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Research article

Multiplicity of solutions for a fractional Schrödinger-Poisson system without (PS) condition

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Abstract: In this paper, we study the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u) & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & x \in \mathbb{R}^3. \end{cases}$$

Using the variant fountain theorem introduced by Zou [32], we get the existence of infinitely many large energy solutions without the Ambrosetti-Rabinowitz's 4-superlinearity condition. Recent results from the literature are extended and improved.

Keywords: fractional Schrödinger-Poisson equation; variant fountain theorem; variational methods
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1. Introduction and the main results

In recent years, a great deal of work has been devoted to the study of semiclassical standing waves for the fractional Schrödinger equation:

$$i\frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + \widetilde{V}(x)\Psi - f(x, |\Psi|) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{1.1}$$

where i is the imaginary unit, $\widetilde{V}(x) = V(x) + E$ is the potential function with the constant E and $f(exp(i\theta)\xi) = exp(i\theta)f(\xi)$ for $\theta, \xi \in \mathbb{R}$ is a nonlinear function.

Equations of the type (1.1) were introduced by Laskin (see [15, 16]) and come from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [7, 21]).

An interesting Schrödinger equation class is when it describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. That is a fractional nonlinear Schrödinger-Poisson system (also called fractional Schrödinger-Maxwell system):

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + \widetilde{V}(x)\Psi + \phi\Psi - f(x, |\Psi|) & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ (-\Delta)^s \phi = |\Psi|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

A solution of the form $(e^{-iEt}u(x), \phi(x))$ is called a standing wave and $(e^{-iEt}u(x), \phi(x))$ is a solution of (1.2) if and only if $(u(x), \phi(x))$ satisfies

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

For the local case, that is $s = 1$, Eq (1.3) gives back the classical Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

which was proposed by Benci and Fortunato [6] in 1998 on a bounded domain, and is related to the Hartree equation ([18]). Recently, in order to better simulate the interaction effect among many particles in quantum mechanics, systems of a nonlinear version of the Schrödinger equation coupled with a Poisson equation have begun to receive much attention, we refer the interested readers to see [1,4,5,8,12,31] and the references therein.

In the nonlocal case, that is, when $s \in (0, 1)$, to the best of our knowledge, there is few reference on existence for the fractional Schrödinger-Poisson systems, maybe because the standard techniques that were developed for the local Laplacian do not work immediately. For some existence results we refer to [3,10,11,13,17,19–22,26–30] and the references therein.

Motivated by an evident and increasing interest in the current literatures on fractional elliptic problems, the aim of our paper is finding infinitely many large energy solutions (i.e. high energy solutions). Such a type of problems is classical and one of the main difficulties is the lack of compactness for Sobolev embedding theorem for the whole space $\mathbb{R}^3$ case. Motivated by the approach used in [23], we will establish the existence results for (1.1), under the following assumptions for potential $V \in L^1_{loc}(\mathbb{R}^3)$ and nonlinear term f:

(V) $\inf_{x\in\mathbb{R}^3} V(x) > 0$. Moreover, for any $M > 0$, there exists $r > 0$ such that

$$\lim_{|y|\to\infty} meas\{x \in \mathbb{R}^3 : |x - y| \le r, V(x) \le M\} = 0.$$

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist $c > 0$, $p \in (4, 2^*_s = \frac{6}{3-2s})$ such that

$$|f(x, u)| \le c(1 + |u|^{p-1}) \text{ for all } x \in \mathbb{R}^3,\ u \in \mathbb{R}.$$

(f_2) $\lim_{|u|\to 0} \frac{f(x,u)}{|u|} = 0$ uniformly for $x \in \mathbb{R}^3$ and $f(x, u)u \ge 0$ for $u > 0$.

(f_3) $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^4} = +\infty$ uniformly for $x \in \mathbb{R}^3$, where $F(x, u) = \int_0^u f(x, t)dt$.

(f_4) $G(x,u) := \frac{1}{4}f(x,u)u - F(x,u) \to \infty$ as $|u| \to \infty$ uniformly for $x \in \mathbb{R}^3$.
(f_5) $f(x,-u) = -f(x,u)$ for any $x \in \mathbb{R}^3$, $u \in \mathbb{R}$.

Theorem 1.1. *Under the assumptions* (V) *and* $(f_1) - (f_5)$*, problem* (1.3) *has infinitely many solutions* $\{(u_n, \phi_n)\}$ *in* $H^s(\mathbb{R}^3) \times \mathcal{D}^{s,2}(\mathbb{R}^3)$ *satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2)dx - \frac{1}{4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_n|dx + \frac{1}{2}\int_{\mathbb{R}^3}\phi_n|u_n|^2dx - \int_{\mathbb{R}^3}F(x,u_n)dx \to +\infty.$$

Remark 1.1. *In this paper, we don't adopt the well know Ambrosetti-Rabinowitz's 4-superlinearity condition:*

$$\exists\, \mu > 4 \text{ such that } 0 \le \mu F(x,u) \le uf(x,u), \text{ for every } x \in \mathbb{R}^3. \tag{1.5}$$

This condition was used in [13] as a variant Ambrosetti-Rabinowitz type condition which was originally introduced by Ambrosetti and Rabinowitz in [2] and is still presented in many works which is used to guarantee the boundedness of (PS) *sequence of the corresponding functional.*

Remark 1.2. *There are functions satisfying the conditions of Theorem 1.1, but not satisfying the Ambrosetti-Rabinowitz type growth condition* (1.5)*. In fact, let* $f(x,u) = u^3 ln(1+|u|)$*. Simple computation yields that*

$$\frac{1}{4}f(x,u)u - F(x,u) = \frac{1}{4}\int_0^{|u|}\frac{\tau^4}{1+\tau}d\tau \to \infty \;\; as\ |u| \to \infty,$$

and (f_4) *follows. However,* f *does not satisfy condition* (1.5)*. Indeed, assume by contradiction that there is some* $\mu > 4$ *such that* $\mu F(x,u) \le uf(x,u)$ *for* $|u|$ *large. Then we have the following inequality*

$$\begin{aligned}
\mu F(x,u) &= \frac{\mu}{4}u^4 ln(1+|u|) - \frac{\mu}{4}(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}u^2 - |u| + ln(1+|u|))\\
&= u^4 ln(1+|u|)(\frac{\mu}{4} - \frac{\mu}{16 ln(1+|u|)}) + \frac{\mu}{4}(\frac{1}{3}|u|^3 - \frac{1}{2}u^2 + |u| - ln(1+|u|))\\
&\le u^4 ln(1+|u|)\\
&= uf(x,u),
\end{aligned}$$

for $|u|$ *large, but this impossible in view of* $\mu > 4$.

Throughout this paper, we denote $|\cdot|_r$ the usual norm of the space $L^r(\mathbb{R}^3)$, $1 \le r < \infty$. C or $C_i (i = 1,2,\cdots)$ denote some positive constants whose values may change from line to line.$a_n \rightharpoonup a$ and $a_n \to a$ mean the weak and strong convergence, respectively, as $n \to \infty$.

2. Preliminary results

Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some skrtchs of the fractional order Sobolev spaces and the complete introduction can be found in [9]. We recall that, for any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3}(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} := \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \Big)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^3)}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_s}(\mathbb{R}^3)$ is continuous and for any $s \in (0,1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{2^*_s(\mathbb{R}^3)}}$$

The fractional laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3.$$

Also $(-\Delta)^s u$ can be equivalently represented [9] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}} dy, \ \forall x \in \mathbb{R}^3$$

where

$$C(s) = \Big(\int_{\mathbb{R}^3} \frac{(1-cos\xi_1)}{|\xi|^{3+2s}} d\xi \Big)^{-1}, \ \xi = (\xi_1,\xi_2,\xi_3).$$

Also, by the Plancherel formular in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)} \|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2}.$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined below

$$u \longmapsto \Big(\int_{\mathbb{R}^3} |u|^2 dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \Big)^{\frac{1}{2}};$$
$$u \longmapsto \Big(\int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi \Big)^{\frac{1}{2}};$$
$$u \longmapsto \Big(\int_{\mathbb{R}^3} |u|^2 dx + \|(-\Delta)^{\frac{s}{2}}u\|_2^2 \Big)^{\frac{1}{2}}$$

are equivalent.

In view of the presence of potential $V(x)$, we introduce the subspace

$$E = \Big\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \Big\},$$

which is a Hilbert space equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \int_{\mathbb{R}^3} V(x)u^2 dx.$$

Since $V(x)$ is bounded from below, the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2_s^*]$. Moreover, from [23], we know that the embedding is also compact for any $r \in [2, 2_s^*)$ under the assumption (V).

It is clear that system (1.3) is the Euler-Lagrange equations of the functional $J : E \times \mathcal{D}^{s,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u, \phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx, \tag{2.1}$$

It is easy to see that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [6]. First of all, for a fixed $u \in E$, there exists a unique $\phi_u^s \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^s\phi = u^2 \text{ in } \mathbb{R}^3.$$

We can write an integral expression for ϕ_u^s in the form

$$\phi_u^s(x) = C_s \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2s}}dy, \quad \forall\, x \in \mathbb{R}^3,$$

which is called s-Riesz potential (see [14]), where

$$C_s = \frac{1}{\pi^{\frac{3}{2}}}\frac{\Gamma(3 - 2s)}{2^{2s}\Gamma(s)}.$$

Then the system (1.3) can be reduced to the first equation with ϕ represented by the solution of the fractional Poisson equation. This is the basic strategy of solving (1.3). To be more precise about the solution ϕ of the fractional Poisson equation, we collect some useful Lemmas.

Lemma 2.1. *[24] For any $u \in H^s(\mathbb{R}^3)$ and $s \geq \frac{3}{4}$, we have*
(i) $\phi_u^s \geq 0$;
(ii) $\phi_u^s : H^s(\mathbb{R}^3) \to \mathcal{D}^{s,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets;
(iii) $\|\phi_u^s\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u^s u^2 dx \leq S_s^2\|u\|^4_{\frac{12}{3+2s}}$;
(iv) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^s \rightharpoonup \phi_u^s$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$;
(v) If $u_n \to u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^s \to \phi_u^s$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^s u^2 dx$.

It follows from Lemma 2.1 that the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx$$

is well defined on E and $I \in C^1(E, \mathbb{R})$. Moreover, $(u, \phi) \in E \times \mathcal{D}^{s,2}(\mathbb{R}^3)$ is a solution of (1.3) if and only if $u \in E$ is a critical point of the functional I, and $\phi = \phi_u^s$.

Since we don't assume (1.5), the verification of (PS) condition becomes complicated, so we use the following variant fountain theorem introduced in [32] to handle the problem.

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$ and

$$B_k = \{u \in Y_k, \|u\| \leq \rho_k\}, \quad N_k = \{u \in Z_k, \|u\| = r_k\} \text{ for } \rho_k > r_k > 0.$$

Consider the following C^1 functional $\Phi_\lambda : E \to \mathbb{R}$ defined by:

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

We assume that

(F_1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$.
(F_2) $B(u) \geq 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$, or
(F_3) $B(u) \leq 0$ for all $u \in E$; $B(u) \to -\infty$ as $\|u\| \to \infty$.

For $k \geq 2$, let

$$c_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)),$$

$$b_k(\lambda) := \inf_{u\in Z_k, \|u\|=r_k} \Phi_\lambda(u),$$

$$a_k(\lambda) := \max_{u\in Y_k, \|u\|=\rho_k} \Phi_\lambda(u),$$

where $\Gamma_k := \{\gamma \in C(B_k, E), \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$.

Lemma 2.2. *([32])Assume (F_1) and (F_2) (or (F_3)) hold. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1, 2]$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in [1, 2]$. Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty \subseteq E$ such that*

$$\sup_n \|u_n^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_n^k(\lambda)) \to 0 \text{ and } \Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \text{ as } n \to \infty.$$

3. Proof of Theorem 1.1

We choose an orthogonal basis $\{e_j\}$ of E and let $X_j := \mathbb{R}e_j$, then define $Y_k = \bigoplus_{j=0}^k X_j$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$. Consider the family of functionals $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \lambda \int_{\mathbb{R}^3} F(x, u)dx := A(u) - \lambda B(u),$$

for $\lambda \in [1, 2]$. Then $B(u) \geq 0$ for all $u \in E$, $A(u) \to \infty$ as $\|u\| \to \infty$, and $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. And it is easy to see that I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$.

Lemma 3.1. *Under the assumption $(f_1) - (f_3)$, there exist $\rho_k > r_k > 0$ such that*

(i) $a_k(\lambda) := \max_{u\in Y_k, \|u\|=\rho_k} I_\lambda(u) < 0,$
(ii) $b_k(\lambda) := \inf_{u\in Z_k, \|u\|=r_k} I_\lambda(u) > 0.$

Proof. By condition (f_3), for any $M > 0$, there exists a constant $\delta = \delta(M) > 0$, such that for all $x \in \mathbb{R}^3$, $|u| > \delta$, we have

$$f(x, u)u \geq M|u|^4 \text{ and } F(x, u) \geq \frac{1}{4}M|u|^4. \tag{3.1}$$

From (f_1) and (f_2), there exists $M_1 = M_1(\delta) = M_1(M) > 0$ such that for all $x \in \mathbb{R}^3$, $0 \le |u| \le \delta$, we have

$$f(x,u)u \ge -(M_1+1)|u|^2 \text{ and } F(x,u) \ge -\frac{1}{2}(M_1+1)|u|^2. \tag{3.2}$$

Denote $C := \frac{1}{2}(M_1+1) + \frac{1}{4}M\delta^4$, combining (3.1) and (3.2), we have for all $x \in \mathbb{R}^3$, $u \in \mathbb{R}$

$$F(x,u) \ge \frac{1}{4}M|u|^4 - C|u|^2.$$

For $u \in Y_k$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u)dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{C_1}{4}\|u\|^4 - \frac{\lambda}{4}M|u|_4^4 + \lambda C|u|_2^2 \\ &\le \frac{1}{2}\|u\|^2 + \frac{C_1}{4}\|u\|^4 - \frac{\lambda}{4}MC_2\|u\|^4 + \lambda CC_3\|u\|^2, \end{aligned}$$

where we use the equivalence of all norms on the finite dimensional subspace Y_k.

Choose M large enough such that

$$\frac{C_1}{4} - \frac{1}{4}\lambda MC_2 < 0.$$

Then we can choose $\|u\| = \rho_k > 0$ large enough such that

$$a_k(\lambda) := \max_{u\in Y_k, \|u\|=\rho_k} I_\lambda(u) < 0.$$

On the other hand, by $(f_1), (f_2)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $x \in \mathbb{R}^3$, $u \in \mathbb{R}$,

$$F(x,u) \le \varepsilon|u|^2 + C_\varepsilon|u|^p.$$

Define

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} |u|_p.$$

Then, for any $u \in Z_k$ and $\varepsilon > 0$ small enough, one has

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \lambda\varepsilon|u|_2^2 - \lambda C_\varepsilon|u|_p^p \\ &\ge \frac{1}{4}\|u\|^2 - \lambda C_\varepsilon\beta_k^p\|u\|^p. \end{aligned}$$

Choosing $r_k = (\lambda C_\varepsilon p\beta_k^p)^{\frac{1}{2-p}}$, then for $u \in Z_k$ with $\|u\| = r_k$,

$$I_\lambda(u) \ge (4\lambda C_\varepsilon p\beta_k^p)^{\frac{1}{2-p}}(\frac{1}{4} - \frac{1}{4p}) := b_k^*,$$

which means that $b_k(\lambda) := \inf_{u\in Z_k, \|u\|=r_k} I_\lambda(u) \ge b_k^* \to \infty$ due to $\beta_k \to 0$ as $k \to \infty$([25]). □

Thus, by Lemma 2.2, for a.e. $\lambda \in [1,2]$, there exits a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \ \ I'_\lambda(u_n^k(\lambda)) \to 0 \text{ and } \ I_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \geq b_k(\lambda) \geq b_k^* \text{ as } n \to \infty,$$

where $c_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} I_\lambda(\gamma(u))$. Moreover, since $c_k(\lambda) \leq \sup_{u\in B_k} I_\lambda := c_k^*$ and the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is compact for any $r \in [2, 2_s^*)$, by standard argument, $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ has a convergent subsequence. Suppose $u_n^k(\lambda) \to u^k(\lambda)$ as $n \to \infty$. We get $I'_\lambda(u^k(\lambda)) = 0$ and $I_\lambda(u^k(\lambda)) \in [b_k^*, c_k^*]$. So, when $\lambda_n \to 1$, with $\lambda_n \in [1,2]$, we find a sequence $u^k(\lambda_n)$ (denoted by u_n^k for simplicity) satisfying $I'_{\lambda_n}(u_n^k) = 0$ and $I_{\lambda_n}(u_n^k) \in [b_k^*, c_k^*]$.

Lemma 3.2. *Under the assumption Theorem 1.1, the sequence $\{u_n^k\}$ is bounded in E.*

Proof. Seeking a contradiction we suppose that $\|u_n^k\| \to \infty$. Let $v_n = \frac{u_n^k}{\|u_n^k\|}$. Then, up to a subsequence, we get

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } E,\\ &v_n \to v \text{ in } L^r(\mathbb{R}^3), 2 \leq r < 2_s^*,\\ &v_n(x) \to v(x) \ a.e. \ x \in \mathbb{R}^3. \end{aligned}$$

There are two possible cases: (i) $v \neq 0$ in E; (ii) $v = 0$ in E.

In Case (i), it follows from $I'_{\lambda_n}(u_n^k) = 0$ that

$$\int_{\mathbb{R}^3} \frac{f(x,u_n^k)u_n^k}{\|u_n^k\|^4} dx \leq C.$$

On the other hand, by Fatou's lemma and conditions $(f_3), (f_4)$, we obtain

$$\int_{\mathbb{R}^3} \frac{f(x,u_n^k)u_n^k}{\|u_n^k\|^4} dx = \int_{\{v_n(x)\neq 0\}} \frac{f(x,u_n^k)u_n^k}{|u_n^k|^4} |v_n|^4 dx \to \infty,$$

which is a contradiction.

In Case (ii), by $(f_1), (f_2)$ and (f_4), we know that there is a constant $C_1 > 0$ such that

$$\frac{1}{4} f(x,u_n^k)u_n^k - F(x,u_n^k) \geq -C_1 |u_n^k|^2$$

for all $x \in \mathbb{R}^3$ and all positive integral numbers n. consequently,

$$\frac{1}{\|u_n^k\|^2}(I_{\lambda_n}(u_n^k) - \frac{1}{4}\langle I'_{\lambda_n}(u_n^k), u_n^k\rangle) \geq \frac{1}{4} + \frac{\lambda_n}{\|u_n^k\|^2} \int_{\mathbb{R}^3} (\frac{1}{4} f(x,u_n^k)u_n^k - F(x,u_n^k))dx \geq \frac{1}{4} - 2C_1|v_n|_2^2.$$

Taking limit in the above inequality, we get $0 \geq \frac{1}{4}$, a contradiction. Hence $\{u_n^k\}$ is bounded in E and the conclusion follows. ◻

Proof of Theorem 1.1. Combining Lemma 3.1-3.2, and the fact that the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is compact for any $r \in [2, 2_s^*)$, a standard argument implies that there exists a convergent subsequence of $\{u_n^k\}$ in E. We suppose that $u_n^k \to u^k$ in E, as $n \to \infty$. Obviously,

$$I(u_n^k) = I_{\lambda_n}(u_n^k) + (\lambda_n - 1)\int_{\mathbb{R}^3} F(x,u_n^k)dx.$$

Since $\{u_n^k\} \subset E$ is bounded, $\int_{\mathbb{R}^3} F(x, u_n^k)dx$ stays bounded as $n \to \infty$. Recalling that $I_{\lambda_n}(u_n^k) \in [b_k^*, c_k^*]$, we get

$$I(u^k) = \lim_{n\to\infty} I(u_n^k) \in [b_k^*, c_k^*].$$

On the other hand, we have

$$\langle I'(u_n^k), v\rangle = \langle I'_{\lambda_n}(u_n^k), v\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} f(x, u_n^k)v dx \text{ for all } v \in E.$$

Since $I'_{\lambda_n}(u_n^k) = 0$, $\{u_n^k\}$ is bounded, the above equality implies that

$$\lim_{n\to\infty}\langle I'(u_n^k), v\rangle = 0 \text{ for all } v \in E.$$

Since $I \in C^1(E)$, we have $I'(u_n^k) \to I'(u^k)$ in E^*. Thus, for any $v \in E$,

$$\langle I'(u_n^k) - I'(u^k), v\rangle \le \|I'(u_n^k) - I'(u^k)\|\|v\| \to 0.$$

This means $\langle I'(u^k), v\rangle = 0$ for any $v \in E$, i.e. $I'(u^k) = 0$ in E^*. By $b_k^* \to \infty$, we know that $\{u^k\}_{k=1}^{\infty}$ is an unbounded sequence of critical points of functional I. This completes the proof of Theorem 1.1.

4. Conclusions

In this paper, we use the variant fountain theorem to prove the existence of infinitely many large energy solutions without the Ambrosetti-Rabinowitz's 4-superlinearity condition. Recent results of the literatures are extended and improved.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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