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*Research article*

# On the reflexive edge strength of the circulant graphs

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**Abstract:** A labeling of a graph is an assignment that carries some sets of graph elements into numbers (usually the non negative integers). The total $k$-labeling is an assignment $f_e$ from the edge set to the set $\{1, 2, ..., k_e\}$ and assignment $f_v$ from the vertex set to the set $\{0, 2, 4, ..., 2k_v\}$, where $k = max\{k_e, 2k_v\}$. An edge irregular reflexive $k$-labeling of the graph $G$ is the total $k$-labeling, if distinct edges have distinct weights, where the edge weight is defined as the sum of label of that edge and the labels of the end vertices. The minimum $k$ for which the graph $G$ has an edge irregular reflexive $k$-labeling is called the reflexive edge strength of the graph $G$, denoted by $res(G)$. In this paper we study the edge reflexive irregular $k$-labeling for some cases of circulant graphs and determine the exact value of the reflexive edge strength for several classes of circulant graphs.

**Keywords:** edge irregular reflexive labeling; reflexive edge strength; circulant graphs
**Mathematics Subject Classification:** 05C78, 05C12, 05C90

## 1. Introduction

Let $G$ be a connected, simple, and undirected graph with a vertex set $V(G)$ and edge set $E(G)$. In graph theory, graph labeling is a topic that is interesting to study. Graph labelings were introduced for first time by Sedláček in 1963 [32], the various methods for labeling of the edges of graph have been studied by stewart in [33], and then formulated by Kotig and Rosa in [27]. In 1988, Chartrand et al. [20] defined irregular labeling for a graph $G$. Informally, the irregular labeling defined as follows. Let $G(V, E)$ be a graph, the mapping $f : E \to \{1, 2, 3, ..., k\}$ is called irregular $k$-labeling of $G$, if every different vertices $u$ and $v$ have distinct weights, that is:

$$\sum_{x \in V} f(ux) \neq \sum_{y \in V} f(vy).$$

The irregularity strength of $G$, denoted by $s(G)$, is known as the minimum positive natural number $k$ which $G$ has an irregular $k$-labeling. Therefore, there are many researchers who give certain results of the irregular labeling (see [1, 7, 9, 18, 21, 22, 26, 27, 32, 33, 36]). The natural extension of irregularity strength of a graph $G$, Baća, Jendrol, Miller, and Ryan [11] introduced the other types of irregular labeling based on the total labeling, namely the vertex irregular total k-labeling and edge irregular total k-labeling. Some results on the vertex irregular total k-labeling and edge irregular total k-labeling can be found in [2–5, 8, 10, 12, 13, 16, 23, 28, 30]. In [11] Baća et al. defined the concept of edge irregular total $k$-labeling as a labeling of vertices and edges of $G$, $f : V \bigcup E \rightarrow \{1, 2, ..., k\}$ such that for any different edges $xy$, $x^{'}y^{'}$ have distinct weights i.e. $wt(xy) \neq wt(x^{'}y^{'})$ where $wt(xy) = f(x) + f(xy) + f(y)$. The minimum $k$ for which $G$ has an edge irregular total $k$-labeling is namely the total edge irregularity strength of $G$, denoted by $tes(G)$. Furthermore, Ryan et al. [31] generalized the concept of edge irregular total $k$-labeling of a graph to be an edge irregular reflexive $k$-labeling.

For a graph $G$ the total $k$-labeling defined the map $f_e : E(G) \rightarrow \{1, 2, 3, ..., k_e\}$ and $f_v : V(G) \rightarrow \{0, 2, 4, ..., 2k_v\}$, where $k = max\{k_e, 2k_v\}$. The total $k$-labeling is called an edge irregular reflexive $k$-labeling , if for every two distinct edges $xy$ and $x^{'}y^{'}$ of $G$, $wt(xy) \neq wt(x^{'}y^{'})$, where $wt(xy) = f_v(x) + f_e(xy) + f_v(y)$. The minimum value of $k$ for which such labeling exists is known as the reflexive edge strength of graph $G$ and is denoted by $res(G)$. Over the last years, $res(G)$ has been investigated for different family of graphs (see [6, 14, 35, 38]). The notion of reflexive edge strength was defined in [35]. The following lemma estimate the lower bound of the reflexive edge strength of graph $G$.

**Lemma 1.1.** *( [35]) For every graph G,*

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & |E(G)| \not\equiv 2, 3 \ (mod\ 6), \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & |E(G)| \equiv 2, 3 \ (mod\ 6)\ . \end{cases}$$

Also, Baća et al. [14] suggested the following conjecture:

**Conjecture 1.1.** *( [14]) Let G be a simple graph with maximum degree $\triangle = \triangle(G)$. Then:*

$$res(G) = max\{\lfloor \frac{\triangle + 2}{2} \rfloor, \lceil \frac{|E(G)|}{3} \rceil + r\}$$

*where $r = 1$ for $|E(G)| \equiv 2, 3$ (mod 6), and zero otherwise.*

## 2. Circulant graphs

The circulant graphs are another important family of graphs that can be used to construct local area networks see [17]. Suppose $1 \leq s_1 \leq s_2 \leq ... \leq s_m \leq \frac{n}{2}$, where $n$ and $s_j$, $j = 1, 2, 3, ..., m$ are positive numbers. The circulant graph $C_n(s_1, s_2, ..., s_m)$ is the graph on vertex set $V = \{x_0, x_1, ..., x_{n-1}\}$ and edge set $E = \{x_i x_{i+s_j} : i = 0, 1, 2, ..., n-1 \text{ and } j = 1, 2, ..., m\}$ where $i + s_j$ is taken modulo $n$ see [15]. It is clear to see that $C_n(s_1, s_2, ..., s_m)$ is a $r$-regular, where $r = 2m - 1$ if $s_m = \frac{n}{2}$ with $n(2m - 1)/2$ edges on other hand $r = 2m$ with $mn$ edges if $s_m < \frac{n}{2}$. The circulant graphs have a large number of applications and uses in telecommunication network VLSI design, parallel and distributed computing. The properties of circulant graphs; such as bipartitness, planarity, Hamiltonicity and colourability have been studied in [19, 24, 25, 29, 34, 37]. In this paper we investigate the existence of the reflexive edge irregularity strength for some classes of circulant graphs.

## 3. Main results

In this section, we discuss the edge irregular reflexive k-labeling for some classes of circulant graphs.

**Theorem 1.** *Let $C_n(1,2)$, be a circulant graph on $n \geq 4$ vertices. Then*

$$res(C_n(1,2)) = \begin{cases} 4, & n = 4, \\ 5, & n = 5, 6, \\ \lceil \frac{2n}{3} \rceil, & n \equiv 0, 2 \text{ (mod 3) and } n \geq 8, \\ \lceil \frac{2n}{3} \rceil + 1, & n \equiv 1 \text{ (mod 3) and } n \geq 7 \,. \end{cases}$$

*Proof.* Let $C_n(1,2)$ be a circulant graph with vertex set $V(C_n(1,2)) = \{x_i : 1 \leq i \leq n\}$ and edge set $E(C_n(1,2)) = \{x_i x_{i+j} : 1 \leq i \leq n, 1 \leq j \leq 2\}$, where indices are taken modulo $n$. First note that $C_n(1,2)$ isomorphic to the wheel $W_3$ and so from [22] Ryan proved that $res(W_3) = 4$, so $res(C_n(1,2)) = 4$. By the same arguments that used in [22], we can find that $res(C_5(1,2)) \geq 5$ and $res(C_6(1,2)) \geq 5$. The corresponding labelings for $C_n(1,2)$, $n = 4, 5, 6, 7, 8, 9$, are shown in Figure 1.

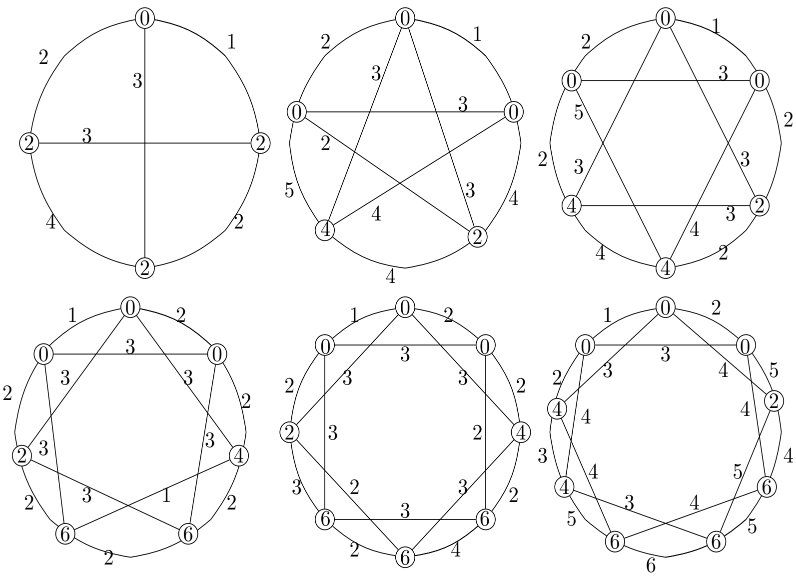

**Figure 1.** The reflexive edge irregular $k$-labeling of $C_n(1,2), n = 4, 5, 6, 7, 8, 9$.

For $n \geq 10$ and based on Lemma 1.1, we show the following lower bound for the circulant $C_n(1,2)$ : $res(C_n(1,2)) \geq k = \lceil \frac{2n}{3} \rceil$ if $n \equiv 0, 2$ (mod 3) and $res(C_n(1,2)) \geq k = \lceil \frac{2n}{3} \rceil + 1$ if $n \equiv 1$ (mod 3). Then, to convince our proof we define a total labeling of $C_n(1,2)$ such as:

$$f(x_i) = \begin{cases} 0, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{4} \rfloor, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ 2\lfloor \frac{k}{2} \rfloor, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{6} \rfloor, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n. \end{cases}$$

$$f(x_ix_{i+1})=\begin{cases} i, & 1\le i\le \lfloor\frac{n}{3}\rfloor-1,\\ 2n-2\lfloor\frac{n}{2}\rfloor- \\ -2\lfloor\frac{k}{4}\rfloor-2, & i=\lfloor\frac{n}{3}\rfloor,\\ 2n-2\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor- \\ -4\lfloor\frac{k}{4}\rfloor+i, & \lfloor\frac{n}{3}\rfloor+1\le i\le \lfloor\frac{n}{2}\rfloor-1,\\ 2n-2\lfloor\frac{n}{3}\rfloor- \\ -2(\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{4}\rfloor)+1, & i=\lfloor\frac{n}{2}\rfloor,\\ 2n-\lfloor\frac{n}{2}\rfloor-2\lfloor\frac{n}{3}\rfloor- \\ -4\lfloor\frac{k}{2}\rfloor+3+i, & \lfloor\frac{n}{2}\rfloor+1\le i\le \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-1,\\ 2n-2\lfloor\frac{n}{3}\rfloor- \\ -2(\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{6}\rfloor)-2, & i=\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor,\\ \lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor- \\ -4\lfloor\frac{k}{6}\rfloor+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1\le i\le n-1,\\ 2\lfloor\frac{n}{3}\rfloor-2\lfloor\frac{k}{6}\rfloor-2, & i=n. \end{cases}$$

$$f(x_ix_{i+2})=\begin{cases} \lfloor\frac{n}{3}\rfloor-1+i, & 1\le i\le \lfloor\frac{n}{3}\rfloor-2,\\ 2n-2\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor- \\ -2\lfloor\frac{k}{4}\rfloor+i, & \lfloor\frac{n}{3}\rfloor-1\le i\le \lfloor\frac{n}{3}\rfloor,\\ 2n-\lfloor\frac{n}{2}\rfloor-2\lfloor\frac{n}{3}\rfloor- \\ -4\lfloor\frac{k}{4}\rfloor-1+i, & \lfloor\frac{n}{3}\rfloor+1\le i\le \lfloor\frac{n}{2}\rfloor-2 \text{ and}\\ & \lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor\ge 2\ ,\\ 2n-2\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor- \\ -2(\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{4}\rfloor)+3+i, & \lfloor\frac{n}{2}\rfloor-1\le i\le \lfloor\frac{n}{2}\rfloor,\\ 2n-\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor- \\ -4\lfloor\frac{k}{2}\rfloor+2+i, & \lfloor\frac{n}{2}\rfloor+1\le i\le \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-2,\\ 2n-3\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor- \\ -2(\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{6}\rfloor)+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-1\le i\le \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor,\\ i-4\lfloor\frac{k}{6}\rfloor, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1\le i\le n-2 \text{ and}\\ & \lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor\ge 3, \text{ where } n \text{ is even},\\ 2\lfloor\frac{n}{3}\rfloor-2\lfloor\frac{n}{2}\rfloor- \\ -2\lfloor\frac{k}{6}\rfloor+i, & n-1\le i\le n. \end{cases}$$

Evidently, the vertices of $C_n(1,2)$ labeled with even numbers. Now we will compute the weights of edges under the labeling $f$:
The edge set of $C_n(1,2)$ can be split into the eight mutually disjoint subsets, $A_\ell$, $1\le \ell\le 8$ as follows:
For $1\le j\le 2$.

- $A_1=\{x_ix_{i+j} : 1\le i\le \lfloor\frac{n}{3}\rfloor-j\}$ the set of all edges which has end vertices labeled with 0.
- $A_2=\{x_ix_{i+j} : \lfloor\frac{n}{3}\rfloor-j+1\le i\le \lfloor\frac{n}{3}\rfloor\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor\frac{k}{4}\rfloor$.
- $A_3=\begin{cases} \{x_{\lfloor\frac{n}{3}\rfloor+1}x_{\lfloor\frac{n}{3}\rfloor+2}\}, & \text{if } \lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor=2.\\ \{x_ix_{i+j} : \lfloor\frac{n}{3}\rfloor+1\le i\le \lfloor\frac{n}{2}\rfloor-j,\}, & \text{if } \lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor\ge 3 \end{cases}$
  the set of all edges which has end vertices labeled with $2\lfloor\frac{k}{4}\rfloor$.

- $A_4 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + 1 - j \le i \le \lfloor \frac{n}{2} \rfloor\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{4} \rfloor$ and $2\lfloor \frac{k}{2} \rfloor$.
- $A_5 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - j\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$.
- $A_6 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - j + 1 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$ and $2\lfloor \frac{k}{6} \rfloor$.
- $A_7 = \begin{cases} \{x_{n-1}x_n\}, & \text{if } n \text{ is even and } \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2. \\ \\ \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \le i \le n - j\}, & \text{otherwise.} \end{cases}$
  the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{6} \rfloor$.
- $A_8 = \{x_i x_{i+j} : n - j + 1 \le i \le n\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{6} \rfloor$.

Hence, the edge weights under the labeling $f$ are the following:

1. The weights of the edges of the set $A_1$, admit the successive integers from the set $A_1 = \{1, 2, ..., 2\lfloor \frac{n}{3} \rfloor - 3\}$ .
2. The weights of the edges of the set $A_2$, admit the successive integers from the set $A_2 = \{2n - 2\lfloor \frac{n}{2} \rfloor - 2, 2n - 2\lfloor \frac{n}{2} \rfloor - 1, 2n - 2\lfloor \frac{n}{2} \rfloor\}$
3. The weight of the edge of the set $A_3$, admit the integer $\{2n - 2\lfloor \frac{n}{2} \rfloor + 1, ..., 2n - 2\lfloor \frac{n}{3} \rfloor - 3\}$,
4. The weights of the edges of the set $A_4$, admit the successive integers from the set $A_4 = \{2n - 2\lfloor \frac{n}{3} \rfloor + 1, 2n - 2\lfloor \frac{n}{3} \rfloor + 2, 2n - 2\lfloor \frac{n}{3} \rfloor + 3\}$
5. The weights of the edges of the set $A_5$, admit the successive integers from the set $A_5 = \{2n - 2\lfloor \frac{n}{3} \rfloor + 4, ..., 2n\}$
6. The weights of the edges of the set $A_6$, admit the successive integers from the set $A_6 = \{2n - 2\lfloor \frac{n}{3} \rfloor - 2, 2n - 2\lfloor \frac{n}{3} \rfloor - 1, 2n - 2\lfloor \frac{n}{3} \rfloor\}$
7. The weights of the edges of the set $A_7$, admit the successive integers from the set $A_7 = \{2\lfloor \frac{n}{3} \rfloor + 1, ..., 2n - 2\lfloor \frac{n}{2} \rfloor - 3\}$
8. The weights of the edges of the set $A_8$, admit the successive integers from the set $A_8 = \{2\lfloor \frac{n}{3} \rfloor - 2, 2\lfloor \frac{n}{3} \rfloor - 1, 2\lfloor \frac{n}{3} \rfloor\}$

This mean that for $n \ge 10$, the edge weights are from the set $W = \{1, 2, 3, ..., 2n\}$. Thus, we can easily see that all integers in $W$ are distinct. □

**Theorem 2.** *Let $C_n(1, 2, 3)$, be a circulant graph on $n \ge 6$ vertices. Then*

$$res(C_n(1,2,3)) = \begin{cases} n, & n \text{ is even,} \\ n+1, & n \text{ is odd.} \end{cases}$$

*Proof.* According to the fact that the circulant $C_n(1, 2, 3)$ has $3n$ edges if $n \ge 7$, then using the Lemma 1.1, we obtain the lower bound for the circulant $C_n(1, 2, 3)$ as following: $res(C_n(1, 2, 3)) \ge n$ for $n$ is even and $res(C_n(1, 2, 3)) \ge n + 1$ for $n$ is odd. To prove the equality, it suffices to prove the existence of an optimal total labeling of $C_n(1, 2, 3)$ such as:
The corresponding labelings for $C_n(1, 2, 3)$, $n = 6, 7, 8, 9, 10$, are shown in Figure 2. Suppose $k = n$ for

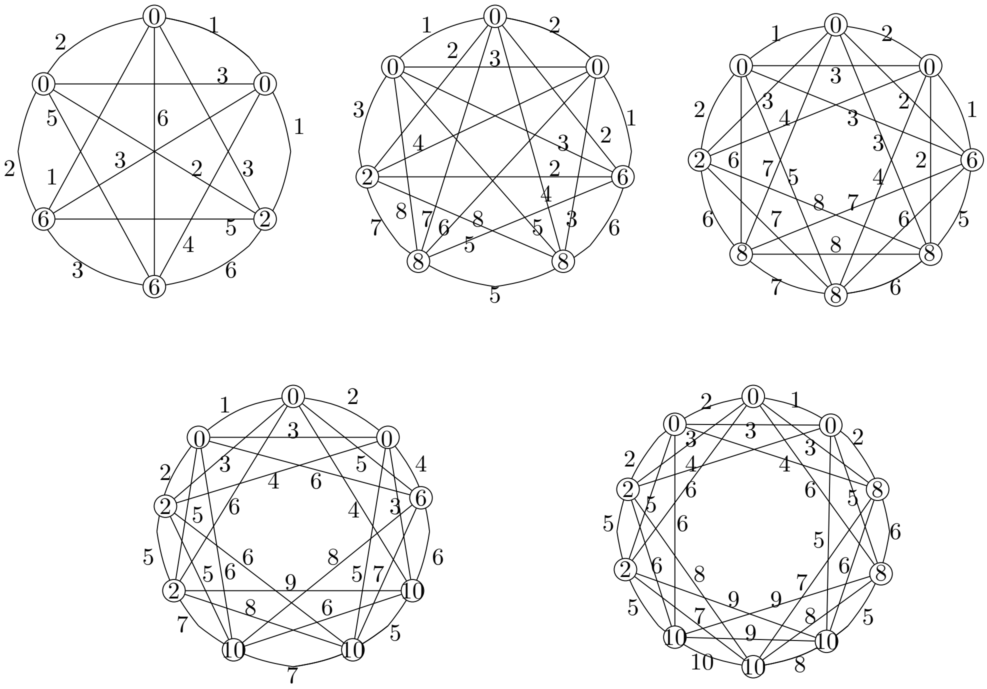

**Figure 2.** The reflexive edge irregular $k$-labeling of $C_n(1,2,3), n = 6,7,8,9,10$.

$n$ is even and $k = n+1$ for $n$ is odd. Hence, for $n \geq 11$ we have the following labelings:

$$f(x_i) = \begin{cases} 0, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{4} \rfloor + 2, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ k, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{8} \rfloor, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n. \end{cases}$$

**Case 1.** If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$.
Firstly, for $n$ is even, it is only realizable when $n = 12$, then the Figure 3, shows labelings of vertices and edges along with their weights.
Secondly, for $n$ is odd.

$$f(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1, \\ 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor + 2, & i = \lfloor \frac{n}{3} \rfloor, \\ 3\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{4} \rfloor + 6, & i = \lfloor \frac{n}{3} \rfloor + 1, \\ 3\lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{4} \rfloor + 15, & i = \lfloor \frac{n}{3} \rfloor + 2, \\ 2\lfloor \frac{n}{3} \rfloor - 2k + 19 + i, & \lfloor \frac{n}{3} \rfloor + 3 \leq i \leq 2\lfloor \frac{n}{3} \rfloor + 1, \\ 3\lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{8} \rfloor + 11, & i = 2\lfloor \frac{n}{3} \rfloor + 2, \\ \lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{8} \rfloor - 2 + i, & 2\lfloor \frac{n}{3} \rfloor + 3 \leq i \leq 2\lfloor \frac{n}{3} \rfloor + 4, \\ 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 5, & i = n. \end{cases}$$

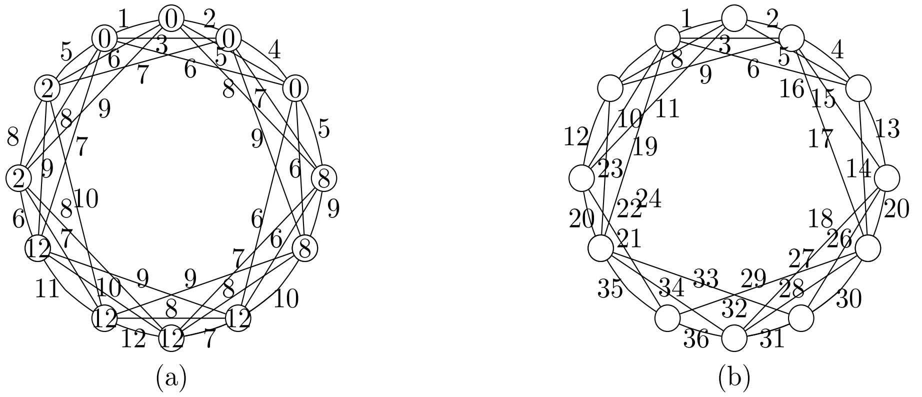


**Figure 3.** A reflexive irregular 12-labeling of $C_{12}(1,2,3)$ and its edge weights.

$$f(x_ix_{i+2}) = \begin{cases} \lfloor \frac{n}{3} \rfloor - 1 + i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 2, \\ 2\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor + 4 + i, & \lfloor \frac{n}{3} \rfloor - 1 \le i \le \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{4} \rfloor + & \\ +15 + i, & \lfloor \frac{n}{3} \rfloor + 1 \le i \le \lfloor \frac{n}{3} \rfloor + 2, \\ 3\lfloor \frac{n}{3} \rfloor - 2k + 18 + i, & \lfloor \frac{n}{3} \rfloor + 3 \le i \le 2\lfloor \frac{n}{3} \rfloor, \\ \lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{8} \rfloor + & \\ +11 + i, & 2\lfloor \frac{n}{3} \rfloor + 1 \le i \le 2\lfloor \frac{n}{3} \rfloor + 2, \\ 3\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{8} \rfloor + 3, & i = 2\lfloor \frac{n}{3} \rfloor + 3, \\ 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 10 + i, & 2\lfloor \frac{n}{3} \rfloor + 4 \le i \le n. \end{cases}$$

$$f(x_ix_{i+3}) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor - 3 + i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 3 \text{ and } \lfloor \frac{n}{3} \rfloor \ge 4\ , \\ 2\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor + 7 + i, & \lfloor \frac{n}{3} \rfloor - 2 \le i \le \lfloor \frac{n}{3} \rfloor - 1, \\ 3\lfloor \frac{n}{3} \rfloor - k + 9, & i = \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{4} \rfloor + & \\ +17 + i, & \lfloor \frac{n}{3} \rfloor + 1 \le i \le \lfloor \frac{n}{3} \rfloor + 2, \\ 4\lfloor \frac{n}{3} \rfloor - 2k + 16 + i, & \lfloor \frac{n}{3} \rfloor + 3 \le i \le 2\lfloor \frac{n}{3} \rfloor - 1, \\ \lfloor \frac{n}{3} \rfloor - k - 2\lfloor \frac{k}{8} \rfloor + & \\ +14 + i, & 2\lfloor \frac{n}{3} \rfloor \le i \le 2\lfloor \frac{n}{3} \rfloor + 2, \\ \lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 5 + i, & 2\lfloor \frac{n}{3} \rfloor + 3 \le i \le n, \end{cases}$$

**Case 2.** If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \ge 3$
For $1 \le i \le \lfloor \frac{n}{2} \rfloor$

$$f(x_ix_{i+1}) = \begin{cases} i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 1, \\ 3n - 3\lfloor \frac{n}{2} \rfloor - & \\ -2\lfloor \frac{k}{4} \rfloor - 7, & i = \lfloor \frac{n}{3} \rfloor, \\ 3n - 3\lfloor \frac{n}{2} \rfloor - & \\ -\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{4} \rfloor - 4 + i, & \lfloor \frac{n}{3} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1, \\ 3n - 3\lfloor \frac{n}{3} \rfloor - & \\ k - 2\lfloor \frac{k}{4} \rfloor - 1, & i = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

For $\lfloor\frac{n}{2}\rfloor+1\le i\le n$

$$f(x_ix_{i+1})=\begin{cases}3n-3\lfloor\frac{n}{3}\rfloor-\\ -\lfloor\frac{n}{2}\rfloor-2k+6+i, & \lfloor\frac{n}{2}\rfloor+1\le i\le\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-1,\\ 3n-3\lfloor\frac{n}{3}\rfloor-\\ -k-2\lfloor\frac{k}{8}\rfloor-5, & i=\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor,\\ 2\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -4\lfloor\frac{k}{8}\rfloor+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1\le i\le 2\lfloor\frac{n}{2}\rfloor,\\ 3\lfloor\frac{n}{3}\rfloor-2\lfloor\frac{k}{8}\rfloor-5, & i=n.\end{cases}$$

$$f(x_ix_{i+2})=\begin{cases}\lfloor\frac{n}{3}\rfloor-1+i, & 1\le i\le\lfloor\frac{n}{3}\rfloor-2,\\ 3n-3\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor-\\ -2\lfloor\frac{k}{4}\rfloor-5+i, & \lfloor\frac{n}{3}\rfloor-1\le i\le\lfloor\frac{n}{3}\rfloor,\\ 3n-2\lfloor\frac{n}{2}\rfloor-2\lfloor\frac{n}{3}\rfloor-\\ -4\lfloor\frac{k}{4}\rfloor-5+i, & \lfloor\frac{n}{3}\rfloor+1\le i\le\lfloor\frac{n}{2}\rfloor-2,\\ 3n-3\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -k-2\lfloor\frac{k}{4}\rfloor+1+i, & \lfloor\frac{n}{2}\rfloor-1\le i\le\lfloor\frac{n}{2}\rfloor,\\ 3n-2\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -2k+5+i, & \lfloor\frac{n}{2}\rfloor+1\le i\le\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-2,\\ 3n-4\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -k-2\lfloor\frac{k}{8}\rfloor-3+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-1\le i\le\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor,\\ \lfloor\frac{n}{3}\rfloor-4\lfloor\frac{k}{8}\rfloor+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1\le i\le 2\lfloor\frac{n}{2}\rfloor-1,\\ 3\lfloor\frac{n}{3}\rfloor-2\lfloor\frac{n}{2}\rfloor-\\ -2\lfloor\frac{k}{8}\rfloor-4+i, & 2\lfloor\frac{n}{2}\rfloor\le i\le n.\end{cases}$$

$$f(x_ix_{i+3})=\begin{cases}2\lfloor\frac{n}{3}\rfloor-3+i, & 1\le i\le\lfloor\frac{n}{3}\rfloor-3,\\ 3n-3\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor-\\ -2\lfloor\frac{k}{4}\rfloor-2+i, & \lfloor\frac{n}{3}\rfloor-2\le i\le\lfloor\frac{n}{3}\rfloor,\\ 3n-\lfloor\frac{n}{2}\rfloor-3\lfloor\frac{n}{3}\rfloor-\\ -4\lfloor\frac{k}{4}\rfloor-7+i, & \lfloor\frac{n}{3}\rfloor+1\le i\le\lfloor\frac{n}{2}\rfloor-3 \text{ and}\\ & \lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor\ge 4,\\ 3n-3\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -k-2\lfloor\frac{k}{4}\rfloor+4+i, & \lfloor\frac{n}{2}\rfloor-2\le i\le\lfloor\frac{n}{2}\rfloor,\\ 3n-\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -2k+3+i, & \lfloor\frac{n}{2}\rfloor+1\le i\le\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-3,\\ 3n-4\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor-\\ -k-2\lfloor\frac{k}{8}\rfloor+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor-2\le i\le\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor,\\ 2\lfloor\frac{n}{2}\rfloor-2\lfloor\frac{n}{3}\rfloor-\\ -4\lfloor\frac{k}{8}\rfloor-1+i, & \lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1\le i\le 2\lfloor\frac{n}{2}\rfloor-2,\\ n+3\lfloor\frac{n}{3}\rfloor-4\lfloor\frac{n}{2}\rfloor-\\ -2\lfloor\frac{k}{8}\rfloor-2+i, & 2\lfloor\frac{n}{2}\rfloor-1\le i\le n.\end{cases}$$

Obviously $f$ is $k$-labeling and the vertices are labeled with even numbers. For the edge weights of $x_ix_{i+j}$, $1\le i\le n, 1\le j\le 3$ in $C_n(1,2,3)$ under the labeling $f$ we have:

In Case 1, the edge set of $C_n(1,2,3)$ can be split into the nine mutually disjoint subsets, $A_\ell$, $1 \le \ell \le 9$ as follows:

1. $A_1 = \{x_i x_{i+j} : 1 \le i \le \lfloor \frac{n}{3} \rfloor - j, 1 \le j \le 3\}$ the set of all edges which has end vertices labeled with 0.
2. $A_2 = \{x_i x_{i+j} : \lfloor \frac{n}{3} \rfloor - j + 1 \le i \le \lfloor \frac{n}{3} \rfloor, 1 \le j \le 2\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{4} \rfloor + 2$.
3. $A_3 = \{x_{\lfloor \frac{n}{3} \rfloor + 1} x_{\lfloor \frac{n}{3} \rfloor + 2}\}$ the set of only one edge which has end vertices labeled with $2\lfloor \frac{k}{4} \rfloor + 2$,
4. $A_4 = \{x_i x_{i+j} : \lfloor \frac{n}{3} \rfloor + 3 - j \le i \le \lfloor \frac{n}{3} \rfloor + 2, 1 \le j \le 2\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{4} \rfloor + 2$ and $2\lfloor \frac{k}{2} \rfloor$.
5. $A_5 = \{x_i x_{i+j} : \lfloor \frac{n}{3} \rfloor + 3 \le i \le 2\lfloor \frac{n}{3} \rfloor, 1 \le j \le 2\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$.
6. $A_6 = \{x_i x_{i+j} : 2\lfloor \frac{n}{3} \rfloor + 3 - j \le i \le 2\lfloor \frac{n}{3} \rfloor + 2, 1 \le j \le 3\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$ and $2\lfloor \frac{k}{8} \rfloor$.
7. $A_7 = \{x_i x_{i+j} : 2\lfloor \frac{n}{3} \rfloor + 3 \le i \le 2\lfloor \frac{n}{3} \rfloor + 5 - j, 1 \le j \le 2\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{8} \rfloor$.
8. $A_8 = \{x_i x_{i+j} : 2\lfloor \frac{n}{3} \rfloor + 6 - j \le i \le 2\lfloor \frac{n}{3} \rfloor + 5, 1 \le j \le 3\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{8} \rfloor$.
9. $A_9 = \{x_{\lfloor \frac{n}{3} \rfloor} x_{\lfloor \frac{n}{3} \rfloor + 3}\}$ the set of only edge which has end vertices labeled with 0 and $2\lfloor \frac{k}{2} \rfloor$.

Thus, we obtain the edge weights as follows:

1. The weights of the edges of the set $A_1$, admit the successive integers from the set $A_1 = \{1, 2, ..., 3\lfloor \frac{n}{3} \rfloor - 6\}$ .
2. The weights of the edges of the set $A_2$, admit the successive integers from the set $A_2 = \{3\lfloor \frac{n}{3} \rfloor + 4, ..., 3\lfloor \frac{n}{3} \rfloor + 8\}$
3. The weight of the edge of the set $A_3$, admits the integer $\{3\lfloor \frac{n}{3} \rfloor + 10\}$,
4. The weights of the edges of the set $A_4$, admit the successive integers from the set $A_4 = \{3\lfloor \frac{n}{3} \rfloor + 17, ..., 2\lfloor \frac{n}{3} \rfloor + 21\}$
5. The weights of the edges of the set $A_5$, admit the successive integers from the set $A_5 = \{3\lfloor \frac{n}{3} \rfloor + 22, ..., 6\lfloor \frac{n}{3} \rfloor + 15\}$
6. The weights of the edges of the set $A_6$, admit the successive integers from the set $A_6 = \{3\lfloor \frac{n}{3} \rfloor + 11, ..., 3\lfloor \frac{n}{3} \rfloor + 16\}$
7. The weights of the edges of the set $A_7$, admit the successive three integers from the set $A_7 = \{3\lfloor \frac{n}{3} \rfloor + 1, 3\lfloor \frac{n}{3} \rfloor + 2, 2\lfloor \frac{n}{3} \rfloor + 3\}$
8. The weights of the edges of the set $A_8$, admit the successive integers from the set $A_8 = \{3\lfloor \frac{n}{3} \rfloor - 5, ..., 3\lfloor \frac{n}{3} \rfloor\}$
9. The weight of the edge of the set $A_9$, admit the integer $\{3\lfloor \frac{n}{3} \rfloor + 9\}$.

In Case 2, the set of edges can be divided into eight mutually disjoint subsets as in the proof of Theorem 1.1. Therefore, the edge weights under the labeling $f$ are the following:

1. The weights of the edges of the set $A_1$, admit the successive integers from the set $A_1 = \{1, 2, ..., 3\lfloor \frac{n}{3} \rfloor - 6\}$ .
2. The weights of the edges of the set $A_2$, admit the successive integers from the set $A_2 = \{3n - 3\lfloor \frac{n}{2} \rfloor - 5, ..., 3n - 3\lfloor \frac{n}{2} \rfloor\}$

3. The weights of the edges of the set $A_3$, admit the successive integers from the set $\{3n - 3\lfloor \frac{n}{2} \rfloor + 1, ..., 3n - 3\lfloor \frac{n}{3} \rfloor - 6\}$,
4. The weights of the edges of the set $A_4$, admit the successive integers from the set $A_4 = \{3n - 3\lfloor \frac{n}{3} \rfloor + 1, ..., 3n - 3\lfloor \frac{n}{3} \rfloor + 6\}$
5. The weights of the edges of the set $A_5$, admit the successive integers from the set $A_5 = \{3n - 3\lfloor \frac{n}{3} \rfloor + 7, ..., 3n\}$
6. The weights of the edges of the set $A_6$, admit the successive integers from the set $A_6 = \{3n - 3\lfloor \frac{n}{3} \rfloor - 5, ..., 3n - 3\lfloor \frac{n}{3} \rfloor\}$
7. The weights of the edges of the set $A_7$, admit the successive integers from the set $A_7 = \{3\lfloor \frac{n}{3} \rfloor + 1, ..., 3n - 3\lfloor \frac{n}{2} \rfloor - 6\}$
8. The weights of the edges of the set $A_8$, admit the successive integers from the set $A_8 = \{3\lfloor \frac{n}{3} \rfloor - 5, ..., 3\lfloor \frac{n}{3} \rfloor\}$

Moreover, for $n$ is odd and $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$ the edge weights are from the set $W = \{1, 2, 3, ..., 6\lfloor \frac{n}{3} \rfloor + 15\}$ and for $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \geq 3$, the edge weights are from the set $W = \{1, 2, 3, ..., 3n\}$. It is not difficult to see that all numbers in set $W$ are different. □

**Theorem 3.** *Let $C_n(1, 2, 3, 4, 5)$, be a circulant graph on $n \geq 15$ vertices. Then*

$$res(C_n(1, 2, 3, 4, 5)) = \begin{cases} \lceil \frac{5n}{3} \rceil, & n \equiv 0, 1, 2, 5 \ (mod\ 6), \\ \lceil \frac{5n}{3} \rceil + 1, & n \equiv 3, 4 \ (mod\ 6)\ . \end{cases}$$

*Proof.* The corresponding labelings for $C_n(1, 2, 3, 4, 5)$, $n = 10, 11, 12, 13, 14$, are shown in Figure 4. From Lemma 1.1, we get the following lower bound for the circulant $C_n(1, 2, 3, 4, 5)$ : $res(C_n(1, 2, 3, 4, 5)) \geq k = \lceil \frac{5n}{3} \rceil$ if $n \equiv 0, 1, 2, 5 \pmod 6$ and $res(C_n(1, 2, 3, 4, 5)) \geq k = \lceil \frac{5n}{3} \rceil + 1$ if $n \equiv 3, 4 \pmod 6$. Now, we will prove that:

$$res(C_n(1, 2, 3, 4, 5)) \leq \begin{cases} \lceil \frac{5n}{3} \rceil, & n \equiv 0, 1, 2, 5 \pmod 6, \\ \lceil \frac{5n}{3} \rceil + 1, & n \equiv 3, 4 \pmod 6\ . \end{cases}$$

For this we construct the $f$-labeling on $C_n(1, 2, 3, 4, 5)$ as follows:
For $n \geq 15$ we have the following labelings:

$$f(x_i) = \begin{cases} 0, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{4} \rfloor + 4, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ 2\lfloor \frac{k}{2} \rfloor, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 2\lfloor \frac{k}{8} \rfloor, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n. \end{cases}$$

Moreover, for the labeling of the edges $x_i x_{i+1}$, $1 \leq i \leq n$ we distinguish two cases.
**Case 1.** If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \leq 4$.
Firstly, for $n$ is odd.

$$f(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1, \\ 5\lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{k}{4} \rfloor - 13, & i = \lfloor \frac{n}{3} \rfloor, \\ 4\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 1) - & \\ -4\lfloor \frac{k}{4} \rfloor + 7 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, \\ 5\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 9) - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 17, & i = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

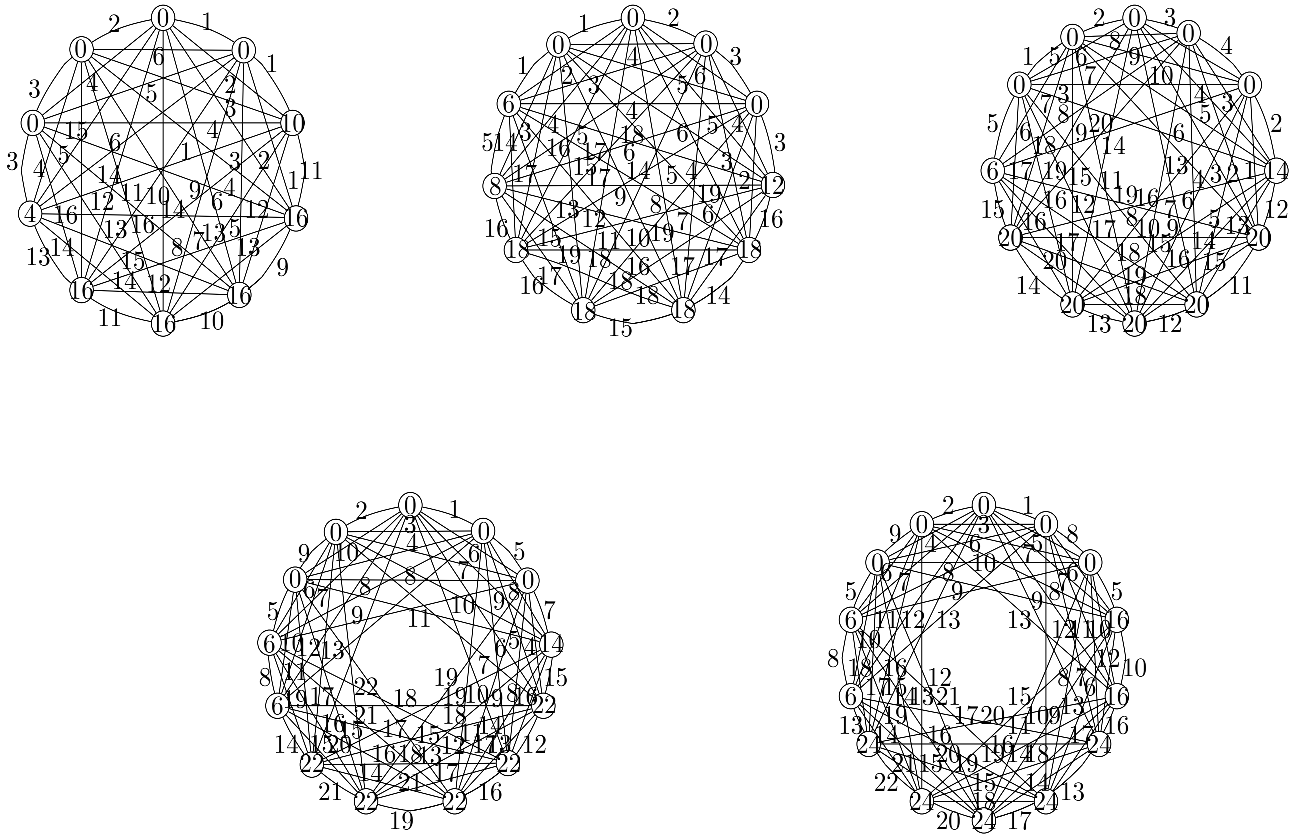

**Figure 4.** The reflexive edge irregular $k$-labeling of $C_n(1,2,3,4,5), n = 10, 11, 12, 13, 14$.

For $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n$.

$$f(x_i x_{i+1}) = \begin{cases} 9\lfloor \frac{n}{2} \rfloor - 5\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + 20 + i, & \lfloor \frac{n}{2} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 1, \\ 5\lfloor \frac{n}{3} \rfloor + (\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor)^2 - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) + 16, & i = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 3\lfloor \frac{n}{3} \rfloor - \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 7) - & \\ -4\lfloor \frac{k}{8} \rfloor - 10 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \le i \le 2\lfloor \frac{n}{2} \rfloor, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 14, & i = n. \end{cases}$$

$$f(x_i x_{i+2}) = \begin{cases} \lfloor \frac{n}{3} \rfloor - 1 + i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 2, \\ 5\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor - 11 + i, & \lfloor \frac{n}{3} \rfloor - 1 \le i \le \lfloor \frac{n}{3} \rfloor. \end{cases}$$

For $\lfloor \frac{n}{3} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor$.
If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$.

$$f(x_i x_{i+2}) = 4\lfloor \frac{n}{3} \rfloor - 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 28 + i.$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \geq 3$.

$$f(x_ix_{i+2}) = \begin{cases} 4\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 3) - \\ -4\lfloor \frac{k}{4} \rfloor + 6 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2, \\ 4\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 7) - \\ -2(\lfloor \frac{k}{4} \rfloor + \lfloor \frac{k}{2} \rfloor) + 19 + i, & \lfloor \frac{n}{2} \rfloor - 1 \leq i \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

For $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n$.

$$f(x_ix_{i+2}) = \begin{cases} 9\lfloor \frac{n}{2} \rfloor - 4\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + 19 + i, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 2, \\ 4\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor)^2 - \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) + 18 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 3\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(9 - \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor) - \\ -4\lfloor \frac{k}{8} \rfloor - 10 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq 2\lfloor \frac{n}{2} \rfloor - 1, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 13 + i, & 2\lfloor \frac{n}{2} \rfloor \leq i \leq n. \end{cases}$$

$$f(x_ix_{i+3}) = 2\lfloor \frac{n}{3} \rfloor - 3 + i, \qquad 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 3.$$

For $\lfloor \frac{n}{3} \rfloor - 2 \leq i \leq \lfloor \frac{n}{3} \rfloor$.
If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$.

$$f(x_ix_{i+3}) = \begin{cases} 4\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor + 2 + i, & \lfloor \frac{n}{3} \rfloor - 2 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{2} \rfloor + 10, & i = \lfloor \frac{n}{3} \rfloor. \end{cases}$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \geq 3$.

$$f(x_ix_{i+3}) = 5\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor - 8 + i.$$

For $\lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.
If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$.

$$f(x_ix_{i+3}) = 4\lfloor \frac{n}{3} \rfloor - 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 30 + i.$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 3$

$$f(x_ix_{i+3}) = 4\lfloor \frac{n}{3} \rfloor - 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 37 + i.$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 4$

$$f(x_ix_{i+3}) = \begin{cases} 4\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{4} \rfloor + 22 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 3, \\ 4\lfloor \frac{n}{3} \rfloor - 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 44 + i, & \lfloor \frac{n}{2} \rfloor - 2 \leq i \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

For $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq 2\lfloor \frac{n}{2} \rfloor + 1$.
If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2$

$$f(x_ix_{i+3}) = 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 14 + i.$$

If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor \geq 3$.

$$f(x_i x_{i+3}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor + \frac{(\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor)}{2}(11 - \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor)- & \\ -4\lfloor\frac{k}{8}\rfloor - 11 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq 2\lfloor\frac{n}{2}\rfloor - 2, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 10 + i, & 2\lfloor\frac{n}{2}\rfloor - 1 \leq i \leq n. \end{cases}$$

$$f(x_i x_{i+4}) = 3\lfloor\frac{n}{3}\rfloor - 6 + i, \qquad 1 \leq i \leq \lfloor\frac{n}{3}\rfloor - 4.$$

For $\lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{3}\rfloor$.
If $\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 2$.

$$f(x_i x_{i+4}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 5 + i, & \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{3}\rfloor - 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 12 + i, & \lfloor\frac{n}{3}\rfloor - 1 \leq i \leq \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 3$.

$$f(x_i x_{i+4}) = \begin{cases} 4\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{k}{4}\rfloor - 1 + i, & \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{3}\rfloor - 1, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 18, & i = \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 4$.

$$f(x_i x_{i+4}) = 4\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{k}{4}\rfloor + i.$$

For $\lfloor\frac{n}{3}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor$.

$$f(x_i x_{i+4}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 32 + i, & \text{if } \lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 40 + i, & \text{if } \lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 3, \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 48 + i, & \text{if } \lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 4. \end{cases}$$

For $\lfloor\frac{n}{2}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor$.

$$f(x_i x_{i+4}) = 9\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 14 + i, \qquad \lfloor\frac{n}{2}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 4.$$

If $\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor = 2$.

$$f(x_i x_{i+4}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) + 27 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 13, & i = \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor \geq 3$.

$$\begin{aligned} f(x_i x_{i+4}) = {} & 4\lfloor\frac{n}{3}\rfloor - \lfloor\frac{n}{2}\rfloor + (\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor)^2- \\ & - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) + 25 + i, \qquad \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor. \end{aligned}$$

For $\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \le i \le n$.
If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 2$.

$$f(x_i x_{i+4}) = 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 11 + i.$$

If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_i x_{i+4}) = 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 12 + i.$$

If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_i x_{i+4}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{8}\rfloor + 5 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \le i \le 2\lfloor\frac{n}{2}\rfloor - 3, \\ 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 14 + i, & 2\lfloor\frac{n}{2}\rfloor - 2 \le i \le n. \end{cases}$$

For $1 \le i \le \lfloor\frac{n}{3}\rfloor$.
If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 2$.

$$f(x_i x_{i+5}) = \begin{cases} 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 3 + i, & 1 \le i \le 2, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 11 + i, & 3 \le i \le \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_i x_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 10 + i, & 1 \le i \le \lfloor\frac{n}{3}\rfloor - 5, \text{ and } \lfloor\frac{n}{3}\rfloor \ge 6, \\ 4\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 3 + i, & \lfloor\frac{n}{3}\rfloor - 4 \le i \le \lfloor\frac{n}{3}\rfloor - 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 20 + i, & \lfloor\frac{n}{3}\rfloor - 1 \le i \le \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_i x_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 10 + i, & 1 \le i \le \lfloor\frac{n}{3}\rfloor - 5, \\ 4\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 5 + i, & \lfloor\frac{n}{3}\rfloor - 4 \le i \le \lfloor\frac{n}{3}\rfloor - 1, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 25, & i = \lfloor\frac{n}{3}\rfloor. \end{cases}$$

For $\lfloor\frac{n}{3}\rfloor + 1 \le i \le \lfloor\frac{n}{2}\rfloor$.

$$f(x_i x_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 34 + i, & \text{if } \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 43 + i, & \text{if } \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 3, \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 52 + i, & \text{if } \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 4. \end{cases}$$

For $\lfloor\frac{n}{2}\rfloor + 1 \le i \le \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor$.
If $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor = 2$

$$f(x_i x_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) + 26 + i, & \lfloor\frac{n}{2}\rfloor + 1 \le i \le \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 11 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 1 \le i \le \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 3$

$$f(x_i x_{i+5}) = \begin{cases} 8\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + & \\ +37 + i, & \lfloor \frac{n}{2} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 5 \text{ and } \lfloor \frac{n}{3} \rfloor \ge 6, \\ 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{2} \rfloor - & \\ -2\lfloor \frac{k}{8} \rfloor + 36 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 4 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 1, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{2} \rfloor + 21, & i = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor. \end{cases}$$

If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 4$

$$f(x_i x_{i+5}) = \begin{cases} 8\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + 46 + i, & \lfloor \frac{n}{2} \rfloor + 1 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 5, \\ 3\lfloor \frac{n}{3} \rfloor - 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) + 42 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 4 \le i \le \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor. \end{cases}$$

For $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \le i \le n$.

$$f(x_i x_{i+5}) = \begin{cases} 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 8 + i, & \text{if } \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 2, 3, \\ 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 9 + i, & \text{if } \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor = 4. \end{cases}$$

Secondly, for $n$ is even.

$$f(x_i x_{i+1}) = \begin{cases} i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 1, \\ \frac{5n}{2} - 2\lfloor \frac{k}{4} \rfloor - 18, & i = \lfloor \frac{n}{3} \rfloor, \\ 4\lfloor \frac{n}{3} \rfloor - \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(1 - \frac{n}{2} + \lfloor \frac{n}{3} \rfloor) - & \\ -4\lfloor \frac{k}{4} \rfloor + 7 + i, & \lfloor \frac{n}{3} \rfloor + 1 \le i \le \frac{n}{2} - 1, \\ 5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor + 9) - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 12, & i = \frac{n}{2}, \\ \frac{9n}{2} - 5\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + 15 + i, & \frac{n}{2} + 1 \le i \le \frac{n}{2} + \lfloor \frac{n}{3} \rfloor - 1, \\ 6\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + (\frac{n}{2} - \lfloor \frac{n}{3} \rfloor)^2 - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) + 16, & i = \frac{n}{2} + \lfloor \frac{n}{3} \rfloor, \\ 3\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(9 - \frac{n}{2} + \lfloor \frac{n}{3} \rfloor) - & \\ -4\lfloor \frac{k}{8} \rfloor - 15 + i, & \frac{n}{2} + \lfloor \frac{n}{3} \rfloor + 1 \le i \le n - 1, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 14, & i = n. \end{cases}$$

$$f(x_i x_{i+2}) = \begin{cases} \lfloor \frac{n}{3} \rfloor - 1 + i, & 1 \le i \le \lfloor \frac{n}{3} \rfloor - 2, \\ \frac{5n}{2} - \lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{4} \rfloor - 16 + i, & \lfloor \frac{n}{3} \rfloor - 1 \le i \le \lfloor \frac{n}{3} \rfloor, \\ 4\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor + 1) - & \\ -4\lfloor \frac{k}{4} \rfloor + 6 + i, & \lfloor \frac{n}{3} \rfloor + 1 \le i \le \frac{n}{2} - 2, \\ 4\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor + 7) - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 14 + i, & \frac{n}{2} - 1 \le i \le \frac{n}{2}, \\ \frac{9n}{2} - 4\lfloor \frac{n}{3} \rfloor - 4\lfloor \frac{k}{2} \rfloor + 14 + i, & \frac{n}{2} + 1 \le i \le \frac{n}{2} + \lfloor \frac{n}{3} \rfloor - 2, \\ 5\lfloor \frac{n}{3} \rfloor - n + (\frac{n}{2} - \lfloor \frac{n}{3} \rfloor)^2 - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) + 18 + i, & \frac{n}{2} + \lfloor \frac{n}{3} \rfloor - 1 \le i \le \frac{n}{2} + \lfloor \frac{n}{3} \rfloor, \\ 3\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(11 - \frac{n}{2} + \lfloor \frac{n}{3} \rfloor) - & \\ -4\lfloor \frac{k}{8} \rfloor - 16 + i, & \frac{n}{2} + \lfloor \frac{n}{3} \rfloor + 1 \le i \le n - 2, \\ 5\lfloor \frac{n}{3} \rfloor - n - 2\lfloor \frac{k}{8} \rfloor - 11 + i, & n - 2 \le i \le n. \end{cases}$$

For $1 \le i \le \lfloor\frac{n}{3}\rfloor$.

$$f(x_ix_{i+3}) = \begin{cases} 2\lfloor\frac{n}{3}\rfloor - 3 + i, & 1 \le i \le \lfloor\frac{n}{3}\rfloor - 3, \\ \frac{5n}{2} - \lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor - 13 + i, & \lfloor\frac{n}{3}\rfloor - 2 \le i \le \lfloor\frac{n}{3}\rfloor. \end{cases}$$

For $\lfloor\frac{n}{3}\rfloor + 1 \le i \le \frac{n}{2}$.
If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+3}) = 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{4}\rfloor + \lfloor\frac{k}{2}\rfloor) + 32 + i.$$

If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+3}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{4}\rfloor + 18 + i, & \lfloor\frac{n}{3}\rfloor + 1 \le i \le \frac{n}{2} - 3 \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{4}\rfloor + \lfloor\frac{k}{2}\rfloor) + 39 + i, & \lfloor\frac{n}{2}\rfloor - 2 \le i \le \frac{n}{2}\ . \end{cases}$$

For $\frac{n}{2} + 1 \le i \le \frac{n}{2} + \lfloor\frac{n}{3}\rfloor$.

$$f(x_ix_{i+3}) = \begin{cases} \frac{9n}{2} - 3\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 12 + i, & \frac{n}{2} + 1 \le i \le \frac{n}{2} + \lfloor\frac{n}{3}\rfloor - 3 \\ 5\lfloor\frac{n}{3}\rfloor - n + (\frac{n}{2} - \lfloor\frac{n}{3}\rfloor)^2 - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) + 21 + i, & \frac{n}{2} + \lfloor\frac{n}{3}\rfloor - 2 \le i \le \frac{n}{2} + \lfloor\frac{n}{3}\rfloor\ . \end{cases}$$

For $\frac{n}{2} + \lfloor\frac{n}{3}\rfloor + 1 \le i \le n$.
If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+3}) = 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 15 + i.$$

If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+3}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{8}\rfloor + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \le i \le n - 3 \\ 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 17 + i, & n - 2 \le i \le n\ . \end{cases}$$

For $1 \le i \le \lfloor\frac{n}{3}\rfloor$.

$$f(x_ix_{i+4}) = 3\lfloor\frac{n}{3}\rfloor - 6 + i, \qquad 1 \le i \le \lfloor\frac{n}{3}\rfloor - 4.$$

If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+4}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 6 + i, & \lfloor\frac{n}{3}\rfloor - 3 \le i \le \lfloor\frac{n}{3}\rfloor - 1 \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 13, & i = \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+4}) = 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 11 + i, \qquad \lfloor\frac{n}{3}\rfloor - 3 \le i \le \lfloor\frac{n}{3}\rfloor.$$

For $\lfloor\frac{n}{3}\rfloor + 1 \le i \le \frac{n}{2} + \lfloor\frac{n}{3}\rfloor$.

$$f(x_ix_{i+4}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 35 + i, & \text{if } \frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 3 \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 43 + i, & \text{if } \frac{n}{2} - \lfloor\frac{n}{3}\rfloor = 4. \end{cases}$$

For $\frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor$.
If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+4}) = \begin{cases} 7\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 36 + i, & \frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 4 \\ 3\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor)+ & \\ +28 + i, & \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 3 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 1 \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 14, & i = \frac{n}{2}+\lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+4}) = \begin{cases} 7\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 45 + i, & \frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 4 \\ 3\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor)+ & \\ +33 + i, & \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 3 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor. \end{cases}$$

For $\frac{n}{2}+\lfloor\frac{n}{3}\rfloor + 1 \le i \le n$.

$$f(x_ix_{i+4}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 12 + i, & \text{if } \frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 3 \\ 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{8}\rfloor - 13 + i, & \text{if } \frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 4. \end{cases}$$

For $1 \le i \le \lfloor\frac{n}{3}\rfloor$.
If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 10 + i, & 1 \le i \le \lfloor\frac{n}{3}\rfloor - 5 \text{ and } \lfloor\frac{n}{3}\rfloor \ge 6, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 10 + i, & \lfloor\frac{n}{3}\rfloor - 4 \le i \le \lfloor\frac{n}{3}\rfloor - 2, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 16 + i, & \lfloor\frac{n}{3}\rfloor - 1 \le i \le \lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 10 + i, & 1 \le i \le \lfloor\frac{n}{3}\rfloor - 5, \\ 4\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{4}\rfloor + 16 + i, & \lfloor\frac{n}{3}\rfloor - 4 \le i \le \lfloor\frac{n}{3}\rfloor - 1, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 20, & i = \lfloor\frac{n}{3}\rfloor. \end{cases}$$

For $\lfloor\frac{n}{3}\rfloor + 1 \le i \le \frac{n}{2}$.

$$f(x_ix_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 38 + i, & \text{if } \frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 3 \\ 4\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 47 + i, & \text{if } \frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 4. \end{cases}$$

For $\frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor$.
If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 3$.

$$f(x_ix_{i+5}) = \begin{cases} 8\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 32 + i, & \frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 5 \text{ and } \lfloor\frac{n}{3}\rfloor \ge 6, \\ 3\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor)+ & \\ +32 + i, & \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 4 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 2, \\ 3\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 15 + i, & \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor. \end{cases}$$

If $\frac{n}{2}-\lfloor\frac{n}{3}\rfloor = 4$.

$$f(x_ix_{i+5}) = \begin{cases} 8\lfloor\frac{n}{3}\rfloor - 4\lfloor\frac{k}{2}\rfloor + 41 + i, & \frac{n}{2}+1 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 5, \\ 3\lfloor\frac{n}{3}\rfloor - 2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor)+ & \\ +38 + i, & \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 4 \le i \le \frac{n}{2}+\lfloor\frac{n}{3}\rfloor - 1, \\ 5\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{k}{2}\rfloor + 21, & i = \frac{n}{2}+\lfloor\frac{n}{3}\rfloor. \end{cases}$$

For $\frac{n}{2} + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n$.

$$f(x_i x_{i+5}) = 3\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 9 + i.$$

**Case 2.** If $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \geq 5$.

$$f(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1, \\ 5n - 5\lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{k}{4} \rfloor - 18, & i = \lfloor \frac{n}{3} \rfloor, \\ 5n - 5\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - & \\ -4\lfloor \frac{k}{4} \rfloor - 8 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, \\ 5n - 5\lfloor \frac{n}{3} \rfloor - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) - 3, & i = \lfloor \frac{n}{2} \rfloor, \\ 5n - 5\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - & \\ -4\lfloor \frac{k}{2} \rfloor + 15 + i, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 1, \\ 5n - 5\lfloor \frac{n}{3} \rfloor - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) - 14, & i = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ 4\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - & \\ -4\lfloor \frac{k}{8} \rfloor + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n - 1, \\ 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - 14, & i = n. \end{cases}$$

$$f(x_i x_{i+2}) = \begin{cases} \lfloor \frac{n}{3} \rfloor - 1 + i, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 2, \\ 5n - 5\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - & \\ -2\lfloor \frac{k}{4} \rfloor - 16 + i, & \lfloor \frac{n}{3} \rfloor - 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, \\ 5n - 4\lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{n}{3} \rfloor - & \\ -4\lfloor \frac{k}{4} \rfloor - 9 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2, \\ 5n - \lfloor \frac{n}{2} \rfloor - 5\lfloor \frac{n}{3} \rfloor - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) - 1 + i, & \lfloor \frac{n}{2} \rfloor - 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ 5n - \lfloor \frac{n}{2} \rfloor - 4\lfloor \frac{n}{3} \rfloor - & \\ -4\lfloor \frac{k}{2} \rfloor + 14 + i, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 2, \\ 5n - \lfloor \frac{n}{2} \rfloor - 6\lfloor \frac{n}{3} \rfloor - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{8} \rfloor) - 12 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, \\ n - 2\lfloor \frac{n}{3} \rfloor + 3\lfloor \frac{n}{3} \rfloor - & \\ -4\lfloor \frac{k}{8} \rfloor - 1 + i, & \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n - 2, \\ -n + 5\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{k}{8} \rfloor - & \\ -12 + i, & n - 1 \leq i \leq n. \end{cases}$$

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$

$$f(x_i x_{i+3}) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor - 3 + i, & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - 3, \\ 5n - 5\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - & \\ -2\lfloor \frac{k}{4} \rfloor - 13 + i, & \lfloor \frac{n}{3} \rfloor - 2 \leq i \leq \lfloor \frac{n}{3} \rfloor, \\ 5n - 3\lfloor \frac{n}{2} \rfloor - 3\lfloor \frac{n}{3} \rfloor - & \\ -4\lfloor \frac{k}{4} \rfloor - 11 + i, & \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 3, \\ 5n - \lfloor \frac{n}{2} \rfloor - 5\lfloor \frac{n}{3} \rfloor - & \\ -2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor) + 2 + i, & \lfloor \frac{n}{2} \rfloor - 2 \leq i \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

For $\lfloor\frac{n}{2}\rfloor + 1 \leq i \leq n$

$$f(x_ix_{i+3}) = \begin{cases} 5n - \lfloor\frac{n}{2}\rfloor - 3\lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{2}\rfloor + 15 + i, & \lfloor\frac{n}{2}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 3, \\ 5n - \lfloor\frac{n}{2}\rfloor - 6\lfloor\frac{n}{3}\rfloor - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) - 9 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 2 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor, \\ 2n - 3\lfloor\frac{n}{2}\rfloor + 2\lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{8}\rfloor - 3 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq n - 3, \\ -n + 5\lfloor\frac{n}{3}\rfloor - & \\ -2\lfloor\frac{k}{8}\rfloor - 9 + i, & n - 2 \leq i \leq n. \end{cases}$$

$$f(x_ix_{i+4}) = \begin{cases} 3\lfloor\frac{n}{3}\rfloor - 6 + i, & 1 \leq i \leq \lfloor\frac{n}{3}\rfloor - 4, \\ 5n - 5\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - & \\ -2\lfloor\frac{k}{4}\rfloor - 9 + i, & \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{3}\rfloor, \\ 5n - 2\lfloor\frac{n}{2}\rfloor - 4\lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{4}\rfloor - 14 + i, & \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor - 4, \\ 5n - \lfloor\frac{n}{2}\rfloor - 5\lfloor\frac{n}{3}\rfloor - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 6 + i, & \lfloor\frac{n}{2}\rfloor - 3 \leq i \leq \lfloor\frac{n}{2}\rfloor, \\ 5n - \lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{2}\rfloor + 9 + i, & \lfloor\frac{n}{2}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 4, \\ 5n - \lfloor\frac{n}{2}\rfloor - 6\lfloor\frac{n}{3}\rfloor - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) - 5 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 3 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor, \\ 3n - 4\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{8}\rfloor - 6 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq n - 4, \\ -n + 5\lfloor\frac{n}{3}\rfloor - & \\ -2\lfloor\frac{k}{8}\rfloor - 5 + i, & n - 3 \leq i \leq n. \end{cases}$$

$$f(x_ix_{i+5}) = \begin{cases} 4\lfloor\frac{n}{3}\rfloor - 10 + i, & 1 \leq i \leq \lfloor\frac{n}{3}\rfloor - 5, \\ 5n - 5\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - & \\ -2\lfloor\frac{k}{4}\rfloor - 4 + i, & \lfloor\frac{n}{3}\rfloor - 4 \leq i \leq \lfloor\frac{n}{3}\rfloor, \\ 5n - \lfloor\frac{n}{2}\rfloor - 5\lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{4}\rfloor - 18 + i, & \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor - 5 \text{ and } \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor \geq 6, \\ 5n - 6\lfloor\frac{n}{3}\rfloor - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{4}\rfloor) + 6 + i, & \lfloor\frac{n}{2}\rfloor - 4 \leq i \leq \lfloor\frac{n}{2}\rfloor, \\ 5n - \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - & \\ -4\lfloor\frac{k}{2}\rfloor + 5 + i, & \lfloor\frac{n}{2}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 5, \\ 5n - \lfloor\frac{n}{2}\rfloor - 6\lfloor\frac{n}{3}\rfloor - & \\ -2(\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{8}\rfloor) + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor - 4 \leq i \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor, \\ 4n - 5\lfloor\frac{n}{2}\rfloor - & \\ -4\lfloor\frac{k}{8}\rfloor - 10 + i, & \lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor + 1 \leq i \leq n - 5 \text{ and} \\ & \lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor \geq 6, \text{ where } n \text{ is even}, \\ -n + 5\lfloor\frac{n}{3}\rfloor - & \\ -2\lfloor\frac{k}{8}\rfloor + i, & n - 4 \leq i \leq n. \end{cases}$$

Hence, we can see that $f$ is $k$-labeling and all vertices are labeled with even numbers.
Furthermore, we will estimate the weights of edges under the labeling $f$ as follows:
In Case 1, the edge set of $C_n(1,2,3,4,5), n \geq 15$ can be split into nine mutually disjoint subsets, $A_\ell, 1 \leq \ell \leq 9$ as follows:

- $A_1 = \begin{cases} \{x_i x_{i+j} : 1 \leq i \leq 5 - j, 1 \leq j \leq 4\}, & \text{if } \lfloor \frac{n}{3} \rfloor = 5. \\ \{x_i x_{i+j} : 1 \leq i \leq \lfloor \frac{n}{3} \rfloor - j, 1 \leq j \leq 5\}, & \text{if } \lfloor \frac{n}{3} \rfloor \geq 6 \end{cases}$
  the set of all edges which has end vertices labeled with 0.
- $A_2 = \{x_i x_{i+j} : \lfloor \frac{n}{3} \rfloor - j + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{4} \rfloor + 2$,
- $A_3 = \{x_i x_{i+j} : \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - j, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 1\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{4} \rfloor + 2$,
- $A_4 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + 1 - j \leq i \leq \lfloor \frac{n}{2} \rfloor, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{4} \rfloor + 2$ and $2\lfloor \frac{k}{2} \rfloor$.
- $A_5 = \begin{cases} \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 5 - j, 1 \leq j \leq 4\}, & \text{if } \lfloor \frac{n}{3} \rfloor = 5. \\ \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - j, 1 \leq j \leq 5\}, & \text{if } \lfloor \frac{n}{3} \rfloor \geq 6 \end{cases}$
  the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$.
- $A_6 = \{x_i x_{i+j} : n - j \leq i \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor, 1 \leq j \leq n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{2} \rfloor$ and $2\lfloor \frac{k}{8} \rfloor$.
- $A_7 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n - j, 1 \leq j \leq n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 1\}$ the set of all edges which has end vertices labeled with $2\lfloor \frac{k}{8} \rfloor$.
- $A_8 = \{x_i x_{i+j} : n - j + 2 \leq i \leq n, 1 \leq j \leq n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{8} \rfloor$.
- $A_9 = \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor - j + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 1 \leq j \leq 5\} \cup \{x_i x_{i+j} : \lfloor \frac{n}{2} \rfloor - j + 2 \leq i \leq n, n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 1 \rfloor \leq j \leq 5\}$ the set of all edges which has end vertices labeled with 0 and $2\lfloor \frac{k}{2} \rfloor$.

Thus, for $n$ is odd the edge weights under the labeling $f$ are the following:

1. The weights of the edges of the set $A_1$, receive the consecutive integers from the set $A_1 = \{1, 2, ..., 5\lfloor \frac{n}{3} \rfloor - 15\}$.
2. The weights of the edges of the set $A_2$, receive the consecutive integers from the set $A_2 = \{5\lfloor \frac{n}{2} \rfloor - 9, ..., 5\lfloor \frac{n}{2} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(11 - \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor) - 10\}$.
3. The weights of the edges of the set $A_3$, receive the consecutive integers from the set $\{5\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 1) + 16, ..., 5\lfloor \frac{n}{3} \rfloor + (\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor)^2 + 15\}$.
4. The weights of the edges of the set $A_4$, receive the consecutive integers from the set $A_4 = \{5\lfloor \frac{n}{2} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} - \lfloor \frac{n}{3} \rfloor - 1) + 21, ..., 10\lfloor \frac{n}{2} \rfloor - 5\lfloor \frac{n}{3} \rfloor + 20\}$.
5. The weights of the edges of the set $A_5$, receive the consecutive integers from the set $A_5 = \{10\lfloor \frac{n}{2} \rfloor - 5\lfloor \frac{n}{3} \rfloor + 21, ..., 10\lfloor \frac{n}{2} \rfloor + 5\}$.
6. The weights of the edges of the set $A_6$, receive the consecutive integers from the set $A_6 = \{5\lfloor \frac{n}{3} \rfloor + (\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor)^2 + 16, ..., 5\lfloor \frac{n}{2} \rfloor + \frac{\lfloor \frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor - 1) + 20\}$.
7. The weights of the edges of the set $A_7$, receive the consecutive integers from the set $A_7 = \{5\lfloor \frac{n}{2} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - 1) - 9, ..., 5\lfloor \frac{n}{2} \rfloor - 10\}$.

8. The weights of the edges of the set $A_8$, receive the consecutive integers from the set $A_8 = \{5\lfloor \frac{n}{3} \rfloor - 14, ..., 5\lfloor \frac{n}{2} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - 1) - 10\}$.
9. The weights of the edges of the set $A_9$, receive the consecutive integers from the set $A_9 = \{5\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(11 - \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor) - 9, ..., 5\lfloor \frac{n}{3} \rfloor + \frac{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor + 1) + 15\}$.

Also, for $n$ is even the edge weights under the labeling $f$ are the following:

1. The weights of the edges of the set $A_1$, receive the consecutive integers from the set $A_1 = \{1, 2, ..., 5\lfloor \frac{n}{3} \rfloor - 15\}$.
2. The weights of the edges of the set $A_2$, receive the consecutive integers from the set $A_2 = \{\frac{5n}{2} - 14, ..., 5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + 21) - 15\}$.
3. The weights of the edges of the set $A_3$, receive the consecutive integers from the set $\{5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor - 1) + 16, ..., 6\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + (\frac{n}{2} - \lfloor \frac{n}{3} \rfloor)^2 + 15\}$.
4. The weights of the edges of the set $A_4$, receive the consecutive integers from the set $A_4 = \{5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor + 9) + 16, ..., 5n - 5\lfloor \frac{n}{3} \rfloor + 15\}$.
5. The weights of the edges of the set $A_5$, receive the consecutive integers from the set $A_5 = \{5n - 5\lfloor \frac{n}{3} \rfloor + 16, ..., 5n\}$.
6. The weights of the edges of the set $A_6$, receive the consecutive integers from the set $A_6 = \{6\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + (\frac{n}{2} - \lfloor \frac{n}{3} \rfloor)^2 + 16, ..., 5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor + 9) + 15\}$.
7. The weights of the edges of the set $A_7$, receive the consecutive integers from the set $A_7 = \{\frac{n}{2} + 4\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + 9) - 14, ..., \frac{5n}{2} - 15\}$.
8. The weights of the edges of the set $A_8$, receive the consecutive integers from the set $A_8 = \{5\lfloor \frac{n}{3} \rfloor - 14, ..., \frac{n}{2} + 4\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + 9) - 15\}$.
9. The weights of the edges of the set $A_9$, receive the consecutive integers from the set $A_9 = \{5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\lfloor \frac{n}{3} \rfloor - \frac{n}{2} + 21) - 14, ..., 5\lfloor \frac{n}{3} \rfloor + \frac{\frac{n}{2} - \lfloor \frac{n}{3} \rfloor}{2}(\frac{n}{2} - \lfloor \frac{n}{3} \rfloor - 1) + 15\}$.

In Case 2, using the similar arguments as in the proof of Theorem 1.1, the edge set of $C_n(1, 2, 3, 4, 5)$ can be divided into eight mutually disjoint subsets, so we can find the edge weights as follows:

1. The weights of the edges of the set $A_1$, receive the consecutive integers from the set $A_1 = \{1, 2, ..., 5\lfloor \frac{n}{3} \rfloor - 15\}$.
2. The weights of the edges of the set $A_2$, receive the consecutive integers from the set $A_2 = \{5n - 5\lfloor \frac{n}{2} \rfloor - 14, ..., 5n - \lfloor \frac{n}{2} \rfloor\}$.
3. The weights of the edges of the set $A_3$, receive the consecutive integers from the set $\{5n - 5\lfloor \frac{n}{2} \rfloor + 1, ..., 5n - 5\lfloor \frac{n}{3} \rfloor - 15\}$.
4. The weights of the edges of the set $A_4$, receive the consecutive integers from the set $A_4 = \{5n - 5\lfloor \frac{n}{3} \rfloor + 1, ..., 5n - 5\lfloor \frac{n}{3} \rfloor + 15\}$.
5. The weights of the edges of the set $A_5$, receive the consecutive integers from the set $A_5 = \{5n - 5\lfloor \frac{n}{3} \rfloor - 5\lfloor \frac{n}{3} \rfloor + 16, ..., 5n\}$.
6. The weights of the edges of the set $A_6$, receive the consecutive integers from the set $A_6 = \{5n - 5\lfloor \frac{n}{3} \rfloor - 14, ..., 5n - 5\lfloor \frac{n}{3} \rfloor\}$.
7. The weights of the edges of the set $A_7$, receive the consecutive integers from the set $A_7 = \{5\lfloor \frac{n}{3} \rfloor + 1, ..., 5n - 5\lfloor \frac{n}{2} \rfloor - 15\}$.

8. The weights of the edges of the set $A_8$, receive the consecutive integers from the set $A_8 = \{5\lfloor \frac{n}{3} \rfloor - 14, ..., 5\lfloor \frac{n}{3} \rfloor\}$.

We can see that all vertex labels are even numbers. Furthermore, for $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \leq 4$ the edge weights are from the set $W = \{1, 2, ..., 10\lfloor \frac{n}{2} \rfloor + 5\}$, when $n$ is odd and from the set $W = \{1, 2, ..., 5n\}$, when $n$ is even. Also for $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{3} \rfloor \geq 5$, the edge weights are from the set $W = \{1, 2, ..., 5n\}$. Finally we can easily check that all numbers in the set $W$ are distinct. □

## 4. Conclusions

In this paper we discuss an edge irregular reflexive k-labeling for three families of the circulant graphs $C_n(1, 2), C_n(1, 2, 3)$ and $C_n(1, 2, 3, 4, 5)$ including their exact values of the reflexive edge strength. For $n \geq 8$ we proved that $res(C_n(1, 2)) = \lceil \frac{2n}{3} \rceil$ for $n \equiv 0, 2 \pmod 3$ and $res(C_n(1, 2)) = \lceil \frac{2n}{3} \rceil + 1$ for $n \equiv 1 \pmod 3$. Also we proved that $res(C_n(1, 2, 3)) = n$ for $n$ is even and $res(C_n(1, 2, 3)) = n + 1$ for $n$ is odd, where $n \geq 6$. Moreover we showed that $res(C_n(1, 2, 3, 4, 5)) = \lceil \frac{5n}{3} \rceil + 1$ for $n \equiv 3, 4 \pmod 6$, $n \geq 15$ and $res(C_n(1, 2, 3, 4, 5)) = \lceil \frac{5n}{3} \rceil$, otherwise.

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### Conflict of interest

The authors declare that they have no competing interest.

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