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Research article

Biharmonic submanifolds of Kaehler product manifolds

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Abstract: In this paper, the authors have established the necessary and sufficient conditions for the submanifolds of Kaehler product manifolds to be biharmonic. Moreover, the magnitude of scalar curvature for the hypersurfaces in a product of two unit spheres has been derived. Also, for the same product, the magnitude of the mean curvature vector for Lagrangian submanifolds has been estimated. Finally, the non-existence condition for totally complex Lagrangian submanifolds in a product of unit sphere and a hyperbolic space has been proved.

Keywords: biharmonic submanifolds; Kaehler product manifolds; totally real submanifolds; Lagrangian submanifolds; mean curvature
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1. Introduction

For any two manifolds (M,g) and (N,h), a harmonic map ψ is the critical point of the energy functional defined as

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dv_g.$$

The natural generalization of the harmonic maps was given by J. Eells and J. H. Sampson [1]. The established map ψ is called biharmonic if it is the critical point of energy functional

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 dv_g.$$

with $\tau(\psi) = tr(\nabla d\psi)$ as the vanishing tensor field for any harmonic map. For the above established E_2, the first and second variation was studied by G. Y. Jiang [2]. For the same bi-harmonic functional, the associated Euler-Lagrange equation is $\tau_2(\psi) = 0$, where $\tau_2(\psi)$ is called bi-tension field and is defined as

$$\tau_2(\psi) = \Delta\tau(\psi) - tr(R^N(d\psi, \tau(\psi))d\psi.$$

In the above equation, Δ is the rough Laplacian acting on the sections of $\psi^{-1}(TN)$ and R^N is the curvature tensor for N. For any $V \in \Gamma(\psi^{-1}(TN))$ and X,Y $\in \Gamma(TN)$, the definitions of Δ and R^N are given by

$$\Delta V = tr(\Delta^2 V),$$

$$R^N(X, Y) = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}.$$

A large number of studies have been done on biharmonic submanifolds [3–8]. It is a general fact that every harmonic map is biharmonic, but the vice-versa isn't true. The biharmonic maps, which are not harmonic, are called proper-biharmonic maps. If the harmonic map ψ is isometric immersion from the manifold (M, g) into (N, h), then the manifold M is called minimal submanifold of N. From the definition of proper biharmonic maps, it can be concluded that these are those submanifolds that aren't harmonic. Biharmonic submanifolds in different ambient spaces for different space forms have been extensively studied in the last few decades. Caddeo R. et al. [9] studied biharmonic submanifolds in spheres. Fetcu D. et al. [10–12] studied these submanifolds in complex, Sasakian and the product of sphere and real line space forms. J. Roth and A. Upadhyay [13, 14] studied the biharmonic submanifolds on product spaces and generalized space forms. Chen B. Y. proved Chen's biharmonic conjecture stating that biharmonic surfaces do not exist in any Euclidean space with parallel normalized mean curvature vectors [15]. Yu F. et al. proved the same conjecture for hypersurfaces in R^5 [16].

The present study establishes the necessary and sufficient conditions for the submanifolds of Kaehler product manifolds to be biharmonic. Our future work then combines the work done in this paper with the techniques of singularity theory presented in [17–20]. We have derived the magnitude of scalar curvature for the hypersurfaces in a product of two spheres. We have also estimated the magnitude of the mean curvature vector for Lagrangian submanifolds in a product of two spheres. Finally, we proved the non-existence condition for totally complex Lagrangian submanifolds in a product of unit sphere and hyperbolic space.

2. Preliminaries

Let $\hat{M}^n$ and $\hat{M}^p$ be any Kehlerian manifolds of dimensions n (real dimension $2n$) and p (real dimension $2p$) respectively. Let us further assume J_n and J_p denote the almost complex structures of $\hat{M}^n$ and $\hat{M}^p$, respectively. Suppose, $\hat{M}^n$ and $\hat{M}^p$ are complex space forms with constant holomorphic sectional curvatures c_1 and c_2, respectively. The Riemannian curvature tensor $\hat{R}_n$ of $\hat{M}^n(c_1)$ is given by

$$\hat{R}_n(X, Y)Z = \tfrac{1}{4}c_1[g_n(Y, Z)X - g_n(X, Z)Y]$$
$$+ \tfrac{1}{4}c_1[g_n(J_nY, Z)J_nX - g_n(J_nX, Z)J_nY + 2g_n(X, J_nY)J_nZ].$$

Similarly, the Riemannian curvature tensor $\hat{R}_p$ of $\hat{M}^p(c_2)$ is given by

$$\hat{R}_p(X, Y)Z = \tfrac{1}{4}c_2[g_p(Y, Z)X - g_p(X, Z)Y]$$
$$+ \tfrac{1}{4}c_2[g_p(J_pY, Z)J_pX - g_p(J_pX, Z)J_pY + 2g_p(X, J_pY)J_pZ].$$

For any generalized submanifold M of any complex space form N, the almost complex structure J induces the existence of four operators on M, namely

$$j : TM \to TM, k : TM \to NM, l : NM \to TM, m : NM \to NM,$$

defined for all X ∈ TM (tangent bundle) and $\zeta \in NM$ (normal bundle) by

$$\begin{aligned} JX &= jX + kX, \\ J\zeta &= l\zeta + m\zeta. \end{aligned} \tag{2.1}$$

Since J is the almost complex structure, it satisfies $J^2 = -Id$. For any X, Y tangent to N, we also have $g(JX, Y) = -g(X, JY)$. Using the above properties of J, the relations for the operators, j, k, l and m are given as

$$j^2X + lkX + X = 0, \tag{2.2}$$

$$m^2\zeta + kl\zeta + \zeta = 0, \tag{2.3}$$

$$jl\zeta + lm\zeta = 0, \tag{2.4}$$

$$kjX + mkX = 0, \tag{2.5}$$

$$g(kX, \zeta) + g(X, l\zeta) = 0. \tag{2.6}$$

for all X ∈ $\Gamma(TM)$ and $\zeta \in \Gamma(NM)$. Also, j and m are skew-symmetric.

Now, let us consider the Kaehler product manifold $\hat{M}^n(c_1) \times \hat{M}^p(c_2)$ denoted by $\hat{M}$. If P and Q denote projection operators of the tangent spaces of $\hat{M}^n(c_1)$ and $\hat{M}^p(c_2)$, then we always have $P^2 = P$, $Q^2 = Q$ and $PQ = QP$. If we put $F = P - Q$, the properties of P and Q establish $F^2 = I$. This F is almost product structure of $\hat{M}^n(c_1) \times \hat{M}^p(c_2)$. Moreover, we define a Riemannian metric g on $\hat{M}$ as

g(X,Y) = $g_n(PX, PY) + g_p(QX, QY)$.

Where X and Y are vector fields on $\hat{M}$. It further follows, $g(FX, Y) = g(X, FY)$. If we put $JX = J_nPX + J_pQX$, we get $J_nP = PJ$, $J_pQ = QJ$, $FJ = JF$, $g(JX, JY) = g(X, Y)$, $\hat{\nabla}J = 0$. Thus J is the Kaehlerian structure on $\hat{M}$. The Riemannian curvature tensor $\hat{R}$ of the product manifold $\hat{M}$ is given as [21]

$$\begin{aligned} \hat{R}(X, Y)Z = \frac{c_1 + c_2}{16}[&g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY \\ &+ 2g(X, JY)JZ + g(FY, Z)FX - g(FX, Z)FY + g(FJY, Z)FJX - g(FJX, Z)FJY \\ &+ g(FZ, JY)FJZ] + \frac{c_1 - c_2}{16}[g(FY, Z)X - g(FX, Z)Y + g(Y, Z)FX - g(X, Z)FY \\ &+ g(FJY, Z)JX - g(FJX, Z)JY + g(JY, Z)FJX - g(JX, Z)FJY \\ &+ 2g(FX, JY)JZ + 2g(X, JY)JFZ]. \end{aligned} \tag{2.7}$$

The product structure F induces the existence of four operators:

$$f : TM \to TM, h : TM \to NM, s : NM \to TM \text{ and } t : NM \to NM,$$

defined for all X ∈ TM (tangent bundle) and $\zeta \in NM$ (normal bundle) by

$$\begin{aligned} FX &= fX + hX, \\ F\zeta &= s\zeta + t\zeta. \end{aligned} \tag{2.8}$$

These four operators follow the following relations

$$f^2X + shX = X, \tag{2.9}$$

$$t^2\zeta + hs\zeta = \zeta, \tag{2.10}$$

$$fs\zeta + st\zeta = 0, \tag{2.11}$$

$$hfX + thX = 0, \tag{2.12}$$

$$g(hX, \zeta) = g(X, s\zeta). \tag{2.13}$$

for all X ∈ $\Gamma(TM)$ and $\zeta \in \Gamma(NM)$. Also, f and t are symmetric.

3. Results

The first theorem gives necessary and sufficient condition for the manifold to be biharmonic.

Theorem 3.1. *Let M be a u-dimensional submanifold of the Kaehler product manifold* $\hat{M} = \hat{M}^n(c_1) \times \hat{M}^p(c_2)$ *with A, B and H, respectively denoting the shape operator, second fundamental form and mean curvature vector. Then, this submanifold is biharmonic if and only if the following equations are satisfied:*

$$\begin{aligned} &-\nabla^{\perp}H + tr(B(.,A_H.)) + \frac{c_1+c_2}{16}[-uH + 3klH + hsH - tr(f)tH + 2(hjflH \\ &+ tkflH + hjsmH + tksmH) - tr(fj+sk)(hlH + tmH)] + \frac{c_1-c_2}{16}[-tr(f)H \\ &- utH + 3(kflH + ksmH) - tr(fj+sk)(mH) + 3(hjlH + tklH)] = 0. \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{u}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + \frac{c_1+c_2}{8}[3jlH + fsH - tr(f)sH \\ &+ 2(fjflH + skflH + fjsmH + sksmH) - tr(fj+sk)(flH + smH)] \\ &+ \frac{c_1-c_2}{8}[sH - usH + 3(jflH + jsmH) - tr(fj+sk)(lH) + 3(fjlH + sklH)] = 0. \end{aligned} \tag{3.2}$$

Proof. The equations of biharmonicity have been already established in [12,22,23]. Projection of the equation $\tau(\psi) = 0$ on both tangential and normal bundles establishes the following equations

$$\begin{aligned} -\nabla^{\perp}H + tr(B(.,A_H.)) + tr(\bar{R}(.,H.)^{\perp} &= 0,\\ \frac{u}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + 2tr(\bar{R}(.,H.)^{\top} &= 0. \end{aligned} \tag{3.3}$$

Suppose that $\{X_i\}_{i=1}^{u}$ is a local orthonormal frame for TM, then by using the Eq 2.7 of curvarture tensor $\bar{R}$, we have

$$tr(\bar{R}(.,H.) = \sum_{i=1}^{u} \bar{R}(X_i,H)X_i, \tag{3.4}$$

$\implies tr(\bar{R}(.,H.) = \sum_{i=1}^{u} \{\frac{c_1+c_2}{16}[g(H,X_i)X_i - g(X_i,X_i)H + g(JH,X_i)JX_i$

$- g(JX_i,X_i)JH + 2g(X_i,JH)JX_i + g(FH,X_i)FX_i - g(FX_i,X_i)FH$

$+ g(FJH,X_i)FJX_i - g(FJX_i,X_i)FJH + g(FX_i,JH)FJX_i]$

$+ \frac{c_1-c_2}{16}[g(FH,X_i)X_i - g(FX_i,X_i)H + g(H,X_i)FX_i - g(X_i,X_i)FH$

$+ g(FJH,X_i)JX_i - g(FJX_i,X_i)JH + g(JH,X_i)FJX_i - g(JX_i,X_i)FJH$

$+ 2g(FX_i,JH)JX_i + 2g(X_i,JH)JFX_i]\}$,

Introducing the established sets of four operators, j, k, l and m and f, h, s and t for J and F respectively, we get the simplified equation as

$tr(\bar{R}(.,H.) = \frac{c_1+c_2}{16}[-uH + \sum_{i=1}^{u} g(lH,X_i)JX_i + \sum_{i=1}^{u} 2g(X_i,lH)JX_i$

$+ F(FH)^{\top} - tr(f)FH + FJ(FJH)^{\top} - tr(fj+sk)FJH + FJ(FJH)^{\top}]$

$+ \frac{c_1-c_2}{16}[(FH)^{\top} - tr(f)H - uFH + J(FJH)^{\top} - tr(fj+sk)JH$

$+ \sum_{i=1}^{u} g(lH,X_i)FJX_i + 2J(FJH)^{\top} + \sum_{i=1}^{u} 2g(X_i,lH)JFX_i]$,

or $tr(\bar{R}(.,H.) = \frac{c_1+c_2}{16}[-uH + 3JlH + fsH + hsH - tr(f)sH - tr(f)tH$

$+ 2FJ(flH + smH) - tr(fj+sk)FJH]$

$+ \frac{c_1-c_2}{16}[sH - tr(f)H - uFH + J(flH + smH) - tr(fj+sk)JH +$

$3FJlH + 2J(flH + smH)]$,

$\implies tr(\bar{R}(.,H.) = \frac{c_1+c_2}{16}[-uH + 3jlH + 3klH + fsH + hsH - tr(f)sH - tr(f)tH$

$+ 2(fjflH + hjflH + skflH + tkflH + fjsmH + hjsmH + sksmH + tksmH)$

$- tr(fj + sk)(flH + hlH + smH + tmH)]$

$+ \frac{c_1-c_2}{16}[sH - tr(f)H - ush - utH + 3(jflH + kflH + jsmH + ksmH)$

$- tr(fj + sk)(lH + mH) + 3(fjlH + hjlH + sklH + tklH)].$

By identification of tangential and normal parts, we get the required equations. □

Corollary 3.2. *If M is a u-dimensional totally real submanifold of the Kaehler product manifold* $\hat{M} = \hat{M}^n(c_1) \times \hat{M}^p(c_2)$*. Then, this submanifold is biharmonic if and only if the following equations are satisfied*

$$\begin{aligned} &- \nabla^{\perp}H + tr(B(.,A_H.)) + \frac{c_1+c_2}{16}[-uH + 3klH + hsH - tr(f)tH \\ &+ 2(tkflH + tksmH) - tr(sk)(hlH + tmH)] + \frac{c_1-c_2}{16}[-tr(f)H \\ &- utH + 3(kflH + ksmH) - tr(sk)(mH) + 3(tklH)] = 0. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\frac{u}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + \frac{c_1+c_2}{8}[fsH - tr(f)sH \\ &+ 2(skflH + sksmH) - tr(sk)(flH + smH)] \\ &+ \frac{c_1-c_2}{8}[sH - usH - tr(sk)(lH) + 3(sklH)] = 0. \end{aligned} \tag{3.6}$$

Proof. If M is a totally real submanifold, then we know that for any $X \in \Gamma(TM)$, we have

$$JX = kX,$$

In other words, $jX = 0$. Using this fact in Theorem 3.1, we get the required equations. □

Corollary 3.3. ***a)***: *If M is any hypersurface of the Kaehler product manifold*

$$\hat{M} = \hat{M}^p(c_1) \times \hat{M}^{n-p}(c_2).$$

Then, M is biharmonic if and only if the following equations are satisfied

$$\begin{aligned} &- \nabla^{\perp}H + tr(B(.,A_H.)) + \frac{c_1+c_2}{16}[-(n-2)H + hsH \\ &- tr(f)tH + 2(hjflH + tkflH) - tr(fj + sk)(hlH)] \\ &+ \frac{c_1-c_2}{16}[-tr(f)H - (n-1)tH + 3(kflH) + 3(tklH)] = 0. \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\frac{n-1}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + \frac{c_1+c_2}{8}[fsH - tr(f)sH \\ &+ 2(fjflH + skflH) - tr(fj + sk)(flH)] + \frac{c_1-c_2}{8}[sH \end{aligned} \tag{3.8}$$

$$-(n-1)sH+3(jflH)-tr(fj+sk)(lH)-3sH]=0.$$

***b)**: If M is any totally real hypersurface of the Kaehler product manifold*

$$\hat{M}=\hat{M}^p(c_1)\times\hat{M}^{n-p}(c_2).$$

Then, M is biharmonic if and only if the following equations are satisfied:

$$\begin{aligned}&-\nabla^{\perp}H+tr(B(.,A_H.))+\frac{c_1+c_2}{16}[-(n-2)H+hsH-tr(f)tH+2(tkflH)\\&-tr(sk)(hlH)]+\frac{c_1-c_2}{16}[-tr(f)H-(n-1)tH+3(kflH)+3(tklH)]=0.\end{aligned}\tag{3.9}$$

$$\begin{aligned}&\frac{n-1}{2}grad|H|^2+2tr(A_{\nabla^{\perp}H}(.))+\frac{c_1+c_2}{8}[fsH-tr(f)sH+2(skflH)-tr(sk)(flH)]\\&+\frac{c_1-c_2}{8}[sH-(n-1)sH-tr(sk)(lH)-3sH]=0.\end{aligned}\tag{3.10}$$

Proof. a): For any hypersurface M, J maps normal vectors to tangent vectors as such $m=0$. Using this fact with the Eqs 2.3 and 2.4 for H, we get the required equations from Theorem 3.1.
b): For any totally real hypersurface M, we have $j=0$ and $m=0$. □

Corollary 3.4. *If M is a u-dimensional Lagrangian manifold of the Kaehler product manifold*

$$\hat{M}=\hat{M}^n(c_1)\times\hat{M}^p(c_2).$$

Then, M is biharmonic if and only if the following equations are satisfied

$$\begin{aligned}&-\nabla^{\perp}H+tr(B(.,A_H.))+\frac{c_1+c_2}{16}[-(u+3)H+hsH-tr(f)tH\\&+2(tkflH)-tr(sk)(hlH)]+\frac{c_1-c_2}{16}[-tr(f)H-utH+3(kflH)+3(tklH)]=0.\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\frac{u}{2}grad|H|^2+2tr(A_{\nabla^{\perp}H}(.))+\frac{c_1+c_2}{8}[fsH-tr(f)sH\\&+2(skflH)-tr(sk)(flH)]+\frac{c_1-c_2}{8}[sH-usH-tr(sk)(lH)-3(sH)]=0.\end{aligned}\tag{3.12}$$

Proof. If M is a Lagrangian manifold , then $j=0$ and $m=0$. Using this fact with Eq 2.3, we get the required equations from Theorem 3.1. □

From now on, the authors will consider the ambient space to be product of two 2-spheres of same radius (for simplicity radius equals 1 unit). The reason for taking 2-sphere follows from [24] as it is the only sphere which accepts Kaehler structure. In the following equations, we will have

$$\frac{c_1+c_2}{16}=\frac{c_1}{8}=\frac{1}{8}\text{ and }\frac{c_1-c_2}{8}=b=0.$$

To estimate the magnitude of mean curvature vector and scalar curvature, the authors will further assume the cases where F will map the whole of tangent bundle or normal vectors to respective bundles only. The reason being the equations involve the product of almost complex structure J and product structure F. As such it isn't possible to get simpler equations involving dimensions of submanifolds and mean curvature vector only.

Proposition 3.5. *Let M be any hypersurface of* $S^2 \times S^2$ *with non-zero constant mean curvature such that* $FX \in \Gamma(TM^{\perp})$ *and* $FN \in \Gamma(TM)$ *for any* $X \in \Gamma(TM)$ *and* $N \in \Gamma(TM^{\perp})$*. Then M is biharmonic if we have*

$$|B|^2 = \frac{1}{8}\left[1 + \frac{1}{|H|^2} tr(sk)\langle FJH, H\rangle\right]. \tag{3.13}$$

Proof. By the established hypothesis on F, we have $f = 0$ and $t = 0$. Using these equations along with Eqs 2.9 and 2.10 in Eq 3.7, we get

$$-\nabla^{\perp} H + tr(B(.,A_H.)) - \frac{1}{8}\left[H + tr(sk)(hlH)\right] = 0, \tag{3.14}$$

Since M is a hypersurface, the above equation becomes,

$$tr(B(.,A_H.)) - \frac{1}{8}\left[H + tr(sk)(hlH)\right] = 0, \tag{3.15}$$

Since $tr(B(.,A_H.))= |B|^2 H$, on further simplifying, we get,

$$|B|^2 H^2 = \frac{1}{8}\left[H^2 + tr(sk)\langle hlH, H\rangle\right], \tag{3.16}$$

or

$$|B|^2 = \frac{1}{8}\left[1 + \frac{1}{|H|^2} tr(sk)\langle FJH, H\rangle\right]. \tag{3.17}$$

□

Remark 3.6. It can be easily concluded from above proposition that there doesn't exist any hypersurface of $S^2 \times S^2$ when $FX \in \Gamma(TM^{\perp})$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ for

$$tr(sk)\langle FJH, H\rangle + |H|^2 \leq 0.$$

The above proposition can be used to derive the value of scalar curvature for biharmonic hypersurface M when $FX \in \Gamma(TM^{\perp})$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

Proposition 3.7. *Let M be any proper-biharmonic hypersurface of* $S^2 \times S^2$ *with non-zero constant mean curvature such that* $FX \in \Gamma(TM^{\perp})$ *and* $FN \in \Gamma(TM)$ *for any* $X \in \Gamma(TM)$ *and* $N \in \Gamma(TM^{\perp})$*. Then the scalar curvature* τ *of M is given by*

$$\tau_M = \frac{1}{8}[5 + tr(sk)^2 - \frac{1}{|H|^2} tr(sk)\langle FJH, H\rangle] + 3|H|^2.$$

Proof. By the equation of Gauss, we have,

$$\tau_M = \sum_{i,j=1}^{n-1} \langle \bar{R}(X_i, X_j)X_j, X_i\rangle - |B|^2 + (n-1)|H|^2,$$

The curvature tensor $\hat{R}$ for $S^2 \times S^2$ is given by Eq 2.7 with

$$\frac{c_1 + c_2}{16} = \frac{c_1}{8} = \frac{1}{8} \text{ and } \frac{c_1 - c_2}{8} = 0.$$

And,

$$\langle \hat{R}(X_i, X_j)X_j, X_i\rangle = \frac{1}{8}[1 + \langle FX_j, X_j\rangle\langle FX_i, X_i\rangle \\ - \langle FX_i, X_j\rangle^2 + \langle FJX_j, X_j\rangle\langle FX_i, X_i\rangle], \tag{3.18}$$

Since $FX_i \in \Gamma(TM^{\perp})$ and $f = 0$. We have

$$\sum_{i,j=1}^{n-1}\langle \hat{R}(X_i, X_j)X_j, X_i\rangle = \frac{1}{8}[6 + tr(sk)^2]. \tag{3.19}$$

Using the value of $|B|^2$ gives the required equation. □

Proposition 3.8. *Let M be any totally complex-hypersurface of $S^2 \times S^2$ with non-zero constant mean curvature such that $FX \in \Gamma(TM^{\perp})$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. Then for trivially biharmonic M, we have*

$$|B|^2 = \frac{1}{8}. \tag{3.20}$$

Proof. By the established hypothesis on F, we have $f = 0$ and $t = 0$. Using these equations along with Eqs 2.9 and 2.10 in Theorem 3.1, we get

$$-\nabla^{\perp}H + tr(B(.,A_H.)) - \frac{1}{8}H = 0, \tag{3.21}$$

Since M is a hypersurface, the above equation becomes

$$tr(B(.,A_H.)) - \frac{1}{8}H = 0. \tag{3.22}$$

Since $tr(B(.,A_H.))= |B|^2H$. On further simplifying, we get the required equation. □

Proposition 3.9. *Let M be any proper-biharmonic totally complex-hypersurface of $S^2 \times S^2$ with non-zero constant mean curvature such that $FX \in \Gamma(TM^{\perp})$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. Then the scalar curvature τ of M is given as*

$$\tau_M = \frac{1}{8}[5 + tr(sk)^2] + 3|H|^2. \tag{3.23}$$

Proof. By the equation of Gauss, we have

$$\tau_M = \sum_{i,j=1}^{n-1}\langle \hat{R}(X_i, X_j)X_j, X_i\rangle - |B|^2 + (n-1)|H|^2,$$

The curvature tensor $\hat{R}$ for $S^2 \times S^2$ is given by Eq 2.7 with

$$\frac{c_1 + c_2}{16} = \frac{c_1}{8} = \frac{1}{8} \text{ and } \frac{c_1 - c_2}{8} = 0.$$

Then,

$$\langle \hat{R}(X_i, X_j)X_j, X_i\rangle = \frac{1}{8}[1 + \langle FX_j, X_j\rangle\langle FX_i, X_i\rangle - \langle FX_i, X_j\rangle^2 + \langle FJX_j, X_j\rangle\langle FX_i, X_i\rangle]. \tag{3.24}$$

Since $FX_i \in \Gamma(TM^\perp)$ and $f = 0$. We have

$$\sum_{i,j=1}^{n-1}\langle \hat{R}(X_i, X_j)X_j, X_i\rangle = \frac{1}{8}[6 + tr(sk)^2]. \tag{3.25}$$

Using the value of $|B|^2$ gives the required equation. □

Corollary 3.10. *Let M be u-dimensional Lagrangian submanifold of $S^2 \times S^2$ with non-zero constant mean curvature such that $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. Let us further assume $[tr(sk)\langle FJH, H\rangle] \geq 0$ Then we have*

a): If M is a proper-biharmonic, then $0 < |H|^2 \leq \frac{u+2}{8u}$.
b): If $|H|^2 = \frac{u+2}{8u}$, then M is biharmonic if and only if it is pseudo-umbilical manifold, $\nabla^\perp H = 0$ and $tr(sk)$=0.

Proof. By the given hypothesis for F, we have $f = 0$ and $t = 0$.

Implementing the above conditions along with Eq 2.9 in Corollary 3.4 a), we get,

$$-\Delta^\perp H + tr(B(.,A_H.)) - \frac{1}{8}[(u + 2)Htr(sk)(hlH)] = 0. \tag{3.26}$$

By taking the inner product with H, we get

$$-\langle\Delta^\perp H, H\rangle + |A_H|^2 - \frac{1}{8}[(u + 2)|H|^2 + tr(sk)\langle FJH, H\rangle)] = 0, \tag{3.27}$$

where A_H is the shape operator associated with mean curvature vector H.

Using Bochner formula, we get

$$\frac{1}{8}(u + 2)|H|^2 = |A_H|^2 + |\nabla^\perp H|^2 + \frac{1}{8}tr(sk)\langle FJH, H\rangle). \tag{3.28}$$

By the Cauchy-Schwarz inequality, we have $|A_H|^2 \geq u|H|^4$. Using this fact, we have

$$\begin{aligned}\frac{1}{8}(u + 2)|H|^2 &\geq u|H|^4 + |\nabla^\perp H|^2 + \frac{1}{8}tr(sk)\langle FJH, H\rangle) \\ &\geq u|H|^4 + \frac{1}{8}tr(sk)\langle FJH, H\rangle) \geq u|H|^4.\end{aligned} \tag{3.29}$$

Since H is a non-zero constant, we have

$$0 < |H|^2 \leq \frac{u+2}{8u}.$$

If $|H|^2 \leq \frac{u+2}{8u}$ and M is proper-biharmonic, all of the above inequalities become equalities. Thus, we have $\nabla^\perp H|^2 = 0$ and $tr(sk) = 0$ as FJ is an isometry. Since the Cuachy-Schwarz inequality becomes equality, we have M as pseudo-umbilical. □

Remark 3.11. The cases for which $FX \in \Gamma(TM)$ and $FN \in \Gamma(TM^{\perp})$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ establish the results comparable to those established in this paper. The proofs of all those results follow a similar procedure; thus, they haven't been discussed here.

Finally, we discuss a non-existence case for the product of a unit sphere and a hyperbolic space. Out of all the discussed cases, the non-existence result can be found only for totally-complex Lagrangian submanifolds. Same has been discussed here:

Proposition 3.12. *There doesn't exist any proper biharmonic totally complex Lagrangian submanifold (dimension* ≥ 2*) with parallel mean curvature in* $S^2 \times H^{n-2}$ *such that* $FX \in \Gamma(TM^{\perp})$ *and* $FN \in \Gamma(TM)$ *for any* $X \in \Gamma(TM)$ *and* $N \in \Gamma(TM^{\perp})$.

Proof. Since mean curvature H is parallel and not identically zero. Therefore, FH isn't zero identically. M is trivially biharmonic, according to Theorem 3.1, we have

$$\begin{aligned}&\frac{u}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + \frac{c_1+c_2}{8}[fsH - tr(f)sH] \\ &+ \frac{c_1-c_2}{8}[sH - usH - 3(sH)] = 0.\end{aligned} \tag{3.30}$$

For the above equation, we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 2$,

or

$$\frac{u}{2}grad|H|^2 + 2tr(A_{\nabla^{\perp}H}(.)) + \frac{1}{4}[-(u+2)sH] = 0. \tag{3.31}$$

Using the hypothesis, we have $sH = 0$ or $FH = 0$, which isn't possible. □

4. Conclusions

We established the necessary and sufficient conditions for the submanifolds of Kaehler product manifolds to be biharmonic. And we derived the magnitude of scalar curvature for the hypersurfaces in a product of two unit spheres. Also, for the same product, the magnitude of the mean curvature vector for Lagrangian submanifolds has been estimated. Finally, we proved the non-existence condition for totally complex Lagrangian submanifolds in a product of unit sphere and a hyperbolic space.

Conflict of interest

The authors declare no conflict of interest.

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