AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 9100–9108.
DOI:10.3934/math.2021528
Received: 26 April 2021
Accepted: 11 June 2021
Published: 17 June 2021

Ground state solution for a class of magnetic equation with general convolution nonlinearity

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Abstract: In this paper, we consider the following magnetic Laplace nonlinear Choquard equation

$$-\Delta_A u + V(x)u = (I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u, \text{ in } \mathbb{R}^N,$$

where $u : \mathbb{R}^N \to C$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a vector potential, $N \geq 3$, $\alpha \in (N-2, N)$, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar potential function and I_α is a Riesz potential of order $\alpha \in (N-2, N)$. Under certain assumptions on $A(x)$, $V(x)$ and $f(t)$, we prove that the equation has at least a ground state solution by variational methods.

Keywords: magnetic Laplace operator; ground state solution; Nehari manifold
Mathematics Subject Classification: 35A15, 35J35, 35J60, 35R11

1. Introduction

In this article, we study the following magnetic Laplace nonlinear Choquard equation

$$\begin{cases} -\Delta_A u + V(x)u = (I_\alpha * F(|u|))\frac{f(|u|)}{|u|}u, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\Delta_A u := (\nabla - iA)^2 u$ is the magnetic Laplace operator. Here $u : \mathbb{R}^N \to C$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a vector magnetic potential, $N \geq 3$, $F(t) = \int_0^t f(s)\mathrm{d}s$, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar potential function and I_α is a Riesz potential whose order is $\alpha \in (N-2, N)$ defined by $I_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}$, where Γ is the Gamma function. $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a continuous, bounded potential function satisfying:

(V1) $\inf_{\mathbb{R}^N} V(x) > 0$,

(V2) there exist a constant $V_\infty > 0$ such that for all $x \in \mathbb{R}^N$,

$$0 < V(x) \leq \liminf_{|y|\to+\infty} V(y) = V_\infty < +\infty.$$

We also suppose A satisfies:
(A1) $\liminf_{|x|\to+\infty} A(x) = A_\infty$,
(A2) $A \in L^\upsilon(\mathbb{R}^N, \mathbb{R}^N), \upsilon > N \geq 3$,
(AV) $|A(y)|^2 + V(y) < |A_\infty|^2 + V_\infty$.
Moreover, we assume that the function $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies:
$(f1)$ $f(t) = o(t^{\frac{\alpha}{N}})$ as $t \to 0$,
$(f2)$ $\lim_{|t|\to+\infty} \frac{f(t)}{t^{\frac{\alpha+2}{N-2}}} = 0$,
$(f3)$ $\frac{f(t)}{t}$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$.
$(f4)$ $f(t)$ is increasing on $\mathbb{R}$.

It should be noted that there is a lot of literature on the competition phenomena for elliptic equations without magnetic potential in different situations, i.e. $A \equiv 0$. Actually, when $A \equiv 0$ it conduces to the Choquard equation. There is a huge collections of articles on the subject and some good reviews of the Choquard equation can be found in [1–9].

On the other hand, there are works concerning the following Schrödinger equations with magnetic field recently:

$$-\Delta_A u + V(x)u = |u|^{p-2}u, \text{ in } \Omega \subset R^N, \ N \geq 2. \tag{1.2}$$

Here $u : \Omega \to C$, $-\Delta_A u := (-i\nabla + A)^2 u$, $2 < p \leq 2^*$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or 2. Besides, $A : \Omega \to \mathbb{R}^N$ and $V : \Omega \to \mathbb{R}$ are smooth.

To the best of our knowledge, the first paper in which problem (1.2) has been studied maybe Esteban-Lions [10]. They have used the concentration-compactness principle and minimization arguments to prove the existence of solutions for $N = 2$ and $N = 3$. More recently, applying constrained minimization and a minimax-type argument, Arioli-Szulkin [11] considered the equation in a magnetic filed. They established the existence of nontrivial solutions both in the critical and in the subcritical case, provided that some technical conditions relating to A and V were assumed. We also refer to [12, 13] for other results related to problem (1.1) in the presence of the magnetic field when the nonlinearity has a subcritical growth. Besides, we must mention the works [14, 15] for the critical case and also refer to the recent papers [16–18] for the study of various classes of PDEs with magnetic potential.

Inspired by the above works, we want to research the Eq (1.2) with general convolution term as the right-hand side, i.e. Eq (1.1). Our aim of this paper is to prove the existence of a ground state solution for problem (1.1), that is a nontrivial solution with minimal energy.

Notice that if we define

$$\tilde{f}(t) = \begin{cases} \frac{f(t)}{t}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

our assumptions assure that $\tilde{f}(t)$ is continuous. Therefore, Eq (1.1) can be rewritten in the form

$$-\Delta_A u + V(x)u = (I_\alpha * F(|u|))\tilde{f}(|u|)u. \tag{1.3}$$

The right-hand side of problem (1.3) generalizes the term $(\frac{1}{|x|^\alpha} * |u|^p)|u|^{p-2}u$, which was studied by Cingolani, Clapp and Secchi in [19]. Similar problems were also studied in [20–22]. Especially, it is worth mentioning that in [23], the authors obtained the ground state solution of the following Eq (1.4)

$$-(\nabla + iA)^2u + V(x)u = (\frac{1}{|x|^\alpha} * F(|u|))\frac{f(|u|)}{|u|}u, \text{ in } \mathbb{R}^N, \tag{1.4}$$

which can be rewritten in the form

$$-(\nabla + iA)^2u + V(x)u = (\frac{1}{|x|^\alpha} * F(|u|))\tilde{f}(|u|)u. \tag{1.5}$$

They considered the "limit problem" of problem (1.4), then by the splitting lemma, they proved that (1.4) has at least a ground state solution. In our paper, we improve the growth condition of f to the critical case, and generalizes the convolution term to a more general case. Most importantly, we get the ground state solution in a more straightforward way which is completely different from [23].

Our main result is as follows:

Theorem 1.1. *If $\alpha \in (N-2, N)$, (A1), (A2), (V1), (V2), (AV) are valid, and $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies (f1)-(f3), then problem(1.1) has at least a ground state solution.*

Now we define $\nabla_A u = -i\nabla u - Au$ and consider the space

$$H^1_{A,V} = \{u \in L^2(\mathbb{R}^N, C) : \nabla_A u \in L^2(\mathbb{R}^N, C)\}$$

equipped with scalar product

$$\langle u, v\rangle_{A,V} = \Re e \int_{\mathbb{R}^N} (\nabla_A u \cdot \overline{\nabla_A v} + V(x)u\bar{v})\mathrm{d}x.$$

Therefore

$$||u||^2_{A,V} = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V(x)|u|^2)\mathrm{d}x$$

which is an equivalent norm to the norm obtained by considering $V \equiv 1$, see [6].

Hereafter for the convenience of narration, we will use the following notations:

- $L^r(\mathbb{R}^N)(1 \le r < \infty)$ denotes the Lebesgue space in which the norm is defined as follows

$$|u|_r = (\int_{\mathbb{R}^N} |u|^r \mathrm{d}x)^{1/r},$$

- $C, C_\varepsilon, C_1, C_2, \ldots$ denote positive constants which are possibly different in different lines.

2. Preliminaries

In this section, we will give some very important inequalities and lemmas.

Lemma 2.1. [10] *Assume $u \in H^1_{A,V}$, then $|u| \in H^1(\mathbb{R}^N)$ and the diamagnetic inequality holds $\left|\nabla|u|(x)\right| \le |\nabla_A u(x)|$.*

Remark 2.2. It is well known that the embedding $H^1_{A,V} \hookrightarrow L^r(\mathbb{R}^N, C)$ is continuous for $r \in [1, 2^*]$.

Lemma 2.3. *Assume (f1)–(f4) hold, then we have*

(1) *for all* $\varepsilon > 0$, *there is a* $C_\varepsilon > 0$ *such that* $|f(t)| \leq \varepsilon|t|^{\frac{\alpha}{N}} + C_\varepsilon|t|^{\frac{\alpha+2}{N-2}}$ *and* $|F(t)| \leq \varepsilon|t|^{\frac{N+\alpha}{N}} + C_\varepsilon|t|^{\frac{N+\alpha}{N-2}}$,
(2) *for all* $\varepsilon > 0$, *there is a* $C_\varepsilon > 0$ *such that for every* $p \in (2, 2^*)$, $|F(t)| \leq \varepsilon(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha}{N-2}}) + C_\varepsilon|t|^{\frac{p(N+\alpha)}{2N}}$, *and* $|F(t)|^{\frac{2N}{N+\alpha}} \leq \varepsilon(|t|^2 + |t|^{\frac{2N}{N-2}}) + C_\varepsilon|t|^p$,
(3) *for any* $s \neq 0$, $sf(s) > 2F(s)$ *and* $F(s) > 0$.

Proof. One can easily obtain the results by elementary calculation.□

Lemma 2.4. [23] *Let* $O \subset \mathbb{R}^N$ *be any open set, for* $1 < p < \infty$, *and* $\{f_n\}$ *be a bounded sequence in* $L^p(O, C)$ *such that* $f_n(x) \rightharpoonup f(x)$ *a.e., then* $f_n(x) \rightharpoonup f(x)$.

Lemma 2.5. [23] *Suppose that* $u_n \rightharpoonup u_0$ *in* $H^1_{A,V}(\mathbb{R}^N, C)$, *and* $u_n(x) \to u_0(x)$ *a.e. in* $\mathbb{R}^N$, *then* $I_\alpha * F(|u_n(x)|) \rightharpoonup I_\alpha * F(|u_0(x)|)$ *in* $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$.

Corollary 2.6. *Suppose that* $u_n \rightharpoonup u_0$ *in* $H^1_{A,V}(\mathbb{R}^N, C)$, *then* $\Re e \int_{\mathbb{R}^N} I_\alpha * F(|u_n|)\tilde{f}(|u_n|)u_n\overline{\varphi} \to \Re e \int_{\mathbb{R}^N} I_\alpha * F(|u_0|)\tilde{f}(|u_0|)u_0\overline{\varphi}$ *for* $\varphi \in C_c^\infty(\mathbb{R}^N, C)$.

Lemma 2.7. *(Hardy-Littlewood-Sobolev inequality [6]). Let* $0 < \alpha < N$, $p, q > 1$ *and* $1 \leq r < s < \infty$ *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \ \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

(1) *For any* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$, *one has*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \leq C(N, \alpha, p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}.$$

(2) *For any* $f \in L^r(\mathbb{R}^N)$ *one has*

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C(N, \alpha, r)\|f\|_{L^r(\mathbb{R}^N)}.$$

Remark 2.8. By Lemma 2.3 (1), Lemma 2.7 (1) and Sobolev imbedding theorem, we can get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x \right| &\leq C|F(u)|^2_{\frac{2N}{N+\alpha}} \\ &\leq C\Big[\int_{\mathbb{R}^N} (|u|^{\frac{N+\alpha}{N}} + |u|^{\frac{N+\alpha}{N-2}})^{\frac{(2N)}{N+\alpha}} \mathrm{d}x \Big]^{\frac{N+\alpha}{N}} \\ &\leq C\Big[\int_{\mathbb{R}^N} (|u|^2 + |u|^{\frac{2N}{N-2}})\mathrm{d}x \Big]^{\frac{N+\alpha}{N}} \\ &\leq C(\|u\|_{A,V}^{\frac{2N+2\alpha}{N}} + \|u\|_{A,V}^{\frac{2N+2\alpha}{N-2}}). \end{aligned} \tag{2.1}$$

3. Variational formulation

The energy functional associated to problem (1.1) is given by:

$$J_{A,V}(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla_A u|^2 + V(x)u^2]\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(|u|))F(|u|)\mathrm{d}x. \tag{3.1}$$

The derivative of the energy functional $J_{A,V}(u)$ is given by

$$\langle J'_{A,V}(u), \varphi \rangle = \langle u, \varphi \rangle_{A,V} - \Re e \int_{\mathbb{R}^N} (I_\alpha * F(|u|))\tilde{f}(|u|)u\overline{\varphi}\mathrm{d}x. \tag{3.2}$$

Thus,

$$\langle J'_{A,V}(u), u\rangle = \int_{\mathbb{R}^N}[|\nabla_A u|^2 + V(x)u^2]\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * F(|u|))f(|u|)|u|\mathrm{d}x. \tag{3.3}$$

Now, we can prove the following results.

Lemma 3.1. *The functional $J_{A,V}$ possesses the mountain-pass geometry, that is*

(1) *there exist $\rho, \delta > 0$ such that $J_{A,V} \geq \delta$ for all $\|u\| = \rho$;*
(2) *for any $u \in H^1_{A,V}(\mathbb{R}^N, C)\backslash\{0\}$, there exist $\tau \in (0, +\infty)$ such that $\|\tau u\| > \rho$ and $J_{A,V}(\tau u) < 0$.*

Proof. (1) By Lemma 2.7 (1) and Lemma 2.3, one can get

$$J_{A,V}(u) \geq \frac{1}{2}\|u\|^2_{A,V} - C(\|u\|^{\frac{2N+2\alpha}{N}}_{A,V} + \|u\|^{\frac{2N+2\alpha}{N-2}}_{A,V}).$$

Thus there exist $\rho, \delta > 0$ such that $J_{A,V} \geq \delta$ for all $\|u\| = \rho > 0$ small enough.
(2) For any fixed $u_0 \in H^1_{A,V}\backslash\{0\}$, consider the function $g_{u_0}(t) : (0, +\infty) \to \mathbb{R}$ given by

$$g_{u_0}(t) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(I_\alpha * F(\frac{t|u_0|}{\|u_0\|_{A,V}})\Big)F(\frac{t|u_0|}{\|u_0\|_{A,V}})\mathrm{d}x, \tag{3.4}$$

then

$$\begin{aligned} g'_{u_0}(t) &= \int_{\mathbb{R}^N}\Big(I_\alpha * F(\frac{t|u_0|}{\|u_0\|_{A,V}})\Big)f(\frac{t|u_0|}{\|u_0\|_{A,V}})\frac{|u_0|}{\|u_0\|_{A,V}}\mathrm{d}x \\ &= \frac{4}{t}\int_{\mathbb{R}^N}\frac{1}{2}\Big(I_\alpha * F(\frac{t|u_0|}{\|u_0\|_{A,V}})\Big)\frac{1}{2}f(\frac{t|u_0|}{\|u_0\|_{A,V}})\frac{t|u_0|}{\|u_0\|_{A,V}}\mathrm{d}x \\ &\geq \frac{4}{t}g_{u_0}(t) > 0, (t > 0). \end{aligned} \tag{3.5}$$

Thus, $lng_{u_0}(t)\Big|_1^{\tau\|u_0\|_{A,V}} \geq 4lnt\Big|_1^{\tau\|u_0\|_{A,V}}$. So $\frac{g_{u_0}(\tau\|u_0\|_{A,V})}{g_{u_0}(1)} \geq (\|u_0\|_{A,V})^4$ which implies that $g_{u_0}(\tau\|u_0\|_{A,V}) \geq M(\|u_0\|_{A,V})^4$ for a constant $M > 0$. Then we can get

$$J_{A,V}(\tau u_0) = \frac{\tau^2}{2}\|u_0\|^2_{A,V} - g_{u_0}(\tau\|u_0\|_{A,V}) \leq C_1\tau^2 - C_2\tau^4 \tag{3.6}$$

yields that $J_{A,V}(\tau u_0) < 0$ when τ is large enough.□

Hence we can define the mountain-pass level of $J_{A,V}$:

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{A,V}(\gamma(t)) > 0,$$

where: $\Gamma = \{\gamma \in C([0, 1], H^1_{A,V}(\mathbb{R}^N, C)) : \gamma(0) = 0, J_{A,V}(\gamma(1)) < 0\}$.

Now we recall the Nehari manifold

$$\mathcal{N}_\alpha := \{u \in H^1_{A,V}(\mathbb{R}^N, C)\backslash\{0\} : \langle J'_{A,V}(u), u\rangle = 0\}.$$

Let $c_\alpha = \inf_{u\in\mathcal{N}_\alpha} J_{A,V}(u)$, Moreover by the similar argument as Chapter 4 [24], we have the following characterization

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{A,V}(\gamma(t)) = c_\alpha = \inf_{u\in\mathcal{N}_\alpha} J_{A,V}(u) = c^* = \inf_{u\in H^1_{A,V}(\mathbb{R}^N,C)\backslash\{0\}}\max_{t\geq 0} J_{A,V}(tu).$$

Remark 3.2. If we set $\Phi(t) = \frac{1}{2}\|tu\|^2_{A,V} - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(|tu|))F(|tu|)\mathrm{d}x$, the proof of Lemma 3.1 assures that $\Phi(t) > 0$ for t small enough, and $\Phi(t) < 0$ for t large enough. Besides $g'_u(t) > 0$ if $t > 0$, we can get that $\max_{t\geq 0}\Phi(t)$ is achieved at a unique $t_u > 0$. Furthermore, $\Phi'(t_u) = 0$ implies that $t_u u \in \mathcal{N}_\alpha$ and the map $u \to t_u(u \neq 0)$ is continuous.

4. Ground state solution for problem (1.1)

In this section, we prove the main theorem.

Proof of Theorem 1.1. Let $\{u_n\}$ be minimizing sequence given as a consequence of Lemma 3.1 i.e. $\{u_n\} \subset H^1_{A,V}$ such that $J'_{A,V}(u_n) \to 0$, $J_{A,V}(u_n) \to c$, where $c = c_\alpha = \inf_{u\in\mathcal{N}_\alpha} J_{A,V}(u) = c^* = \inf_{u\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} J_{A,V}(tu)$. Then we have

$$\begin{aligned} c_\alpha + o(1) &= J_{A,V}(u_n) - \frac{1}{4}\langle J'_{A,V}(u_n), u_n\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla_A u_n|^2 + V(x)|u_n|^2]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(|u_n|))[f(|u_n|)|u_n| - 2F(|u_n|)]\mathrm{d}x \\ &\geq \frac{1}{4}\|u_n\|^2_{A,V}. \end{aligned} \tag{4.1}$$

Consequence, $\{u_n\}$ is bounded. Then by standard methods we can get the convergence of $\{u_n\}$.
Next, let $\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2\mathrm{d}x$. We claim $\delta > 0$. On the contrary, by Lions' concentration compactness principle, we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. By Lemma 2.3(2), for any $\varepsilon > 0$ there exist a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} &\limsup_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x \\ &\leq C\limsup_{n\to\infty}\Big[\varepsilon(\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x + \int_{\mathbb{R}^N}|u_n|^{\frac{2N}{N-2}}\mathrm{d}x) + C_\varepsilon\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x\Big]^{\frac{N+\alpha}{N}} \\ &\leq C\Big[\varepsilon C_1 + C_\varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x\Big]^{\frac{N+\alpha}{N}} \\ &= C(\varepsilon C_2)^{\frac{N+\alpha}{N}}. \end{aligned}$$

Note that ε is arbitrary, we get

$$\int_{\mathbb{R}^N}(I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x = o(1).$$

Combining with $J'_{A,V}(u_n) \to 0$, we can get

$$\begin{aligned} o(1) &= \langle J'_{A,V}(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N}[|\nabla_A u_n|^2 + V(x)u_n^2]\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x, \end{aligned} \tag{4.2}$$

which implies that

$$\int_{\mathbb{R}^N}[|\nabla_A u_n|^2 + V(x)u_n^2]\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * F(|u_n|))f(|u_n|)|u_n|\mathrm{d}x + o(1) = 2o(1) \tag{4.3}$$

Then we have $\int_{\mathbb{R}^N}[|\nabla_A u_n|^2 + V(x)|u_n|^2]\mathrm{d}x \to 0$, which implies $u_n \to 0$ in $H^1_{A,V}$. We deduce that $c_\alpha = 0$, which contradicts to the fact that $c_\alpha > 0$. Hence $\delta > 0$ and there exist $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^p \mathrm{d}x \geq \frac{\delta}{2} > 0$. We set $v_n(x) = u_n(x + y_n)$, then $\|u_n\| = \|v_n\|$, $\int_{B_1(0)} |v_n|^p \mathrm{d}x > \frac{\delta}{2}$ and $J_{A,V}(v_n) \to c_\alpha = c$, $J'_{A,V}(v_n) \to 0$. Thus there exist a $v_0 \neq 0$ such that

$$\begin{cases} v_n \rightharpoonup v_0 \text{ in } H^1_{A,V}, \\ v_n \to v_0 \text{ in } L^s(\mathbb{R}^N), \ \forall\ s \in [2, 2^*) \\ v_n \to v_0 \ a.e. \text{ on } \mathbb{R}^N. \end{cases}$$

Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have $0 = \langle J'_{A,V}(v_n), \varphi\rangle + o(1) = \langle J'_{A,V}(v_0), \varphi\rangle$, which means v_0 is a solition of Eq (1.1).

On the other hand, combining with the Fatou Lemma, we can obtain

$$\begin{aligned} c_\alpha &= J_{A,V}(v_n) - \frac{1}{4}\langle J'_{A,V}(v_n), v_n\rangle + o(1) \\ &= \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla_A v_n|^2 + V(x)|v_n|^2]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(|v_n|))[f(|v_n|)|v_n| - 2F(|v_n|)]\mathrm{d}x + o(1) \\ &\geq \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla_A v_0|^2 + V(x)|v_0|^2]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(|v_0|))[f(|v_0|)|v_0| - 2F(|v_0|)]\mathrm{d}x + o(1) \\ &= J_{A,V}(v_0) - \frac{1}{4}\langle J'_{A,V}(v_0), v_0\rangle + o(1) \\ &= J_{A,V}(v_0) + o(1). \end{aligned} \tag{4.4}$$

At the same time, we know $c_\alpha \leq J_{A,V}(v_0)$ by the definition of c_α. Then we can deduce that v_0 is a ground state solution of Eq (1.1).□

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grant No. 11771198, 11361042, 11901276).

Conflict of interest

There is no conflict of interest.

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