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Research article

Maximizing the goal-reaching probability before drawdown with borrowing constraint

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Abstract: We study the optimal investment problem in a constrained financial market, where the proportion of borrowed amount to the current wealth level is no more than a given constant. The objective is to maximize the goal-reaching probability before drawdown, namely, the probability that the value of the wealth process reaches the safe level before hitting a lower dynamic barrier. The financial market consists of a risk-free asset and multiple risky assets. By the construction of auxiliary market and convex analysis, we relax the borrowing constraint and investigate the new optimization problem in an auxiliary market, where there is no such borrowing constraint. Then, we find the relationship between the optimal results in auxiliary market and those in constrained market. The explicit expressions for the optimal investment strategy and the maximum goal-reaching probability before drawdown are derived in closed-form. Finally, we provide some numerical examples to show the effect of model parameters on the behaviors of investor.

Keywords: optimal investment; borrowing constraint; support function; stochastic optimal control; goal-reaching probability before drawdown
Mathematics Subject Classification: 49L20, 91G10, 93E20

1. Introduction

The theory of optimal investment can date back to the seminal works of Merton [14]. From then on, the optimal investment problems have been paid great attention by the scholars all over the world and the stochastic control theory has been widely used in the literature of investment. The popular criteria include minimizing the probability of ruin, maximizing the expected utility of terminal wealth, mean-variance criterion, etc. See, for example, Browne [5], Hipp & Plum [8], Promislow & Young [15], B*ä*uerle [4], Zhang et al. [22], Yuan et al. [20], Sun et al. [17].

In the recent years, drawdown, which describes the event that the value of wealth drops to its historic higher-water mark becomes a frequently quoted risk metric. The occurrence of drawdown will result in larger portfolio losses, or even a long-term recession. Hence, the fund managers tend to make the investment decisions with low drawdown risks. Research on drawdown model is increasing rapidly in the past few years. See for example, Angoshtari et al. [2] and Chen et al. [7] both derived the minimum drawdown probability in a lifetime investment problem. Angoshtari et al. [1] and Han & Liang [11] considered an optimal investment problem in an infinite-time horizon. Under the thinning dependence/common shock structure, Han et al. [9, 10] studied the optimal reinsurance problem, which minimized the probability of drawdown with dependent risks.

Most of the existing literature about drawdown solved the problems without borrowing constraint. However, the borrowing constraints do exist and the investor is not allowed to freely borrow money without any limitations. A few scholars have advocated and investigated the optimal investment problems under different borrowing constraints. For example, Bayraktar & Young [3] considered the problem of minimizing the probability of lifetime ruin, in which the individual continuously consumed either a constant dollar amount or a constant proportion of the wealth under the higher borrowing rate. Luo [13] derived the minimum ruin probability for an insurance company with proportional reinsurance and investment under the limited leverage rate constraint. Both of those results were obtained for only one risky asset in the financial market. For the multiple risky assets, Karatzas & Shreve [12] considered the optimal investment under the utility maximization framework. Yener [18] computed the optimal investment strategy with borrowing and short selling constraints, which minimized the probability of lifetime ruin. For more research on borrowing constraints, we refer readers to Yener [19], Yuan et al. [21], etc.

Inspired by the above-mentioned work, we focus on an optimal investment problem with borrowing constraint, where the proportion of the borrowed amount to the current wealth level is no more than a given constant. The investor aims to maximize the goal-reaching probability before drawdown, i.e., the probability that value of the wealth process reaches safe level before some fixed proportion of its maximum value to date. To make the optimization problem tractable, we introduce an auxiliary market and allow the trading to be done as if there is no borrowing constraint in it. After finding the relationship between the constrained and auxiliary markets, we derive the optimal investment strategy under borrowing constraint by dynamic programming and convex analysis. We observe that the optimal investment strategy of maximizing goal-reaching probability before drawdown will also maximize goal-reaching probability before ruin. This phenomenon reveals that the drawdown levle only affects the evaluation of value function while it exerts no impact on the investor's optimal strategy.

As far as we concerned, only Yuan et al. [21] investigated the minimum drawdown probability with borrowing constraints for one risky asset. The current paper extends the model of Yuan et al. [21] and includes it as a special case. The main contribution of this paper is threefold: (*i*) compared with the early work of [21], we further consider the case of multiple risky assets, where the method of truncation in [21] does not apply anymore. Therefore, following the analysis of Karatzas & Shreve [12] and Haluk Yener [18], an auxiliary market is introduced to tackle the constrained problem; (*ii*) the ruin level set in our paper is no longer fixed, which is totally different from Yener [18, 19]. A more general criterion, that is, maximizing the goal-reaching probability before drawdown is investigated in our paper, which covers the minimizing the probability of ruin/drawdown as a special case. (*iii*) the effect of model parameters on the optimal investment strategy and borrowing constraint point is investigated in our paper.

The remainder of this paper is organized as follows. In Section 2, the constrained market and the auxiliary market are presented. By the stochastic control theory, explicit expressions for the optimal strategy and corresponding maximum goal-reaching probability before drawdown are obtained in Sections 3. In section 4, we introduce some numerical examples to show the impact of model parameters. Finally, we conclude the paper in Section 5.

2. Model and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, where $\mathbb{P}$ is the real world probability measure and $\mathcal{F}_t$ stands for the information available until time t. Assume that all Brownian motions introduced below are well-defined and adapted processes defined on the complete probability space.

2.1. The constrained market

The investor is allowed to invest in multiple risk assets (stocks), which evolve as

$$dS_i(t) = S_i(t)\Big[\mu_i dt + \sum_{j=1}^{N} \sigma_{ij} dB_j(t)\Big], \text{ for } i = 1, 2, \cdots, N,$$

where μ_i and σ_{ij} $(i, j = 1, 2, \cdots, N)$ are all given constants; $B(t) = (B_1(t), \cdots, B_N(t))'$ is a standard N-dimensional Brownian motion. Meanwhile, the investor can invest a non-negative amount in a risk-free asset that earns interest at the constant rate r. Hence, the market price of risk is defined by

$$\zeta = \sigma^{-1}(\mu - r1_N),$$

where $1_N = (1, \cdots, 1)'$, $\mu = (\mu_1, \cdots, \mu_N)'$, $\sigma = (\sigma_1, \cdots, \sigma_N)'$ and $\sigma_i = (\sigma_{i,1}, \cdots, \sigma_{iN})'$ for $i = 1, \cdots, N$.

In our model, we denote the fractions of the wealth invested in the risky assets by $\pi(t) = (\pi_1(t), \cdots, \pi_N(t))'$ and the fraction $(1 - \pi(t)'1_N)$ is then invested in the risk-free asset. We can see that $\pi(t)'1_N < 1$ means that saving is existent; if $\pi(t)'1_N > 1$, it means that the investor has to borrow money with risk-free rate r from the monetary market and invest all the wealth in the risky assets. However, borrowing amount is not limitless. Therefore, we suppose that the proportion of the borrowed amount to the current wealth level is no more than a given constant k $(k \geq 0)$ in this paper. Besides, the investor's consumption process is assumed as $C(X^\pi(t)) = aX^\pi(t) + b$, where a $(0 \leq a < r)$ and b $(b > 0)$ are two given constants. With the admissible strategy $\pi(t)$ and consumption process, the wealth process satisfies the stochastic differential equation:

$$dX^\pi(t) = [(r - a)X^\pi(t) + \pi(t)'(\mu - r1_N)X^\pi(t) - b]dt + \pi(t)'\sigma X^\pi(t)dB(t), \tag{2.1}$$

where $X^\pi(0) = x$. Before introducing the optimization objective, we state formally the definition of admissible strategies.

Definition 2.1 (Admissible strategy in constrained market)**.** *A control $\pi(t)$ is said to be admissible if it satisfies the following conditions:*
(i) $\pi(t)'1_N \leq 1 + k$ for all $t > 0$;

(*ii*) *$\pi(t)$ is $\mathcal{F}_t$-progressively measurable and $E(\int_0^\infty \|\pi(t)\|^2 dt) < \infty$ where $\|\pi(t)\| = \sum_i^n \pi_i^2(t)$;*
(*iii*) *associated with $\pi(t)$, the state equation* (2.1) *has a unique strong solution.*

Let $\mathcal{A}_c$ be the set of all admissible strategies in the constrained market. Define the maximum wealth value $M(t)$ at time t by

$$M(t) = \max\left\{ \sup_{0\le s\le t} X^\pi(s),\ M(0)\right\},$$

where $M(0) = m > 0$. Note that we set $x > m$ to allow the wealth process to have a financial past. Drawdown is the first time when the wealth process reaches $\alpha \in [0, 1)$ times its maximum value. The hitting time τ_α is defined by $\tau_\alpha = \inf\{t \ge 0|X^\pi(t) \le \alpha M(t)\}$. If $\alpha = 0$, the drawdown level is the same as the ruin level 0.

Besides, if the value of wealth is large enough, say, at least $x^* = b/(r - a)$, then the investor can invest all the wealth in the risk-free asset and the wealth process becomes non-decreasing. As a result, drawdown will never happen in this case. For this reason, we call x^* safe level as given in Angoshtari et al. [1] and Han et al. [11]. The first hitting time to x^* is defined as $\tau_{x^*} = \inf\{t \ge 0|X^\pi(t) \ge x^*\}$. We set $\tau_\alpha = \infty$ ($\tau_{x^*} = \infty$) if the wealth process (2.1) never hits the level $\alpha M(t)$ (x^*).

Now assume that the investor is interested in maximizing the probability of reaching the safe level x^* before drawdown (hitting $\alpha M(t)$). We denote the maximum goal-reaching probability before drawdown by $V(x, m)$, which depends on the initial wealth x and maximum (past) value m. Specifically, V is the maximum probability of $\tau^\pi_{x^*} < \tau^\pi_\alpha$, thus, we derive the performance function as

$$J^\pi(x, m) = \mathrm{P}_{x,m}(\tau_{x^*} < \tau_\alpha) = \mathrm{E}_{x,m}(1_{\{\tau_{x^*}<\tau_\alpha\}}).$$

Here, $\mathrm{P}_{x,m}$ and $\mathrm{E}_{x,m}$ denote the probability and expectation conditional on $X^\pi(0) = x$ and $M(0) = m$, respectively,. Then, the corresponding value function is given by

$$V(x, m) = \sup_{\pi\in\mathcal{A}_c} J^\pi(x, m).$$

Note that if $x \le \alpha m$, then $V(x, m) = 0$, and if $x > x^*$ and $x > \alpha m$, then $V(x, m) = 1$. It remains for us to determine the maximum goal-reaching probability before drawdown on the demain

$$O = \{(x, m) \in (\mathbb{R}^+)^2|\alpha m \le x \le \min(m, x^*), \alpha m < x^*\}^*,$$

that is the topic of our paper.

2.2. The auxiliary market

When considering the constraint that the proportion of borrowed amount to the current wealth level is no more than a non-negative constant k, the financial market becomes incomplete. Along the same lines in Karatzas & Shreve [12], we introduce a vector of fictitious parameters $\nu \in \mathbb{R}^N$ to complete the market. After the completion of the market, we can allow the trading to be done as if there is no borrowing constraint in the auxiliary market. Hence, for each $\nu \in \mathbb{R}^N$, we obtain an optimal investment strategy π^*_ν. Then, it will be shown that if we find the ν^*, which minimizes the value function of

*Note that if $\alpha m = x^*$, then technically drawdown has occurred, but the investor could keep its wealth at $\alpha m = x^*$, thereafter by only investing in the risk-free asset. Therefore, we avoid this ambiguous case by assuming $\alpha m < x^*$ throughout.

the auxiliary market over a certain region, the optimal investment strategy $\pi^*_{\nu^*}$ is indeed the optimal investment strategy π^* under the borrowing constraint.

Since we are investigating the borrowing constraint in this paper, one can define the constraint set of the investment strategy as

$$\mathcal{K} = \left\{\pi \in \mathbb{R}^N \middle| \pi' 1_N \le 1 + k\right\}. \tag{2.2}$$

Then, for such closed convex set $\mathcal{K}$, the support function of the convex set $-\mathcal{K}$ is denoted by

$$\delta(\nu) = \sup_{\pi \in \mathcal{K}} (-\pi' \nu), \ \nu \in \mathbb{R}^N,$$

with $\delta(0) = 0$ and the effective domain is given by $\tilde{\mathcal{K}} = \{\nu \in \mathbb{R}^N | \delta(\nu) < \infty\}$. $\tilde{\mathcal{K}}$ can be seen as the barrier cone of $-\tilde{\mathcal{K}}$ (Rockafellar [16], page 114). We have $\delta(\nu) + \pi' \nu \ge 0$, $\forall \nu \in \tilde{\mathcal{K}}$, if and only if $\pi \in \mathcal{K}$. Meanwhile, according to the definition of support function, the effective domain $\tilde{\mathcal{K}}$ can be written as

$$\tilde{\mathcal{K}} = \{\nu \in \mathbb{R}^N | \nu_1 = \nu_2 = \cdots = \nu_N \le 0\},$$

and $\delta(\nu) = -(1 + k)\nu_1 \ge 0$ on $\tilde{\mathcal{K}}$ for some $\nu_1 \le 0$. The fictitious process ν is used to pull the investment strategy back into the constrained set. On the one hand, if the borrowing constraint are not violated, the support function remains 0 and has no effect on the optimal strategy. On the other hand, if the borrowing constraint is violated, it becomes positive and turns optimal unconstrained strategy into constrained strategy.

For any $\nu \in \tilde{\mathcal{K}}$, we define the assets of the auxiliary market as

$$\begin{cases} dS_0^\nu(t) = (r + \delta(\nu(t)))S_0^\nu(t)dt, \\ dS_i^\nu(t) = S_i^\nu(t)\Big[(\mu_i + \nu_i(t) + \delta(\nu(t)))dt + \sum_{j=1}^N \sigma_{ij} dB_j(t)\Big], \ \text{for} \ i = 1, 2, \cdots, N, \end{cases} \tag{2.3}$$

and the market price of risk in the auxiliary market becomes

$$\zeta_\nu(t) = \zeta + \sigma^{-1}\nu(t). \tag{2.4}$$

The wealth process of the investor in this auxiliary market follows the dynamics

$$dX_\nu^\pi(t) = [(r + \delta(\nu) - a)X_\nu^\pi(t) + \pi_\nu(t)'(\mu + \nu(t) - r1_N)X_\nu^\pi(t) - b]dt + \pi_\nu(t)'\sigma X_\nu^\pi(t)dB(t), \tag{2.5}$$

where $X_\nu^\pi(0) = x$ and the definition of admissible strategy in the auxiliary market is given by

Definition 2.2 (Admissible strategy in auxiliary market)**.** *A control $\pi_\nu(t)$ is said to be admissible if it satisfies the following conditions:*
(*i*) *$\pi_\nu(t)$ is $\mathcal{F}_t$-progressively measurable and $E(\int_0^\infty \|\pi_\nu(t)\|^2 dt) < \infty$;*
(*ii*) *associated with $\pi_\nu(t)$, the state equation* (2.5) *has a unique strong solution.*

Let $\mathcal{A}_\nu$ be the set of all admissible strategies in the auxiliary market. Meanwhile, the hitting times τ_α^ν and $\tau_{x^*}^\nu$ in the auxiliary market are modified to

$$\tau_\alpha^\nu = \inf\{t \ge 0 : X_\nu^\pi(t) \le \alpha M_\nu(t)\}, \qquad \tau_{x^*}^\nu = \inf\{t \ge 0 : X_\nu^\pi(t) \ge x^*\}.$$

Define the candidate value function as

$$V^{\nu}(x,m) = \sup_{\pi_\nu \in \mathcal{A}_\nu} P_{x,m}(\tau^{\nu}_{x^*} < \tau^{\nu}_{\alpha}).$$

Based on the above discussion, we have the following lemma to show the relationship between the value function $V(x,m)$ and candidate value function $V^{\nu}(x,m)$.

Lemma 2.1. *If $\nu^* \in \tilde{\mathcal{K}}$ satisfies the equality $\delta(\nu^*) + \pi^{'}\nu^* = 0$, we have*

$$V(x,m) = V^{\nu^*}(x,m) = \inf_{\nu \in \tilde{\mathcal{K}}} V^{\nu}(x,m) = \inf_{\nu \in \tilde{\mathcal{K}}} \sup_{\pi_\nu \in \mathcal{A}_\nu} \mathrm{P}_{x,m}(\tau^{\nu}_{x^*} < \tau^{\nu}_{\alpha}).$$

Proof. This proof can be outlined from the Remark 8.2 in Karatzas & Shreve [12] or the Appendix in Yener [19], hence, we omit it here. □

3. The optimal results

To solve the problem, we split the optimization problem into two different cases: $m \geq x^*$ and $m < x^*$. Recall the safe level x^*, when $m \geq x^*$, the wealth level will reach x^* before m and $M_\nu(t) = m$ holds almost surely for all $t \geq 0$, which leads to a fixed drawdown level αm. However, when $m < x^*$, the maximum process $M_\nu(t)$ may increase above m under some case, i.e., the drawdown level we set is not necessarily a fixed one.

3.1. HJB equation and verification theorem

Before showing the HJB equation for the characterization of the candidate value function V_ν and corresponding optimal strategy, we denote

$$C^{2,1}(\mathcal{O}) = \{\phi(x,m) | V^{\nu}(x,\cdot) \text{ is twice continuously differentiable on } \mathcal{O},$$
$$\text{and } V^{\nu}(\cdot,m) \text{ is once continuously differentiable on } \mathcal{O}\}.$$

It follows from the standard arguments that the $C^{2,1}(\mathcal{O})$ candidate value function $V^{\nu}(x,m)$ satisfies the following HJB equation:

$$\inf_{\nu \in \tilde{\mathcal{K}}} \sup_{\pi_\nu \in \mathcal{A}_\nu} \mathcal{L}^{\pi} V^{\nu}(x,m) = 0, \tag{3.1}$$

where

$$\mathcal{L}^{\pi} V^{\nu}(x,m) = [(r + \delta(\nu) - a)x + \pi^{'}_{\nu}(\mu + \nu - r1_N)x - b]V^{\nu}_x(x,m) + \frac{1}{2}\|\sigma^{'}\pi_\nu\|^2 x^2 V^{\nu}_{xx}(x,m). \tag{3.2}$$

The notations $V^{\nu}_x(x,m)$ and $V^{\nu}_{xx}(x,m)$ stand for the partial derivatives with respect to the first variable. Now we are going to give the verification theorem that a classical solution to the HJB equation, subject to the boundary conditions, is the value function.

Applying the methods of Chen et al. [7], Angoshtari et al. [1], and Han et al. [11], we can obtain the following verification theorem directly, thus we omit the proof here.

Theorem 3.1 (Verification Theorem)**.** *Suppose that $V^{\nu}(x,m):\mathcal{O}\to[0,1]$ is a bounded, continuous function, which satisfies the following conditions:*
(i) $V^{\nu}(x,m)\in C^{2,1}(\mathcal{O})$ is an increasing and concave function with respect to x;
(ii) $\partial V^{\nu}(m,m)/\partial m=0$ if $m<x^*$;
(iii) $V^{\nu}(\alpha m,m)=0$ and $V^{\nu}(x^*,m)=1$;
(iv) $V^{\nu^*}(x,m)$ is the solution of the HJB Eq (3.1) for some $(\nu^*,\pi^*_{\nu^*})\in\tilde{\mathcal{K}}\times\mathcal{A}_c$.
Then, we have $V^{\nu^*}(x,m)=V(x,m)$ on $\mathcal{O}$, and $\pi^*_{\nu*}$ is the optimal investment strategy under the borrowing constraint.

3.2. Maximizing the goal-reaching probability before drawdown when $m\ge x^*$

According to Theorem 3.1, we now proceed to focus on the following boundary-value problem and try to find a solution of the Eq (3.1):

$$\begin{cases}\inf\limits_{\nu\in\tilde{\mathcal{K}}}\sup\limits_{\pi_\nu\in\mathcal{A}_\nu}\left\{[(r+\delta(\nu)-a)x+\pi_\nu'(\mu+\nu-r1_N)x-b]V_x^\nu+\dfrac{1}{2}\|\sigma'\pi_\nu\|^2x^2V_{xx}^\nu\right\}=0,\\ V^\nu(\alpha m,m)=0,\qquad V^\nu(x^*,m)=1,\end{cases}\tag{3.3}$$

for $\alpha m\le x\le x^*\le m$.

Assume that $V^\nu(x,m)\in C^{2,1}(\mathcal{O})$, $V_x^\nu(x,m)>0$ and $V_{xx}^\nu(x,m)<0$. From the first-order condition, we obtain the maximizer $\pi_\nu^*(x)$ as

$$\pi_\nu^*(x)=-(\sigma')^{-1}\zeta_\nu(x)\frac{V_x^\nu}{xV_{xx}^\nu},\tag{3.4}$$

where $\zeta_\nu(x)$ is given by (2.4). Inserting (3.4) into (3.3) yields

$$\inf_{\nu\in\tilde{\mathcal{K}}}\left\{[(r+\delta(\nu)-a)x-b]V_x^\nu-\frac{1}{2}\parallel\zeta+\sigma^{-1}\nu\parallel^2\frac{(V_x^\nu)^2}{V_{xx}^\nu}\right\}=0.$$

Again, in view of the first-order condition, the minimum point of the fictitious parameter is $\nu^*=\nu_1^*1_N$ and ν_1^* is given by

$$\nu_1^*=\begin{cases}0, & \text{if } x>-\dfrac{D}{1+k}\dfrac{V_x^\nu}{V_{xx}^\nu},\\ -\dfrac{1}{K}\left((1+k)x\dfrac{V_{xx}^\nu}{V_x^\nu}+D\right), & \text{if } x\le-\dfrac{D}{1+k}\dfrac{V_x^\nu}{V_{xx}^\nu},\end{cases}\tag{3.5}$$

where $K=\mathbf{1}_N'(\sigma\sigma')^{-1}1_N$ and $D=\zeta'\sigma^{-1}1_N$. Given the maximum fictitious process ν^*, the optimal investment strategy $\pi^*_{\nu^*}(x)$ can be rewritten as

$$\pi^*_{\nu^*}=\begin{cases}-(\sigma\sigma')^{-1}(\mu-r1_N)\dfrac{V_x^0}{xV_{xx}^0}, & \text{if } x>-\dfrac{D}{1+k}\dfrac{V_x^{\nu^*}}{V_{xx}^{\nu^*}},\\ -(\sigma\sigma')^{-1}(\mu-r1_N-\nu^*)\dfrac{V_x^{\nu^*}}{xV_{xx}^{\nu^*}}, & \text{if } x\le-\dfrac{D}{1+k}\dfrac{V_x^{\nu^*}}{V_{xx}^{\nu^*}}.\end{cases}\tag{3.6}$$

One can easily verify the optimality of $(\nu^*,\pi^*_{\nu^*})$. Due to the different expressions for the optimal investment strategy in different regions, we need to discuss the following two cases.

Case 1: $x > -\frac{D}{1+k}\frac{V_x^{\nu^*}}{V_{xx}^{\nu^*}}$. We can see that $1_N^{'}\pi_{\nu^*}^* < 1+k$, that is, the borrowing constraint is not violated. Substituting $\pi_{\nu^*}^* = -(\sigma\sigma^{'})^{-1}(\mu - r1_N)\frac{V_x^{\nu^*}}{xV_{xx}^{\nu^*}}$ and $\nu^* = 0$ into the HJB Eq (3.3), then we have

$$\xi_0(x) = \frac{V_x^0}{V_{xx}^0} = \frac{2[(r-a)x - b]}{\parallel \zeta \parallel^2}, \tag{3.7}$$

and $1_N^{'}\pi_0^*(x) < 1 + k$ yields $x_0 < x < x^*$, where x_0 is given by

$$x_0 = \frac{2Db}{(1+k) \parallel \zeta \parallel^2 + 2D(r-a)} < x^*. \tag{3.8}$$

Case 2: $x \leq -\frac{D}{1+k}\frac{V_x^{\nu^*}}{V_{xx}^{\nu^*}}$. We can see that $1_N^{'}\pi_{\nu^*}^* = 1 + k$. Inserting $\pi_{\nu^*}^* = -\frac{1}{K}((1+k)x\frac{V_{xx}^{\nu}}{V_x^{\nu}} + D)$ and $\nu^* = -\frac{1}{K}((1+k)x\frac{V_{xx}^{\nu}}{V_x^{\nu}} + D)1_N$ into the HJB Eq (3.3), we obtain

$$\xi_{\nu^*}(x) = \frac{V_x^{\nu^*}}{V_{xx}^{\nu^*}} = -\frac{[b - (r - a + \frac{D}{K})x] + \sqrt{\theta(x)}}{\parallel \zeta \parallel^2 - \frac{D^2}{K}}, \tag{3.9}$$

where $\theta(x)$ is defined as

$$\theta(x) = \left[b + \left(r - a + \frac{D}{K}\right)x\right]^2 + \frac{(1+k)x^2}{K}\left(\parallel \zeta \parallel^2 - \frac{D^2}{K}\right),$$

and $1_N^{'}\pi_{\nu^*}^*(x) \geq 1 + k$ holds for $\alpha m \leq x \leq x_0$.

Remark 3.1. *One can see that if the wealth level decreases upon x_0, the borrowing constraint works and the investor maintains the maximum leverage ratio $1 + k$. Thus, we call x_0 borrowing constraint level.*

In the following theorem, according to the above analysis, we summarize the optimal results for the case of $m \geq x^*$. Then combining with the verification theorem, we can verify that the resulting function are indeed the maximum goal-reaching probability before drawdown on O.

Theorem 3.2. *Suppose that $m \geq x^*$. Let $\xi_0(x)$, $\xi_{\nu^*}(x)$, x_0 be given in (3.7), (3.9) and (3.8), respectively. Then, for any $x \in [\alpha m, x^*]$, the maximum goal-reaching probability before drawdown for the wealth process (2.1) is given by*

$$V(x,m) = \begin{cases} \dfrac{g_1(x,m)}{g_2(x^*,m)}, & \alpha m \leq x < \max(\alpha m, x_0), \\ \dfrac{g_2(x,m)}{g_2(x^*,m)}, & \max(\alpha m, x_0) \leq x \leq x^*, \end{cases}$$

where the functions $g_i(x,m)$ $(i = 1, 2)$ are given by

$$\begin{cases} g_1(x,m) = \displaystyle\int_{\alpha m}^{x} \exp\Big\{ -2\int_{\alpha m}^{y} \xi_{\nu^*}(\omega)^{-1} d\omega\Big\} dy, \\ g_2(x,m) = \displaystyle\int_{\alpha m}^{\alpha m \vee x_0} \exp\Big\{ -2\int_{\alpha m}^{y} \xi_{\nu^*}(\omega)^{-1} d\omega\Big\} dy \\ \qquad + \displaystyle\int_{\alpha m \vee x_0}^{x} \exp\Big\{ -2(\int_{\alpha m}^{\alpha m \vee x_0} \xi_{\nu^*}(\omega)^{-1} + \int_{\alpha m \vee x_0}^{y} \xi_0(\omega)^{-1}) d\omega\Big\} dy. \end{cases} \tag{3.10}$$

The optimal investment strategy is

$$\pi^*(x) = \begin{cases} -(\sigma^{'})^{-1}\zeta_{\nu^*}(x)\dfrac{\xi_{\nu^*}(x)}{x}, & \alpha m \le x < \max(\alpha m, x_0), \\ -(\sigma^{'})^{-1}\zeta\dfrac{\xi_0(x)}{x}, & \max(\alpha m, x_0) \le x \le x^*, \end{cases} \tag{3.11}$$

where ν^* is given by

$$\nu^*(x) = \begin{cases} -\dfrac{1}{K}\Big(\dfrac{(1+k)x}{\xi_{\nu^*}(x)} + D\Big)\mathbf{1}_N, & \alpha m \le x < \max(\alpha m, x_0), \\ 0 \cdot 1_N, & \max(\alpha m, x_0) \le x \le x^*. \end{cases} \tag{3.12}$$

Proof. We only give the proof under the case of $\max(\alpha m, x_0) \le m < x^*$, the other case $\alpha m \le m < \max(\alpha m, x_0)$ can be proved similarly. Integrating $\xi(x)$, which is defined by (3.14) from αm to x yields

$$V^{\nu^*}(x, m) = \phi_1(m) \cdot g_2(x, m),$$

where $g_2(x, m)$ is given by (3.10) and $\phi_1(m)$ is an unknown function to be determined.

Based on the boundary condition $V^{\nu^*}(x^*, m) = 1$, we have $\phi_1(m) = 1/g_2(x^*, m)$. One can see that the function $V^{\nu^*}(x, m)$ satisfies the conditions (i), (iii) and (iv) in Theorem 3.1. While condition (ii) is moot because $m \ge x^*$. Hence, $V^{\nu^*}(x, m)$ equals the value function $V(x, m)$, which completes the proof. □

Remark 3.2. *From* (3.11)*, we can see that the optimal investment strategy is only dependent on the value of wealth x, while it has nothing to do with the drawdown proportion α. Hence, similar to Yuan et al. [21] and Han et al. [9, 10], the optimal investment strategy for maximizing the goal-reaching probability before drawdown coincides with the one for maximizing the goal-reaching probability before ruin until drawdown happens.*

Remark 3.3. *Furthermore, from the optimal investment strategy* (3.11)*, it is shown that the borrowing constraint violates when the wealth process decreases to the drawdown level. In fact, only when the wealth condition keeps deteriorating, the investor chooses to gamble on the risky asset in order to avoid the appearance of drawdown. When the value of wealth is close to the safe level, the investor becomes cautious and invests less in the risky asset.*

3.3. Maximizing the goal-reaching probability before drawdown when $m < x^*$

In the previous subsection 3.2, we derive the maximum goal-reaching probability before drawdown and the corresponding optimal strategy for the case of $m \ge x^*$. In this subsection, we will consider the problem for the case of $m < x^*$.

Again, following from the verification theorem, we turn to the following boundary-value problem

$$\begin{cases} \inf\limits_{\nu \in \tilde{\mathcal{K}}} \sup\limits_{\pi_\nu \in \mathcal{A}_\nu} \Big\{[(r + \delta(\nu) - a)x + \pi_\nu^{'}(\mu + \nu - r1_N)x - b]V_x^\nu + \dfrac{1}{2}\|\sigma^{'}\pi_\nu\|^2 x^2 V_{xx}^\nu\Big\} = 0, \\ V^\nu(\alpha m, m) = 0, \quad V^\nu(x^*, m) = 1, \quad \partial V^\nu(m, m)/\partial m = 0, \end{cases} \tag{3.13}$$

for $\alpha m \le x \le m < x^*$. We can construct the solution which is actually the value function according to Theorem 3.1.

To simplify further analysis, we define the function $\xi(x)$ as follows

$$\xi(x) = \begin{cases} \xi_{\nu^*}(x)^{-1}, & \alpha m \leq x < \max(\alpha m, x_0), \\ \xi_0(x)^{-1}, & \max(\alpha m, x_0) \leq x \leq m < x^*, \end{cases} \tag{3.14}$$

with $\xi_{\nu^*}(x)$ and $\xi_0(x)$ given by (3.9) and (3.7) , respectively. From (3.3) and (3.13), it is obvious that the only difference is the boundary condition. As a result, the HJB equation (3.13) is equivalent to

$$\begin{cases} \dfrac{V^\nu_{xx}}{V^\nu_x} = \xi(x), \\ V^\nu(\alpha m, m) = 0, \quad V^\nu(x^*, m) = 1, \quad \partial V^\nu(m, m)/\partial m = 0, \end{cases}$$

Based on the above results, we can conclude the optimal results for the case of $m < x^*$ in the following theorem.

Theorem 3.3. *Suppose that $m < x^*$. Let $g_i(x, m)$ be given in* (3.10)*, $\xi_0(x)$, $\xi_{\nu^*}(x)$, x_0 be given in* (3.7)*,* (3.9) *and* (3.8)*, respectively. Then,*

(i) *if* $\max(\alpha m, x_0) \leq m < x^*$, *for any $x \in [\alpha m, x^*]$, the maximum goal-reaching probability before drawdown for the wealth process* (2.1) *is given by*

$$V(x, m) = \begin{cases} \exp\Big\{ - \displaystyle\int_m^{x^*} f_2(y) dy \Big\} \cdot \dfrac{g_1(x, m)}{g_2(x^*, x^*)}, & \alpha m \leq x < \max(\alpha m, x_0), \\ \exp\Big\{ - \displaystyle\int_m^{x^*} f_2(y) dy \Big\} \cdot \dfrac{g_2(x, m)}{g_2(x^*, x^*)}, & \max(\alpha m, x_0) \leq x < x^*, \end{cases}$$

where

$$f_2(y) = \begin{cases} \alpha\Big[\dfrac{1}{g_2(y, y)} - 2\xi_{\nu^*}(\alpha y)^{-1} \Big], & x_0 \leq \alpha m, \\ \alpha\Big[\dfrac{1}{g_2(y, y)} - 2\xi_0(\alpha y)^{-1} \Big], & \alpha m < x_0; \end{cases}$$

(ii) *if $\alpha m \leq m < \max(\alpha m, x_0)$, for any $x \in [\alpha m, x^*]$, the maximum goal-reaching probability before drawdown for the wealth process* (2.1) *is given by*

$$V(x, m) = \exp\Big\{ \Big(\int_m^{x_0} -f_1(y) - \int_{x_0}^{x^*} f_2(y) \Big) dy \Big\} \cdot \frac{g_1(x, m)}{g_2(x^*, x^*)},$$

where

$$f_1(y) = \alpha\Big[\frac{1}{g_1(y, y)} - 2\xi_0(\alpha y)^{-1} \Big].$$

The optimal investment strategy π^* and fictitious process ν^* are given by (3.11) *and* (3.12)*, respectively.*

Proof. Similar to the proof of Theorem 3.2, we only give the proof under the case of $\max(\alpha m, x_0) \leq m < x^*$. Integrating $\xi(x)$ from αm to x, we have

$$V^{\nu^*}(x, m) = \phi_1(m) \cdot g_2(x, m),$$

where $g_2(x, m)$ is given by (3.10) and $\phi_1(m)$ is a function of m to be determined.

Differentiating V^{ν^*} w.r.t m, we have

$$\begin{aligned}\frac{\partial V^{\nu^*}}{\partial m} &= \frac{\partial \phi_1(m)}{\partial m} \cdot g_2(x,m) + \phi_1(m) \cdot \frac{\partial g_2(x,m)}{\partial m} \\ &= \frac{\partial \phi_1(m)}{\partial m} \cdot g_2(x,m) + \phi_1(m) \cdot (2\alpha\xi(\alpha m)^{-1} \cdot g_2(x,m) - \alpha).\end{aligned}$$

According to the boundary condition $\partial V^{\nu}/\partial m = 0$, it follows that

$$\frac{\partial \phi_1(m)/\partial m}{\phi_1(m)} = \alpha\Big(- 2\xi(\alpha m)^{-1} + \frac{1}{g_2(m,m)}\Big).$$

We integrate both sides of this equation from m to x^*, which yields

$$\phi_1(m) = \phi_1(x^*) \cdot \exp\Big\{ - \int_m^{x^*} f_2(y)dy\Big\}.$$

Following the condition $V^{\nu^*}(x^*, x^*) = 1$, we get $\phi_1(x^*) = g_2(x^*, x^*)^{-1}$. One can easily verify that the function V^{ν^*} satisfies the conditions of Theorem 3.1, which completes the proof. □

Remark 3.4. *Comparing the expressions for optimal results under the cases of $m \geq x^*$ and $m < x^*$, it is plain to find that the optimal investment strategy remains the same, but the value functions make a difference due to the extra boundary condition.*

Remark 3.5. *Furthermore, if we set $N = 1$, that is, there is only one risky asset in the financial market, the optimal investment strategy can reduce to the one in Yuan et al. [21]. Meanwhile, the value function for minimum probability of drawdown $\bar{V}(x,m)$ has a representation as*

$$\bar{V}(x,m) = 1 - V(x,m),$$

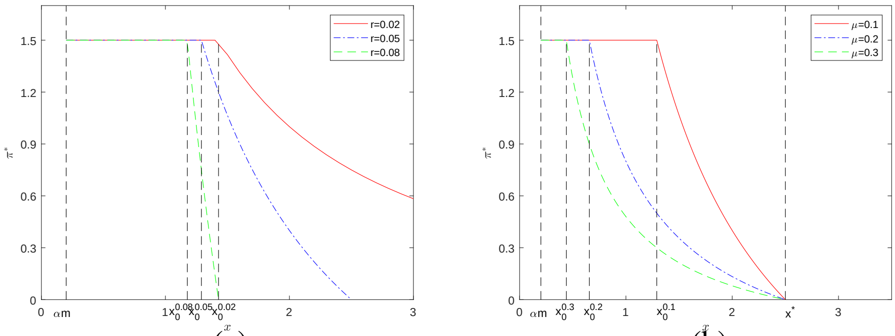
which can be seen from

$$\bar{V}(x,m) = \sup_{\nu\in\tilde{\mathcal{K}}} \inf_{\pi_\nu\in\mathcal{A}_\nu} \mathrm{P}_{x,m}(\tau_\alpha^\nu < \tau_{x^*}^\nu) = \sup_{\nu\in\tilde{\mathcal{K}}} \inf_{\pi_\nu\in\mathcal{A}_\nu} [1 - \mathrm{P}_{x,m}(\tau_{x^*}^\nu < \tau_\alpha^\nu)] = 1 - V(x,m).$$

4. Numerical examples

In this section, we investigate the effect of model parameters on the optimal investment strategy. To make the things simple, we set $N = 1$ in this section, that is, there is one risky asset in the market. Unless otherwise stated, we set the basic parameters as $\alpha = 0.2, k = 0.5, r = 0.05, \mu = 0.1, \sigma = 0.1, a = 0.01, b = 0.1$ and $m = 1$.

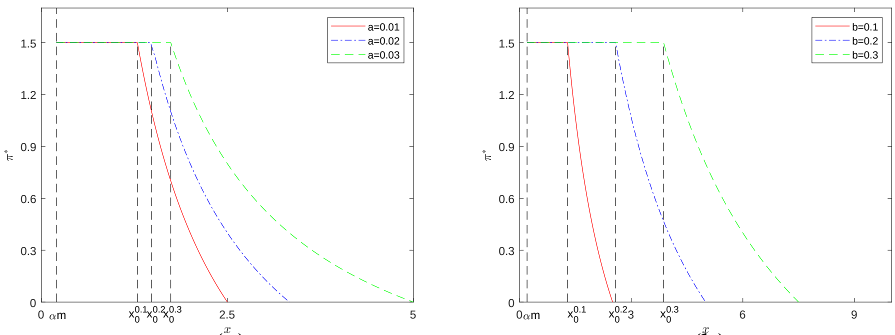
Figure 1 (a) shows that higher risk-free rate indicates the lower borrowing constraint level. The higher risk-free rate means lower risk premium. Thus, the investor chooses to retain the higher leverage rate in a worse condition. On the other hand, as μ increases, the amount put in the risky asset decreases, which can be seen in Figure 1 (b). This is because the higher risk premium makes the investor borrowing money at a lower wealth level.



(a) (b)

Figure 1. The influence of r and μ on the optimal investment strategy.

The Figure 2 gives that how the parameters of consumption function affect the optimal investment strategy. It can be seen that the increase in the consumption function, i.e., parameters a and b leads directly of the increase of the optimal investment strategy and borrowing constraint level, which is a natural consequence. As a fact, in order to avoiding drawdown, the higher consumption will drive the investor to invest more in the risky-asset, which makes the borrowing constraint level become higher in return.



(a) (b)

Figure 2. The influence of a and b on the optimal investment strategy.

The Figure 3 (a) illustrates that how the parameter k affects the optimal investment strategy and borrowing constraint level. With higher permissible borrowing ratio, the investor is influenced by the borrowing constraint at a lower wealth level. In Figure 3 (b), we can see that the optimal investment strategy has nothing to do with the volatility of risky asset, which is a rather surprising result. In fact, based the expression of (3.11), the optimal constrained investment strategy for only one risky asset follows that

$$\pi^*(x) = \begin{cases} 1+k, & \alpha m \le x < \max(\alpha m, x_1), \\ \dfrac{2(r-a)}{\mu - r}\Big(x^* - x\Big), & \max(\alpha m, x_1) \le x \le x^*, \end{cases}$$

where x_1 is given by

$$x_1 = x^* - \frac{(1+k)(\mu - r)}{2(r-a)}.$$

This behavior is not the first time in the optimal investment problems, which can be also found in Browne [6]. However, this does not hold when there are multiple risky assets in which to invest in (see (3.11) for details).

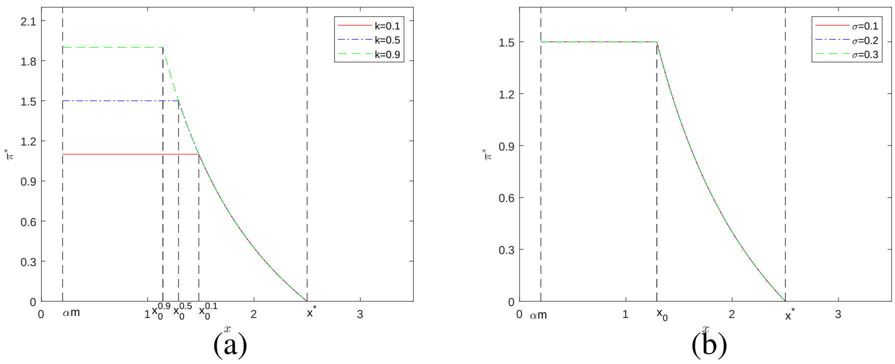


Figure 3. The influence of k and σ on the optimal investment strategy.

5. Conclusions

In this paper, we consider the optimal investment problem with borrowing constraint under the criterion of maximizing the goal-probability before drawdown. Using the technique of stochastic control theory and convex analysis, we derive the optimal investment strategy and the corresponding value function. By analysis, we that the behavior of borrowing typically occurs with a lower wealth level and the borrowing constraint level strongly depend on the parameters of the risky asset, as well as the consumption process. For the further research, one can extend the model to the case with jump in the process of risky asset. Meanwhile, some other objective functions can be applied, such as utility maximization or mean-variance criteria. All of these problems are meaningful and challenging, and they are our future directions.

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Conflict of interest

The authors declare that they have no competing interests.

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