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Research article

Local existence and lower bound of blow-up time to a Cauchy problem of a coupled nonlinear wave equations

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Abstract: In this paper, we consider a Cauchy problem of a coupled linearly-damped wave equations with nonlinear sources. In the whole space, we establish the local existence and show that there are solutions with negative initial energy that blow up in a finite time. Moreover, under some conditions on the initial data, we estimate a lower bound of that time.

Keywords: Cauchy problem; blow up; negative initial energy; nonlinear source; lower bound
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1. Introduction

In this work, we consider the following Cauchy problem of a coupled linearly-damped wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda_1 u_t(x,t) = f_1(u,v), & \text{in } \quad \mathbb{R}^n \times (0,\infty), \\ v_{tt} - \Delta v + \lambda_2 v_t(x,t) = f_2(u,v), & \text{in } \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \qquad x \in \mathbb{R}^n, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \qquad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where λ_1, λ_2 are two positive constants. The functions u_0, u_1, v_0, v_1 are the initial data to be specified later.

A single equation of problem (1.1), in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$u_{tt} - \Delta u + a u_t |u_t|^m = b|u|^\gamma u, \quad \Omega \times (0,\infty), \tag{1.2}$$

$m,\gamma \geq 0$, it is well known that, for $a = 0$, the source term $bu|u|^{\gamma}$, ($\gamma > 0$) causes finite time blow up of solutions with negative initial energy (see [1]). The interaction between the damping and the source terms was first considered by Levine [2, 3] in the linear damping case ($m = 0$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [4] extended Levine's result to the nonlinear damping case ($m > 0$). In their work, the authors introduced a different method and showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended to solutions with negative initial energy by Messaoudi [5] and others. For results of same nature, we refer the reader to Levine and Serrin [6], Vitillaro [7], Messaoudi and Said-Houari [8] and Messaoudi [9, 10].

For problem (1.2) in $\mathbb{R}^n$, we mention, among others, the work of Levine Serrin and Park [11]. Where the authors established a global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. Todorova [12, 13], showed that restriction in the case of compactly supported initial data is crucial for the blow-up result and it is not essential for the global existence. The local existence theorem could be proved without the requirement for the compact support of the data. Messaoudi [14], consider the Cauchy problem for the nonlinearly damped wave equation with nonlinear source

$$u_{tt} - \Delta u + au_t|u_t|^m = b|u|^{\gamma}u, \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where $a, b > 0$ and $m, \gamma > 2$. He proved that given any time $T > 0$, there exist always initial data with sufficiently negative initial energy, for which the solution blows up in time $\leq T$. This result improves an earlier one by Todorova [12]. Zhou [15], considered the Cauchy problem for a nonlinear wave equation with linear damping and source terms. He proved that the solution blows up in finite time even for vanishing initial energy if the initial datum u_0, u_1 satisfies $\int_{\mathbb{R}^n} u_0u_1dx \geq 0$.

In [16], Kafini and Messaoudi considered the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1}u, & x \in \mathbb{R}^n \quad , t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n \end{cases}$$

with negative initial energy and

$$\int_0^t g(s)ds < \frac{2p-2}{2p-1} \quad \text{and} \quad \int_{\mathbb{R}^n} u_0u_1dx \geq 0,$$

they proved a finite-time blow-up result. In [17], the same authors proved a blow-up result to a coupled system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f_1(u,v), & \text{in } \mathbb{R}^n \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds = f_2(u,v), & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

Systems in a bounded domains of $\mathbb{R}^n$ have been extensively studied by many authors. Messaoudi and Houari [18], established a global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. Agre and Rammaha [19] studied the following system

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-1} = f_1(u,v) \\ v_{tt} - \Delta v + v_t|v_t|^{r-1} = f_2(u,v), \end{cases} \tag{1.4}$$

in $\Omega \times (0, T)$ with initial and boundary conditions of Dirichlet type. They proved several results concerning local and global existence of a weak solution and showed that any weak solution with negative initial energy blows up in finite time. For more results on different systems, we refer to Messaoudi et al. [20], Santos [21], Cavalcanti et al. [22] and Wu [23].

Looking at our system in (1.1) and the system in (1.3), they differ by the type of damping only. But the proof is completely different due to nature of the viscoelastic damping. Although the damping in (1.4) is nonlinear, but it is proved in a bounded domain $\Omega \subset \mathbb{R}^n$ and for $n = 1, 2, 3$ only. Also the method is different.

Our aim is to study (1.1) and establish the local existence of solutions, the blow-up result in a finite time using the concavity method introduced by Levine in [2]. In addition, we obtain the lower bound of the blow-up time. To achieve this goal some conditions have to be imposed on the source functions f_1 and f_2 and the initial data as well. The paper is organized as follows. In section 2, we present conditions needed for our results and the proof of the local existence. In section 3, we present the statement and proof of the main result.

2. Local existence

We start with the following theorem and assumptions.

Theorem 2.1. [24] *(Sobolev, Gagliardo, Nirenberg)*
Suppose that $1 \leq p < n$. *If*

$$u \in W^{1,p}(\mathbb{R}^n), \quad \text{then} \quad u \in L^{p*}(\mathbb{R}^n),$$

with

$$\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}. \tag{2.1}$$

Moreover there exists a constant C=C(N, p) such that

$$\|u\|_{p^*} \leq C\|\nabla u\|_p, \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 2.2. [24] *Suppose that* $1 \leq p < n$. *Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad \forall q \in [p, p^*]$$

with continuous injection.

(G1) There exists a function $I(u, v) \geq 0$ such that

$$\frac{\partial I}{\partial u} = f_1(u, v), \qquad \frac{\partial I}{\partial v} = f_2(u, v).$$

(G2) There exists a constant $\rho > 2$ such that

$$\int_{\mathbb{R}^n} [u f_1(u, v) + v f_2(u, v) - \rho I(u, v)] dx \geq 0. \quad \forall u, v \in H^1(\mathbb{R}^n).$$

(G3) There exists a constant $d > 0$ such that

$$|f_1(\chi, \phi)| \leq d\left(|\chi|^{\beta_1} + |\phi|^{\beta_2}\right), \quad \forall (\chi, \phi) \in \mathbb{R}^2,$$

$$|f_2(\chi,\phi)| \;\leq\; d\left(|\chi|^{\beta_3}+|\phi|^{\beta_4}\right), \quad \forall(\chi,\phi)\in\mathbb{R}^2,$$

where

$$\beta_i > 2 \quad \text{if} \quad n=1,2 \quad \text{and} \quad \frac{n+2}{n}\leq\beta_i\leq\frac{n}{n-2},\ n\geq 3, \quad i=1,2,3,4. \tag{2.2}$$

(G4) Further assume that $f_1, f_2 \in C^1(\mathbb{R}^n)$ such that

$$|\nabla f_i(u,v)| \leq C\left(|u|^{\beta-1}+|v|^{\beta-1}+1\right), \ \ \forall u,v\in\mathbb{R}, \quad i=1,2,$$

and $\beta = \max\{\beta_i,\ i=1,2,3,4\} > 1$.

Note that condition (G3) and (G4) are sufficient to establish the existence of a local solution of (1.1) in an interval $(0,T]$ (see [20, 25–27]) and not needed for the blow-up result.

An example of a function that satisfying the other conditions (G1) and (G2) is

$$I(u,v)=\frac{\alpha}{\rho}\,|u-v|^{\rho},$$

where we have

$$\tfrac{\partial I}{\partial u} = f_1(u,v) = \alpha\,|u-v|^{\rho-2}\,(u-v),$$

$$\tfrac{\partial I}{\partial v} = f_2(u,v) = -\alpha\,|u-v|^{\rho-2}\,(u-v).$$

To start the proof, we set $\phi = u_t$, $\psi = v_t$ and denote by

$$\begin{aligned}
&\Phi = (u,\phi,v,\psi)^T,\\
&\Phi(0) = \Phi_0 = (u_0,u_1,v_0,v_1)^T,\\
&J(\Phi) = (0,f_1,0,f_2)^T.
\end{aligned}$$

Therefore (1.1) can be rewritten as an initial value problem:

$$\begin{cases} \partial_t\Phi + \mathcal{A}\Phi = J(\Phi)\\ \Phi(0)=\Phi_0, \end{cases} \tag{2.3}$$

where the linear operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -\phi \\ -\Delta u + \lambda_1\phi \\ -\psi \\ -\Delta v + \lambda_2\psi \end{pmatrix}. \tag{2.4}$$

The state space of Φ is the Hilbert space

$$\mathcal{H} = \left[H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right]^2,$$

equipped with the inner product

$$\left\langle \Phi,\widetilde{\Phi}\right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^n}\left(\nabla u.\nabla\widetilde{u} + \phi\widetilde{\phi} + \nabla v.\nabla\widetilde{v} + \psi\widetilde{\psi}\right)dx,$$

for all $\Phi = (u, \phi, v, \psi)^T$ and $\widetilde{\Phi} = \left(\widetilde{u}, \widetilde{\phi}, \widetilde{v}, \widetilde{\psi}\right)^T$ in $\mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{\Phi \in \mathcal{H} : u, v \in H^2(\mathbb{R}^n),\ \phi, \psi \in H^1(\mathbb{R}^n)\right\}.$$

Then, we have the following local existence result.

Theorem 2.3. *Assume that (G4) holds. Then for any $\Phi_0 \in \mathcal{H}$, problem (2.3) has a unique weak solution $\Phi \in C(\mathbb{R}^+; \mathcal{H})$.*

Proof. First, for all $\Phi \in D(\mathcal{A})$, we have

$$\begin{aligned} &\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} \\ = & -\int_{\mathbb{R}^n} \nabla\phi . \nabla u dx + \int_{\mathbb{R}^n} (-\Delta u + \lambda_1 \phi)\, \phi dx - \int_{\mathbb{R}^n} \nabla\psi . \nabla v dx \\ & + \int_{\mathbb{R}^n} (-\Delta v + \lambda_2 \psi)\, \psi dx \\ = & \ \lambda_1 \int_{\mathbb{R}^n} |\phi|^2 dx + \lambda_2 \int_{\mathbb{R}^n} |\psi|^2 dx \geq 0. \end{aligned} \tag{2.5}$$

Therefore, $\mathcal{A}$ is a monotone operator.

To show that $\mathcal{A}$ is maximal, we prove that for each

$$G = (g_1, g_2, g_3, g_4)^T \in \mathcal{H},$$

there exists $V = (u, \phi, v, \psi)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})\, V = G$. That is,

$$\begin{cases} u - \phi = g_1 \\ \phi - \Delta u + \lambda_1 \phi = g_2 \\ v - \psi = g_3 \\ \psi - \Delta v + \lambda_2 \psi = g_4. \end{cases} \tag{2.6}$$

From $(2.6)_1$ and $(2.6)_3$, we have $\phi = u - g_1$ and $\psi = v - g_3$. Thus, $(2.6)_2$ and $(2.6)_4$ give

$$\begin{aligned} u - \frac{1}{1+\lambda_1}\Delta u &= \frac{1}{1+\lambda_1} g_2 + g_1, \\ v - \frac{1}{1+\lambda_2}\Delta v &= \frac{1}{1+\lambda_2} g_4 + g_3. \end{aligned}$$

Now we define, over $\left[H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\right]^2$, the bilinear form

$$B\left((u, v), (\widetilde{u}, \widetilde{v})\right) = \int_{\mathbb{R}^n} \left(u\widetilde{u} + v\widetilde{v} + \frac{1}{1+\lambda_1} \nabla u . \nabla \widetilde{u} + \frac{1}{1+\lambda_2} \nabla v . \nabla \widetilde{v}\right) dx,$$

and over $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, the linear form

$$L((\widetilde{u}, \widetilde{v})) = \int_{\mathbb{R}^n} \left[\left(\frac{1}{1+\lambda_1} g_2 + g_1\right)\widetilde{u} + \left(\frac{1}{1+\lambda_2} g_4 + g_2\right)\widetilde{v}\right] dx.$$

It is easy to verify that B is continuous and coercive over $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and L is continuous on $H^1(\mathbb{R}^n)$. Then, Lax-Milgram theorem implies that the equation

$$B((u,v),(\widetilde{u},\widetilde{v})) = L((\widetilde{u},\widetilde{v})), \quad \forall(\widetilde{u},\widetilde{v}) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n), \tag{2.7}$$

has a unique solution $(u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. Hence,

$$\phi = u - g_1 \in H^1(\mathbb{R}^n) \quad \text{and} \quad \psi = v - g_3 \in H^1(\mathbb{R}^n)\,.$$

Thus, $V \in \mathcal{H}$.

Using (2.7) for $\widetilde{v} = 0$, we get

$$\int_{\mathbb{R}^n} \left(u\widetilde{u} + \frac{1}{1+\lambda_1} \nabla u . \nabla \widetilde{u} \right) dx = \int_{\mathbb{R}^n} \left(\frac{1}{1+\lambda_1} g_2 + g_1 \right) \widetilde{u} dx \quad \forall \widetilde{u} \in H^1(\mathbb{R}^n). \tag{2.8}$$

The elliptic regularity theory, then, implies that $u \in H^2(\mathbb{R}^n)$ and, in addition, Green's formula and $(2.6)_2$ give

$$\int_{\mathbb{R}^n} \left[u - \frac{1}{1+\lambda_1} \Delta u - \left(\frac{1}{1+\lambda_1} g_2 + g_1 \right) \right] \widetilde{u} = 0, \quad \forall \widetilde{u} \in H^1(\mathbb{R}^n). \tag{2.9}$$

Hence,

$$u - \frac{1}{1+\lambda_1} \Delta u = \left(\frac{1}{1+\lambda_1} g_2 + g_1 \right) \in L^2(\mathbb{R}^n)\,.$$

Similarly, using (2.7) for $\widetilde{u} = 0$, we get that $v \in H^2(\mathbb{R}^n)$ and

$$v - \frac{1}{1+\lambda_2} \Delta u = \left(\frac{1}{1+\lambda_2} g_4 + g_3 \right) \in L^2(\mathbb{R}^n)\,.$$

Therefore,

$$V = (u, \phi, v, \psi)^T \in D(\mathcal{A})\,.$$

Consequently, $I + \mathcal{A}$ is surjective and then $\mathcal{A}$ is maximal.

Finally, we show that $J : \mathcal{H} \to \mathcal{H}$ is locally Lipchitz. Hence, we estimate

$$\begin{aligned} & \left\| J(\Phi) - J(\widetilde{\Phi}) \right\|_{\mathcal{H}}^2 \\ = \;& \|(0, f_1(u,v) - f_1(\widetilde{u},\widetilde{v}), 0, f_2(u,v) - f_2(\widetilde{u},\widetilde{v}))\|_{\mathcal{H}}^2 \\ = \;& \|f_1(u,v) - f_1(\widetilde{u},\widetilde{v})\|_{L^2}^2 + \|f_2(u,v) - f_2(\widetilde{u},\widetilde{v})\|_{L^2}^2 \\ = \;& \int_{\mathbb{R}^n} |f_1(u,v) - f_1(\widetilde{u},\widetilde{v})|^2 \, dx + \int_{\mathbb{R}^n} |f_2(u,v) - f_2(\widetilde{u},\widetilde{v})|^2 \, dx. \end{aligned} \tag{2.10}$$

To handle the first term of the right-hand side of (2.10), we use the mean value theorem to get

$$|f_1(u,v) - f_1(\widetilde{u},\widetilde{v})| \le |\nabla f_1(u,v)| \left(|u - \widetilde{u}| + |v - \widetilde{v}| \right). \tag{2.11}$$

Therefore, (G4) implies

$$\int_{\mathbb{R}^n} |f_1(u,v) - f_1(\widetilde{u},\widetilde{v})|^2 \, dx \le C \int_{\mathbb{R}^n} \left(|u-\widetilde{u}|^2 + |v-\widetilde{v}|^2 \right) \times \left(|u|^{2(\beta-1)} + |\widetilde{u}|^{2(\beta-1)} + |v|^{2(\beta-1)} + |\widetilde{v}|^{2(\beta-1)} + 1 \right) dx. \tag{2.12}$$

All terms in (2.12) are estimated in the same manner. As $u, \widetilde{u}, v, \widetilde{v} \in H^1(\mathbb{R}^n)$, we use Theorem 2.1, Corollary 2.2 and Hölder's inequality, to obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} |u-\widetilde{u}|^2 \, |u|^{2(\beta-1)} \, dx \\ \le \;& C \left(\int_{\mathbb{R}^n} |u-\widetilde{u}|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \left(\int_{\mathbb{R}^n} |u|^{n(\beta-1)} \, dx \right)^{\frac{2}{n}} \\ \le \;& C \, \|u-\widetilde{u}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \, \|u\|^{2(\beta-1)}_{L^{n(\beta-1)}(\mathbb{R}^n)} \\ \le \;& C \, \|u-\widetilde{u}\|^2_{H^1(\mathbb{R}^n)} \left(\|u\|_{L^{n(\beta-1)}(\mathbb{R}^n)} \right)^{2(\beta-1)} \\ \le \;& C \, \|u-\widetilde{u}\|^2_{H^1(\mathbb{R}^n)} \left(\|u\|_{H^1(\mathbb{R}^n)} \right)^{2(\beta-1)} . \end{aligned} \tag{2.13}$$

Hence

$$\int_{\mathbb{R}^n} |f_1(u,v) - f_1(\widetilde{u},\widetilde{v})|^2 \, dx \le C \left(\|u-\widetilde{u}\|^2_{H^1(\mathbb{R}^n)} + \|v-\widetilde{v}\|^2_{H^1(\mathbb{R}^n)} \right). \tag{2.14}$$

Also, similar estimations yield to

$$\int_{\mathbb{R}^n} |f_2(u,v) - f_2(\widetilde{u},\widetilde{v})|^2 \, dx \le C \left(\|u-\widetilde{u}\|^2_{H^1(\mathbb{R}^n)} + \|v-\widetilde{v}\|^2_{H^1(\mathbb{R}^n)} \right). \tag{2.15}$$

Inserting (2.14), (2.15) in (2.10), we have

$$\left\| J(\Phi) - J(\widetilde{\Phi}) \right\|^2_{\mathcal{H}} \le C \left\| \Phi - \widetilde{\Phi} \right\|^2_{\mathcal{H}} . \tag{2.16}$$

Therefore, J is locally Lipchitz. Thanks to the theorems in Komornik [28] (See also Pazy [29]), the proof is completed.

The energy functional associated to the above system is given by

$$E(t) := \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 \right) - \int_{\mathbb{R}^n} I(u,v) dx, \quad t \ge 0. \tag{2.17}$$

Consequently, we present the following lemma.

Lemma 2.4. *The solution of* (1.1) *satisfies,*

$$E'(t) \le -\lambda \int_{\mathbb{R}^n} \left(|u_t|^2 + |v_t|^2 \right) dx \le 0, \tag{2.18}$$

where $\lambda = \min\{\lambda_1, \lambda_2\}$.

Proof. Multiplying Eq $(1.1)_1$ by u_t, equation $(1.1)_2$ by v_t and integrating over $\mathbb{R}^n$ then summing up we get the result.

Finally, we set

$$F(t) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|u(x,t)|^2 + |v(x,t)|^2\right)dx + \frac{\lambda}{2}L(t) + \frac{1}{2}\beta(t+t_0)^2, \tag{2.19}$$

for $t_0 > 0$ and $\beta > 0$ to be chosen later and $L(t)$ is define by

$$\begin{aligned} L(t) = \textstyle\int_0^t \int_{\mathbb{R}^n}\left(|u(x,s)|^2 + |v(x,s)|^2\right)dxds \\ + (T-t)\textstyle\int_{\mathbb{R}^n}\left(|u_0(s)|^2 + |v_0(s)|^2\right)ds. \end{aligned} \tag{2.20}$$

3. Blow up

In this section we state and prove our main result.

Theorem 3.1. *Assume that (G1)–(G3) hold and the initial data*

$$(u_0, v_0), (u_1, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

satisfying,

$$E(0) = \frac{1}{2}\left(\|u_1\|_2^2 + \|\nabla u_0\|_2^2 + \|v_1\|_2^2 + \|\nabla v_0\|_2^2\right) - \int_{\mathbb{R}^n} I(u_0, v_0)dx < 0. \tag{3.1}$$

Then the corresponding solution of (1.1) blows up in finite time.

Before the proof of this theorem, we need the following technical lemmas.

Lemma 3.2. *Along the solution of (1.1), the functional $L(t)$ defined in (2.20) satisfies,*

$$L'(t) = 2\int_0^t \int_{\mathbb{R}^n} (uu_t(x,s) + vv_t(x,s))\,dxds \tag{3.2}$$

and

$$L''(t) = 2\int_{\mathbb{R}^n} (uu_t(x,s) + vv_t(x,s))\,dxds. \tag{3.3}$$

Proof. A direct differentiation of (2.20) yields

$$\begin{aligned} L'(t) &= \int_{\mathbb{R}^n}\left(|u(x,s)|^2 + |v(x,s)|^2\right)dx - \int_{\mathbb{R}^n}\left(|u_0(s)|^2 + |v_0(s)|^2\right)ds \\ &= 2\int_0^t \int_{\mathbb{R}^n} (uu_t(x,s) + vv_t(x,s))\,dxds \end{aligned}$$

and

$$L''(t) = 2\int_{\mathbb{R}^n} (uu_t(x,s) + vv_t(x,s))\,dxds.$$

Lemma 3.3. *Along the solution of (1.1), we estimate,*

$$\begin{aligned} \int_{\mathbb{R}^n}(uu_{tt} + vv_{tt})dx &= \left(\frac{\rho}{2} - 1\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad + \frac{\rho}{2}\left(\int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} v_t^2 dx\right) \end{aligned}$$

$$-\rho E(t) - \lambda\left(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} vv_t dx\right). \tag{3.4}$$

Proof. Multiply Eq $(1.1)_1$ by u and Eq $(1.1)_2$ by v and integrate by parts over $\mathbb{R}^n$ to get

$$\int_{\mathbb{R}^n}(uu_{tt} + vv_{tt})dx = -\int_{\mathbb{R}^n}(|\nabla u|^2 + |\nabla v|^2)dx - \lambda_1 \int_{\mathbb{R}^n} uu_t dx$$

$$- \lambda_2 \int_{\mathbb{R}^n} vv_t dx + \rho \int_{\mathbb{R}^n} I(u, v)dx.$$

Exploiting (2.17) to substitute for $\int_{\mathbb{R}^n} I(u, v)dx$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(uu_{tt} + vv_{tt})dx &= \left(\frac{\rho}{2} - 1\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad + \frac{\rho}{2}\left(\int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} v_t^2 dx\right) \\ &\quad - \rho E(t) - \lambda\left(\int_{\mathbb{R}^n} uu_t dx + \int_{\mathbb{R}^n} vv_t dx\right).\end{aligned}$$

Lemma 3.4. *Along the solution of (1.1), we estimate, for any $\epsilon > 0$,*

$$\begin{aligned} J &= \left(\int_{\mathbb{R}^n}(u_t u + v_t v)dx + \beta(t + t_0) + \frac{\lambda}{2}L'(t)\right)^2 \\ &\leq F(t)\left[(1 + \epsilon)\left(\int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} v_t^2 dx\right) + 2\left(1 + \frac{1}{\epsilon}\right)(\beta - \lambda E(t))\right]. \end{aligned} \tag{3.5}$$

Proof. By using Young's inequality, we have

$$\begin{aligned} J &\leq \left(\int_{\mathbb{R}^n}(u_t u + v_t v)dx\right)^2 + \left(\beta(t + t_0) + \frac{\lambda}{2}L'(t)\right)^2 \\ &\quad + 2\left[\beta(t + t_0) + \frac{\lambda}{2}L'(t)\right]\int_{\mathbb{R}^n}(u_t u + v_t v)dx \\ &\leq \left(\int_{\mathbb{R}^n}(u_t u + v_t v)dx\right)^2 + \left(\beta(t + t_0) + \frac{\lambda}{2}L'(t)\right)^2 \\ &\quad + 2\left[\frac{\epsilon}{2}\left(\int_{\mathbb{R}^n}(u_t u + v_t v)dx\right)^2 + \frac{1}{2\epsilon}\left(\beta(t + t_0) + \frac{\lambda}{2}L'(t)\right)^2\right] \\ &\leq (1 + \epsilon)\left(\int_{\mathbb{R}^n}(u_t u + v_t v)dx\right)^2 + \left(1 + \frac{1}{\epsilon}\right)\left(\beta(t + t_0) + \frac{\lambda}{2}L'(t)\right)^2 \end{aligned}$$

and using Cauchy-Schwartz inequality yields

$$\begin{aligned} J &\leq (1 + \epsilon)\left(\int_{\mathbb{R}^n} u^2 dx + \int_{\mathbb{R}^n} v^2 dx\right)\left(\int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} v_t^2 dx\right) \\ &\quad + 2\left(1 + \frac{1}{\epsilon}\right)\left[\beta^2(t + t_0)^2 + \left(\frac{\lambda}{2}L'(t)\right)^2\right]. \end{aligned}$$

Recalling (3.2), we estimate

$$\begin{aligned}(L'(t))^2 &= \left(2\int_0^t\int_{\mathbb{R}^n}(uu_t(x,s)+vv_t(x,s))\,dxds\right)^2 \\ &\leq 4\left(\int_0^t\left(\|u(\tau)\|_2^2+\|v(\tau)\|_2^2\right)d\tau\right)\left(\int_0^t\left(\|u_t(\tau)\|_2^2+\|v_t(\tau)\|_2^2\right)d\tau\right).\end{aligned} \tag{3.6}$$

Using (2.18) and (3.1), we have

$$E(t) \leq E(0) < 0.$$

Hence,

$$\int_0^t\left(\|u_t(\tau)\|_2^2+\|v_t(\tau)\|_2^2\right)d\tau \leq \frac{1}{\lambda}\left[E(0)-E(t)\right] \leq -\frac{1}{\lambda}E(t). \tag{3.7}$$

Using (2.19) and (3.7), estimation (3.6) becomes

$$\left(\frac{\lambda}{2}L'(t)\right)^2 \leq -\lambda F(t)E(t).$$

Hence,

$$\begin{aligned}J &\leq \left[(1+\epsilon)F(t)\left(\int_{\mathbb{R}^n}u_t^2dx+\int_{\mathbb{R}^n}v_t^2dx\right)\right]+2\left(1+\frac{1}{\epsilon}\right)F(t)\left[\beta-\lambda E(t)\right] \\ &\leq F(t)\left[(1+\epsilon)\left(\int_{\mathbb{R}^n}u_t^2dx+\int_{\mathbb{R}^n}v_t^2dx\right)+2\left(1+\frac{1}{\epsilon}\right)(\beta-\lambda E(t))\right].\end{aligned}$$

Proof of theorem 3.1. By differentiating F in (2.19) twice we get

$$F'(t)=\int_{\mathbb{R}^n}(u_tu+v_tv)dx+\frac{\lambda}{2}L'(t)+\beta(t+t_0) \tag{3.8}$$

and

$$F''(t)=\int_{\mathbb{R}^n}(u_{tt}u+v_{tt}v)dx+\int_{\mathbb{R}^n}(|u_t|^2+|v_t|^2)dx+\frac{\lambda}{2}L''(t)+\beta. \tag{3.9}$$

Inserting (3.3) and (3.4) in (3.9), yield

$$F''(t) \geq \left(\frac{\rho}{2}+1\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\left(\frac{\rho}{2}-1\right)\left[\|\nabla u\|_2^2+\|\nabla v\|_2^2\right]-\rho E(t)+\beta. \tag{3.10}$$

We then define

$$G(t) := F^{-\gamma}(t),$$

for $\gamma > 0$ to be chosen properly.

Differentiating G twice we arrive at

$$G'(t)=-\gamma F^{-(\gamma+1)}(t)F'(t) \quad \text{and} \quad G''(t)=-\gamma F^{-(\gamma+2)}(t)Q(t),$$

where

$$Q(t)=F(t)F''(t)-(\gamma+1)F'^2(t). \tag{3.11}$$

Using (3.5), the last term of (3.11) takes the form

$$\begin{aligned} & -(\gamma+1)\left(\int_{\mathbb{R}^n}(uu_t+vv_t)dx+\frac{\lambda}{2}L'(t)+\beta(t+t_0)\right)^2 \\ \geq\ & -(\gamma+1)F(t)\left[(1+\epsilon)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+2\left(1+\frac{1}{\epsilon}\right)(\beta-\lambda E(t))\right]. \end{aligned}$$

Recalling (3.10), estimation (3.11) becomes

$$\begin{aligned} Q(t) \geq\ & F(t)\left[\left(\frac{\rho}{2}-1\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\right] \\ & +F(t)\left[\left(\frac{\rho}{2}+1-(1+\varepsilon)(\gamma+1)\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\right] \\ & +F(t)\left[-\rho E(t)-2(\gamma+1)\left(1+\frac{1}{\epsilon}\right)(\beta-\lambda E(t))\right], \qquad \forall\varepsilon>0. \end{aligned} \tag{3.12}$$

It is known, from (G2), that $\frac{\rho}{2}>1$. Consequently, we choose $\varepsilon<\frac{\rho}{4}$ and γ so that

$$0<\gamma<\frac{\rho-4\varepsilon}{4(1+\varepsilon)}.$$

Hence, (3.12) becomes

$$Q(t)\geq F(t)\left\{-E(t)\left[\rho-2(\gamma+1)\left(1+\frac{1}{\epsilon}\right)\lambda\right]-2\beta(\gamma+1)\left(1+\frac{1}{\epsilon}\right)\right\},\ \ \forall\varepsilon>0. \tag{3.13}$$

Now, the choice of

$$\lambda<\frac{\rho}{2(\gamma+1)\left(1+\frac{1}{\epsilon}\right)}$$

will make

$$k=\rho-2(\gamma+1)\left(1+\frac{1}{\epsilon}\right)\lambda>0.$$

Using the fact that

$$-kE(t)>-kE(0)>0,$$

we infer, from (3.13), that

$$Q(t)\geq F(t)\left[-kE(0)-2\beta(\gamma+1)\left(1+\frac{1}{\epsilon}\right)\right]. \tag{3.14}$$

Then for β small enough we conclude, from (3.14), that

$$Q(t)\geq 0, \qquad \forall t\geq 0$$

and

$$G''(t)\leq 0, \qquad \forall t\geq 0.$$

Therefore, G' is decreasing. By choosing t_0 large enough we get

$$F'(0) = \int_{\mathbb{R}^n} (u_0 u_1 + v_0 v_1) dx + \beta t_0 > 0,$$

hence $G'(0) < 0$.

Finally Taylor expansion of G yields

$$G(t) \leq G(0) + tG'(0), \quad \forall t \geq 0,$$

which shows that $G(t)$ vanishes at a time $t_m \leq -\frac{G(0)}{G'(0)}$. Consequently $F(t)$ must become infinite at time t_m.

4. Lower bound for the blow-up time

In this section, we estimate a lower bound for the blow-up time through the following theorem.

Theorem 4.1. *Assume that (G1)–(G3) hold and the initial data*

$$(u_0, v_0), (u_1, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

are with compact support. Assume further u is a solution of (1.1) which blows up at a finite time T^*. Then there exists a positive constant C such that

$$\int_{\psi(0)}^{\infty} \frac{dy}{y + Cy^r} \leq T^*,$$

where $r = \max\{\beta_i,\ i = 1, 2, 3, 4\} \geq 1$ *and*

$$\psi(t) = (L + t)^{-2r} \int_{\mathbb{R}^n} I(u, v) dx.$$

Before we start the proof, we exploit the finite wave speed of propagation to have the following lemmas.

Lemma 4.2. *If the initial data $u_0, u_1$are with compact support then for any solution u of (1.1), we have*

$$\|u(t)\|_2 \leq C(L + t)\|\nabla u(t)\|_2.$$

Proof. In Theorem 2.1, if we let $p = 2$ then we have $p^* = \frac{2n}{n-2}, \quad n \geq 3$ and

$$\|u\|_{p^*} \leq C\|\nabla u\|_2.$$

Now

$$\int_{\mathbb{R}^n} |u|^2\, dx = \int_{B(L+t)} |u|^2\, dx,$$

where $L > 0$ is such that

$$\text{Supp}\{u_0(x), u_1(x)\} \subset B(L)$$

and $B(L + t)$ is the ball, with radius $L + t$, centered at the origin. Using Hölder inequality, we get

$$\int_{\mathbb{R}^n} |u|^2\, dx \quad \leq \quad \left(\int_{B(L+t)} 1 dx \right)^{1-\frac{2}{p^*}} \left(\int_{B(L+t)} (|u|^2)^{\frac{p^*}{2}} dx \right)^{\frac{2}{p^*}}$$

$$\leq \quad C(L+t)^2\|u(t)\|_{p^*}^2,$$

or

$$\|u(t)\|_2 \leq C(L+t)\|u(t)\|_{p^*} \leq C(L+t)\|\nabla u(t)\|_2.$$

Hence, the result follows.

Proof of Theorem 4.1. A direct differentiation of $\psi(t)$ yields

$$\begin{aligned}\psi'(t) &= -2r\,(L+t)^{-2r-1}\int_{\mathbb{R}^n} I(u,v)dx \\ &\quad +(L+t)^{-2r}\int_{\mathbb{R}^n}\left(u_t I_u(u,v)+v_t I_v(u,v)\right)dx \\ &\leq (L+t)^{-2r}\int_{\mathbb{R}^n}\left(u_t I_u(u,v)+v_t I_v(u,v)\right)dx.\end{aligned}$$

Using Young's inequality and (G1), we have

$$\begin{aligned}(L+t)^{2r}\,\psi'(t) &\leq \frac{1}{2}\int_{\mathbb{R}^n}\left(u_t^2+v_t^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^n}\left(I_u^2+I_v^2\right)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\left(u_t^2+v_t^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^n}\left(f_1^2+f_2^2\right)dx.\end{aligned}$$

Using (G3), we get

$$(L+t)^{2r}\,\psi'(t) \leq \frac{1}{2}\int_{\mathbb{R}^n}\left(u_t^2+v_t^2\right)dx+d\int_{\mathbb{R}^n}\left(|u|^r+|v|^r\right)^2 dx. \tag{4.1}$$

To estimate the last term of (4.1), we have two cases:

Case I: For $|u| \leq |v|$.

Using Theorem 2.1 and Corollary 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left(|u|^r+|v|^r\right)^2 dx &\leq 4\int_{\mathbb{R}^n}|v|^{2r}dx \leq C\left((L+t)^2\int_{\mathbb{R}^n}|\nabla v|^2 dx\right)^r \\ &\leq C\left((L+t)^2\left(\|\nabla v\|_2^2+\|\nabla u\|_2^2\right)\right)^r.\end{aligned}$$

Case II: For $|u| \geq |v|$.

Similar to case I, we have

$$\int_{\mathbb{R}^n}\left(|u|^r+|v|^r\right)^2 dx \leq 4\int_{\mathbb{R}^n}|u|^{2r}dx \leq C\left((L+t)^2\left(\|\nabla v\|_2^2+\|\nabla u\|_2^2\right)\right)^r.$$

Thus, estimation (4.1) becomes

$$(L+t)^{2r}\,\psi'(t) \leq \frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+C\left((L+t)^2\left(\|\nabla v\|_2^2+\|\nabla u\|_2^2\right)\right)^r. \tag{4.2}$$

Recalling the fact that $E(t) \leq E(0) < 0$, we deduce, from (2.17), that

$$\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2 \leq 2\int_{\mathbb{R}^n} I(u,v)dx. \tag{4.3}$$

Inserting (4.3) in (4.2), gives

$$(L+t)^{2r}\psi'(t) \leq \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + C\left((L+t)^2\int_{\mathbb{R}^n} I(u,v)dx\right)^r. \tag{4.4}$$

Again (2.17) yields

$$\|u_t\|_2^2 + \|v_t\|_2^2 \leq 2\int_{\mathbb{R}^n} I(u,v)dx \leq 2\,(L+t)^{2r}\psi(t),$$

thus, from (4.4), we have

$$(L+t)^{2r}\psi'(t) \leq (L+t)^{2r}\psi(t) + C\left((L+t)^2\psi(t)\right)^r, \tag{4.5}$$

or

$$\psi'(t) \leq \psi(t) + C\,(\psi(t))^r.$$

If we use the substitution $y = \psi(t)$ and solve (4.5) over $(0, T^*)$, then we reach

$$\int_{\psi(0)}^{\infty} \frac{dy}{y + Cy^r} \leq T^*.$$

This completes the proof.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. J. Ball, Remarks on blow up and nonexistence theorems for nonlinear evolution's equations, *Quart. J. Math. Oxford*, **28** (1977), 473–486.
2. H. A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$, *T. Am. Math. Soc.*, **192** (1974), 1–21.
3. H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equation, *SIAM J. Math. Anal.*, **5** (1974), 138–146.
4. V. Georgiev, G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, *J. Differ. Equations*, **109** (1994), 295–308.
5. S. A. Messaoudi, Blow up in a nonlinearly damped wave equation, *Mathematische Nachrichten*, **231** (2001), 1–7.

6. H. A. Levine, J. Serrin, A global nonexistence theorem for quasilinear evolution equation with dissipation, *Arch. Rational Mech. Anal.*, **137** (1997), 341–361.

7. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Rational Mech. Anal.*, **149** (1999), 155–182.

8. S. A. Messaoudi, B. Said-Houari, Blow up of solutions of a class of wave equations with nonlinear damping and source terms, *Math. Method. Appl. Sci.*, **27** (2004), 1687–1696.

9. S. A. Messaoudi, Blow up and global existence in a nonlinear viscoelastic wave equation, *Mathematische Nachrichten*, **260** (2003), 58–66.

10. S. A. Messaoudi, Blow up of solutions with positive initial energy in a nonlinear viscoelastic equation, *J. Math. Anal. Appl.*, **320** (2006), 902–915.

11. H. A. Levine, S. Ro Park, J. Serrin, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation, *J. Math. Anal. Appl.*, **228** (1998), 181–205.

12. G. Todorova, Cauchy problem for a nonlinear wave with nonlinear damping and source terms, *C. R. Acad. Sci. Paris Ser. I*, **326** (1998), 191–196.

13. G. Todorova, Stable and unstable sets for the Cauchy problem for a nonlinear wave with nonlinear damping and source terms, *J. Math. Anal. Appl.*, **239** (1999), 213–226.

14. S. A. Messaoudi, Blow up in the Cauchy problem for a nonlinearly damped wave equation, *Commun. Pure Appl. Anal.*, **7** (2003), 379–386.

15. Z. Yong, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in $\mathbb{R}^n$, *Appl. Math. Lett.*, **18** (2005), 281–286.

16. M. Kafini, S. Messaoudi, A blow-up result in a Cauchy viscoelastic problem, *Appl. Math. Lett.*, **21** (2008), 549–553.

17. M. Kafini, S. Messaoudi, A blow-up result for a viscoelastic system in $\mathbb{R}^n$, *Electron. J. Differ. Eq.*, **113** (2007), 1–7.

18. S. A. Messaoudi, H. Belkacem, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *J. Math. Anal. Appl.*, **365** (2010), 277–287.

19. K. Agre, M. Rammaha, Systems of nonlinear wave equations with damping and source terms, *Differ. Integral Equ.*, **19** (2006), 1235–1270.

20. S. A. Messaoudi, N-e. Tatar, Uniform stabilization of solutions of a nonlinear system of viscoelastic equations, *Appl. Anal.*, **87** (2008), 247–263.

21. M. L. Santos, Decay rates for solutions of a system of wave equations with memory, *Electron. J. Differ. Eq.*, **38** (2002), 1–17.

22. C. O. Alves, M. M. Cavalcanti, V. N. D. Cavalcanti, M. A. Rammaha, D. Toundykov, On existence, uniform decay rates and blow up for solutions of systems of nonlinear wave equations with damping and source terms, *Discrete Cont. Dyn. S*, **2** (2009), 583–608.

23. S. T. Wu, Blow-up of solutions for an integro-differential equation with a nonlinear source, *Electron. J. Differ. Eq.*, **45** (2006), 1–9.

24. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, 2011.

25. S. H. Belkacem, Exponential growth of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *Z. Angew. Math. Phys.*, **62** (2011), 115–133.

26. W. Liu, Uniform decay of solutions for a quasilinear system of viscoelastic equations, *Nonlinear Anal.*, **71** (2009), 2257–2267.

27. M. I. Mustafa, Wellposedness and asymptotic behavior of acoupled system of nonlinear viscoelastic equations, *Nonlinear Anal. Real.*, **13** (2012), 452–463.

28. V. Komornik, *Exact controllability and stabilization: the multiplier method*, Elsevier Masson, 1994.

29. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer-Verlag, 1983.