



Research article

On Simpson type inequalities for generalized strongly preinvex functions via (p, q) -calculus and applications

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Abstract: In this paper, we establish a new (p, q) -integral identity. Then, the obtained result is employed to derive (p, q) -integral Simpson type inequalities involving generalized strongly preinvex functions. Moreover, our results are also used to study some special cases and some examples are given to illustrate the investigated results.

Keywords: Simpson inequality; strongly preinvex function; (p, q) -calculus; (p, q) -differentiable function; (p, q) -integrable function

Mathematics Subject Classification: 05A30, 26A51, 26D10, 26D15

1. Introduction

Quantum calculus, also called q -calculus, is the study of calculus without limits. In the beginning study of the q -calculus, Newton's infinite series was established by Euler (1707/1783). Then, Jackson [1] relied on the knowledge of Euler to define q -derivative and q -integral of a continuous function on the interval $(0, \infty)$, based on q -calculus of infinite series, in 1910. In q -calculus, the main objective is to obtain the q -analogues of mathematical objects recaptured by taking $q \rightarrow 1^-$. In recent years, the q -calculus has attracted interest because it can be applied in various fields such as mathematics and physics, see [2–16] for more details and the references cited therein. In 2002, Kac and Cheung [17] summarized the basic theoretical concept of the q -calculus in their book.

In 2013, Tariboon and Ntouyas [18] defined the new q -derivative and q -integral of a continuous function on finite interval and proved their basic properties. Moreover, they investigated the existence and uniqueness results of initial value problems for first and second order impulsive q -difference equations. Then, these definitions have been studied in various inequalities, for example, Simpson type inequalities, Newton type inequalities, Hermite-Hadamard inequalities, Ostrowski inequalities, and Fejr type inequalities, see [19–25] for more details and the references cited therein.

Post-quantum calculus, also called (p, q) -calculus, is another generalization of the q -calculus on the interval $(0, \infty)$. The (p, q) -calculus consists of two-parameter quantum calculus (p and q -numbers) which are independent. The (p, q) -calculus was first introduced by Chakrabarti and Jagannathan [26] in 1991. Then, the new (p, q) -derivative and (p, q) -integral of a continuous function on finite interval were improved by Tun and Gv [27, 28] in 2016. In (p, q) -calculus, we obtain q -calculus formula for case of $p = 1$, and then get classical formula for case of $q \rightarrow 1^-$. Based on (p, q) -calculus, many literatures have been published by many researchers, see [29–39] for more details and the references cited therein.

Several generalizations and extensions of convexity have been studied via various techniques in several directions. In 1981, Hanson [40] presented that a significant generalization of convex functions was that of invex functions. His initial result inspired many literatures and expanded the role and applications of invexity in both areas of pure and applied sciences [41–44]. A class of convex functions, called a preinvex function, was presented by Ben-Israel and Mond [45] in 1986. Moreover, the basic properties of the preinvex functions and their role in optimization were introduced by Weir and Mond [46] in 1988. In recent years, these concepts have been studied by many researchers in various fields, see [47–51] for more details and some researchers have studied the preinvex functions via q -calculus, see [52–57] for more details. After that, Noor et al. [58] studied a new class of generalized convex functions, called a strongly preinvex function, in 2006. Then, Deng et al. [59] studied strongly preinvex functions via q -calculus of Simpson type inequalities in 2019.

Mathematical inequalities play important roles in the study of pure and applied mathematics [60–62]. One of the most interesting inequalities is Simpson type inequalities. Simpson's rules, developed by Simpson (17101761), are techniques for the numerical integration and the numerical estimation of definite integrals. Then, there were a lot of results on Simpsions type inequalities studied by many researchers, see [63–71] for more details and the references cited therein. Simpsions quadrature (Simpsons 1/3 rule) is formulated as follows:

$$\int_a^b f(x)dx \approx \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right],$$

see [72] for more details. The estimation of Simpson inequality is as follows:

Theorem 1.1. [72] *If $f : [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable function on (a, b) and*

$$\|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty,$$

then

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^5.$$

Motivated by the above mentioned reports, we propose to study some new properties of Simpson type inequalities for the generalized strongly preinvex functions via (p, q) -calculus.

The rest of the paper is organized as follows. In Section 2 contains some basic knowledge and notation used in the next sections. In Section 3, we give some properties of Simpson type inequalities via (p, q) -calculus. In Section 4, we display some examples to illustrate the applications of the (p, q) -calculus for Simpson type inequalities. In the final section, we summarize our results.

2. Preliminaries

In this section, we give basic knowledge used in our work. Throughout this paper, let $[a, b] \subseteq \mathbb{R}$ be an interval with $a < b$ and $0 < q < p \leq 1$ be constants.

Definition 2.1. [48] A set $\mathcal{K} \subset \mathbb{R}$ is said to be invex with respect to $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, if

$$y + \lambda\eta(x, y) \in \mathcal{K}, \quad (2.1)$$

holds for all $x, y \in \mathcal{K}$ and $\lambda \in [0, 1]$.

Definition 2.2. [48] A function f on the invex set $\mathcal{K} \subset \mathbb{R}$ is said to be preinvex with respect to $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, if

$$f(y + \lambda\eta(x, y)) \leq (1 - \lambda)f(y) + \lambda f(x) \quad (2.2)$$

holds for all $x, y \in \mathcal{K}$ and $\lambda \in [0, 1]$.

In Definition 2.2, if $\eta(x, y) = x - y$, then (2.2) reduces to

$$f((1 - \lambda)y + x) \leq (1 - \lambda)f(y) + \lambda f(x),$$

which is the convex functions, see [73–76] for more details.

Definition 2.3. [58] A function f on the invex set $\mathcal{K} \subset \mathbb{R}$ is said to be strongly preinvex with respect to $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and modulus $\mu > 0$, if

$$f(y + \lambda\eta(x, y)) \leq (1 - \lambda)f(y) + \lambda f(x) - \mu\lambda(1 - \lambda)\eta^2(x, y) \quad (2.3)$$

holds for all $x, y \in \mathcal{K}$ and $\lambda \in [0, 1]$.

Definition 2.4. [59] A function f on the invex set $\mathcal{K} \subset \mathbb{R}$ is said to be generalized strongly preinvex with respect to $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and modulus $\mu \geq 0$, if

$$f(y + \lambda\eta(x, y)) \leq (1 - \lambda)f(y) + \lambda f(x) - \mu\lambda(1 - \lambda)\eta^2(x, y) \quad (2.4)$$

holds for all $x, y \in \mathcal{K}$ and $\lambda \in [0, 1]$.

In Definition 2.4, if $\mu = 0$, then the generalized strongly preinvex functions reduce to the preinvex functions as defined in Definition 2.2.

Definition 2.5. [27, 28] If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and $0 < q < p \leq 1$, then the (p, q) -derivative of function f at $t \in [a, b]$ is defined by

$$\begin{aligned} {}_aD_{p,q}f(t) &= \frac{f(pt + (1-p)a) - f(qt + (1-q)a)}{(p-q)(t-a)}, \quad t \neq a, \\ {}_aD_{p,q}f(a) &= \lim_{t \rightarrow a} {}_aD_{p,q}f(t). \end{aligned} \quad (2.5)$$

The function f is said to be (p, q) -differentiable function on $[a, b]$ if ${}_aD_{p,q}f(t)$ exists for all $t \in [a, b]$.

In Definition 2.5, if $p = 1$, then ${}_aD_{1,q}f(t) = {}_aD_qf(t)$, and (2.5) reduces to

$$\begin{aligned} {}_aD_qf(t) &= \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, \quad t \neq a, \\ {}_aD_qf(a) &= \lim_{t \rightarrow a} {}_aD_qf(t), \end{aligned} \quad (2.6)$$

which is the q -derivative of function f defined on $[a, b]$, see [77–79] for more details. In addition, if $a = 0$, then ${}_0D_qf(t) = D_qf(t)$, and (2.6) reduces to

$$\begin{aligned} D_qf(t) &= \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \\ D_qf(a) &= \lim_{t \rightarrow 0} D_qf(t), \end{aligned} \quad (2.7)$$

which is the q -derivative of function f defined on $[0, b]$, see [17] for more details.

Definition 2.6. [27, 28] If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and $0 < q < p \leq 1$, then the (p, q) -integral of function f at $t \in [a, b]$ is defined by

$$\int_a^b f(t) {}_a d_{p,q}t = (p-q)(b-a) \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}b + \left(1 - \frac{q^j}{p^{j+1}}\right)a\right). \quad (2.8)$$

The function f is said to be (p, q) -integrable function on $[a, b]$ if $\int_a^b f(t) {}_a d_{p,q}t$ exists for all $t \in [a, b]$.

If $a = 0$, then (2.8) is the (p, q) -integral on $[0, b]$ which can be expressed as:

$$\int_0^b f(x) d_{p,q}x = (p-q)b \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}b\right). \quad (2.9)$$

In addition, If $p = 1$, then (2.9) reduces to

$$\int_0^b f(x) d_qx = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b), \quad (2.10)$$

which is the q -Jackson integral, see [17] for more details.

Theorem 2.1. [27] If $f, g : [a, b]$ are continuous functions, $c \in [a, b]$, $s \in \mathbb{R}$, then the following identities hold:

- (i) $\int_a^b (f(t) + g(t)) {}_a d_{p,q} t = \int_a^b f(t) {}_a d_{p,q} t + \int_a^b g(t) {}_a d_{p,q} t$;
(ii) $\int_a^b s f(t) {}_a d_{p,q} t = s \int_a^b f(t) {}_a d_{p,q} t$;
(iii) $\int_c^b f(t) {}_a d_{p,q} t = \int_a^b f(t) {}_a d_{p,q} t - \int_a^c f(t) {}_a d_{p,q} t$.

Lemma 2.1. [27] For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following expression holds:

$$\int_a^b (x - a)^\alpha {}_a d_{p,q} x = \frac{p - q}{p^{\alpha+1} - q^{\alpha+1}} (b - a)^{\alpha+1}.$$

3. Main results

In this section, we establish a new (p, q) -integral identity. The defined identity is then used to derive the (p, q) -integral inequalities of Simpson type for generalized strongly preinvex function.

Lemma 3.1. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$. If ${}_a D_{p,q} f$ is a (p, q) -integrable function on $[a, a + \eta(b, a)]$, then

$$\begin{aligned} & \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{p\eta(b, a)} \int_a^{a + p\eta(b, a)} f(t) {}_a d_{p,q} t \\ & = \eta(b, a) \int_0^1 \Psi(t, q) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t, \end{aligned} \quad (3.1)$$

where

$$\Psi(t, q) = \begin{cases} qt - \frac{1}{6}, & \text{for } 0 \leq t < \frac{1}{2}; \\ qt - \frac{5}{6}, & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proof. It is not difficult to see that

$$\int_0^1 \Psi(t, q) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t = Q_1 + Q_2, \quad (3.2)$$

where

$$Q_1 = \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t,$$

and

$$Q_2 = \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) {}_a d_{p,q} t.$$

Using Definitions 2.5, 2.6, and Theorem 2.1, we have

$$\begin{aligned} Q_1 & = \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t \\ & = \int_0^{\frac{1}{2}} qt {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t - \frac{1}{6} \int_0^{\frac{1}{2}} {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{1}{2}} q \frac{f(a + p\eta(b, a)) - f(a + q\eta(b, a))}{(p - q)\eta(b, a)} d_{p,q}t - \frac{1}{6} \int_0^{\frac{1}{2}} \frac{f(a + p\eta(b, a)) - f(a + q\eta(b, a))}{t(p - q)\eta(b, a)} d_{p,q}t \\
&= \frac{1}{2\eta(b, a)} \left[\sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) - \sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^{j+1}}{2p^{j+1}} \eta(b, a)\right) \right] \\
&\quad - \frac{1}{6\eta(b, a)} \left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) - \sum_{j=0}^{\infty} f\left(a + \frac{q^{j+1}}{2p^{j+1}} \eta(b, a)\right) \right] \\
&= \frac{1}{2\eta(b, a)} \left[\frac{q}{p} \sum_{j=0}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) - \sum_{j=1}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) \right] \\
&\quad - \frac{1}{6\eta(b, a)} \left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) - \sum_{j=1}^{\infty} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) \right] \\
&= \frac{1}{2\eta(b, a)} \left[\frac{q}{p} f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{p - q}{p} \sum_{j=1}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) \right] \\
&\quad - \frac{1}{6\eta(b, a)} \left[f\left(\frac{2a + \eta(b, a)}{2}\right) - f(a) \right] \\
&= \frac{q}{2p\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{p - q}{2p\eta(b, a)} \sum_{j=0}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) \\
&\quad + \frac{p - q}{2p\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{1}{6\eta(b, a)} \left[f\left(\frac{2a + \eta(b, a)}{2}\right) - f(a) \right] \\
&= \frac{1}{3\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) + \frac{f(a)}{6\eta(b, a)} - \frac{1}{\eta(b, a)} \int_0^{\frac{1}{2}} f(a + p\eta(b, a)) d_{p,q}t. \tag{3.3}
\end{aligned}$$

Then, we find that

$$\begin{aligned}
Q_2 &= \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \\
&= \int_0^1 \left(qt - \frac{5}{6} \right) {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t - \int_0^{\frac{1}{2}} \left(qt - \frac{5}{6} \right) {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \\
&= \int_0^1 qt {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t - \frac{5}{6} \int_0^1 {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \\
&\quad - \left(\int_0^{\frac{1}{2}} qt {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t - \frac{5}{6} \int_0^{\frac{1}{2}} {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \right). \tag{3.4}
\end{aligned}$$

Consider

$$\begin{aligned}
&\int_0^1 qt {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t - \frac{5}{6} \int_0^1 {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \\
&= \int_0^1 q \frac{f(a + p\eta(b, a)) - f(a + q\eta(b, a))}{(p - q)\eta(b, a)} d_{p,q}t - \frac{5}{6} \int_0^1 \frac{f(a + p\eta(b, a)) - f(a + q\eta(b, a))}{t(p - q)\eta(b, a)} d_{p,q}t
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\eta(b, a)} \left[\sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) - \sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^{j+1}}{p^{j+1}} \eta(b, a)\right) \right] \\
&\quad - \frac{5}{6\eta(b, a)} \left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) - \sum_{j=0}^{\infty} f\left(a + \frac{q^{j+1}}{p^{j+1}} \eta(b, a)\right) \right] \\
&= \frac{1}{\eta(b, a)} \left[\frac{q}{p} \sum_{j=0}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) - \sum_{j=1}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{2p^j} \eta(b, a)\right) \right] \\
&\quad - \frac{5}{6\eta(b, a)} \left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) - \sum_{j=1}^{\infty} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) \right] \\
&= \frac{1}{\eta(b, a)} \left[\frac{q}{p} f(a + \eta(b, a)) - \frac{p-q}{p} \sum_{j=1}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) \right] - \frac{5(f(a + \eta(b, a)) - f(a))}{6\eta(b, a)} \\
&= \frac{qf(a + \eta(b, a))}{p\eta(b, a)} - \frac{p-q}{p\eta(b, a)} \sum_{j=0}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{p^j} \eta(b, a)\right) + \frac{(p-q)f(a + \eta(b, a))}{p\eta(b, a)} \\
&\quad - \frac{5(f(a + \eta(b, a)) - f(a))}{6\eta(b, a)} \\
&= \frac{f(a + \eta(b, a))}{6\eta(b, a)} + \frac{5f(a)}{6\eta(b, a)} - \frac{1}{\eta(b, a)} \int_0^1 f(a + p\eta(b, a)) d_{p,q}t.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} qt {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t - \frac{5}{6} \int_0^{\frac{1}{2}} {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t \\
&= -\frac{1}{3\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) + \frac{5f(a)}{6\eta(b, a)} - \frac{1}{\eta(b, a)} \int_0^{\frac{1}{2}} f(a + p\eta(b, a)) d_{p,q}t.
\end{aligned}$$

From (3.4), we obtain

$$\begin{aligned}
Q_2 &= \frac{f(a + \eta(b, a))}{6\eta(b, a)} + \frac{1}{3\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) \\
&\quad - \frac{1}{\eta(b, a)} \int_0^1 f(a + p\eta(b, a)) d_{p,q}t + \frac{1}{\eta(b, a)} \int_0^{\frac{1}{2}} f(a + p\eta(b, a)) d_{p,q}t. \tag{3.5}
\end{aligned}$$

Substituting (3.3) and (3.5) in (3.2), we have

$$\begin{aligned}
&\int_0^1 \Psi(t, q) {}_aD_{p,q}f(a + t\eta(b, a)) d_{p,q}t = Q_1 + Q_2 \\
&= \frac{f(a)}{6\eta(b, a)} + \frac{2}{3\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) + \frac{f(a + \eta(b, a))}{6\eta(b, a)} - \frac{1}{\eta(b, a)} \int_0^1 f(a + p\eta(b, a)) d_{p,q}t \\
&= \frac{1}{6} \left[\frac{f(a)}{\eta(b, a)} + \frac{4}{\eta(b, a)} f\left(\frac{2a + \eta(b, a)}{2}\right) + \frac{f(a + \eta(b, a))}{\eta(b, a)} \right] - \frac{1}{p\eta^2(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p,q}t.
\end{aligned}$$

Multiplying the above equality with $\eta(b, a)$, we obtain the required result. Therefore, the proof is completed. \square

Remark 3.1. If $p = 1$, then (3.1) reduces to

$$\begin{aligned} & \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(t) {}_a d_{p, q} t \\ &= \eta(b, a) \int_0^1 \Psi(t, q) {}_a D_q f(a + t\eta(b, a)) d_q t, \end{aligned}$$

where

$$\Psi(t, q) = \begin{cases} qt - \frac{1}{6}, & \text{for } 0 \leq t < \frac{1}{2}; \\ qt - \frac{5}{6}, & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

which appeared in [59].

Theorem 3.1. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$. If $|{}_a D_{p, q} f|$ is a (p, q) -integrable function and a generalized strongly preinvex function with modulus $\mu \geq 0$, then

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p, q} t \right| \\ & \leq \eta(b, a) \left[[A_1(p, q) + A_4(p, q)] |{}_a D_{p, q} f(a)| + [A_2(p, q) + A_5(p, q)] |{}_a D_{p, q} f(b)| \right. \\ & \quad \left. - \mu(A_3(p, q) + A_6(p, q))\eta^2(b, a) \right], \end{aligned} \quad (3.6)$$

where $A_i(p, q)$, $i = 1, 2, 3, \dots, 6$ are defined by

$$\begin{aligned} A_1(p, q) &= \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p, q} t \\ &= \begin{cases} \frac{2p^3 - p^2 + 2pq - 2p^2q + 2q^2 - 2pq^2 - 4q^3}{24(p+q)(p^2 + pq + q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{36q^5 + 18pq^4 - 6q^4 + 18p^2q^3 + 6pq^3 - 12q^3 + 33p^2q^2 - 18p^3q^2 - 12pq^2 - 2q^2 - 2pq + 2q - 2p^2 + 2p}{216q^2(p+q)(p^2 + pq + q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases} \end{aligned}$$

$$\begin{aligned} A_2(p, q) &= \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_{p, q} t \\ &= \begin{cases} \frac{p^2 - 2pq - 2q^2}{24(p+q)(p^2 + pq + q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{18q^4 + 18pq^3 - 9p^2q^2 + 2q^2 + 2pq - 2q + 2p^2 - 2p}{216q^2(p+q)(p^2 + pq + q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases} \end{aligned}$$

$$A_3(p, q) = \int_0^{\frac{1}{2}} t(1-t) \left| qt - \frac{1}{6} \right| d_{p, q} t$$

$$\begin{aligned}
& \left\{ \begin{array}{l} \frac{2p^4 - p^3 - 4p^3q + 2p^2q - 2p^2q^2 + 2pq^2 - 4pq^3 + 2q^3 - 4q^4}{48(p+q)(p^2+q^2)(p^2+pq+q^2)}, \quad \text{for } 0 < q < \frac{1}{3}; \\ \left[\frac{108q^7 + 108pq^6 - 54q^6 + 54p^2q^5 - 54pq^5 + 12q^5 + 108p^3q^4}{-54p^2q^4 + 12pq^4 - 12q^4 - 54p^4q^3 + 27p^3q^3 + 24p^2q^3} \right. \\ \left. \frac{-12pq^3 - 2q^3 + 12p^3q^2 - 12p^2q^2 - 2pq^2}{+2q^2 + 12p^4q - 12p^3q - 2p^2q + 2pq - 2p^3 + 2p^2} \right] \\ \frac{\quad}{1296q^3(p+q)(p^2+q^2)(p^2+pq+q^2)}, \quad \text{for } \frac{1}{3} \leq q < 1, \end{array} \right. \\
A_4(p, q) &= \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \\
& \left\{ \begin{array}{l} \frac{10p^3 - 15p^2 + 2p^2q + 6pq + 2p^2q^2 + 6q^2 - 8q^3}{24(p+q)(p^2+pq+q^2)}, \quad \text{for } 0 < q < \frac{5}{6}; \\ \left[\frac{-270pq^4 + 282q^4 - 270p^2q^3 + 582pq^3 - 300q^3}{-270p^3q^2 + 825p^2q^2 - 300pq^2 - 250q^2 - 250p^2 + 250p} \right] \\ \frac{\quad}{216q^2(p+q)(p^2+pq+q^2)}, \quad \text{for } \frac{5}{6} \leq q < 1, \end{array} \right. \\
A_5(p, q) &= \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \\
& \left\{ \begin{array}{l} \frac{5p^2 - 2pq - 2q^2}{8(p+q)(p^2+pq+q^2)}, \quad \text{for } 0 < q < \frac{5}{6}; \\ \frac{18q^4 + 18pq^3 - 225p^2q^2 + 250q^2 + 250pq - 250q + 250p^2 - 250p}{216q^2(p+q)(p^2+pq+q^2)}, \quad \text{for } \frac{5}{6} \leq q < 1, \end{array} \right. \\
A_6(p, q) &= \int_{\frac{1}{2}}^1 t(1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \\
& \left\{ \begin{array}{l} \left[\frac{30p^4 - 35p^3 - 12p^3q + 10p^2q + 18p^2q^2}{+10pq^2 - 12pq^3 + 10q^3 - 12q^4} \right] \\ \frac{\quad}{48(p+q)(p^2+q^2)(p^2+pq+q^2)}, \quad \text{for } 0 < q < \frac{5}{6}; \\ \left[\frac{108q^7 + 108pq^6 - 162q^6 - 1242p^2q^5 - 162pq^5}{+1500q^5 - 1500q^4 + 108p^3q^4 - 162p^2q^4 + 1500pq^4} \right. \\ \left. \frac{-1350p^4q^3 + 1215p^3q^3 + 3000p^2q^3 - 1500pq^3 - 1250q^3}{+1500p^3q^2 - 1500p^2q^2 - 1250pq^2 + 1250q^2 + 1500p^4q} \right. \\ \left. \frac{-1500p^3q - 1250p^2q + 1250pq - 1250p^3 + 1250p^2}{1296q^3(p+q)(p^2+q^2)(p^2+pq+q^2)} \right], \quad \text{for } \frac{5}{6} \leq q < 1. \end{array} \right.
\end{aligned}$$

Proof. Using Lemma 3.1 and Definition 2.4, we have

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p,q} t \right|$$

$$\begin{aligned}
&= \left| \eta(b, a) \int_0^1 \Psi(t, q) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,qt} \right| \\
&= \eta(b, a) \left| \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,qt} + \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) {}_a d_{p,qt} \right| \\
&\leq \eta(b, a) \left[\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))| d_{p,qt} + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))| {}_a d_{p,qt} \right] \\
&\leq \eta(b, a) \left[\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| \left((1-t) |{}_a D_{p,q} f(a)| + t |{}_a D_{p,q} f(b)| - \mu t(1-t)\eta^2(b, a) \right) d_{p,qt} \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| \left((1-t) |{}_a D_{p,q} f(a)| + t |{}_a D_{p,q} f(b)| - \mu t(1-t)\eta^2(b, a) \right) {}_a d_{p,qt} \right] \\
&= \eta(b, a) \left[|{}_a D_{p,q} f(a)| \left(\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} + \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,qt} \right) \right. \\
&\quad \left. + |{}_a D_{p,q} f(b)| \left(\int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_{p,qt} + \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_a d_{p,qt} \right) \right. \\
&\quad \left. - \mu \eta^2(b, a) \left(\int_0^{\frac{1}{2}} t(1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} + \int_{\frac{1}{2}}^1 t(1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,qt} \right) \right].
\end{aligned}$$

Using Definition 2.6, Theorem 2.1 and Lemma 2.1, we obtain the required result. Therefore, the proof is completed. \square

Remark 3.2. If $p = 1$, then (3.6) reduces to

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(t) {}_a d_q t \right| \\
&\leq \eta(b, a) \left[[A_1(q) + A_4(q)] |{}_a D_q f(a)| + [A_2(q) + A_5(q)] |{}_a D_q f(b)| \right. \\
&\quad \left. - \mu A_3(q) + A_6(q)\eta^2(b, a) \right],
\end{aligned}$$

where $A_i(q)$, $i = 1, 2, 3, \dots, 6$ are defined by

$$A_1(q) = \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_q t = \begin{cases} \frac{1 - 4q^3}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{36q^3 + 12q^2 + 12q + 1}{216(1+q)(1+q+q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases}$$

$$A_2(q) = \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_q t = \begin{cases} \frac{1 - 2q - 2q^2}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{18q^2 + 18q - 7}{216(1+q)(1+q+q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases}$$

$$\begin{aligned}
A_3(q) &= \int_0^{\frac{1}{2}} t(1-t) \left| qt - \frac{1}{6} \right| d_q t = \begin{cases} \frac{1 - 2q - 2q^3 - 4q^4}{48(1+q)(1+q^2)(1+q+q^2)}, & \text{for } 0 \leq q < \frac{1}{3} \\ \frac{108q^4 + 54q^3 + 12q^2 + 54q - 17}{1296(1+q)(1+q^2)(1+q+q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases} \\
A_4(q) &= \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_a d_q t = \begin{cases} \frac{-5 + 8q + 8q^2 - 8q^3}{24(1+q)(1+pq+q^2)}, & \text{for } 0 < q < \frac{5}{6}; \\ \frac{12q^2 + 12q + 5}{216(1+q)(1+q+q^2)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases} \\
A_5(q) &= \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_a d_q t = \begin{cases} \frac{5 - 2q - 2q^2}{8(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{5}{6}; \\ \frac{18q^2 + 18q + 25}{216(1+q)(1+q+q^2)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases} \\
A_6(q) &= \int_{\frac{1}{2}}^1 t(1-t) \left| qt - \frac{5}{6} \right| {}_a d_q t = \begin{cases} \frac{-5 - 2q + 28q^2 - 2q^3 - 12q^4}{48(1+q)(1+q^2)(1+q+q^2)}, & \text{for } 0 \leq q < \frac{5}{6} \\ \frac{108q^4 - 54q^3 + 96q^2 - 54q + 115}{1296(1+q)(1+q^2)(1+q+q^2)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases}
\end{aligned}$$

which appeared in [59].

Theorem 3.2. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$. If $|{}_a D_{p,q} f|^r$ is a (p, q) -integrable function and a generalized strongly preinvex function with modulus $\mu \geq 0$ and $r > 1$, then

$$\begin{aligned}
& \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p,q} t \right| \\
& \leq \eta(b, a) \left[(B_1(p, q)^{1-1/r}) \left(A_1(p, q) |{}_a D_{p,q} f(a)|^r + A_2(p, q) |{}_a D_{p,q} f(b)|^r - \mu A_3(p, q) \eta^2(b, a) \right)^{1/r} \right. \\
& \quad \left. + (B_2(p, q)^{1-1/r}) \left(A_4(p, q) |{}_a D_{p,q} f(a)|^r + A_5(p, q) |{}_a D_{p,q} f(b)|^r - \mu A_6(p, q) \eta^2(b, a) \right)^{1/r} \right], \quad (3.7)
\end{aligned}$$

where $A_i(p, q)$, $i = 1, 2, 3, \dots, 6$ are given in Theorem 3.1 and $B_j(p, q)$, $j = 1, 2$ are defined by

$$B_1(p, q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,q} t = \begin{cases} \frac{p - 2q}{12(p+q)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{6q^2 - 3pq + 2q + 2p - 2}{36q(p+q)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases}$$

and

$$B_2(p, q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t = \begin{cases} \frac{5p - 4q}{12(p + q)}, & \text{for } 0 < q < \frac{5}{6}; \\ \frac{-45pq + 50q + 50p - 50}{36q(p + q)}, & \text{for } \frac{5}{6} \leq q < 1. \end{cases}$$

Proof. Using Lemma 3.1, Definition 2.4 and the Hlder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p,q} t \right| \\ &= \left| \eta(b, a) \int_0^1 \Psi(t, q) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t \right| \\ &= \eta(b, a) \left| \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) d_{p,q} t + \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) {}_a D_{p,q} f(a + t\eta(b, a)) {}_a d_{p,q} t \right| \\ &\leq \eta(b, a) \left[\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))| d_{p,q} t + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))| {}_a d_{p,q} t \right] \\ &\leq \eta(b, a) \left[\left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,q} t \right)^{1-1/r} \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))|^r d_{p,q} t \right)^{1/r} \right. \\ &\quad \left. + \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \right)^{1-1/r} \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_{p,q} f(a + t\eta(b, a))|^r {}_a d_{p,q} t \right)^{1/r} \right] \\ &\leq \eta(b, a) \left[\left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,q} t \right)^{1-1/r} \right. \\ &\quad \times \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| \left((1-t) |{}_a D_{p,q} f(a)|^r + t |{}_a D_{p,q} f(b)|^r - \mu t(1-t)\eta^2(b, a) \right) d_{p,q} t \right)^{1/r} \\ &\quad \left. + \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \right)^{1-1/r} \right. \\ &\quad \times \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| \left((1-t) |{}_a D_{p,q} f(a)|^r + t |{}_a D_{p,q} f(b)|^r - \mu t(1-t)\eta^2(b, a) \right) {}_a d_{p,q} t \right)^{1/r} \Big] \\ &= \eta(b, a) \left[\left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,q} t \right)^{1-1/r} \times \left(|{}_a D_{p,q} f(a)|^r \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,q} t \right. \right. \\ &\quad \left. \left. + |{}_a D_{p,q} f(b)|^r \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_{p,q} t - \mu \eta^2(b, a) \int_0^{\frac{1}{2}} t(1-t) \left| qt - \frac{1}{6} \right| d_{p,q} t \right)^{1/r} \right. \\ &\quad \left. + \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \right)^{1-1/r} \times \left(|{}_a D_{p,q} f(a)|^r \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \right. \right. \\ &\quad \left. \left. + |{}_a D_{p,q} f(b)|^r \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t - \mu \eta^2(b, a) \int_{\frac{1}{2}}^1 t(1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,q} t \right)^{1/r} \Big]. \end{aligned}$$

Using Definition 2.6, Theorem 2.1 and Lemma 2.1, we obtain the required result. Therefore, the proof is completed. \square

Remark 3.3. If $p = 1$, then (3.7) reduces to

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(t) {}_a d_q t \right| \\ & \leq \eta(b, a) \left[(B_1(q))^{1-1/r} (A_1(q) |{}_a D_q f(a)|^r + A_2(q) |{}_a D_q f(b)|^r - \mu A_3(q) \eta^2(b, a))^{1/r} \right. \\ & \quad \left. + (B_2(q))^{1-1/r} (A_4(q) |{}_a D_q f(a)|^r + A_5(q) |{}_a D_q f(b)|^r - \mu A_6(q) \eta^2(b, a))^{1/r} \right], \end{aligned}$$

where $A_i(q)$, $i = 1, 2, 3, \dots, 6$ are given in Remark (3.2) and $B_j(q)$, $j = 1, 2$ are defined by

$$B_1(q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_q t = \begin{cases} \frac{1-2q}{12(1+q)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{6q-1}{36(1+q)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases}$$

and

$$B_2(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_q t = \begin{cases} \frac{5-4q}{12(1+q)}, & \text{for } 0 < q < \frac{5}{6}; \\ \frac{5}{36(1+q)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases}$$

which appeared in [59].

4. Applications

It is worth noting that if $\mu = 0$ in Definition 2.4, then the generalized strongly preinvex functions reduce to the preinvex functions. Moreover, if $\eta(x, y) = x - y$, then the preinvex functions reduce to the convex functions. Here, we give some examples to illustrate the applications of our main results.

In the following, we show a new result of the preinvex function, which can be obtained directly by taking $\mu = 0$ in Theorem 3.1.

Corollary 4.1. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$. If $|{}_a D_{p,q} f|$ is a (p, q) -integrable function and a preinvex function, then

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(a + \eta(b, a))}{2} + 2f\left(\frac{2a + \eta(b, a)}{2}\right) \right] - \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p,q} t \right| \\ & \leq \eta(b, a) \left[[A_1(p, q) + A_4(p, q)] |{}_a D_{p,q} f(a)| + [A_2(p, q) + A_5(p, q)] |{}_a D_{p,q} f(b)| \right], \end{aligned} \quad (4.1)$$

where $A_1(p, q)$, $A_2(p, q)$, $A_4(p, q)$, and $A_5(p, q)$ are given in Theorem 3.1.

Remark 4.1. If $p = 1$, then (4.1) reduces to

$$\left| \frac{1}{3} \left[\frac{f(a) + f(a + \eta(b, a))}{2} + 2f\left(\frac{2a + \eta(b, a)}{2}\right) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(t) {}_a d_q t \right| \leq \eta(b, a) \left[[A_1(q) + A_4(q)] |{}_a D_q f(a)| + [A_2(q) + A_5(q)] |{}_a D_q f(b)| \right], \quad (4.2)$$

where $A_1(q), A_2(q), A_4(q)$, and $A_5(q)$ are given in Remark 3.2, which appeared in [59].

If $\eta(b, a) = b - a$, then from (4.1) we have the following Corollary.

Corollary 4.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function. If $|{}_a D_{p,q} f|$ is a (p, q) -integrable function and a convex function, then

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(t) {}_a d_{p,q} t \right| \leq (b-a) \left[[A_1(p, q) + A_4(p, q)] |{}_a D_{p,q} f(a)| + [A_2(p, q) + A_5(p, q)] |{}_a D_{p,q} f(b)| \right], \quad (4.3)$$

where $A_1(p, q), A_2(p, q), A_4(p, q)$, and $A_5(p, q)$ are given in Theorem 3.1.

Remark 4.2. If $p = 1$, then (4.3) reduces to

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t) {}_a d_{p,q} t \right| \leq (b-a) \left[[A_1(q) + A_4(q)] |{}_a D_q f(a)| + [A_2(q) + A_5(q)] |{}_a D_q f(b)| \right], \quad (4.4)$$

where $A_1(q), A_2(q), A_4(q)$, and $A_5(q)$ are given in Remark 3.2, which appeared in [59].

In addition, if $q \rightarrow 1^-$ in (4.4) and we use the basic properties of q -derivative and q -integral (see [17, 48])

$$\lim_{q \rightarrow 1} {}_a D_q f(t) = f'(t), \quad \lim_{q \rightarrow 1} \int_a^b f(t) {}_a d_q t = \int_a^b f(t) {}_a dt,$$

with the equalities

$$\lim_{q \rightarrow 1} (A_1(q) + A_4(q)) = \lim_{q \rightarrow 1} \left(\frac{1 + 12q + 12q^2 + 36q^3}{216(1+q)(1+q+q^2)} + \frac{12q^2 + 12q + 5}{216(1+q)(1+q+q^2)} \right) = \frac{5}{72},$$

and

$$\lim_{q \rightarrow 1} (A_2(q) + A_5(q)) = \lim_{q \rightarrow 1} \left(\frac{18q^2 + 18q - 7}{216(1+q)(1+q+q^2)} + \frac{18q^2 + 18q + 25}{216(1+q)(1+q+q^2)} \right) = \frac{5}{72},$$

then we obtain the inequality

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} [|f'(a)| + |f'(b)|],$$

which appeared in [80].

Theorem 4.1. ([81], Theorem 1) *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. If $|{}_aD_q f|$ is a convex function and a q -integrable function with $0 < q < 1$, then*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t) {}_a d_q t \right| \\ & \leq \frac{(b-a)}{12} \left[\frac{2q^2 + 2q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_{p,q} f(b)| + \frac{1}{3} \cdot \frac{6q^3 + 4q^2 + 4q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_{p,q} f(a)| \right]. \end{aligned} \quad (4.5)$$

Now, we give the following example to assert that the left side of (4.5) is correct, but the right side of (4.5) is not correct.

Example 4.1. *Define function $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 1 - x$. Then $|{}_a D_q f(x)| = |{}_a D_q(1 - x)| = 1$ is a convex function and a q -integrable function on $[0, 1]$. Then f satisfies the conditions of Theorem 4.1 with $q = \frac{1}{2}$, so the left side of (4.5) becomes*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & = \left| \frac{1}{6} \left[f(0) + 4f\left(\frac{0+1}{2}\right) + f(1) \right] - \frac{1}{1-0} \int_0^1 (1-x) {}_0 d_{\frac{1}{2}} x \right| \\ & = \left| \frac{1}{6} [1 + 2 + 0] - \int_0^1 (1-x) {}_0 d_{\frac{1}{2}} x \right| = \left| \frac{1}{2} - \frac{1}{3} \right| = \frac{1}{6} \end{aligned}$$

and the right side of (4.5) becomes

$$\begin{aligned} & \frac{(b-a)}{12} \left[\frac{2q^2 + 2q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_{p,q} f(b)| + \frac{1}{3} \cdot \frac{6q^3 + 4q^2 + 4q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_{p,q} f(a)| \right] \\ & = \frac{(1-0)}{12} \left[\frac{2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 1}{\frac{1}{8} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 1} \cdot |1| + \frac{1}{3} \cdot \frac{6 \cdot \frac{1}{8} + 4 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} + 1}{\frac{1}{8} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 1} \cdot |1| \right] = \frac{7}{54}. \end{aligned}$$

This implies that

$$\frac{1}{6} \not\leq \frac{7}{54}.$$

Therefore, inequality (4.5) is not correct.

Remark 4.3. ([81], Lemmas 4 and 5) *The established inequality (4.5) gives the results involving q -integrals, $0 < q < 1$, as follows*

$$\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{1}{6} \right| d_q t = \frac{36q^3 + 12q^2 + 12q + 1}{216(q^3 + 2q^2 + 2q + 1)}, \quad (4.6)$$

and

$$\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| d_q t = \frac{5 + 12q + 12q^2}{216(q^3 + 2q^2 + 2q + 1)}. \quad (4.7)$$

However, the equality (4.6) is not correct for the case of $0 < q < \frac{1}{3}$, but is correct for $\frac{1}{3} \leq q < 1$. Equality (4.7) is not correct for the case of $0 < q < \frac{5}{6}$, but is correct for $\frac{5}{6} \leq q < 1$.

Deng et al. [59] modified the equalities (4.6) and (4.7) to be valid for $0 < q < \frac{1}{3}$ and $0 < q < \frac{5}{6}$, respectively, as follows

$$\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{qt} = \frac{1-4q^3}{24(q^3+2q^2+2q+1)}, \quad (4.8)$$

and

$$\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| d_{qt} = \frac{-5+8q+8q^2-8q^3}{216(q^3+2q^2+2q+1)}. \quad (4.9)$$

In the following, we show a new result involving (p, q) -integrals of equalities (4.8) and (4.9) for $0 < q < \frac{1}{3}$ and $0 < q < \frac{5}{6}$, respectively. If $p = 1$, then we give the correct results of quantum integral inequalities.

Lemma 4.1. *If $0 < q < p \leq 1$ are constants, then*

$$\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} = \frac{2p^3 - p^2 + 2pq - 2p^2q + 2q^2 - 2pq^2 - 4q^3}{24(q^3 + 2pq^2 + 2p^2q + p^3)} \quad (4.10)$$

holds for all $0 < q < \frac{1}{3}$, and

$$\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| d_{p,qt} = \frac{10p^3 - 15p^2 + 2p^2q + 6pq + 2pq^2 + 6q^2 - 8q^3}{24(q^3 + 2pq^2 + 2p^2q + p^3)} \quad (4.11)$$

holds for all $0 < q < \frac{5}{6}$.

Proof. Using Definition 2.6, Theorem 2.1 and Lemma 2.1, we have

$$\begin{aligned} \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} &= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,qt} - \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_{p,qt} \\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{6} - qt \right) d_{p,qt} - \int_0^{\frac{1}{2}} t \left(\frac{1}{6} - qt \right) d_{p,qt} \\ &= \frac{p-2q}{12(p+q)} - \frac{p^2-2pq-2q^2}{24(p+q)(p^2+pq+q^2)} \\ &= \frac{2p^3 - p^2 + 2pq - 2p^2q + 2q^2 - 2pq^2 - 4q^3}{24(q^3 + 2pq^2 + 2p^2q + p^3)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| d_{p,qt} &= \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| d_{p,qt} - \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| d_{p,qt} \\ &= \int_{\frac{1}{2}}^1 \left(\frac{5}{6} - qt \right) d_{p,qt} - \int_{\frac{1}{2}}^1 t \left(\frac{5}{6} - qt \right) d_{p,qt} \end{aligned}$$

$$\begin{aligned}
&= \left(\int_0^1 \left(\frac{5}{6} - qt \right) d_{p,qt} - \int_0^{\frac{1}{2}} \left(\frac{5}{6} - qt \right) d_{p,qt} \right) \\
&\quad - \left(\int_0^1 t \left(\frac{5}{6} - qt \right) d_{p,qt} - \int_0^{\frac{1}{2}} t \left(\frac{5}{6} - qt \right) d_{p,qt} \right) \\
&= \frac{5p - 4q}{12(p + q)} - \frac{15p^2 - 6pq - 6q^2}{24(p + q)(p^2 + pq + q^2)} \\
&= \frac{10p^3 - 15p^2 + 2p^2q + 6pq + 2pq^2 + 6q^2 - 8q^3}{24(q^3 + 2p^2q + 2p^2q + p^3)}.
\end{aligned}$$

Therefore, the proof is completed. \square

Next, we show a new result involving (p, q) -integrals of equalities (4.6) and (4.7) for $\frac{1}{3} \leq q < 1$ and $\frac{5}{6} \leq q < 1$, respectively. If $p = 1$, then we give the correct results of quantum integral inequalities.

Lemma 4.2. *If $0 < q < p \leq 1$ are constants, then*

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} \\
&= \frac{1}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)} \cdot [36q^5 + 18pq^4 - 6q^4 + 18p^2q^3 + 6pq^3 \\
&\quad - 12q^3 + 33p^2q^2 - 18p^3q^2 - 12pq^2 - 2q^2 - 2pq + 2q - 2p^2 + 2p]
\end{aligned} \tag{4.12}$$

holds for all $\frac{1}{3} \leq q < 1$, and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| d_{p,qt} \\
&= \frac{1}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)} \cdot [-270pq^4 + 282q^4 - 270p^2q^3 + 582pq^3 \\
&\quad - 300q^3 - 270p^3q^2 + 825p^2q^2 - 300pq^2 - 250q^2 - 250p^2 + 250p]
\end{aligned} \tag{4.13}$$

holds for all $\frac{5}{6} \leq q < 1$.

Proof. Using Definition 2.6, Theorem 2.1 and Lemma 2.1, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| d_{p,qt} \\
&= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_{p,qt} - \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| d_{p,qt} \\
&= \left(\int_0^{\frac{1}{6q}} \left(\frac{1}{6} - qt \right) d_{p,qt} + \int_{\frac{1}{6q}}^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a d_{p,qt} \right) \\
&\quad - \left(\int_0^{\frac{1}{6q}} t \left(\frac{1}{6} - qt \right) d_{p,qt} + \int_{\frac{1}{6q}}^{\frac{1}{2}} t \left(qt - \frac{1}{6} \right) {}_a d_{p,qt} \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\int_0^{\frac{1}{6q}} \left(\frac{1}{6} - qt \right) d_{p,q}t + \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) d_{p,q}t - \int_0^{\frac{1}{6q}} \left(qt - \frac{1}{6} \right) d_{p,q}t \right) \\
&\quad - \left(\int_0^{\frac{1}{6q}} t \left(\frac{1}{6} - qt \right) d_{p,q}t + \int_0^{\frac{1}{2}} t \left(qt - \frac{1}{6} \right) d_{p,q}t \right) - \int_0^{\frac{1}{6q}} t \left(qt - \frac{1}{6} \right) d_{p,q}t \\
&= \frac{6q^2 - 3pq + 2q + 2p - 2}{36q(p+q)} - \frac{18q^4 + 18pq^3 - 9p^2q^2 + 2q^2 + 2pq - 2q + 2p^2 - 2p}{216q^2(p+q)(p^2 + pq + q^2)} \\
&= \frac{1}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)} \cdot [36q^5 + 18pq^4 - 6q^4 + 18p^2q^3 + 6pq^3 - 12q^3 \\
&\quad + 33p^2q^2 - 18p^3q^2 - 12pq^2 - 2q^2 - 2pq + 2q - 2p^2 + 2p].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_a d_{p,q}t \\
&= \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_a d_{p,q}t - \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_a d_{p,q}t \\
&= \left(\int_{\frac{1}{2}}^{\frac{5}{6q}} \left(\frac{5}{6} - qt \right) {}_a d_{p,q}t + \int_{\frac{5}{6q}}^1 \left(qt - \frac{5}{6} \right) {}_a d_{p,q}t \right) \\
&\quad - \left(\int_{\frac{1}{2}}^{\frac{5}{6q}} t \left(\frac{5}{6} - qt \right) {}_a d_{p,q}t + \int_{\frac{5}{6q}}^1 t \left(qt - \frac{5}{6} \right) {}_a d_{p,q}t \right) \\
&= \frac{-45pq + 50q + 50p - 50}{36q(p+q)} \\
&\quad - \frac{18q^4 + 18pq^3 - 225p^2q^2 + 250q^2 + 250pq - 250q + 250p^2 - 250p}{216q^2(p+q)(p^2 + pq + q^2)} \\
&= \frac{1}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)} \cdot [-270pq^4 + 282q^4 - 270p^2q^3 + 582pq^3 - 300q^3 \\
&\quad - 270p^3q^2 + 825p^2q^2 - 300pq^2 - 250q^2 - 250p^2 + 250p].
\end{aligned}$$

Therefore, the proof is completed. \square

In the following, we provide a modified version involving (p, q) -integral of inequality (4.5).

Corollary 4.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function. If $|{}_a D_{p,q}f|$ is a (p, q) -integrable function and a convex function, then

$$\begin{aligned}
&\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(t) {}_a d_{p,q}t \right| \\
&\leq (b-a) \left[[C_1(p, q)] |{}_a D_{p,q}f(a)| + [C_2(p, q)] |{}_a D_{p,q}f(b)| \right], \tag{4.14}
\end{aligned}$$

where $C_1(p, q)$ and $C_2(p, q)$ are defined by

$$C_1(p, q) = \begin{cases} \frac{-3q^3 + 2q^2 + 2pq + 3p^3 - 4p^2}{6(q^3 + 2pq^2 + 2p^2q + p^3)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{\begin{bmatrix} -36q^5 + 36pq^4 + 48q^4 + 36p^2q^3 + 60pq^3 - 12q^3 + 72p^3q^2 \\ -102p^2q^2 - 12pq^2 - 2q^2 - 2pq + 2q + 2p - 2p^2 \end{bmatrix}}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)}, & \text{for } \frac{1}{3} \leq q < \frac{5}{6}; \\ \frac{\begin{bmatrix} 36q^5 - 252pq^4 + 276q^4 - 252p^2q^3 + 588pq^3 - 312q^3 - 288p^3q^2 \\ +858p^2q^2 - 312pq^2 - 252q^2 - 2pq + 2q - 252p^2 + 252p \end{bmatrix}}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases}$$

and

$$C_2(p, q) = \begin{cases} \frac{-q^2 - pq + 2p^2}{3(q^3 + 2pq^2 + 2p^2q + p^3)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{-36q^4 - 36pq^3 + 126p^2q^2 + 2q^2 + 2pq - 2q + 2p^2 - 2p}{216(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)}, & \text{for } \frac{1}{3} \leq q < \frac{5}{6}; \\ \frac{2q^4 + 2pq^3 - 13p^2q^2 + 14q^2 + 14pq - 14q + 14p^2 - 14p}{12(q^5 + 2pq^4 + 2p^2q^3 + p^3q^2)}, & \text{for } \frac{5}{6} \leq q < 1. \end{cases}$$

Proof. Using Corollary 4.2 to show a simple calculation in the expressions $C_1(p, q) = A_1(p, q) + A_4(p, q)$ and $C_2(p, q) = A_2(p, q) + A_5(p, q)$, where $A_1(p, q)$, $A_2(p, q)$, $A_4(p, q)$, and $A_5(p, q)$ are given in Theorem 3.1, we obtain the inequality (4.14). Therefore, the proof is completed. \square

Remark 4.4. If $p = 1$, then (4.14) reduces to

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t) {}_aD_{p,q}t \right| \leq (b-a) \left[[C_1(p, q)] |{}_aD_{p,q}f(a)| + [C_2(p, q)] |{}_aD_{p,q}f(b)| \right], \quad (4.15)$$

where $C_1(p, q)$ and $C_2(p, q)$ are defined by

$$C_1(q) = \begin{cases} \frac{-3q^3 + 2q^2 + 2q - 1}{6(q^3 + 2q^2 + 2q + 1)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{-9q^3 + 21q^2 + 21q - 11}{54(q^3 + 2q^2 + 2q + 1)}, & \text{for } \frac{1}{3} \leq q < \frac{5}{6}; \\ \frac{6q^3 + 4q^2 + 4q + 1}{36(q^3 + 2q^2 + 2q + 1)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases}$$

and

$$C_2(q) = \begin{cases} \frac{-q^2 - q + 2}{3(q^3 + 2q^2 + 2q + 1)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{-9q^2 - 9q + 32}{54(q^3 + 2q^2 + 2q + 1)}, & \text{for } \frac{1}{3} \leq q < \frac{5}{6}; \\ \frac{2q^2 + 2q + 1}{12(q^3 + 2q^2 + 2q + 1)}, & \text{for } \frac{5}{6} \leq q < 1, \end{cases}$$

which appeared in [59].

Example 4.2. Define function $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 1 - x$. Then $|{}_a D_{p,q} f(x)| = |{}_a D_{p,q}(1 - x)| = 1$ is a convex function and a (p, q) -integrable function on $[0, 1]$. Then f satisfies the conditions of Corollary 4.4 with $p = 1$ and $q = \frac{1}{2}$, so the left side of (4.14) becomes

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x \right| \\ &= \left| \frac{1}{3} \left[\frac{f(0) + f(1)}{2} + 2f\left(\frac{0+1}{2}\right) \right] - \frac{1}{1 \cdot (1-0)} \int_0^{1 \cdot 1 + (1-1)0} f(t) d_{1, \frac{1}{2}} x \right| \\ &= \left| \frac{1}{3} \left[\frac{1}{2} + 1 \right] - \int_0^1 (1-x) d_{1, \frac{1}{2}} x \right| = \left| \frac{1}{2} - \frac{1}{3} \right| = \frac{1}{6}, \end{aligned}$$

and the right side of (4.14) becomes

$$\begin{aligned} & (b-a) \left[[C_1(p, q)] |{}_a D_{p,q} f(a)| + [C_2(p, q)] |{}_a D_{p,q} f(b)| \right] \\ &= (1-0) \left[\left[C_1\left(1, \frac{1}{2}\right) \right] |{}_0 D_{1, \frac{1}{2}} f(0)| + \left[C_2\left(1, \frac{1}{2}\right) \right] |{}_0 D_{1, \frac{1}{2}} f(1)| \right] \\ &= \left[\left[\frac{29}{1134} \right] \cdot |1| + \left[\frac{101}{567} \right] \cdot |1| \right] = \frac{11}{54}. \end{aligned}$$

This implies that

$$\frac{1}{6} \leq \frac{11}{54},$$

which demonstrates the result described in Corollary 4.3.

Corollary 4.4. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a (p, q) -differentiable function on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$. If $|{}_a D_{p,q} f|$ is a (p, q) -integrable function and a generalized strongly preinvex function with modulus $\mu \geq 0$, then

$$\begin{aligned} & \left| \int_a^{a+p\eta(b,a)} f(t) {}_a d_{p,q} t \right| \leq p\eta(b, a) \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \right| \\ & + p\eta^2(b, a) \left[[A_1(p, q) + A_4(p, q)] |{}_a D_{p,q} f(a)| + [A_2(p, q) + A_5(p, q)] |{}_a D_{p,q} f(b)| \right. \\ & \left. - \mu(A_3(p, q) + A_6(p, q))\eta^2(b, a) \right], \end{aligned} \quad (4.16)$$

where $A_i(p, q)$, $i = 1, 2, \dots, 6$ are given in Theorem 3.1.

Proof. We have

$$\begin{aligned} \left| \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p, q} t \right| &\leq \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \right| \\ &+ \left| \frac{1}{p\eta(b, a)} \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p, q} t \right. \\ &\left. - \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \right|. \end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned} \frac{1}{p\eta(b, a)} \left| \int_a^{a+p\eta(b, a)} f(t) {}_a d_{p, q} t \right| &\leq \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \right| \\ &+ \eta(b, a) \left[[A_1(p, q) + A_4(p, q)] |{}_a D_{p, q} f(a)| + [A_2(p, q) \right. \\ &\left. + A_5(p, q)] |{}_a D_{p, q} f(b)| - \mu(A_3(p, q) + A_6(p, q)) \eta^2(b, a) \right] \end{aligned}$$

Multiplying the above equality with $p\eta(b, a)$, we obtain the required result. Therefore, the proof is completed. \square

Remark 4.5. If $p = 1$, then (4.16) reduces to

$$\begin{aligned} \left| \int_a^{a+\eta(b, a)} f(t) {}_a d_q t \right| &\leq \eta(b, a) \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] \right| \\ &+ \eta^2(b, a) \left[[A_1(q) + A_4(q)] |{}_a D_q f(a)| + [A_2(q) + A_5(q)] |{}_a D_q f(b)| \right. \\ &\left. - \mu(A_3(q) + A_6(q)) \eta^2(b, a) \right], \end{aligned}$$

where $A_i(q)$, $i = 1, 2, 3, \dots, 6$ are given in Remark 3.2, which appeared in [59].

5. Conclusions

In this work, we established new Simpson type inequalities via (p, q) -integrals. The presented results in this study generalize and extend some previous inequalities in the literature of Simpson type inequalities. Moreover, the obtained results were used to study some special cases, namely preinvex function and convex function, and some examples were given to illustrate the investigated results.

Author's contributions

All authors contributed equally to this article. They read and approved the final manuscript.

Acknowledgements

This work is supported by the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [grant number B05F630104]. The first author is supported by Development and Promotion of Science and Technology talents project (DPST), Thailand.

Conflict of interest

The authors declare that they have no competing interest.

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