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Research article

Refinements of bounds for the arithmetic mean by new Seiffert-like means

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Abstract: In the article, we present the sharp upper and lower bounds for the arithmetic mean in terms of new Seiffert-like means, which give some refinements of the results obtained in [1]. As applications, two new inequalities for the sine and hyperbolic sine functions will be established.

Keywords: Seiffert-like mean; tangent mean; hyperbolic sine mean; sine mean; hyperbolic tangent mean; arithmetic mean
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1. Introduction

For two positive numbers a and b, four means

$$\mathsf{M}_{\sin}(a,b)=\begin{cases}\dfrac{a-b}{2\sin\left(\frac{a-b}{a+b}\right)} & a\neq b\\ a & a=b\end{cases}, \qquad \text{(sine mean)} \tag{1.1}$$

$$\mathsf{M}_{\tan}(a,b)=\begin{cases}\dfrac{a-b}{2\tan\left(\frac{a-b}{a+b}\right)} & a\neq b\\ a & a=b\end{cases}, \qquad \text{(tangent mean)} \tag{1.2}$$

$$\mathsf{M}_{\sinh}(a,b)=\begin{cases}\dfrac{a-b}{2\sinh\left(\frac{a-b}{a+b}\right)} & a\neq b\\ a & a=b\end{cases}, \qquad \text{(hyperbolic sine mean)} \tag{1.3}$$

and

$$\mathsf{M}_{\tanh}(a,b)=\begin{cases}\dfrac{a-b}{2\tanh\left(\frac{a-b}{a+b}\right)} & a\neq b\\ a & a=b\end{cases}\qquad\text{(hyperbolic tangent mean)}\tag{1.4}$$

are the so-called *Seiffert-like means* introduced by Witkowski [2], which are the means of the form

$$\mathsf{M}_f(a,b)=\begin{cases}\dfrac{|a-b|}{2f\left(\frac{|a-b|}{a+b}\right)}, & a\neq b,\\ a, & a=b,\end{cases}\tag{1.5}$$

where $a,b>0$ and the function $f:(0,1)\mapsto\mathbb{R}$ (called *Seiffert function*) satisfies

$$\frac{x}{1+x}\leq f(x)\leq\frac{x}{1-x}.$$

It is worth mentioning that these new Seiffert-like means have the Seiffert functions sin, tan, sinh and tanh, the inverse counterparts of which can produce the first Seiffert mean [3], second Seiffert mean [4], Neuman-Sándor mean [5] and logarithmic mean [6] by (1.5). In fact, these Seiffert-like means belong essentially to those ones constructed by trigonometric and hyperbolic functions. Such methods to create new means first appeared in [7] by Yang and embodied in several papers [8–10]. For more informations on these means, we refer to the literature in [11–23].

Sharp bounds for the Seiffert-like means and their related special functions have attracted the attention of several researchers [24–26]. In particular, the following chain of inequalities

$$\mathsf{M}_{\tan}(a,b)<\mathsf{M}_{\sinh}(a,b)<\mathsf{A}(a,b)<\mathsf{M}_{\sin}(a,b)<\mathsf{M}_{\tanh}(a,b)$$

had been established in [2] for all $a,b>0$ with $a\neq b$, where $\mathsf{A}(a,b)=(a+b)/2$ is the arithmetic mean.

Very recently, Nowicka and Witkowski [1] proved that the double inequalities

$$\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)<\mathsf{A}(a,b)<\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b),\tag{1.6}$$

$$\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)<\mathsf{A}(a,b)<\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\tag{1.7}$$

hold for all $a,b>0$ with $a\neq b$.

Motivated by (1.6) and (1.7), it makes sense to ask about the optimal parameters $\lambda_1,\lambda_2,\mu_1$ and μ_2 satisfying the following inequalities

$$\begin{aligned}\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{\lambda_1}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1-\lambda_1}&<\mathsf{A}(a,b)\\ &<\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{\mu_1}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1-\mu_1},\\ \left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{\lambda_2}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1-\lambda_2}&<\mathsf{A}(a,b)\\ &<\left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{\mu_2}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1-\mu_2}\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$. This paper aims to answer this question.

2. Lemmas

To prove our main results we need several lemmas, which we present in this section.

Lemma 2.1. ([27, L'Hospital Monotone Rule]) *Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable with $g'(x) \neq 0$ such that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$. If f'/g' is (strictly) increasing (decreasing) on (a, b), then so is f/g .*

The following lemma is a useful tool for dealing with the monotonicity of the ratio of two power series. The first part of Lemma 2.2 is first established by Biernacki and Krzyz [28], while the second part comes from Yang et al. [29, Theorem 2.1]. But we cite the latest version of the second part [30, Lemma 2], where the authors have corrected a bug in the previous version [29, Theorem 2.1].

Lemma 2.2. ([30]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$. Then the following statements hold true:*

(1) If the non-constant sequences $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for all $n \geq 0$, then $h(x)$ is strictly increasing (decreasing) on $(0, r)$;

(2) If for certain $m \in \mathbb{N}$, the sequence $\{a_k/b_k\}_{0\leq k\leq m}$ and $\{a_k/b_k\}_{k\geq m}$ both are non-constant, and they are increasing (decreasing), respectively. Then $h(x)$ is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \geq (\leq)0$. Moreover, if $H_{f,g}(r^-) < (>)0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on (x_0, r).

Let us recall the Taylor series expansions for $\cot x$ and $\csc x$, which can be found in [31].

Lemma 2.3. *For $|x| < \pi$, then we have the Taylor series formulas*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \quad \text{and} \quad \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1},$$

where B_{2n} is the even-index Bernoulli numbers for $n \in \mathbb{N}$.

For the readers' convenience, recall from [31, p.804, 23.1.1] that the Bernoulli numbers B_n may be defined by the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

The first few Bernoulli numbers B_{2k} are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad , B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}$$

with the property $(-1)^{k+1} B_{2k} > 0$ for $k \geq 1$.

Lemma 2.4. ([32]) *For $k \in \mathbb{N}$, the Bernoulli numbers B_{2k} satisfy*

$$\frac{2^{2k-1} - 1}{2^{2k+1} - 1} \frac{(2k+1)(2k+2)}{\pi^2} < \left| \frac{B_{2k+2}}{B_{2k}} \right| < \frac{2^{2k} - 1}{2^{2k+2} - 1} \frac{(2k+1)(2k+2)}{\pi^2}.$$

Some other Taylor series formulas for the functions involving $\cot x$ and $\csc x$ can be obtained from Lemma 2.3 by differentiation.

Lemma 2.5. *Let B_{2n} be the even-index Bernoulli numbers for $n \in \mathbb{N}$. Then*

$$\csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2},$$
$$\csc^2 x \cot x = \frac{1}{x^3} - \sum_{n=1}^{\infty} \frac{n(2n+1)2^{2n+2}}{(2n+2)!}|B_{2n+2}|x^{2n-1},$$
$$\csc x \cot x = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-2}$$

and

$$\csc x \cot^2 x = -\frac{1}{2x} + \frac{1}{x^3} - \sum_{n=1}^{\infty} \frac{(2n+1)\left[(n+1)(2^{2n}-2)|B_{2n}| - n(2^{2n+2}-2)|B_{2n+2}|\right]}{(2n+2)!}x^{2n-1}$$

for $|x| < \pi$.

Proof. Differentiation yields

$$(\cot x)' = -\csc^2 x, \quad (\cot x)'' = 2\csc^2 x \cot x,$$
$$(\csc x)' = -\csc x \cot x, \quad (\csc x)'' = 2\csc x \cot^2 x + \csc x,$$

which in conjunction with Lemma 2.3 gives the desired results. □

Lemma 2.6. *Let $\sigma = [(2+\cos 1)(1+3\cot^2 1 - 3\cot 1)]/[2(1-\cos 1)] = 0.8581\cdots$. Then the function*

$$f(x) = \frac{(2+\cos x)(3x - 2x\sin^2 x - 3\sin x \cos x)}{2x(1-\cos x)\sin^2 x}$$

is strictly increasing from $(0, 1)$ onto $(4/5, \sigma)$.

Proof. Let

$$f_1(x) = 2x\csc x + 6x\csc x\cot^2 x - 2x\cot x + 3x\csc^2 x\cot x - 3\csc^2 x - 6\csc x\cot x + 3$$

and

$$f_2(x) = 2x\csc x - 2x\cot x.$$

Then we clearly see that $f(x) = f_1(x)/f_2(x)$.

By Lemma 2.3 and Lemma 2.5, we can rewrite $f(x)$ in terms of power series as follows

$$f(x) = \frac{\sum_{n=1}^{\infty} a_n x^{2n}}{\sum_{n=1}^{\infty} b_n x^{2n}}, \tag{2.1}$$

where

$$a_n = \frac{(2^{2n}+2)|B_{2n}| + 6(2^{2n}-1)|B_{2n+2}|}{(2n)!}$$

and

$$b_n = \frac{4(2^{2n}-1)}{(2n)!}|B_{2n}|.$$

It can be easily seen from (2.1) and Lemma 2.2(1) that Lemma 2.6 will be proved if we can show that the sequence

$$\left\{\frac{a_n}{b_n} = \frac{2^{2n-1}+1}{2(2^{2n}-1)} + \frac{3}{2}\frac{|B_{2n+2}|}{|B_{2n}|}\right\} \tag{2.2}$$

is strictly increasing for $n \geq 1$.

Simple calculations with (2.2) and Lemma 2.4 yield

$$\begin{aligned}
\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} &= \left[\frac{2^{2n+1}+1}{2(2^{2n+2}-1)} + \frac{3}{2}\frac{|B_{2n+4}|}{|B_{2n+2}|}\right] - \left[\frac{2^{2n-1}+1}{2(2^{2n}-1)} + \frac{3}{2}\frac{|B_{2n+2}|}{|B_{2n}|}\right] \\
&> \left[\frac{2^{2n+1}+1}{2(2^{2n+2}-1)} + \frac{3(n+2)(2n+3)}{\pi^2}\frac{2^{2n+1}-1}{2^{2n+3}-1}\right] \\
&\qquad - \left[\frac{2^{2n-1}+1}{2(2^{2n}-1)} + \frac{3(n+1)(2n+1)}{\pi^2}\frac{2^{2n}-1}{2^{2n+2}-1}\right] \\
&= \frac{3\alpha_n}{4\pi^2(2^{2n-1})(2^{2n+2}-1)(2^{2n+3}-1)},
\end{aligned} \tag{2.3}$$

where

$$\alpha_n = 2^{4n+2}\Big[(4n+5)2^{2n+3} + 6n^2 - 47n - (67+6\pi^2)\Big] - 2^{2n}\Big[24n^2 - 76n - (128+3\pi^2)\Big] - 4(4n+5). \tag{2.4}$$

By using the Bernoulli inequality, we obtain

$$\begin{aligned}
(4n+5)2^{2n+3} + 6n^2 - 47n - (67+6\pi^2) &> 8(4n+5)(2n+1) + 6n^2 - 47n - 127 \\
&= 70n^2 + 65n - 87 > 0
\end{aligned}$$

for $n \geq 1$. According to this with (2.4), it follows that

$$\begin{aligned}
\alpha_n &> 2^{4n+2}(70n^2 + 65n - 87) - 2^{2n}(24n^2 - 76n - 155) - 4(4n+5) \\
&> 2^{2n}\Big[2^4(70n^2 + 65n - 87) - (24n^2 - 76n - 155)\Big] - 4(4n+5) \\
&= 2^{2n}(1096n^2 + 1116n - 1237) - 4(4n+5) \\
&> 2^2(1096n^2 + 1116n - 1237) - 4(4n+5) \\
&= 8(548n^2 + 556n - 621) > 0
\end{aligned}$$

for $n \geq 1$. This together with (2.3) implies that the sequence $\{a_n/b_n\}$ is strictly increasing for $n \geq 1$. So is $f(x)$ from Lemma 2.2(1).

By L'Hopital rule, we obtain

$$\begin{aligned}
f(0^+) &= \lim_{x\to 0^+}\frac{f_1'(x)}{f_2'(x)} = \frac{a_1}{b_1} = \frac{4}{5}, \\
f(1^-) &= \lim_{x\to 1^-} f(x) = \frac{(2+\cos 1)(1 + 3\cot^2 1 - 3\cot 1)}{2(1-\cos 1)} = \sigma.
\end{aligned}$$

This completes the proof. □

Lemma 2.7. *Let* $\tau = [(2 + \cosh 1)(3\coth^2 1 - 3\coth 1 - 1)]/[2(\cosh 1 - 1)] = 0.7603\cdots$. *Then the function*

$$g(x) = \frac{(2 + \cosh x)(3x + 2x\sinh^2 x - 3\sinh x\cosh x)}{2x(\cosh x - 1)\sinh^2 x}$$

is strictly decreasing from $(0, 1)$ *onto* $(\tau, 4/5)$.

Proof. Let

$$g_1(x) = 16x + 10x\cosh x + 8x\cosh(2x) + 2x\cosh(3x) - 3\sinh x - 12\sinh(2x) - 3\sinh(3x)$$

and

$$g_2(x) = 2x[2 - \cosh x - 2\cosh(2x) + \cosh(3x)].$$

Then it is easy to see that $g(x) = g_1(x)/g_2(x)$.

Recall the Taylor series expansions of $\sinh x$ and $\cosh x$ are

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}.$$

According to this, we can rewrite $g(x)$, in terms of power series, as

$$g(x) = \frac{\sum_{n=0}^{\infty} u_n x^{2n}}{\sum_{n=0}^{\infty} v_n x^{2n}}, \tag{2.5}$$

where

$$u_n = \frac{(n+1)2^{2n+8} + (4n+1)3^{2n+4} + 20n + 47}{(2n+5)}$$

and

$$v_n = \frac{2(3^{2n+4} - 2^{2n+5} - 1)}{(2n+4)!}.$$

From (2.5) and Lemma 2.2(1), it suffices to consider the monotonicity of the sequence $\{u_n/v_n\}_{n=0}^{\infty}$. Simple calculations lead to

$$\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = \frac{12\beta_n}{(2n+5)(2n+7)(3^{2n+4} - 2^{2n+5} - 1)(3^{2n+6} - 2^{2n+7} - 1)}, \tag{2.6}$$

where

$$\beta_n = -(40n^2 + 170n + 127)6^{2n+4} + 3^{2n+4}[3^{2n+7} - 2(32n^2 + 176n + 237)] - 2^{2n+4}[2^{2n+10} - (24n^2 + 162n + 239)] - 1.$$

It can be easily verified that

$$\beta_0 = -38400 \quad \text{and} \quad \beta_1 = -2257920. \tag{2.7}$$

We now prove that $\beta_n > 0$ for $n \geq 2$.

By binomial theorem, elementary calculations lead to

$$\begin{aligned}
\frac{\beta_n}{3^{2n+4}} &= -(40n^2+170n+127)2^{2n+4}+3^{2n+7}-2(32n^2+176n+237)\\
&\qquad -\left(\frac{2}{3}\right)^{2n+4}[2^{2n+10}-(24n^2+162n+239)]-\frac{1}{3^{2n+4}}\\
&> -(40n^2+170n+127)2^{2n+4}+3^{2n+7}-2(32n^2+176n+237)\\
&\qquad -\left(\frac{2}{3}\right)^{8}[2^{2n+10}-(24n^2+162n+239)]-\frac{1}{3^{8}}\\
&= 3^{2n+7}-\frac{1}{3^8}\Big[(262440n^2+1115370n+849631)2^{2n+4}\\
&\qquad +418880n^2+2268000n+3048731\Big]\\
&> 3^7\left[2^{2n}+(2n)2^{2n-1}+\frac{2n(2n-1)}{2!}2^{2n-2}+\frac{2n(2n-1)(2n-2)}{3!}2^{2n-3}\right]\\
&\qquad -\frac{1}{3^8}\Big[(262440n^2+1115370n+849631)2^{2n+4}\\
&\qquad +418880n^2+2268000n+3048731\Big]\\
&> \frac{1}{3^8}\left[\frac{1}{4}(9565938n^3-2447253n^2-23553990n+3019244)(1+2n)\right.\\
&\qquad \left.-(418880n^2+2268000n+3048731)\right]\\
&> \frac{(n-2)(42691728+34639615n+42935184n^2+19131876n^3)}{4\cdot 3^8}>0
\end{aligned}$$

for $n \geq 2$. Combining this with (2.6) and (2.7), it follows that u_n/v_n is strictly decreasing for $0 \leq n \leq 2$ and strictly increasing for $n \geq 2$.

Further, differentiation yields

$$\begin{aligned}
H_{g_1,g_2}(x) &= \frac{g_1'(x)}{g_2'(x)}g_2(x)-g_1(x)\\
&= \frac{4x\sinh^2 x(6x\sinh^2 x+8x\cosh x-7\sinh x\cosh x-8\sinh x+7x)}{x+3x\cosh x+\sinh x}\\
&\qquad -\Big[16x+10x\cosh x+8x\cosh(2x)+2x\cosh(3x)\\
&\qquad -3\sinh x-12\sinh(2x)-3\sinh(3x)\Big],
\end{aligned}$$

which gives

$$H_{g_1,g_2}(1) = -0.06789\cdots < 0. \tag{2.8}$$

Lemma 2.2(2) and (2.8) together with the piecewise monotonicity of u_n/v_n lead to the conclusion that $g(x)$ is strictly decreasing on $(0,1)$. Finally, since

$$g(0^+) = \frac{c_0}{d_0} = \frac{4}{5}, \qquad g(1^-) = \frac{(2+\cosh 1)(3\coth^2 1-3\coth 1-1)}{2(\cosh 1-1)} = \tau,$$

the proof is completed. □

3. Main results

Theorem 3.1. *The double inequality*

$$\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{\lambda_1}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1-\lambda_1}<\mathsf{A}(a,b)$$
$$<\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{\mu_1}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1-\mu_1}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\lambda_1\leq 4/5$ *and* $\mu_1\geq\mu_1^*:=[3\log(\sin 1)-\log(\cos 1)]/[3\log\left(\frac{2+\cos 1}{3}\right)-\log(\cos 1)]=0.8386\cdots$.

Proof. Since $\mathsf{M}_{\sin}(a,b),\mathsf{M}_{\tan}(a,b)$ and $\mathsf{A}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we may assume that $a>b>0$.

Let $x=(a-b)/(a+b)\in(0,1)$. Then from (1.1) and (1.2) we clearly see that

$$\frac{\mathsf{M}_{\sin}(a,b)}{\mathsf{A}(a,b)}=\frac{x}{\sin x},\quad \frac{\mathsf{M}_{\tan}(a,b)}{\mathsf{A}(a,b)}=\frac{x}{\tan x}. \tag{3.1}$$

According to (3.1), we obtain

$$\frac{\log\mathsf{A}(a,b)-\log\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]}{\log\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]-\log\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]}$$
$$=\frac{\log\left(\frac{\sin x}{x}\right)-\frac{1}{3}\log(\cos x)}{\log\left(\frac{2+\cos x}{3}\right)-\frac{1}{3}\log(\cos x)}:=\varphi(x). \tag{3.2}$$

Let

$$\varphi_1(x)=\log\left(\frac{\sin x}{x}\right)-\frac{1}{3}\log(\cos x)\quad\text{and}\quad\varphi_2(x)=\log\left(\frac{2+\cos x}{3}\right)-\frac{1}{3}\log(\cos x).$$

Then we clearly see from (3.2) that $\varphi(x)=\varphi_1(x)/\varphi_2(x)$.

Simple calculations lead to

$$\varphi_1(0^+)=\varphi_2(0^+)=0, \tag{3.3}$$
$$\frac{\varphi_1'(x)}{\varphi_2'(x)}=\frac{(2+\cos x)\left(3x-2x\sin^2 x-3\sin x\cos x\right)}{2x(1-\cos x)\sin^2 x}=f(x), \tag{3.4}$$

where $f(x)$ is defined as in Lemma 2.6.

Lemma 2.1 and Lemma 2.6 together with (3.3), (3.4) lead to the conclusion that $\varphi(x)$ is strictly increasing on $(0,1)$.

Therefore, Theorem 3.1 follows easily from (3.2) and the monotonicity of $\varphi(x)$ together with

$$\varphi(0^+)=\lim_{x\to 0^+}\frac{\varphi_1'(x)}{\varphi_2'(x)}=\frac{4}{5},\quad \varphi(1^-)=\frac{3\log(\sin 1)-\log(\cos 1)}{3\log\left(\frac{2+\cos 1}{3}\right)-\log(\cos 1)}=\mu_1^*.$$

□

Theorem 3.2. *The double inequality*

$$\left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{\lambda_2}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1-\lambda_2}<\mathsf{A}(a,b)$$
$$<\left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{\mu_2}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1-\mu_2}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\lambda_2\le\lambda_2^*:=[3\log(\sinh 1)-\log(\cosh 1)]/[3\log\left(\frac{2+\cosh 1}{3}\right)-\log(\cosh 1)]=0.7730\cdots$ *and* $\mu_2\ge 4/5$.

Proof. Since $\mathsf{M}_{\sinh}(a,b)$, $\mathsf{M}_{\tanh}(a,b)$ and $\mathsf{A}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we may assume that $a>b>0$.

Let $x=(a-b)/(a+b)\in(0,1)$. Then it is easy to see from (1.3) and (1.4) that

$$\frac{\mathsf{M}_{\sinh}(a,b)}{\mathsf{A}(a,b)}=\frac{x}{\sinh x},\quad \frac{\mathsf{M}_{\tanh}(a,b)}{\mathsf{A}(a,b)}=\frac{x}{\tanh x}. \tag{3.5}$$

According to (3.1), it follows that

$$\frac{\log\mathsf{A}(a,b)-\log\left[\mathsf{M}_{\sinh}^{2/3}(a,b)\mathsf{M}_{\tanh}^{1/3}(a,b)\right]}{\log\left[\frac{2}{3}\mathsf{M}_{\sinh}(a,b)+\frac{1}{3}\mathsf{M}_{\tanh}(a,b)\right]-\log\left[\mathsf{M}_{\sinh}^{2/3}(a,b)\mathsf{M}_{\tanh}^{1/3}(a,b)\right]}$$
$$=\frac{\log\left(\frac{\sinh x}{x}\right)-\frac{1}{3}\log(\cosh x)}{\log\left(\frac{2+\cosh x}{3}\right)-\frac{1}{3}\log(\cosh x)}:=\phi(x). \tag{3.6}$$

Let

$$\phi_1(x)=\log\left(\frac{\sinh x}{x}\right)-\frac{1}{3}\log(\cosh x)\quad\text{and}\quad\phi_2(x)=\log\left(\frac{2+\cosh x}{3}\right)-\frac{1}{3}\log(\cosh x).$$

Then we clearly see from (3.6) that $\phi(x)=\phi_1(x)/\phi_2(x)$.

Simple calculations lead to

$$\phi_1(0^+)=\phi_2(0^+)=0, \tag{3.7}$$
$$\frac{\phi_1'(x)}{\phi_2'(x)}=\frac{(2+\cosh x)\left(3x+2x\sinh^2 x-3\sinh x\cosh x\right)}{2x(\cosh x-1)\sinh^2 x}=g(x), \tag{3.8}$$

where $g(x)$ is defined as in Lemma 2.7.

Lemma 2.1 and Lemma 2.7 together with (3.7), (3.8) lead to the conclusion that $\phi(x)$ is strictly decreasing on $(0,1)$.

Moreover, by L'Hopital rule and (3.8), one has

$$\phi(0^+)=\lim_{x\to0^+}\frac{\phi_1'(x)}{\phi_2'(x)}=\frac{4}{5},\quad \phi(1^-)=\frac{3\log(\sinh 1)-\log(\cosh 1)}{3\log\left(\frac{2+\cosh 1}{3}\right)-\log(\cosh 1)}=\lambda_2^*. \tag{3.9}$$

Therefore, Theorem 3.2 follows easily from (3.6) and (3.9). □

As a consequence of Theorem 3.1 and Theorem 3.2, new bounds for the sine and hyperbolic sine function are given in the following corollary.

Corollary 3.3. *Let μ_1^* and λ_2^* be defined as in Theorem 3.1 and Theorem 3.2 respectively. Then the double inequalities*

$$\left(\frac{2+\cos x}{3}\right)^{4/5}(\cos x)^{1/15} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\mu_1^*}(\cos x)^{(1-\mu_1^*)/3},$$
$$\left(\frac{2+\cosh x}{3}\right)^{\lambda_2^*}(\cosh x)^{(1-\lambda_2^*)/3} < \frac{\sinh x}{x} < \left(\frac{2+\cosh x}{3}\right)^{4/5}(\cosh x)^{1/15}$$

hold for all $x \in (0,1)$.

4. Conclusions

In the paper, we establish sharp upper and lower bounds for the arithmetic mean in terms of new Seiffert-like means, more precisely, the double inequalities

$$\left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{4/5}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1/5} < \mathsf{A}(a,b)$$
$$< \left[\frac{2}{3}\mathsf{M}_{\sin}(a,b)+\frac{1}{3}\mathsf{M}_{\tan}(a,b)\right]^{\mu_1^*}\left[\mathsf{M}_{\sin}^{2/3}(a,b)\mathsf{M}_{\tan}^{1/3}(a,b)\right]^{1-\mu_1^*}$$

and

$$\left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{\lambda_2^*}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1-\lambda_2^*} < \mathsf{A}(a,b)$$
$$< \left[\frac{1}{3}\mathsf{M}_{\tanh}(a,b)+\frac{2}{3}\mathsf{M}_{\sinh}(a,b)\right]^{4/5}\left[\mathsf{M}_{\tanh}^{1/3}(a,b)\mathsf{M}_{\sinh}^{2/3}(a,b)\right]^{1/5}$$

hold for all $a,b > 0$ with $a \neq b$, where μ_1^* and λ_2^* are given as in Theorem 3.1 and Theorem 3.2, respectively.

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Conflict of interest

The authors declare that they have no competing interests.

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