Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9378–9396.
DOI: 10.3934/math.2021545
Received: 13 December 2020
Accepted: 10 June 2021
Published: 23 June 2021

Research article

Fixed points of nonlinear contractions with applications

Mohammed Shehu Shagari[1], **Qiu-Hong Shi**[2,*], **Saima Rashid**[3,*], **Usamot Idayat Foluke**[4] **and Khadijah M. Abualnaja**[5]

[1] Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria
[2] Department of Mathematics, Huzhou University, Huzhou 313000, China
[3] Department of Mathematics, Government College University, Faisalabad, Pakistan
[4] Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria
[5] Department of Mathematics, Faculty of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: shiqiuhong@zjhu.edu.cn, saimarashid@gcuf.edu.pk.

Abstract: The aim of this paper is to initiate a new concept of nonlinear contraction under the name r-hybrid ψ-contraction and establish some fixed point results for such mappings in the setting of complete metric spaces. The presented ideas herein unify and extend a number of well-known results in the corresponding literature. A few of these special cases are pointed out and analysed. From application point of view, we investigate the existence and uniqueness criteria of solutions to certain functional equation arising in dynamic programming and integral equation of Volterra type. Nontrivial illustrative examples are provided to show the generality and validity of our obtained results.

Keywords: fixed point; ψ-contraction; r-hybrid ψ-contraction; dynamic programming; integral equation
Mathematics Subject Classification: Primary 47H09, 47H10, 90C39, 45D05, 34A12; Secondary 54H25

1. Introduction

Fixed point(fp_t) theory is the epicenter of modern functional analysis with interesting applications in the study of various significant nonlinear phenomena, including convex optimization and minimization [1, 2], variational inequalities [3], fractional calculus [4–8], homotopy perturbation theory [9, 10], analytical chemistry [11], integral inequalities [12–16], Nash equilibrium problems as well as in network bandwidth allocation [17]. In f_{p_t} theory, the contractive conditions on underlying mappings play an important role in finding solutions of f_{p_t} problems. The Banach contraction principle (BC_p) [18] is one of the most known applicable results on fp_t of contraction mappings. This highly celebrated theorem (thr_m), which is an essential tool in several areas of mathematical analysis, surfaced in 1922 in Banach thesis. Due to its usefulness and simplicity, many authors have come up with diverse extensions of the BC_p (e.g. [19–21]). In 2012, Wardowski [22] brought up a notion of contraction, called ψ-contraction and coined a f_{p_t} thr_m which refined the BC_p. Wardowski and Van Dung [23] initiated the idea of ψ-weak contraction and obtained a refinement of ψ-contraction. In [24], Secelean opined that condition (ψ_2) in Wardowski's definition of ψ-contraction can be replaced with an equivalent and subtle one given by (ψ_2') : $\inf \psi = -\infty$. Piri and Kumam [25]launched a variant of Wardowski's thr_m by using the condition (ψ_2'). Cosentino and Vetro [26] toed the direction of ψ-contraction and proved f_{p_t} results of Hardy-Rogers-type. On the other hand, one of the active subfields of f_{p_t} theory that is also presently attracting the foci of investigators is the examination of hybrid contractions. The idea has been shaped in two lanes, viz. first, hybrid contraction deals with those contractions involving both single-valued and multi-valued mappings(mpn) and the second combines both linear and nonlinear contractions. For some articles in this direction, we refer [27–29]. Recently, Karapinar et al. [30] launched the notion of p-hybrid Wardowski contractions. Their results unified and extended several known fixed point theorems due to Wardowski [22], and related results. For other modifications of ψ-contractions and related fixed point theorems, the reader may consult [31–36].

The focus of this article is to bring up a notion called r-hybrid ψ-contraction and establish novel f_{p_t} thr_m in the realm of complete metric space. Our results include as special cases, the fp_t thr_m due to Wardowski [22], Cosentino and Vetro [26], Karapinar [19], Reich [21], and a few others in the corresponding literature. A nontrivial example is provided to indicate the generality of our ideas herein. Moreover, two applications of certain functional eq_n arising in dynamic programming and integral eq_n of Volterra type are provided to show possible usability of our results.

2. Preliminaries

In this section, a handful concepts and results needed in the sequel are recalled. Throughout the article, denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, are the set of real numbers, nonnegative reals and the set of natural numbers, respectively. Moreover, we denote a metric space and a complete metric space by M_s and CM_s, respectively.

$\mho_\psi$ represents the family of functions(fnx) $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$:

(ψ_1) ψ is strictly increasing, that is, for all $\hbar, \wp \in (0, \infty)$, if $\hbar < \wp$ then, $\psi(\hbar) < \psi(\wp)$;
(ψ_2) for every sequence(se_q) $\{\hbar_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $\lim_{n\longrightarrow\infty} \hbar_n = 0$ if and only if $\lim_{n\longrightarrow\infty} \psi(\hbar_n) = -\infty$;
(ψ_3) there exists $\natural \in (0, 1)$ such that $\lim_{n\longrightarrow\infty} \hbar^{\natural}\psi(\hbar) = 0$.

Definition 2.1. [22] Let (Υ, μ) be a M_s. A mapping(mpn) $\Im : \Upsilon \longrightarrow \Upsilon$ is called a ψ-contraction if there exist $\sigma > 0$ and a fnx $\psi \in \mho_\psi$ such that for all $\varsigma, \zeta \in \Upsilon$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi(\mu(\varsigma, \zeta)). \tag{2.1}$$

Example 2.2. [22] Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be defined by $\psi(\hbar) = \ln \hbar$, $\hbar > 0$. Clearly, ψ satisfies (ψ_1) − (ψ_3). Each mpn $\Im : \Upsilon \longrightarrow \Upsilon$ satisfying (2.1) is a ψ-contraction such that for all $\varsigma, \zeta \in \Upsilon$ with $\Im\varsigma \neq \Im\zeta$,

$$\mu(\Im\varsigma, \Im\zeta) \leq e^{-\sigma}\mu(\varsigma, \zeta). \tag{2.2}$$

It is obvious that for all $\varsigma, \zeta \in \Upsilon$ such that $\Im\varsigma = \Im\zeta$, the inequality (2.2) also holds; that is, $\Im$ is a BC_p.

Example 2.3. [22] Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be defined by $\psi(\hbar) = \ln \hbar + \hbar,\ \hbar > 0$, then ψ satisfies $(\psi_1) - (\psi_3)$. Therefore, from Condition (2.1), the mpn $\Im : \Upsilon \longrightarrow \Upsilon$ is of the form

$$\frac{\mu(\Im\varsigma, \Im\zeta)}{\mu(\varsigma, \zeta)} e^{\mu(\Im\varsigma,\Im\zeta)-\mu(\varsigma,\zeta)} \leq e^{-\sigma},$$

for all $\varsigma, \zeta \in \Upsilon$, $\Im\varsigma \neq \Im\zeta$.

Remark 1. From (ψ_1) and (2.1), it is easy to see that if $\Im$ is a ψ-contraction, then $\mu(\Im\varsigma, \Im\zeta) < \mu(\varsigma, \zeta)$ for all $\varsigma, \zeta \in \Upsilon$ such that $\Im\varsigma \neq \Im\zeta$, that is, $\Im$ is a contractive mapping. Hence, every ψ-contraction is a continuous mpn.

Theorem 2.4. *[22] Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a ψ-contraction. Then, $\Im$ has a unique f_{p_t} $u \in \Upsilon$, and for each $\varsigma \in \Upsilon$, the se_q $\{\Im^n\varsigma\}_{n\in\mathbb{N}}$ converges(cvg) to u.*

We design the set of all f_{p_t} of a mpn $\Im$ by $\psi_{i_\varsigma}(\Im)$.

Definition 2.5. [30] Let $\mathcal{M}$ be the family of functions $\psi : (0, \infty) \longrightarrow \mathbb{R}$:

(ψ_a) ψ is strictly increasing;
(ψ_b) there exists $\sigma > 0$ such that for every $\varpi_0 > 0$,

$$\sigma + \lim_{\varpi \longrightarrow \varpi_0} \inf \psi(\varpi) > \lim_{\varpi \longrightarrow \varpi_0} \sup \psi(\varpi).$$

3. Main results

In this section, we launch a new form of nonlinear contraction called r-hybrid ψ-contraction and establish the corresponding f_{p_t} results. Let (Υ, μ) be a metric space and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mpn. For $r \geq 0$ and $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ such that $\sum_{i=1}^{5} a_i = 1$, we define:

$$\Omega_{\Im}^r(\varsigma, \zeta) = \begin{cases} [G(\varsigma, \zeta)]^{\frac{1}{r}}, & \text{for } r > 0,\ \varsigma, \zeta \in \Upsilon, \\ H(\varsigma, \zeta), & \text{for } r = 0,\ \varsigma, \zeta \in \Upsilon \setminus \psi_{i_\varsigma}(\Im), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} G(\varsigma, \zeta) = \ & a_1(\mu(\varsigma, \zeta))^r + a_2(\mu(\varsigma, \Im\varsigma))^r + a_3(\mu(\zeta, \Im\zeta))^r \\ & a_4\left(\frac{\mu(\zeta, \Im\zeta)(1 + \mu(\varsigma, \Im\varsigma))}{1 + \mu(\varsigma, \zeta)}\right)^r \\ & a_5\left(\frac{\mu(\zeta, \Im\varsigma)(1 + \mu(\varsigma, \Im\zeta))}{1 + \mu(\varsigma, \zeta)}\right)^r \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & H(\varsigma, \zeta) \\ & = (\mu(\varsigma, \zeta))^{a_1}(\mu(\varsigma, \Im\varsigma))^{a_2}(\mu(\zeta, \Im\zeta))^{a_3}\left(\frac{\mu(\zeta, \Im\zeta)(1 + \mu(\varsigma, \Im\varsigma))}{1 + \mu(\varsigma, \zeta)}\right)^{a_4}\left(\frac{\mu(\varsigma, \Im\zeta) + \mu(\zeta, \Im\varsigma)}{2}\right)^{a_5}. \end{aligned} \tag{3.3}$$

Definition 3.1. Let (Υ, μ) be a M_s. A mpn $\Im : \Upsilon \longrightarrow \Upsilon$ is called an r-hybrid ψ-contraction if there exist $\psi \in \mathcal{M}$ and $\sigma > 0$ such that for each $r > 0$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi(\Omega^r_{\Im}(\varsigma, \zeta)). \tag{3.4}$$

In particular, if (3.4) holds for $r = 0$, we say that the mpn $\Im$ is a 0-hybrid ψ-contraction.

Remark 2. Every ψ-contraction is an r-hybrid contraction, but the converse is not always true (see Example 3.3). In other words, the class of r-hybrid ψ-contractions is richer.

Theorem 3.2. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be an r-hybrid ψ-contraction for $r > 0$. Then, $\Im$ has a unique f_{p_t} in Υ.*

Proof. Let $\varsigma_0 \in \Upsilon$ be arbitrary, and rename it as $\varsigma_0 := \varsigma$. Note that if $\varsigma_0 = \Im\varsigma_0$, the proof is finished. We develop an iterative se_q $\{\varsigma_n\}_{n\in\mathbb{N}}$ given by $\varsigma_n = \Im\varsigma_{n-1}, n \geq 1$. Without loss of generality, let

$$0 < \mu(\varsigma_{n+1}, \varsigma_n) = \mu(\Im\varsigma_n, \Im\varsigma_{n-1}) \text{ if and only if } \varsigma_{n+1} \neq \varsigma_n,\ n \in \mathbb{N}. \tag{3.5}$$

Taking $\varsigma = \varsigma_{n-1}$ and $\zeta = \varsigma_n$ in (3.1) with $r > 0$, we have

$$\begin{aligned}
\Omega^r_{\Im}(\varsigma_{n-1}, \varsigma_n) = [G(\varsigma_{n-1}, \varsigma_n)]^{\frac{1}{r}} = \Bigg[&a_1(\mu(\varsigma_{n-1}, \varsigma_n))^r + a_2(\mu(\varsigma_{n-1}, \Im\varsigma_{n-1}))^r \\
&+ a_3(\mu(\varsigma_n, \Im\varsigma_n))^r + a_4\left(\frac{\mu(\varsigma_n, \Im\varsigma_n)(1 + \mu(\varsigma_{n-1}, \Im\varsigma_{n-1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r \\
&+ a_5\left(\frac{\mu(\varsigma_n, \Im\varsigma_{n-1})(1 + \mu(\varsigma_{n-1}, \Im\varsigma_n))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r\Bigg]^{\frac{1}{r}} \\
= \Bigg[&a_1(\mu(\varsigma_{n-1}, \varsigma_n))^r + a_2(\mu(\varsigma_{n-1}, \varsigma_n))^r \\
&+ a_3(\mu(\varsigma_n, \varsigma_{n+1}))^r + a_4\left(\frac{\mu(\varsigma_n, \varsigma_{n+1})(1 + \mu(\varsigma_{n-1}, \varsigma_n))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r \\
&+ a_5\left(\frac{\mu(\varsigma_n, \varsigma_n)(1 + \mu(\varsigma_{n-1}, \varsigma_{n+1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r\Bigg]^{\frac{1}{r}} \\
= \Bigg[&a_1(\mu(\varsigma_{n-1}, \varsigma_n))^r + a_2(\mu(\varsigma_{n-1}, \varsigma_n))^r \\
&+ a_3(\mu(\varsigma_n, \varsigma_{n+1}))^r + a_4(\mu(\varsigma_n, \varsigma_{n+1}))^r\Bigg]^{\frac{1}{r}} \\
= \ &[(a_1 + a_2)(\mu(\varsigma_{n-1}, \varsigma_n))^r + (a_3 + a_4)(\mu(\varsigma_n, \varsigma_{n+1}))^r]^{\frac{1}{r}}.
\end{aligned} \tag{3.6}$$

From (3.4) and (3.6), we have

$$\sigma + \psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)) \leq \psi(\Omega^r_{\Im}(\varsigma_{n-1}, \varsigma_n)),$$

that is,

$$\begin{aligned}
\psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)) \leq \ &\psi(\Omega^r_{\Im}(\varsigma_{n-1}, \varsigma_n)) - \sigma \\
= \ &\psi\left([(a_1 + a_2)(\mu(\varsigma_{n-1}, \varsigma_n))^r + (a_3 + a_4)(\mu(\varsigma_n, \varsigma_{n+1}))^r]^{\frac{1}{r}}\right) \\
&- \sigma.
\end{aligned} \tag{3.7}$$

Suppose that $\mu(\varsigma_{n-1}, \varsigma_n) \leq \mu(\varsigma_n, \varsigma_{n+1})$, then, from (3.7),

$$\begin{aligned}\psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)) &\leq \psi\left(\left[(a_1 + a_2 + a_3 + a_4)(\mu(\varsigma_n, \varsigma_{n+1}))^r\right]^{\frac{1}{r}}\right) \\ &\quad - \sigma \\ &\leq \psi((\mu(\varsigma_n, \varsigma_{n+1})^r)^{\frac{1}{r}}) - \sigma \\ &= \psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)) - \sigma \\ &< \psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)),\end{aligned}$$

which is invalid. Therefore, $\max\{\mu(\varsigma_{n-1}, \varsigma_n), \mu(\varsigma_n, \varsigma_{n+1})\} = \mu(\varsigma_{n-1}, \varsigma_n)$, and there exists $\flat \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \mu(\varsigma_{n-1}, \varsigma_n) = \flat. \tag{3.8}$$

Assuming that $\flat > 0$, we have $\lim_{n\longrightarrow\infty} \Omega^r_{\Im}(\varsigma_{n-1}, \varsigma_n) = \flat$, and by (ψ_b), we get

$$\sigma + \psi(\flat) \leq \psi(\flat), \tag{3.9}$$

from which we have $\psi(\flat) \leq \psi(\flat) - \sigma < \psi(\flat)$, a contradiction. Consequently,

$$\lim_{n\longrightarrow\infty} \mu(\varsigma_{n-1}, \varsigma_n) = 0. \tag{3.10}$$

Next, we argue that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy se_q in Υ. For this, assume that there exists $\epsilon > 0$ and se_q $\{n_*(l)\}$, $\{m_*(l)\}$ of positive integers satisfying $n_*(l) > m_*(l)$ with

$$\begin{gathered}\mu(\varsigma_{n_*(l)}, \varsigma_{m_*(l)}) \geq \epsilon \\ \mu(\varsigma_{n_*(l)-1}, \varsigma_{m_*(l)}) < \epsilon.\end{gathered} \tag{3.11}$$

for all $l \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned}\epsilon \leq \mu(\varsigma_{n_*(l)}, \varsigma_{m_*(l)}) \leq&\ \mu(\varsigma_{n_*(l)}, \varsigma_{n_*(l)-1}) \\ &+ \mu(\varsigma_{n_*(l)-1}, \varsigma_{m_*(l)}) \\ <&\ \mu(\varsigma_{n_*(l)}, \varsigma_{n_*(l)-1}) + \epsilon.\end{aligned} \tag{3.12}$$

Letting $n \longrightarrow \infty$ in (3.12), and using (3.10), yields

$$\lim_{n\longrightarrow\infty} \mu(\varsigma_{n_*(l)}, \varsigma_{m_*(l)}) = \epsilon. \tag{3.13}$$

By triangle inequality on Υ, we get

$$\begin{aligned}0 &\leq \left|\mu(\varsigma_{n_*(l)+1}, \varsigma_{m_*(l)+1}) - \mu(\varsigma_{n_*(l)}, \varsigma_{m_*(l)})\right| \\ &\leq \mu(\varsigma_{n_*(l)+1}, \varsigma_{n_*(l)}) - \mu(\varsigma_{m_*(l)}, \varsigma_{m_*(l)+1}).\end{aligned}$$

Hence,

$$\begin{aligned}&\lim_{l\longrightarrow\infty} \left|\mu(\varsigma_{n_*(l)+1}, \varsigma_{m_*(l)+1}) - \mu(\varsigma_{n_*(l)}, \varsigma_{m_*(l)})\right| \\ &\leq \lim_{l\longrightarrow\infty} \left[\mu(\varsigma_{n_*(l)+1}, \varsigma_{n_*(l)}) - \mu(\varsigma_{m_*(l)}, \varsigma_{m_*(l)+)})\right] = 0.\end{aligned} \tag{3.14}$$

It comes up that

$$\lim_{l\longrightarrow\infty}\mu(\varsigma_{n_*(l)+1},\varsigma_{m_*(l)+1}) = \lim_{l\longrightarrow\infty}\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)}) \\ = \epsilon > 0. \tag{3.15}$$

In addition, since

$$\epsilon = \mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)}) \leq \mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)+1}) + \mu(\varsigma_{m_*(l)+1},\varsigma_{m_*(l)}) \\ \leq \mu(\varsigma_{n_*(l)},\varsigma_{n_*(l)+1}) + \mu(\varsigma_{m_*(l)},\varsigma_{n_*(l)+1}),$$

then,

$$\lim_{n\longrightarrow\infty}\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)+1}) = \lim_{n\longrightarrow\infty}\mu(\varsigma_{m_*(l)},\varsigma_{n_*(l)+1}) \\ = \epsilon.$$

Thus, for all $l \geq n_0$, we get

$$\mu(\Im\varsigma_{n_*(l)},\Im\varsigma_{m_*(l)}) = \mu(\varsigma_{n_*(l)+1},\varsigma_{m_*(l)+1}).$$

Therefore, by (3.4), there exists $\sigma > 0$ such that

$$\sigma + \psi(\mu(\varsigma_{n_*(l)+1},\varsigma_{m_*(l)+1})) \leq \psi\left(\Omega^r_{\Im}(\varsigma_{n_*(l)},\varsigma_{m_*(l)})\right), \tag{3.16}$$

where

$$\begin{aligned}\Omega^r_{\Im}(\varsigma_{n_*(l)},\varsigma_{m_*(l)}) = \Bigg[& a_1(\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)}))^r + a_2(\mu(\varsigma_{n_*(l)},\varsigma_{n_*(l)+1}))^r \\ & + a_3(\mu(\varsigma_{m_*(l)},\varsigma_{m_*(l)+1}))^r \\ & + a_4\left(\frac{\mu(\varsigma_{m_*(l)},\varsigma_{m_*(l)+1})(1+\mu(\varsigma_{n_*(l)+1},\varsigma_{n_*(l)+1}))}{1+\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)})}\right) \\ & + a_5\left(\frac{\mu(\varsigma_{m_*(l)},\varsigma_{n_*(l)+1})(1+\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)+1}))}{1+\mu(\varsigma_{n_*(l)},\varsigma_{m_*(l)})}\right)^r\Bigg]^{\frac{1}{r}}.\end{aligned} \tag{3.17}$$

Moreover, since the fnx ψ is nondecreasing, we have

$$\begin{aligned}&\sigma + \lim_{l\longrightarrow\infty}\inf\psi\left(\mu(\varsigma_{n_*(l)+1},\varsigma_{m_*(l)+1})\right) \\ &\leq \sigma + \lim_{l\longrightarrow\infty}\inf\psi\left(\mu(\Im\varsigma_{n_*(l)},\Im\varsigma_{m_*(l)})\right) \\ &\leq \lim_{l\longrightarrow\infty}\inf\psi\left(\Omega^r_{\Im}(\varsigma_{n_*(l)},\varsigma_{m_*(l)})\right) \\ &\leq \lim_{l\longrightarrow\infty}\sup\psi\left(\Omega^r_{\Im}(\varsigma_{n_*(l)},\varsigma_{m_*(l)})\right).\end{aligned} \tag{3.18}$$

From (3.18), we have $\sigma + \psi(\epsilon) \leq \psi(\epsilon)$, that is, $\psi(\epsilon) \leq \psi(\epsilon) - \sigma < \psi(\epsilon)$, a contradiction. This proves that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy se_q in Υ. Since Υ is a CM_s, there exists $u \in \Upsilon$ such that

$\lim_{n\longrightarrow\infty}\mu(\varsigma_n,u)=0$. Thus, there exists a subsequence $\{\varsigma_{n_i}\}$ of $\{\varsigma_n\}_{n\in\mathbb{N}}$ with $\Im\varsigma_{n_i}=\Im u$ for each $i\in\mathbb{N}$ such that

$$\begin{aligned}\mu(u,\Im u) &= \lim_{i\longrightarrow\infty}\mu(\varsigma_{n_i+1},\Im u)\\ &= \lim_{i\longrightarrow\infty}\mu(\Im\varsigma_{n_i},\Im u)=0,\end{aligned}$$

which implies that

$$u=\Im u. \tag{3.19}$$

Assume that (3.19) is not true. Then, there is a number $n_0\in\mathbb{N}$ such that $\mu(\Im\varsigma_n,\Im u)>0$ for all $n\geq n_0$. Now, using (3.4) with $\varsigma=\varsigma_{n-1}$ and $\zeta=u$ with $r>0$, we have

$$\sigma+\psi(\mu(\Im\varsigma_{n-1},\Im u))\leq\psi(\Omega^r_{\Im}(\varsigma_{n-1},u)), \tag{3.20}$$

where

$$\begin{aligned}\Omega^r_{\Im}(\varsigma_{n-1},u) =&\Bigg[a_1(\mu(\varsigma_{n-1},u))^r+a_2(\mu(\varsigma_{n-1},\Im\varsigma_{n-1}))^r+a_3(\mu(u,\Im u))^r\\ &+a_4\left(\frac{\mu(u,\Im u)(1+\mu(\varsigma_{n-1},\Im\varsigma_{n-1}))}{1+\mu(\varsigma_{n-1},u)}\right)^r\\ &+a_5\left(\frac{\mu(u,\Im\varsigma_{n-1})(1+\mu(\varsigma_{n-1},\Im u))}{1+\mu(\varsigma_{n-1},u)}\right)^r\Bigg]^{\frac{1}{r}}\\ =&\Bigg[a_1(\mu(\varsigma_{n-1},u))^r+a_2(\mu(\varsigma_{n-1},\varsigma_n))^r+a_3(\mu(u,\Im u))^r\\ &+a_4\left(\frac{\mu(u,\Im u)(1+\mu(\varsigma_{n-1},\varsigma_n))}{1+\mu(\varsigma_{n-1},u)}\right)^r\\ &+a_5\left(\frac{\mu(u,\varsigma_n)(1+\mu(\varsigma_{n-1},\Im u))}{1+\mu(\varsigma_{n-1},u)}\right)^r\Bigg]^{\frac{1}{r}}.\end{aligned} \tag{3.21}$$

From (3.21), we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\mu(\varsigma_{n-1},\Im u) =\mu(u,\Im u) &= \lim_{n\longrightarrow\infty}\Omega^r_{\Im}(\varsigma_{n-1},u)\\ &= \Big[a_2(\mu(u,\Im u))^r+a_3(\mu(u,\Im u))^r+a_4(\mu(u,\Im u))^r\Big]^{\frac{1}{r}}\\ &= \left[(a_2+a_3+a_4)(\mu(u,\Im u))^r\right]^{\frac{1}{r}}\\ &= (a_2+a_3+a_4)^{\frac{1}{r}}\mu(u,\Im u).\end{aligned} \tag{3.22}$$

Hence, from (3.20) and (3.22), we get

$$\begin{aligned}&\sigma+\lim_{\varpi\longrightarrow\mu(u,\Im u)}\inf\psi((a_2+a_3+a_4)^{\frac{1}{r}}\varpi)\\ \leq\ &\sigma+\lim_{\varpi\longrightarrow\mu(u,\Im u)}\inf\psi(\varpi)\\ <\ &\lim_{\varpi\longrightarrow\mu(u,\Im u)}\sup\psi((a_2+a_3+a_4)^{\frac{1}{r}}\varpi),\end{aligned}$$

which is a contradiction, according to (ψ_b). Thus, $\Im u = u$.

To show that the f_{p_t} of $\Im$ is unique, assume there exists $u^* \in \Upsilon$ with $u \neq u^*$ such that $\Im u^* = u^*$ so that $\mu(u, u^*) = \mu(\Im u, \Im u^*) > 0$. Then, from (3.4), we have

$$\begin{aligned}
\sigma + \psi(\mu(u, u^*)) &= \sigma + \psi(\mu(\Im u, \Im u^*)) \leq \psi(\Omega^r_{\Im}(u, u^*)) \\
&= \psi\Bigg(\Bigg[a_1(\mu(u, u^*))^r + a_2(\mu(u, \Im u))^r + a_3(\mu(u^*, \Im u^*))^r \\
&+ a_4\left(\frac{\mu(u^*, \Im u^*)(1 + \mu(u, \Im u))}{1 + \mu(u, u^*)}\right)^r \\
&+ a_5\left(\frac{\mu(u^*, \Im u)(1 + \mu(u, \Im u^*))}{1 + \mu(u, u^*)}\right)^r\Bigg]^{\frac{1}{r}}\Bigg) \\
&= \psi\Bigg(\Bigg[a_1(\mu(u, u^*))^r + a_5\left(\frac{\mu(u^*, u)(1 + \mu(u, u^*))}{1 + \mu(u, u^*)}\right)^r\Bigg]^{\frac{1}{r}}\Bigg) \\
&= \psi\left([(a_1 + a_5)(\mu(u, u^*))^r]^{\frac{1}{r}}\right) \\
&= \psi\left((a_1 + a_5)^{\frac{1}{r}}\mu(u, u^*)\right) \\
&\leq \psi(\mu(u, u^*)),
\end{aligned}$$

that is,

$$\begin{aligned}
\psi(\mu(u, u^*)) &\leq \psi(\mu(u, u^*)) - \sigma \\
&< \psi(\mu(u, u^*)),
\end{aligned}$$

a contradiction. Therefore, $u = u^*$. □

Example 3.3. Let $\Upsilon = [0, 1]$ and $\mu(\varsigma, \zeta) = |\varsigma - \zeta|$ for all $\varsigma, \zeta \in \Upsilon$. Then, (Υ, μ) is a CM_s. Define $\Im : \Upsilon \longrightarrow \Upsilon$ by

$$\Im\varsigma = \begin{cases} \frac{\varsigma}{6}, & \text{if } \varsigma \in [0, 1) \\ \frac{1}{2}, & \text{if } \varsigma = \frac{1}{4}. \end{cases}$$

Take $r = 2$, $\sigma = \ln\left(\frac{3}{2}\right)$, $a_1 = \frac{1}{4}, a_2 = \frac{36}{49}, a_3 = \frac{3}{196}, a_4 = a_5 = 0$ and $\psi(\varpi) = \ln(\varpi)$ for all $\varpi > 0$. Then, consider the following cases:

Case 1. For $\varsigma, \zeta \in [0, 1)$ with $\varsigma \neq \zeta$, we have $0 < \mu(\Im\varsigma, \Im\zeta) = \frac{|\varsigma-\zeta|}{6}$ and

$$\begin{aligned}
\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) &= \ln\left(\frac{3}{2}\right) + \psi\left(\frac{|\varsigma - \zeta|}{6}\right) \\
&= \ln\left(\frac{3|\varsigma - \zeta|}{12}\right) < \ln\left(\frac{|\varsigma - \zeta|}{2}\right) \\
&= \ln\left(\frac{|\varsigma - \zeta|^2}{4}\right)^{\frac{1}{2}} = \ln\left(a_1(\mu(\varsigma, \zeta))^2\right)^{\frac{1}{2}} \\
&\leq \ln\left(\Omega^2_{\Im}(\varsigma, \zeta)\right).
\end{aligned}$$

Case 2. For $\varsigma \in [0,1)$ and $\zeta = 1$, we have $0 < \mu(\Im\varsigma, \Im\zeta) = \frac{|\varsigma-3|}{6}$, and

$$\begin{aligned}\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) &= \ln\left(\frac{3}{2}\right) + \ln\left(\frac{|\varsigma-3|}{6}\right) \\ &= \ln\left(\frac{|\varsigma-3|}{4}\right) \le \ln\left(\frac{6}{7}\times\frac{1}{2}\right) \\ &= \ln\left(\frac{36}{49}\mu\left(1,\frac{1}{2}\right)^2\right)^{\frac{1}{2}} \\ &\le \ln\left(\Omega^2_{\Im}(\varsigma,\zeta)\right).\end{aligned}$$

Hence, all the assertions of Thr_m 3.2 are satisfied. Consequently, $\Im$ has a unique f_{p_t} in Υ.

Whereas, with $\varsigma = \frac{5}{6}, \zeta = 1$,

$$\mu\left(\Im\left(\frac{5}{6}\right), \Im(1)\right) = \frac{13}{36} > \frac{1}{6} = \mu\left(\frac{5}{6}, 1\right).$$

And, for each $\psi \in \mathcal{M}$, there exists $\sigma > 0$ such that

$$\begin{aligned}\sigma + \psi\left(\mu\left(\Im\left(\frac{5}{6}\right), \Im(1)\right)\right) &= \sigma + \ln\left(\frac{13}{36}\right) \\ &> \ln\left(\frac{1}{6}\right) = \psi\left(\mu\left(\frac{5}{6}, 1\right)\right).\end{aligned}$$

Therefore, $\Im$ is not a ψ-contraction. So, Thr_m 2.4 due to Wardowski [22] is not applicable here.

Remark 3. By taking $a_1 = 1, a_2 = a_3 = a_4 = a_5 = 0$ in the contractive condition (3.4), we obtain the contractive inequality (2.1) due to Wardowski [22].

Theorem 3.4. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a 0-hybrid ψ-contraction. Then, $\Im$ has a f_{p_t} in Υ, provided that for each sequence $\{\hbar_n\}_{n\in\mathbb{N}}$ in $(0,\infty)$, $\lim_{n\longrightarrow\infty}\hbar_n = 0$ if and only if $\lim_{n\longrightarrow\infty}\psi(\hbar_n) = -\infty$.*

Proof. On the same steps as in Thr_m 3.2, we presume that for each $n \in \mathbb{N}$,

$$0 < \mu(\varsigma_{n+1}, \varsigma_n) = \mu(\Im\varsigma_n, \Im\varsigma_{n-1})$$

if and only if $\varsigma_n \ne \varsigma_{n+1}$. Setting $\varsigma = \varsigma_{n-1}$ and $\zeta = \varsigma_n$ in (3.3), we have

$$\begin{aligned}&\Omega^0_{\Im}(\varsigma_{n-1}, \varsigma_n) = H(\varsigma_{n-1}, \varsigma_n) = (\mu(\varsigma_{n-1}, \varsigma_n))^{a_1}(\mu(\varsigma_{n-1}, \Im\varsigma_{n-1}))^{a_2} \\ &\cdot (\mu(\varsigma_n, \Im\varsigma_n))^{a_3}\left(\frac{\mu(\varsigma_n, \Im\varsigma_n)(1+\mu(\varsigma_{n-1}, \Im\varsigma_{n-1}))}{1+\mu(\varsigma_{n-1}, \varsigma_n)}\right)^{a_4} \\ &\cdot\left(\frac{\mu(\varsigma_{n-1}, \Im\varsigma_n) + \mu(\varsigma_n, \Im\varsigma_{n-1})}{2}\right)^{a_5} \\ &= (\mu(\varsigma_{n-1}, \varsigma_n))^{a_1}(\mu(\varsigma_{n-1}, \varsigma_n))^{a_2}(\mu(\varsigma_n, \varsigma_{n+1}))^{a_3} \\ &\cdot\left(\frac{\mu(\varsigma_n, \varsigma_{n+1})(1+\mu(\varsigma_{n-1}, \varsigma_n))}{1+\mu(\varsigma_{n-1}, \varsigma_n)}\right)^{a_4}\left(\frac{\mu(\varsigma_{n-1}, \varsigma_{n+1}) + \mu(\varsigma_n, \varsigma_n)}{2}\right)^{a_5} \\ &= (\mu(\varsigma_{n-1}, \varsigma_n))^{a_1+a_2}(\mu(\varsigma_n, \varsigma_{n+1}))^{a_3+a_4}\left(\frac{\mu(\varsigma_{n-1}, \varsigma_n) + \mu(\varsigma_n, \varsigma_{n+1})}{2}\right)^{a_5}.\end{aligned} \tag{3.23}$$

Combining (3.4) and (3.23), we get

$$\begin{aligned}\sigma + \psi(\mu(\Im\varsigma_{n-1}, \Im\varsigma_n)) &\leq \psi\left(\Omega^0_{\Im}(\varsigma_{n-1}, \varsigma_n)\right)\\ &\leq \psi\Bigg[(\mu(\varsigma_{n-1}, \varsigma_n))^{a_1+a_2}(\mu(\varsigma_n, \varsigma_{n+1}))^{a_3+a_4}\\ &\cdot\left(\frac{\mu(\varsigma_{n-1}, \varsigma_n) + \mu(\varsigma_n, \varsigma_{n+1})}{2}\right)^{a_5}\Bigg]\end{aligned} \tag{3.24}$$

Assume that $\mu(\varsigma_{n-1}, \varsigma_n) \leq \psi(\varsigma_n, \varsigma_{n+1})$, then, (3.24) gives

$$\begin{aligned}\psi(\mu(\varsigma_n, \varsigma_{n+1})) \leq\ & \psi\left[(\mu(\varsigma_n, \varsigma_{n+1}))^{(\Sigma_{i=1}^5 a_i)}\right] - \sigma\\ =\ & \psi(\mu(\varsigma_n, \varsigma_{n+1})) - \sigma\\ <\ & \psi(\mu(\varsigma_n, \varsigma_{n+1})),\end{aligned} \tag{3.25}$$

a contradiction. Hence, $\mu(\varsigma_n, \varsigma_{n+1}) < \mu(\varsigma_{n-1}, \varsigma_n)$, for each $n \in \mathbb{N}$, and there exists $\flat \geq 0$ such that $\lim_{n\longrightarrow\infty} \mu(\varsigma_{n-1}, \varsigma_n) = \flat$. We claim that $\flat = 0$. Otherwise, if $\flat > 0$, then, letting $n \longrightarrow \infty$ in (3.25), yields $\psi(\flat) < \psi(\flat)$, which is not possible. It comes up that

$$\lim_{n\longrightarrow\infty} \mu(\varsigma_{n-1}, \varsigma_n) = 0. \tag{3.26}$$

Now, for each $n \in \mathbb{N}$ and $i \geq 1$, we have

$$\begin{aligned}\Omega^0_{\Im}(\varsigma_n, \varsigma_{n+i}) &= (\mu(\varsigma_n, \varsigma_{n+i}))^{a_1}(\mu(\varsigma_n, \Im\varsigma_n))^{a_2}(\mu(\varsigma_{n+i}, \Im\varsigma_{n+i}))^{a_3}\\ &\cdot\left(\frac{\mu(\varsigma_{n+i}, \Im\varsigma_{n+i})(1 + \mu(\varsigma_n, \Im\varsigma_n))}{1 + \mu(\varsigma_n, \varsigma_{n+i})}\right)^{a_4}\left(\frac{\mu(\varsigma_n, \Im\varsigma_{n+i}) + \mu(\varsigma_{n+i}, \Im\varsigma_n)}{2}\right)^{a_5}\\ &= (\mu(\varsigma_n, \varsigma_{n+i}))^{a_1}(\mu(\varsigma_n, \varsigma_{n+1}))^{a_2}(\mu(\varsigma_{n+i}, \varsigma_{n+i+1}))^{a_3}\\ &\cdot\left(\frac{\mu(\varsigma_{n+i}, \varsigma_{n+i+1})(1 + \mu(\varsigma_n, \varsigma_{n+1}))}{1 + \mu(\varsigma_n, \varsigma_{n+i})}\right)^{a_4}\left(\frac{\mu(\varsigma_n, \varsigma_{n+i+1}) + \mu(\varsigma_{n+i}, \varsigma_{n+1})}{2}\right)^{a_5}.\end{aligned}$$

Using (3.26), we obtain

$$\lim_{n\longrightarrow\infty} \Omega^0_{\Im}(\varsigma_n, \varsigma_{n+i}) = 0. \tag{3.27}$$

Consequently, by hypotheses, $\lim_{n\longrightarrow\infty} \psi(\Omega^0_{\Im}(\varsigma_n, \varsigma_{n+i})) = -\infty$, and, since

$$\sigma + \lim_{n\longrightarrow\infty} \psi(\mu(\varsigma_{n+1}, \varsigma_{n+1+i})) \leq \lim_{n\longrightarrow\infty} \psi(\Omega^0_{\Im}(\varsigma_n, \varsigma_{n+i})),$$

we get $\lim_{n\longrightarrow\infty} \psi(\mu(\varsigma_n, \varsigma_{n+i})) = -\infty$, from which it follows that $\lim_{n\longrightarrow\infty} \mu(\varsigma_n, \varsigma_{n+1}) = 0$. This shows that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy se_q in Υ. Sine Υ is a CM_s, there exists $u \in \Upsilon$ such that $\varsigma_n \longrightarrow u$ $(n \longrightarrow \infty)$. Moreover, it is a routine to check that for $\varsigma = \varsigma_n$ and $\zeta = u$ in (3.3), we have $\Omega^0_{\Im}(\varsigma_n, u) \longrightarrow 0$ $(n \longrightarrow \infty)$. If we assume that there exists a subsequence $\{\varsigma_{n_i}\}$ of $\{\varsigma_n\}_{n\in\mathbb{N}}$ such that $\Im\varsigma_{n_i} = \Im u$, then,

$$\begin{aligned}0 = \lim_{n\longrightarrow\infty} \mu(\Im\varsigma_{n_i}, \Im u) &= \lim_{n\longrightarrow\infty} \mu(\varsigma_{n_i+1}, \Im u)\\ &= \mu(u, \Im u),\end{aligned} \tag{3.28}$$

that is, $u = \Im u$. Hence, let $\mu(\Im\varsigma_n, \Im u) > 0$ for each $n \in \mathbb{N}$. Then, from (3.4), we obtain

$$\sigma + \psi(\mu(\Im\varsigma_n, \Im u)) \leq \psi\left(\Omega^0_{\Im}(\varsigma_n, u)\right). \tag{3.29}$$

Letting $n \longrightarrow \infty$ in (3.29), we have $\lim_{n\longrightarrow\infty} \psi(\mu(\Im\varsigma_n, \Im u)) = -\infty$. Hence, $\Im u = u$, since

$$\mu(u, \Im u) = \lim_{n\longrightarrow\infty} \mu(\Im\varsigma_n, \Im u) = 0.$$

□

In what follows, we derive a few immediate consequences of Thr_m 3.2 and 3.4.

Corollary 1. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mpn. If there exist $\psi \in \mathcal{M}$ and $\sigma > 0$ such that for all $\varsigma, \zeta \in \Upsilon$ with $\varsigma \neq \Im\varsigma$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies*

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi\left(\frac{\mu(\varsigma, \zeta) + \mu(\varsigma, \Im\varsigma)}{2}\right),$$

then, there exists $u \in \Upsilon$ such that $\Im u = u$.

Proof. Take $r = 1$, $a_1 = a_2 = \frac{1}{2}$ and $a_3 = a_4 = a_5 = 0$ in Theorem 3.2. □

Corollary 2. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mapping. If there exist $\psi \in \mathcal{M}$ and $\sigma > 0$ such that for all $\varsigma, \zeta \in \Upsilon \setminus \psi_{i\varsigma}(\Im)$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies*

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi\left(\frac{\mu(\varsigma, \Im\varsigma) + \mu(\zeta, \Im\zeta)}{2}\right),$$

then, there exists $u \in \Upsilon$ such that $\Im u = u$.

Proof. Put $a_1 = a_4 = a_5 = 0$, $a_2 = a_3 = \frac{1}{2}$ and $r = 1$ in Thr_m 3.2. □

Corollary 3. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mpn. If there exist $\psi \in \mathcal{M}$ and $\sigma > 0$ such that for all $\varsigma, \zeta \in \Upsilon$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies*

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi(a_1\mu(\varsigma, \zeta) + a_2\mu(\varsigma, \Im\varsigma) + a_3\mu(\zeta, \Im\zeta)) \tag{3.30}$$

where $\sum_{i=1}^{3} a_i = 1$, then, $\Im$ has a unique fp_t in Υ.

Proof. Take $r = 1$ and $a_4 = a_5 = 0$ in Thr_m 3.2. □

Corollary 4. *Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mpn. If there exist $\psi \in \mathcal{M}$, $\gamma \in (0, 1)$ and $\sigma > 0$ such that for all $\varsigma, \zeta \in \Upsilon$, with $\varsigma \neq \Im\varsigma$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies*

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi\left(\mu(\varsigma, \zeta)^{\gamma}\mu(\varsigma, \Im\varsigma)^{1-\gamma}\right)$$

then, there exists $u \in \Upsilon$ such that $\Im u = u$.

Proof. Set $a_1 = \gamma$, $a_2 = 1 - \gamma$ and $a_3 = a_4 = a_5 = 0$ in Theorem 3.4. □

Remark 4. Following Corollaries 1–4, it is obvious that more particular cases of Thr_m 3.2 and 3.4 can be pointed out.

4. Further consequences

In this section, we show that some well-known f_{p_t} thr_m with metric space structure in the existing literature can be deduced as special cases of our results.

Corollary 5. *[26] Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a single-valued mpn, If there exist $\sigma > 0$ and a mpn $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that for each $\varsigma, \zeta \in \Upsilon$, $\mu(\Im\varsigma, \Im\zeta) > 0$ implies*

$$\sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) \leq \psi(\Lambda_1\mu(\varsigma, \Im\varsigma) + \Lambda_2\mu(\zeta, \Im\zeta))$$

for all nonnegative real numbers $\Lambda_1, \Lambda_2 \in [0, 1)$ with $\sum_{i=1}^{2} \Lambda_i = 1$, then, $\Im$ has a f_{p_t} in Υ.

Proof. Put $a_1 = 0$ and $a_2 = \Lambda_1,\ a_3 = \Lambda_2$ in Corollary 3. □

Definition 4.1. [21] Let (Υ, μ) be a M_s. A single-valued mpn $\Im : \Upsilon \longrightarrow \Upsilon$ is called Reich contraction if there exist $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}_+$ with $\Lambda_1 + \Lambda_2 + \Lambda_3 < 1$ such that for all $\varsigma, \zeta \in \Upsilon$,

$$\mu(\Im\varsigma, \Im\zeta) \leq \Lambda_1\mu(\varsigma, \zeta) + \Lambda_2\mu(\varsigma, \Im\varsigma) + \Lambda_3\mu(\zeta, \Im\zeta). \tag{4.1}$$

Corollary 6. *[21] Let (Υ, μ) be a CM_s and $\Im : \Upsilon \longrightarrow \Upsilon$ be a Reich contraction. Then, $\Im$ has a unique f_{p_t} in Υ.*

Proof. Take $\psi(\varpi) = \ln(\varpi)$ for all $\varpi > 0$ and $\Lambda_i = a_i e^{-\varrho}$ in Corollary 3. □

Definition 4.2. [19] Let (Υ, μ) be a M_s. A mpn $\Im : \Upsilon \longrightarrow \Upsilon$ is called an interpolative Kannan contraction if there exist $\gamma \in (0, 1)$ and $\lambda \in (0, 1)$ such that for all $\varsigma, \zeta \in \Upsilon \setminus \psi_{i\varsigma}(\Im)$,

$$\mu(\Im\varsigma, \Im\zeta) \leq \lambda\left[\mu(\varsigma, \Im\varsigma)^{\gamma}\mu(\zeta, \Im\zeta)^{1-\gamma}.\right] \tag{4.2}$$

Corollary 7. *[19] Let (Υ, μ) be a CM_s and $\Im$ be an interpolative Kannan contraction. Then, $\Im$ has a f_{p_t} in Υ.*

Proof. From (4.2), for all $\varsigma, \zeta \in \Upsilon \setminus \psi_{i\varsigma}(\Im)$ with $\mu(\Im\varsigma, \Im\zeta) > 0$, we have

$$\sigma + \ln(\mu(\Im\varsigma, \Im\zeta)) \leq \ln\left(\left[\mu(\varsigma, \Im\varsigma)^{\gamma}\mu(\zeta, \Im\zeta)^{1-\gamma}\right]\right). \tag{4.3}$$

By taking $\psi(\varpi) = \ln(\varpi)$ for all $\varpi > 0$, (4.3) becomes

$$\begin{aligned} \sigma + \psi(\mu(\Im\varsigma, \Im\zeta)) &\leq \psi\left(\left[\mu(\varsigma, \Im\varsigma)^{\gamma}\mu(\zeta, \Im\zeta)^{1-\gamma}\right]\right) \\ &\leq \psi\left(\Omega_{\Im}^{0}(\varsigma, \zeta)\right), \end{aligned}$$

where $\sigma = \ln\left(\frac{1}{\lambda}\right)$. Therefore, putting $a_1 = a_4 = a_5 = 0$, $a_2 = \gamma$ and $a_3 = 1 - \gamma$, Theorem 3.4 can be applied to find $u \in \Upsilon$ such that $\Im u = u$. □

5. An application in dynamic programming

Mathematical optimization is one of the areas in which the techniques of f_{p_t} theory are generously used. It is a known fact that dynamic programming provides important tools for mathematical optimization and computer programming. In this direction, the problem of dynamic programming with regards to multistage process reduces to solving the functional eq_n:

$$h(\varsigma) = \sup_{\zeta \in G}\{g(\varsigma, \zeta) + S(\varsigma, \zeta, h(\flat(\varsigma, \zeta)))\}, \ \varsigma \in L, \tag{5.1}$$

where $\flat : L \times G \longrightarrow L$, $g : L \times G \longrightarrow \mathbb{R}$ and $S : L \times G \times \mathbb{R} \longrightarrow \mathbb{R}$.

Assume that K and W are Banach spaces, $L \subseteq K$ is a state space and $G \subseteq W$ is a decision space. Precisely, the studied process consists of :

(i) a state space, which is the set of initial state, action and transition model of the process;
(ii) a decision space, which is the set of possible actions that are allowed for the process.

For further details of functional eq_n arising in dynamic programming, the interested reader may consult Bellman and Lee [37]. In this section, we investigate the existence of bounded solution to the functional eq_n (5.1). Let $\Upsilon = B(L)$ be the set of all bounded real-valued functions on L and, for an arbitrary element $p \in \Upsilon$, take $\|p\| = \sup_{\varsigma \in L} |p(\varsigma)|$. Obviously, $(\Upsilon, \|.\|)$ equipped with the metric μ induced by the norm $\|.\|$, via:

$$\mu(p, q) = \|p - q\| = \sup_{\varsigma \in L} |p(\varsigma) - q(\varsigma)| \tag{5.2}$$

for all $p, q \in \Upsilon$, is a Banach space. In fact, the convergence(cvgnce) in Υ with respect to $\|.\|$ is uniform. Hence, if we consider a Cauchy se_q $\{p_n\}_{n\in\mathbb{N}}$ in Υ, then, $\{p_n\}_{n\in\mathbb{N}}$ cvg uniformly to a fnx say p^*, that is also bounded and so $p^* \in \Upsilon$.

Consider a mpn $\Im : \Upsilon \longrightarrow \Upsilon$ defined by

$$\Im(p)(\varsigma) = \sup_{\zeta \in G}\{g(\varsigma, \zeta) + S(\varsigma, \zeta, p(\flat(\varsigma, \zeta)))\} \tag{5.3}$$

for all $p \in \Upsilon$ and $\varsigma \in L$. Clearly, if the fnx g and S are bounded, then $\Im$ is well-defined.

For all $p, q \in \Upsilon$, let

$$\begin{aligned} G^*(p, q) = \Bigg[& \Lambda_1(\mu(p, q))^r + \Lambda_2(\mu(p, \Im p))^r + \Lambda_3(\mu(q, \Im q))^r \\ & \Lambda_4\left(\frac{\mu(q, \Im q)(1 + \mu(p, \Im p))}{1 + \mu(p, q)}\right)^r + \Lambda_5\left(\frac{\mu(q, \Im p)(1 + \mu(p, \Im q))}{1 + \mu(p, q)}\right)^r\Bigg]^{\frac{1}{r}}, \end{aligned} \tag{5.4}$$

where Λ_i $(i = 1, 2, 3, 4, 5)$ are nonnegative real numbers satisfying $\sum_{i=1}^5 \Lambda_i = 1$.

Theorem 5.1. *Let* $\Im : \Upsilon \longrightarrow \Upsilon$ *be a mpn represented in* (5.3) *and suppose that:*

(D_1) $S : L \times G \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $g : L \times G \longrightarrow \mathbb{R}$ *are continuous;*
(D_2) *there exists* $\sigma > 0$ *such that*

$$|S(\varsigma, \zeta, p(\varsigma)) - S(\varsigma, \zeta, q(\varsigma))| \leq e^{-\sigma} G^*(p, q),$$

for all $p, q \in \Upsilon$, where $\varsigma \in L$ and $\zeta \in G$.
Then, the functional eq_n (5.1) has a bounded solution in Υ.

Proof. First, note that (Υ, μ) is a CM_s, where the metric μ is given by (5.2). Let $\varrho > 0$ be an arbitrary real number, $\varsigma \in L$ and $p_1, p_2 \in \Upsilon$. then, there exist $\zeta_1, \zeta_2 \in G$ such that

$$\Im(p_1)(\varsigma) < g(\varsigma, \zeta_1) + S(\varsigma, \zeta_1, p_1(\flat(\varsigma, \zeta_1))) + \varrho, \tag{5.5}$$

$$\Im(p_2)(\varsigma) < g(\varsigma, \zeta_2) + S(\varsigma, \zeta_2, p_2(\flat(\varsigma, \zeta_2))) + \varrho, \tag{5.6}$$

$$\Im(p_1)(\varsigma) \geq g(\varsigma, \zeta_2) + S(\varsigma, \zeta_2, p_1(\flat(\varsigma, \zeta_2))), \tag{5.7}$$

$$\Im(p_2)(\varsigma) \geq g(\varsigma, \zeta_1) + S(\varsigma, \zeta_1, p_2(\flat(\varsigma, \zeta_1))). \tag{5.8}$$

Hence, it follows from (5.5) and (5.8) that

$$\begin{aligned} \Im(p_1)(\varsigma) - \Im(p_2)(\varsigma) &< S(\varsigma, \zeta_1, p_1(\flat(\varsigma, \zeta_1))) - S(\varsigma, \zeta_1, p_2(\flat(\varsigma, \zeta_1))) + \varrho \\ &\leq |S(\varsigma, \zeta_1, p_1(\flat(\varsigma, \zeta_1))) - S(\varsigma, \zeta_1, p_2(\flat(\varsigma, \zeta_1)))| + \varrho \\ &\leq e^{-\sigma} G^*(p_1, p_2) + \varrho, \end{aligned}$$

that is,

$$\Im(p_1)(\varsigma) - \Im(p_2)(\varsigma) \leq e^{-\sigma} G^*(p_1, p_2) + \varrho. \tag{5.9}$$

On similar steps, using (5.6) and (5.7), we get

$$\Im(p_2)(\varsigma) - \Im(p_1)(\varsigma) \leq e^{-\sigma} G^*(p_1, p_2) + \varrho. \tag{5.10}$$

Therefore, from (5.9) and (5.10), we have

$$\left| \Im(p_1)(\varsigma) - \Im(p_2)(\varsigma) \right| \leq e^{-\sigma} G^*(p_1, p_2) + \varrho. \tag{5.11}$$

Taking supremum over all $\varsigma \in L$ in (5.11), yields

$$\mu(\Im(p_1), \Im(p_2)) \leq e^{-\sigma} G^*(p_1, p_2) + \varrho. \tag{5.12}$$

Given that $\varrho > 0$ is arbitrary, then, we deduce from (5.12) that

$$\mu(\Im(p_1), \Im(p_2)) \leq e^{-\sigma} G^*(p_1, p_2). \tag{5.13}$$

So, passing to logarithms in (5.13), gives

$$\sigma + \ln\left(\mu(\Im(p_1), \Im(p_2))\right) \leq \ln\left(G^*(p_1, p_2)\right). \tag{5.14}$$

By defining the fnx $\psi : (0, \infty) \longrightarrow \mathbb{R}$ as $\psi(\varpi) = \ln(\varpi)$ for all $\varpi > 0$, (5.14) becomes

$$\sigma + \psi\left(\Im(p_1), \Im(p_2)\right) \leq \psi\left(G^*(p_1, p_2)\right).$$

Thus, $\Im$ is an r-hybrid ψ-contraction. Consequently, as an application of Thr_m 3.2, we conclude that $\Im$ has a f_{p_t} in Υ, which corresponds to a solution of the functional eq_n (5.1). □

6. Applications to nonlinear Volterra integral equations

F_{pt} for contractive operators on metric spaces are commonly investigated and have gained enormous applications in the theory of differential and integral eq_n (see [34, 36] and references therein). In this subsection, we apply Thr_m 3.2 to discuss the existence and uniqueness of a solution to the following integral eq_n of Volterra type:

$$\varsigma(\varpi) = f(\varpi) + \int_0^{\varpi} L(\varpi, s, \varsigma(s))\mu s, \ \varpi \in [0, \delta] = J, \tag{6.1}$$

where $\delta > 0$, $L : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $f : J \longrightarrow \mathbb{R}$.

Let $\Upsilon = C(J, \mathbb{R})$ be the space of all continuous real-valued fnx defined on J. And, for arbitrary $\varsigma \in \Upsilon$, define $\|\varsigma\|_\sigma = \sup_{\varpi \in J}\{|\varsigma(\varpi)|e^{-\sigma\varpi}\}$, where $\sigma > 0$. It is well-known that $\|.\|_\sigma$ is a norm equivalent to the supremum norm, and Υ equipped with the metric μ_σ defined by

$$\mu_\sigma(\varsigma, \zeta) = \sup\{|\varsigma(\varpi) - \zeta(\varpi)|e^{-\sigma\varpi}\}, \tag{6.2}$$

for all $\varsigma, \zeta \in \Upsilon$, is a Banach space.

Theorem 6.1. *Suppose that:*

$(C1)$ $L : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $f : J \longrightarrow \mathbb{R}$ *are continuous;*
$(C2)$ *there exists* $\sigma > 0$ *such that for all* $s, \varpi \in J$ *and* $\varsigma, \zeta \in \mathbb{R}$,

$$|L(\varpi, s, \varsigma) - L(\varpi, s, \zeta)| \leq \sigma e^{-\sigma}|\varsigma - \zeta|.$$

Then, the integral eq_n (6.1) *has a unique solution in* Υ.

Proof. Note that (Υ, μ_σ) is a CM_s, where the metric μ_σ is given by (6.2). Consider a mpn $\Im : \Upsilon \longrightarrow \Upsilon$ defined by

$$\Im(\varsigma)(\varpi) = f(\varpi) + \int_0^{\varpi} L(\varpi, s, \varsigma(s))\mu s, \ \varsigma \in \Upsilon, \varpi \in J. \tag{6.3}$$

Let $\varsigma, \zeta \in \Upsilon$ such that $\Im\varsigma \neq \Im\zeta$. Then,

$$\begin{aligned}
\left|\Im(\varsigma)(\varpi) - \Im(\zeta)(\varpi)\right| &\leq \int_0^{\varpi} |L(\varpi, s, \varsigma(s)) - L(\varpi, s, \zeta(s))|\mu s \\
&\leq \int_0^{\varpi} \sigma e^{-\sigma}|\varsigma(s) - \zeta(s)|\mu s \\
&= \int_0^{\varpi} \sigma e^{-\sigma}|\varsigma(s) - \zeta(s)|e^{-\sigma s}e^{\sigma s}\mu s \\
&\leq \int_0^{\varpi} \sigma e^{\sigma s}e^{-\sigma}|\varsigma(s) - \zeta(s)|e^{-\sigma s}\mu s \\
&\leq \sigma e^{-\sigma}\|\varsigma - \zeta\|_\sigma \int_0^{\varpi} e^{\sigma s}\mu s \\
&\leq \sigma e^{-\sigma}\|\varsigma - \zeta\|_\sigma \frac{e^{\sigma\varpi}}{\sigma}.
\end{aligned} \tag{6.4}$$

It follows from (6.4) that

$$\left|\Im(\varsigma)(\varpi) - \Im(\zeta)(\varpi)\right| e^{-\sigma\varpi} \leq e^{-\sigma}\|\varsigma - \zeta\|_{\sigma}. \tag{6.5}$$

Taking supremum over all $\varpi \in J$ in (6.5), produces

$$\mu_{\sigma}\left(\Im(\varsigma), \Im(\zeta)\right) \leq e^{-\sigma}\mu_{\sigma}(\varsigma, \zeta). \tag{6.6}$$

Passing to logarithms in (6.6), yields

$$\sigma + \ln\left(\mu_{\sigma}\left(\Im(\varsigma), \Im(\zeta)\right)\right) \leq \ln(\mu_{\sigma}(\varsigma, \zeta)). \tag{6.7}$$

By defining the $fnx\ \psi : (0, \infty) \longrightarrow \mathbb{R}$ as $\psi(\varpi) = \ln(\varpi)$ for all $\varpi > 0$, (6.7) can be rewritten as:

$$\sigma + \psi\left(\mu_{\sigma}\left(\Im(\varsigma), \Im(\zeta)\right)\right) \leq \psi(\mu_{\sigma}(\varsigma, \zeta)).$$

Hence, all the hypotheses of Thr_m 3.2 are satisfied with $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$. Consequently, $\Im$ has a f_{p_t} in Υ, which is the unique solution of the integral eq_n (6.1). □

Example 6.2. Consider the Volterra integral eq_n of the form

$$\varsigma(\varpi) = \frac{\varpi}{1 + \varpi^2} + \int_0^{\varpi} \frac{\varsigma(s)}{25 + (\varsigma(s))^2}\mu s,\ \ \varpi \in [0, \delta], \delta > 0. \tag{6.8}$$

From (6.1) and (6.8), we note that $f(\varpi) = \frac{\varpi}{1+\varpi^2}$ and $L(\varpi, s, \varsigma(s)) = \frac{\varsigma(s)}{25+(\varsigma(s))^2}$ are continuous; that is, condition $(C1)$ of Thr_m 6.1 holds. Moreover,

$$\begin{aligned}
& |L(\varpi, s, \varsigma(s)) - L(\varpi, s, \zeta(s))| \\
\leq\ & \frac{1}{25}|\varsigma(s) - \zeta(s)| \\
\leq\ & (1)e^{-1}|\varsigma(s) - \zeta(s)| = \sigma e^{-\sigma}|\varsigma(s) - \zeta(s)|.
\end{aligned}$$

Hence, condition $(C2)$ is verified. By Thr_m 6.1, it comes up that (6.8) has a unique solution in $\Upsilon = C([0, \delta], \mathbb{R})$.

7. Conclusions

In this work, a novel concept called r-hybrid ψ-contraction has been introduced and some f_{p_t} results for such mpn in the framework of CM_s have been presented. The established f_{p_t} thr_m merge and extend a number of well-known concepts in the corresponding literature. A few of these particular cases have been highlighted and discussed. An example is designed to show the generality and authenticity of our points. From application perspective, we investigated the existence and uniqueness conditions of solutions to certain functional equation arising in dynamic programming and integral equation of Volterra type.

It is noteworthy that the idea of this paper, being established in the setting of a M_s, is fundamental. Hence, it can be improved upon when examined in the structure of b-M_s, F-M_s, G-M_s, modular M_s, and some other pseudo or quasi M_s. It is a familiar fact that construction of fp_t has lots of usefulness; in particular, in transition operators for Cauchy problems of differential equations of either integer and non-integer order. In this direction, the contractive inequalities and functional equations presented here can be studied within the domains of variational inequality and fractional calculus. Furthermore, it is natural to extend the single-valued mappings herein to set-valued mpn within the outlines of either fuzzy or classical mathematics.

Acknowledgments

The authors would like to express their sincere thanks to referees for improving the article and also thanks to the Natural Science Foundation of Zhejiang Province (Grant No.: LQ19A010001) for providing financial assistance to support this research. The authors would like to express their sincere thanks to the support of Taif University Researchers Supporting Project Number (TURSP-2020/217), Taif University, Taif, Saudi Arabia.

This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11971142, 11871202, 11301127, 11701176, 11626101, and 11601485).

Conflict of interest

The authors declare that they have no competing interests.

References

1. Q. L. Dong, Y. C. Tang, Y. J. Cho, T. M. Rassias, Optimal choice of the step length of the projection and contraction methods for solving the split feasibility problem, *J. Global Optim.*, **71** (2018), 341–360.
2. Y. Yao, R. P. Agarwal, M. Postolache, Y. C. Liou, Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem, *Fixed Point Theory Appl.*, **2014** (2014), 183.
3. M. A. Noor, On nonlinear variational inequalities, *Int. J. Math. Math. Sci.*, **14** (1991), 399–402.
4. Y. M. Chu, S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, More new results on integral inequalities for generalized K-fractional conformable integral operators, *Discrete Contin. Dyn. Syst. Ser. S*, 2021. DOI: 10.3934/dcdss.2021063.
5. S. S. Zhou, S. Rashid, A. Rauf, F. Jarad, Y. S. Hamed, K. M. Abualnaja, Efficient computations for weighted generalized proportional fractional operators with respect to a monotone function, *AIMS Math.*, **6** (2021), 8001–8029.
6. S. Rashid, S. Sultana, F. Jarad, H. Jafari, Y. S. Hamed, More efficient estimates via h-discrete fractional calculus theory and applications, *Chaos Solitons Fract.*, **147** (2021), 110981.
7. H. G. Jile, S. Rashid, F. B. Farooq, S. Sultana, Some inequalities for a new class of convex functions with applications via local fractional integral, *J. Funct. Spaces Appl.*, **2021** (2021), 1–17.
8. S. Rashid, S. Parveen, H. Ahmad, Y. M. Chu, New quantum integral inequalities for some new classes of generalized ψ-convex functions and their scope in physical systems, *Open Phys.*, **19** (2021). DOI: 10.1515/phys-2021-0001.
9. D. W. Barnes, L. A. Lambe, A fixed point approach to homological perturbation theory, *Proc. Amer. Math. Soc.*, **112** (1991), 881–892.
10. B. Hazarika, H. M. Srivastava, R. Arab, M. Rabbani, Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it, *J. Comput. Appl. Math.*, **343** (2018), 341–352.

11. M. Abukhaled, S. A. Khuri, A semi-analytical solution of amperometric enzymatic reactions based on Green's functions and fixed point iterative schemes, *J. Electroanal. Chem.*, **792** (2017), 66–71.
12. A. A. El-Deeb, S. Rashid, On some new double dynamic inequalities associated with Leibniz integral rule on time scales, *Adv. Differ. Equ.*, **2021** (2021), 1–22.
13. S. S. Zhou, S. Rashid, S. Parveen, A. O. Akdemir, Z. Hammouch, New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operators, *AIMS Math.*, **6** (2021), 4507–4525.
14. M. Al-Qurashi, S. Rashid, S. Sultana, H. Ahmad, K. A. Gepreel, New formulation for discrete dynamical type inequalities via h-discrete fractional operator pertaining to nonsingular kernel, *Math. Biosci. Eng.*, **18** (2021), 1794–1812.
15. Y. M. Chu, S. Rashid, J. Singh, A novel comprehensive analysis on generalized harmonically Ψ-convex with respect to Raina's function on fractal set with applications, *Math. Methods Appl. Sci.*, 2021. DOI: 10.1002/mma.7346.
16. S. Rashid, F. Jarad, Z. Hammouch, Some new bounds analogous to generalized proportional fractional integral operator with respect to another function, *Discrete. Contin. Dyn. Syst. Ser. S*, 2021. DOI: 10.3934/DCDSS.2021020.
17. H. Iiduka, Fixed point optimization algorithm and its application to network bandwidth allocation, *J. Comput. Appl. Math.*, **236** (2012), 1733–1742.
18. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
19. E. Karapinar, Revisiting the Kannan type contractions via interpolation, *Adv. Theory Nonlinear Anal. Appl.*, **2** (2018), 85–87.
20. E. Karapınar, O. Alqahtani, H. Aydi, On interpolative Hardy-Rogers type contractions, *Symmetry*, **11** (2019), 1–7.
21. S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14** (1971), 121–124.
22. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94.
23. D. Wardowski, N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, *Demonstratio Math.*, **47** (2014), 146–155.
24. N. A. Secelean, Iterated function systems consisting of F-contractions, *Fixed Point Theory Appl.*, **2013** (2013), 1–13.
25. H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2014** (2014), 210.
26. M. Cosentino, P. Vetro, Fixed point results for F-contractive mappings of Hardy-Rogers-type, *Filomat*, **28** (2014), 715–722.
27. M. Alansari, S. S. Mohammed, A. Azam, N. Hussain, On multivalued hybrid contractions with applications, *J. Funct. Spaces Appl.*, **2020** (2020), 1–12.
28. E. Ameer, H. Aydi, M. Arshad, H. Alsamir, M. S. Noorani, Hybrid multivalued type contraction mappings in αK-complete partial b-metric spaces and applications, *Symmetry*, **11** (2019). DOI:10.3390/sym11010086.

29. E. Karapinar, A. Fulga, New hybrid contractions on b-metric spaces, *Mathematics*, **7** (2019), 578.
30. E. Karapinar, H. Aydi, A. Fulga, On p-hybrid Wardowski contractions, *J. Math.*, **2020** (2020), 1–7.
31. J. Ahmad, A. Al-Rawashdeh, A. Azam, New fixed point theorems for generalized F-contractions in complete metric spaces, *Fixed Point Theory Appl.*, **2015** (2015), 80.
32. H. Aydi, E. Karapinar, A. F. Roldán López de Hierro, ω-interpolative Ćirić-Reich-Rus-type contractions, *Mathematics*, **7** (2019), 57.
33. H. Aydi, C. M. Chen, E. Karapınar, Interpolative Ćirić-Reich-Rus type contractions via the Branciari distance, *Mathematics*, **7** (2019), 84.
34. D. Derouiche, H. Ramoul, New fixed point results for F-contractions of Hardy-Rogers type in b-metric spaces with applications, *J. Fixed Point Theory Appl.*, **22** (2020), 1–44.
35. E. Karapinar, R. Agarwal, H. Aydi, Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces, *Mathematics*, **6** (2018), 256.
36. T. Rasham, A. Shoaib, N. Hussain, M. Arshad, S. U. Khan, Common fixed point results for new Ciric-type rational multivalued F-contraction with an application, *J. Fixed Point Theory Appl.*, **20** (2018), 45.
37. R. Bellman, E. S. Lee, Functional equations in dynamic programming, *Aequationes Math.*, **17** (1978), 1–18.

AIMS Press