^{\frac{4\rho+1}{6}} N^{\frac{-2\rho+1}{6}+\epsilon}\right) \\ &+ O\left(x^{\frac{2\rho+2}{3}+\epsilon} N^{-\frac{\rho+1}{3}}\right) + O\left(x^\epsilon\right), \end{aligned} \tag{2.1}$$

where

$$\Sigma(x) = \sum_{n \leq N} c_n n^{-\frac{\rho+2}{3}} x^{\frac{2\rho+1}{3}} \cos\left(6\pi\sqrt[3]{nx} - \frac{\pi\rho}{2}\right).$$

Proof. See Theorem 1.1 in [2]. □

Lemma 2.2. *Let $h \geq 3$, $(i_1, \cdots, i_{h-1}) \in \{0, 1\}^{h-1}$ such that*

$$\sqrt[3]{n_1} + (-1)^{i_1}\sqrt[3]{n_2} + (-1)^{i_2}\sqrt[3]{n_3} + \cdots + (-1)^{i_{h-1}}\sqrt[3]{n_h} \neq 0.$$

Then

$$\left|\sqrt[3]{n_1} + (-1)^{i_1}\sqrt[3]{n_2} + \cdots + (-1)^{i_{h-1}}\sqrt[3]{n_h}\right| \gg \max(n_1, \cdots, n_h)^{-(3^{h-2}-3^{-1})}.$$

Proof. See for example Lemma 2.3 in [21]. □

Lemma 2.3. *If $g(x)$ and $h(x)$ are continuous real-valued functions of x and $g(x)$ is monotonic, then*

$$\int_a^b g(x)h(x)dx \ll \left(\max_{a\le x\le b}|g(x)|\right)\left(\max_{a\le u<v\le b}\left|\int_u^v h(x)dx\right|\right).$$

Proof. See the Lemma 1 in [17]. □

Lemma 2.4. *Suppose $k\ge 3$, $(i_1,\ldots,i_{k-1})\in\{0,1\}^{k-1}$, $(i_1,\ldots,i_{k-1})\ne(0,\ldots,0)$ and*

$$N_1,\ldots,N_k>1,\quad 0<\Delta\ll E^{\frac{1}{3}},\quad E=\max(N_1,\ldots,N_k).$$

Let

$$\mathcal{A}=\mathcal{A}(N_1,\ldots,N_k;i_1,\ldots,i_{k-1};\Delta)$$

denote the number of solutions of the inequality

$$\left|\sqrt[3]{n_1}+(-1)^{i_1}\sqrt[3]{n_2}+\cdots+(-1)^{i_{k-1}}\sqrt[3]{n_k}\right|<\Delta \tag{2.2}$$

with $N_j<n_j\le 2N_j$, $1\le j\le k$. Then

$$\mathcal{A}\ll \Delta E^{-\frac{1}{3}}N_1\cdots N_k+E^{-1}N_1\cdots N_k.$$

Proof. It can be proved similarly as Lemma 2.4 of [21]. Without loss of generality, suppose $E=N_k$. If $n_1,\ldots,n_k$ satisfy (2.2), then

$$\left|\sqrt[3]{n_1}+(-1)^{i_1}\sqrt[3]{n_2}+\cdots+(-1)^{i_{k-2}}\sqrt[3]{n_{k-1}}\right|=(-1)^{i_{k-1}+1}\sqrt[3]{n_k}+\theta\Delta$$

for some $|\theta|<1$. We can get

$$\left(\sqrt[3]{n_1}+(-1)^{i_1}\sqrt[3]{n_2}+\cdots+(-1)^{i_{k-2}}\sqrt[3]{n_{k-1}}\right)^3=(-1)^{i_{k-1}+1}n_k+O\left(\Delta N_k^{\frac{2}{3}}\right).$$

Hence for fixed $n_1,\ldots,n_{k-1}$, the number of n_k is $\ll 1+\Delta N_k^{\frac{2}{3}}$ and thus

$$\mathcal{A}\ll \Delta N_k^{\frac{2}{3}}N_1\cdots N_{k-1}+N_1\cdots N_{k-1}.$$

□

3. The third-power moment of $\Delta_\rho(x, sym^2 f)$

Throughout this paper, n, m and k denote natural numbers. The constants implied in the symbols $\ll$ and O may depend on ϵ.

By (2.1), we know that to establish the asymptotic formula in our theorem, we shall prove, for $H\ge 1$, that

$$\int_H^{2H}\Sigma(x)^2dx=\frac{1}{2}\sum_{n=1}^{\infty}c_n^2 n^{-\frac{2\rho+4}{3}}\int_H^{2H}x^{\frac{4\rho+2}{3}}dx+O\left(H^{\frac{4\rho+5}{3}+\epsilon}N^{-\frac{2\rho+1}{3}}\right), \tag{3.1}$$

and take $N = H^{\frac{6\rho+8}{(8+3\rho)(\rho+1)}}$,

$$\int_H^{2H} \Sigma(x)^3 dx = \frac{3}{4}\cos\left(\frac{\pi\rho}{2}\right) A \int_H^{2H} x^{2\rho+1} dx + O\left(H^{\frac{6\rho^2+21\rho+16}{8+3\rho}+\epsilon}\right). \tag{3.2}$$

Then, since $(a+b)^3 = a^3 + O\left(|b|a^2 + |b|^3\right)$, it follows from (2.1) that

$$\begin{aligned}\int_H^{2H} \Delta_\rho^3(x; sym^2 f) dx =& \left(2^\rho \pi^{\rho+1}\sqrt{3}\right)^{-3} \int_H^{2H} \Sigma(x)^3 dx \\ &+ O\left(H^{\frac{2\rho+2}{3}+\epsilon} N^{-\frac{\rho+1}{3}} \int_H^{2H} \Sigma(x)^2 dx + H^{2\rho+3+\epsilon} N^{-\rho-1}\right) \\ =& \left(2^\rho \pi^{\rho+1}\sqrt{3}\right)^{-3} \frac{3A}{4} \cos\left(\frac{\pi\rho}{2}\right) \int_H^{2H} x^{2\rho+1} dx + O\left(H^{\frac{6\rho^2+21\rho+16}{8+3\rho}+\epsilon}\right)\end{aligned} \tag{3.3}$$

by (3.1) and (3.2). Adding this for $H = T/2, T/2^2, \ldots$, we see that the asymptotic formula in our theorem follows.

We shall now give the details of the proof of (3.2). The proof of (3.1), which we shall omit, employs similar arguments and is simpler.

For simplicity, put

$$r = r(n, m, k) := (nmk)^{-\frac{\rho+2}{3}} c_n c_m c_k, \quad n, m, k \le N, \tag{3.4}$$

and $r = 0$ otherwise. Taking the third power of both sides of $\Sigma(x)$ and for each element $\mathbf{i} = (i_1, i_2) \in \mathbb{I}^2, \mathbb{I} = \{0, 1\}$, using the elementary formula

$$\cos a_1 \cos a_2 \cos a_3 = \frac{1}{4} \sum_{\mathbf{i} \in \mathbb{I}^2} \cos\left(a_1 + (-1)^{i_1} a_2 + (-1)^{i_2} a_3\right),$$

we can write

$$\begin{aligned}\Sigma(x)^3 =& \frac{3}{4} \sum r x^{2\rho+1} \cos\left(6\pi\left(\sqrt[3]{n} + \sqrt[3]{m} - \sqrt[3]{k}\right)\sqrt[3]{x} - \frac{\pi\rho}{2}\right) \\ &+ \frac{1}{4} \sum r x^{2\rho+1} \cos\left(6\pi\left(\sqrt[3]{n} + \sqrt[3]{m} + \sqrt[3]{k}\right)\sqrt[3]{x} - \frac{3\pi\rho}{2}\right) \\ =& S_0(x) + S_1(x) + S_2(x),\end{aligned} \tag{3.5}$$

where

$$S_0(x) := \frac{3}{4}\cos\left(-\frac{\pi\rho}{2}\right) \sum_{\sqrt[3]{n}+\sqrt[3]{m}=\sqrt[3]{k}} r x^{2\rho+1}, \tag{3.6}$$

$$S_1(x) := \frac{3}{4} \sum_{\sqrt[3]{n}+\sqrt[3]{m}\ne\sqrt[3]{k}} r x^{2\rho+1} \cos\left(6\pi\left(\sqrt[3]{n} + \sqrt[3]{m} - \sqrt[3]{k}\right)\sqrt[3]{x} - \frac{\pi\rho}{2}\right), \tag{3.7}$$

$$S_2(x) := \frac{1}{4} \sum r x^{2\rho+1} \cos\left(6\pi\left(\sqrt[3]{n} + \sqrt[3]{m} + \sqrt[3]{k}\right)\sqrt[3]{x} - \frac{3\pi\rho}{2}\right). \tag{3.8}$$

The main term in the right-hand side of (3.2) comes from $S_0(x)$. Indeed, integrating both sides of (3.6) with respect to x over the interval $(H, 2H)$, we have

$$\int_H^{2H} S_0(x)dx = \frac{3}{4}\cos\left(-\frac{\pi\rho}{2}\right)\sum_{\sqrt[3]{n}+\sqrt[3]{m}=\sqrt[3]{k}} r\int_H^{2H} x^{2\rho+1}dx. \tag{3.9}$$

For natural numbers n, m and k, the condition $\sqrt[3]{n}+\sqrt[3]{m}=\sqrt[3]{k}$ holds if and only if n, m and k all have the same cube-free part h, such that $n=\alpha^3 h$, $m=\beta^3 h$, $k=\gamma^3 h$ and $\alpha+\beta=\gamma$. Hence

$$\sum_{\sqrt[3]{n}+\sqrt[3]{m}=\sqrt[3]{k}} r = \sum_{\substack{h\leq N\\ h\ is\ cube-free}} h^{-\rho-2}\sum_{\alpha+\beta\leq\sqrt[3]{\frac{N}{h}}}(\alpha\beta(\alpha+\beta))^{-\rho-2}\,c_{\alpha^3 h}c_{\beta^3 h}c_{(\alpha+\beta)^3 h}.$$

Since $c_{uv}\leq c_u c_v$ and $c_w \ll w^{\epsilon}$, we have

$$\begin{aligned}
&\sum_{\alpha+\beta>\sqrt[3]{\frac{N}{h}}}(\alpha\beta(\alpha+\beta))^{-\rho-2}\,c_{\alpha^3 h}c_{\beta^3 h}c_{(\alpha+\beta)^3 h}\\
&\qquad\ll c_h^3\sum_{\alpha>\frac{1}{2}\sqrt[3]{\frac{N}{h}}}\sum_{\beta\leq\alpha}(\alpha\beta(\alpha+\beta))^{-\rho-2}\,c_{\alpha^3}c_{\beta^3}c_{(\alpha+\beta)^3}\\
&\qquad\ll c_h^3\sum_{\alpha>\frac{1}{2}\sqrt[3]{\frac{N}{h}}}\alpha^{-2\rho-4+\epsilon}\sum_{\beta\leq\alpha}\beta^{-\rho-2}c_\beta^3\\
&\qquad\ll c_h^3\left(\frac{N}{h}\right)^{-\frac{2\rho+3}{3}+\epsilon}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{\sqrt[3]{n}+\sqrt[3]{m}=\sqrt[3]{k}} r &= \sum_{\substack{h\leq N\\ h\ is\ cube-free}} h^{-\rho-2}\sum_{\alpha,\beta=1}^{\infty}(\alpha\beta(\alpha+\beta))^{-\rho-2}\,c_{\alpha^3 h}c_{\beta^3 h}c_{(\alpha+\beta)^3 h}\\
&\quad+O\left(\sum_{h\leq N}h^{-\rho-2}c_h^3\left(\frac{N}{h}\right)^{-\frac{2\rho+3}{3}+\epsilon}\right)\\
&=\sum_{\alpha,\beta=1}^{\infty}(\alpha\beta(\alpha+\beta))^{-\rho-2}\sum_{\substack{h=1\\ h\ is\ cube-free}}^{\infty} h^{-\rho-2}c_{\alpha^3 h}c_{\beta^3 h}c_{(\alpha+\beta)^3 h}\\
&\quad+O\left(\sum_{\alpha,\beta=1}^{\infty}(\alpha\beta(\alpha+\beta))^{-\rho-2}\,c_{\alpha^3}c_{\beta^3}c_{(\alpha+\beta)^3}\sum_{h>N}h^{-\rho-2}c_h^3\right)\\
&\quad+O\left(N^{-\frac{2\rho+3}{3}+\epsilon}\sum_{h\leq N}h^{-\frac{\rho+3}{3}}\right)\\
&=A+O(N^{-\rho-1+\epsilon}).
\end{aligned}$$

Substituting this into (3.9), we have

$$\int_H^{2H} S_0(x)dx = \frac{3}{4}\cos\left(\frac{\pi\rho}{2}\right)A\int_H^{2H} x^{2\rho+1}dx + O\left(H^{2\rho+2}N^{-\rho-1+\epsilon}\right). \tag{3.10}$$

This yields the main term in (3.2) and hence that in the formula in our theorem.

We now proceed to show that the contributions from $S_1(x)$ and $S_2(x)$ are bounded by $H^{\frac{6\rho^2+21\rho+16}{8+3\rho}+\epsilon}$. Applying the Lemma 2.3 we find that for any real numbers $p(\neq 0)$ and q,

$$\begin{aligned}
&\int_H^{2H} x^{2\rho+1}\cos\left(p\sqrt[3]{x}+q\right)dx \\
&\qquad = \int_H^{2H} 3p^{-1}x^{2\rho+\frac{5}{3}}\left(\frac{p}{3x^{\frac{2}{3}}}\cos\left(p\sqrt[3]{x}+q\right)\right)dx \\
&\qquad \ll H^{2\rho+\frac{5}{3}}|p|^{-1}\left|\int_u^v \left(\frac{p}{3x^{\frac{2}{3}}}\cos\left(p\sqrt[3]{x}+q\right)\right)dx\right| \ll H^{2\rho+\frac{5}{3}}|p|^{-1}.
\end{aligned} \tag{3.11}$$

Let us examine $S_2(x)$ first. Integrating both sides of (3.8) and then using (3.11) and (3.4), we have

$$\begin{aligned}
\int_H^{2H} S_2(x)dx &\ll \sum rH^{2\rho+\frac{5}{3}}\left(\sqrt[3]{n}+\sqrt[3]{m}+\sqrt[3]{k}\right)^{-1} \ll H^{2\rho+\frac{5}{3}}\sum_{n\le m\le k\le N} rk^{-\frac{1}{3}} \\
&\ll H^{2\rho+\frac{5}{3}+\epsilon}\sum_{n\le m\le k\le N} n^{-\frac{\rho+2}{3}}m^{-\frac{\rho+2}{3}}k^{-\frac{\rho+3}{3}} \ll H^{2\rho+\frac{5}{3}+\epsilon}N^{\frac{4}{3}-\rho}.
\end{aligned} \tag{3.12}$$

Next we consider $S_1(x)$. For convenience, let

$$\Delta = \sqrt[3]{n}+\sqrt[3]{m}-\sqrt[3]{k}.$$

Using the trivial bound

$$\int_H^{2H} x^{2\rho+1}\cos\left(6\pi\Delta\sqrt[3]{x}-\frac{\pi\rho}{2}\right)dx \ll H^{2\rho+2},$$

where $|\Delta| \le H^{-\frac{8}{24+9\rho}}$ and applying (3.11) when $|\Delta| > H^{-\frac{8}{24+9\rho}}$, we deduce from (3.7) that

$$\begin{aligned}
\int_H^{2H} S_1(x)dx &\ll H^{2\rho+2}\sum_{\substack{0<|\Delta|\le H^{-\frac{8}{24+9\rho}}\\ n\le m}} r + H^{2\rho+\frac{5}{3}}\sum_{\substack{|\Delta|>H^{-\frac{8}{24+9\rho}}\\ n\le m}} r|\Delta|^{-1} \\
&:= H^{2\rho+2}W_1 + H^{2\rho+\frac{5}{3}}W_2.
\end{aligned} \tag{3.13}$$

Now we consider first the sum W_1. For given $n \le m \le N$ there is most one natural number k for which $|\Delta| \le H^{-\frac{8}{24+9\rho}}$, since $\Delta = \sqrt[3]{n}+\sqrt[3]{m}-\sqrt[3]{k}$ and $\left|\Delta\sqrt[3]{N^2}\right| \le H^{-\frac{8}{24+9\rho}}\sqrt[3]{N^2} = o(1)$. Such k, if it exists, must be greater than m. By Lemma 2.2, $0 < |\Delta| \le H^{-\frac{8}{24+9\rho}}$ implies $m^{-\frac{8}{3}} \ll H^{-\frac{8}{24+9\rho}}$, so $m \gg H^{\frac{1}{8+3\rho}}$. Hence, by (3.4),

$$W_1 \ll H^{\epsilon}\sum_{\substack{n\le m\le N\\ m\gg H^{\frac{1}{8+3\rho}}}} (nm)^{-\frac{\rho+2}{3}}m^{-\frac{\rho+2}{3}} \ll H^{\epsilon}\sum_{m\gg H^{\frac{1}{8+3\rho}}} m^{-\rho-1} \ll H^{-\frac{\rho}{8+3\rho}+\epsilon}.$$

Similarly, in the sum W_2, we have $k = \left(\sqrt[3]{n}+\sqrt[3]{m}\right)^3 + O\left(|\Delta|\sqrt[3]{m^2}\right) > m$ and there are $\ll 1 + |\Delta|\sqrt[3]{m^2}$

such k. Hence

$$\begin{aligned}
W_2 &\ll H^{\epsilon} \sum_{\substack{n\le m\le N \\ |\Delta|>H^{-\frac{8}{24+9\rho}}}} n^{-\frac{\rho+2}{3}} m^{-\frac{\rho+2}{3}} |\Delta|^{-1} \left(1+|\Delta|\sqrt[3]{m^2}\right) m^{-\frac{\rho+2}{3}} \\
&\ll H^{\epsilon} \sum_{n\le m\le N} n^{-\frac{\rho+2}{3}} m^{-\frac{2\rho+4}{3}} H^{\frac{8}{24+9\rho}} + H^{\epsilon} \sum_{n\le m\le N} n^{-\frac{\rho+2}{3}} m^{-\frac{2\rho+2}{3}} \\
&\ll H^{\frac{8}{24+9\rho}+\epsilon} + H^{\epsilon} N^{\frac{2}{3}-\rho} \\
&\ll H^{\frac{8}{24+9\rho}+\epsilon}.
\end{aligned}$$

Collecting these estimates we find that

$$\int_H^{2H} S_1(x)dx \ll H^{\frac{6\rho^2+21\rho+16}{8+3\rho}+\epsilon}.$$

Combining this with (3.10) and (3.12), we deduce from (3.2) the formula (3.5). Then according to (3.3), our theorem is proved.

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Conflict of interest

The authors declare no conflict of interest.

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