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Research article

Asymptotic solutions of singularly perturbed integro-differential systems with rapidly oscillating coefficients in the case of a simple spectrum

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Abstract: In this paper, we consider a system with rapidly oscillating coefficients, which includes an integral operator with an exponentially varying kernel. The main goal of the work is to develop the algorithm of Lomov's the regularization method for such systems and to identify the influence of the integral term on the asymptotics of the solution of the original problem. The case of identical resonance is considered, i.e. the case when an integer linear combination of the eigenvalues of a rapidly oscillating coefficient coincides with the points of the spectrum of the limit operator is identical on the entire considered time interval. In addition, the case of coincidence of the eigenvalue of a rapidly oscillating coefficient with the points of the spectrum of the limit operator is excluded. This case is supposed to be studied in our subsequent works. More complex cases of resonance (for example, point resonance) require a more thorough analysis and are not considered in this paper.

Keywords: singularly perturbed; integro-differential equation; regularization of an integral; space of non-resonant solutions; iterative problems; solvability of iterative problems
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1. Introduction

When studying various applied problems related to the properties of media with a periodic structure, it is necessary to study differential equations with rapidly changing coefficients. Equations of this type are often found, for example, in electrical systems under the influence of high frequency external forces. The presence of such forces creates serious problems for the numerical integration of the corresponding differential equations. Therefore, asymptotic methods are usually applied to such equations, the most famous of which are the Feshchenko – Shkil – Nikolenko splitting method

[9–12, 23] and the Lomov's regularization method [18, 20, 21]. The splitting method is especially effective when applied to homogeneous equations, and in the case of inhomogeneous differential equations, the Lomov regularization method turned out to be the most effective. However, both of these methods were developed mainly for singularly perturbed equations that do not contain an integral operator. The transition from differential equations to integro-differential equations requires a significant restructuring of the algorithm of the regularization method. The integral term generates new types of singularities in solutions that differ from the previously known ones, which complicates the development of the algorithm for the regularization method. The splitting method, as far as we know, has not been applied to integro-differential equations. In this article, the Lomov's regularization method [1–8, 13–17, 19, 24] is generalized to previously unexplored classes of integro-differential equations with rapidly oscillating coefficients and rapidly decreasing kernels of the form

$$\begin{gathered} L_\varepsilon z(t,\varepsilon) \equiv \varepsilon \tfrac{dz}{dt} - A(t)z - \varepsilon g(t)\cos\tfrac{\beta(t)}{\varepsilon} B(t)\, z \\ - \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)\, z(s,\varepsilon)ds = h(t), \quad z(t_0,\varepsilon) = z^0, \quad t \in [t_0,T], \end{gathered} \tag{1.1}$$

where $z = \{z_1, z_2\}, h(t) = \{h_1(t), h_2(t)\}, \mu(t) < 0\, (\forall t \in [t_0,T]), g(t)$ is the scalar function, $A(t)$ and $B(t)$ are (2×2)-matrices, moreover $A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$, $\omega(t) > 0, \beta'(t) > 0$ is the frequency of the rapidly oscillating cosine, $z^0 = \left\{z_1^0, z_2^0\right\}$, $\varepsilon > 0$ is a small parameter. It is precisely such a system in the case $\beta(t) = 2\gamma(t), B(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and in the absence of an integral term was considered in [18,20,21].

The functions $\lambda_1(t) = -i\omega(t)$, $\lambda_2(t) = +i\omega(t)$ form the spectrum of the limit operator $A(t)$, the function $\lambda_5(t) = \mu(t)$ characterizes the rapid change in the kernel of the integral operator, and the functions $\lambda_3(t) = -i\beta'(t)$, $\lambda_4(t) = +i\beta'(t)$ are associated with the presence of a rapidly oscillating cosine in the system (1.1). The set $\{\Lambda\} = \{\lambda_1(t), ..., \lambda_5(t)\}$ is called the spectrum of problem (1.1). Such systems have not been considered earlier and in this paper we will try to generalize the Lomov's regularization method [18] to systems of type (1.1).

We introduce the following notations:

$\lambda(t) = (\lambda_1(t), ..., \lambda_5(t))$,

$m = (m_1, ..., m_5)$ is a multi-index with non-negative components m_j, $j = \overline{1,5}$,

$|m| = \sum_{j=1}^{5} m_j$ is the height of multi-index m,

$(m, \lambda(t)) = \sum_{j=1}^{5} m_j \lambda_j(t)$.

The problem (1.1) will be considered under the following conditions:

1) $\omega(t), \mu(t), \beta(t) \in C^\infty([t_0,T], \mathbb{R}), \omega(t) \neq \beta'(t)\, \forall t \in [t_0,T]$,
$g(t) \in \left([t_0,T], \mathbb{C}^1\right), h(t) \in C^\infty\left([t_0,T], \mathbb{C}^2\right)$,
$B(t) \in C^\infty\left([t_0,T], \mathbb{C}^{2\times 2}\right), K(t,s) \in C^\infty\left(\{t_0 \le s \le t \le T\}, \mathbb{C}^{2\times 2}\right)$;

2) the relations $(m, \lambda(t)) = 0$, $(m, \lambda(t)) = \lambda_j(t), j \in \{1, ..., 5\}$ for all multi-indices m with $|m| \ge 2$ or are not fulfilled for any $t \in [t_0,T]$, or are fulfilled identically on the whole segment $[t_0,T]$. In other words, resonant multi-indices are exhausted by the following sets:

$$\begin{gathered} \Gamma_0 = \{m : (m, \lambda(t)) \equiv 0, |m| \ge 2, \forall t \in [t_0,T]\}, \\ \Gamma_j = \left\{m : (m, \lambda(t)) \equiv \lambda_j(t), |m| \ge 2, \forall t \in [t_0,T]\right\}, \ j = \overline{1,5}. \end{gathered}$$

Note that by virtue of the condition $\omega(t) \neq \beta'(t)$, the spectrum $\{\Lambda\}$ of the problem (1.1) is simple.

2. Regularization of the problem (1.1)

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent on t the quantities $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$ and rewrite the system (1.1) in the form

$$\begin{gathered} L_\varepsilon z(t,\varepsilon) \equiv \varepsilon\frac{dz}{dt} - A(t)z - \varepsilon\frac{g(t)}{2}\left(e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1 + e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2\right)B(t)z \\ -\int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t\mu(\theta)d\theta}K(t,s)\,z(s,\varepsilon)ds = h(t), \quad z(t_0,\varepsilon) = z^0, \ \ t\in[t_0,T]. \end{gathered} \tag{2.1}$$

We introduce regularizing variables (see [18])

$$\tau_j = \frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = \overline{1,5} \tag{2.2}$$

and instead of the problem (2.1) we consider the problem

$$L_\varepsilon \tilde{z}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{z}}{\partial t} + \sum_{j=1}^{5}\lambda_j(t)\frac{\partial\tilde{z}}{\partial\tau_j} - A(t)\tilde{z} - \varepsilon\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)\,\tilde{z}$$

$$-\int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t\mu(\theta)d\theta}K(t,s)\,\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds = h(t), \ \tilde{z}(t,\tau,\varepsilon)|_{t=t_0,\tau=0} = z^0, \quad t\in[t_0,T] \tag{2.3}$$

for the function $\tilde{z} = \tilde{z}(t,\tau,\varepsilon)$, where it is indicated (according to (2.2)): $\tau = (\tau_1,...,\tau_5)$, $\psi = (\psi_1,...,\psi_5)$. It is clear that if $\tilde{z} = \tilde{z}(t,\tau,\varepsilon)$ is the solution of the problem (2.3), then the vector function $z = \tilde{z}\left(t,\frac{\psi(t)}{\varepsilon},\varepsilon\right)$ is the exact solution of the problem (2.1), therefore, the problem (2.3) is expansion of the problem (2.1). However, it cannot be considered completely regularized, since the integral term

$$J\tilde{z} \equiv J\left(\tilde{z}(t,\tau,\varepsilon)\,|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t\mu(\theta)d\theta}K(t,s)\,\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$$

has not been regularized in (2.3).

To regularize the integral term, we introduce a class M_ε, asymptotically invariant with respect to the operator $J\tilde{z}$ (see [18], p. 62]). We first consider the space of vector functions $z(t,\tau)$, representable by sums

$$\begin{gathered} z(t,\tau,\sigma) = z_0(t,\sigma) + \textstyle\sum_{i=1}^{5} z_i(t,\sigma)\,e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} z^m(t,\sigma)\,e^{(m,\tau)}, \\ z_0(t,\sigma), z_i(t,\sigma), z^m(t,\sigma) \in C^\infty\left([t_0,T],\mathbb{C}^2\right), i = \overline{1,5}, \ \ 2\le|m|\le N_z, \end{gathered} \tag{2.4}$$

where the asterisk $*$ above the sum sign indicates that in it the summation for $|m|\ge 2$ occurs only over nonresonant multi-indices $m = (m_1,...,m_5)$, i.e. over $m \notin \bigcup_{i=0}^{5}\Gamma_i$. Note that in (2.4) the degree of the polynomial with respect to exponentials e^{τ_j} depends on the element z. In addition, the elements of the space U depend on bounded in $\varepsilon > 0$ constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$, which do not affect the development of the algorithm described below, therefore, henceforth, in the notation of element (2.4) of this space U, we omit the dependence on $\sigma = (\sigma_1,\sigma_2)$ for brevity . We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

The image of the operator J on the element (2.4) of the space U has the form:

$$Jz(t,\tau) = \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} K(t,s)\, z_0(s)\, ds + \sum_{i=1}^{5} \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} K(t,s)\, z_i(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s} \lambda_i(\theta)d\theta} ds$$

$$+ \sum_{2\le|m|\le N_z}^{*} \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} K(t,s)\, z^m(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m,\lambda(\theta))d\theta} ds$$

$$= \int_{t_0}^{t} e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} K(t,s) z_0(s) ds + e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} K(t,s) z_5(s) ds$$

$$+ \sum_{i=1, i\neq 5}^{5} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} K(t,s)\, z_i(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta} ds$$

$$+ \sum_{2\le|m|\le N_z}^{*} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} K(t,s)\, z^m(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta} ds.$$

Integrating in parts, we have

$$J_0(t,\varepsilon) = \int_{t_0}^{t} K(t,s)\, z_0(s)\, e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} ds = \varepsilon \int_{t_0}^{t} \frac{K(t,s)\, z_0(s)}{-\lambda_5(s)} de^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta}$$

$$= \varepsilon \frac{K(t,s)\, z_0(s)}{-\lambda_5(s)} e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta}\Big|_{s=t_0}^{s=t} - \varepsilon \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s)\, z_0(s)}{-\lambda_5(s)}\right) e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} ds$$

$$= \varepsilon \left[\frac{K(t,t_0)\, z_0(t_0)}{\lambda_5(t_0)} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} - \frac{K(t,t)\, z_0(t)}{\lambda_5(t)}\right] + \varepsilon \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s)\, z_0(s)}{\lambda_5(s)}\right) e^{\frac{1}{\varepsilon}\int_s^t \lambda_5(\theta)d\theta} ds.$$

Continuing this process further, we obtain the decomposition

$$J_0(t,\varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^{\nu+1} \left[\left(I_0^{\nu}(K(t,s)\, z_0(s))\right)_{s=t_0} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} - \left(I_0^{\nu}(K(t,s)\, z_0(s))\right)_{s=t}\right],$$
$$I_0^0 = \tfrac{1}{\lambda_5(s)}\cdot, I_0^{\nu} = \tfrac{1}{\lambda_5(s)} \tfrac{\partial}{\partial s} I_0^{\nu-1} (\nu \ge 1).$$

Next, apply the same operation to the integrals:

$$J_{5,i}(t,\varepsilon) = e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} K(t,s)\, z_i(s)\, e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta} ds$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} \frac{K(t,s) z_i(s)}{\lambda_i(s)-\lambda_5(s)} de^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta}$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \left[\frac{K(t,s) z_i(s)}{\lambda_i(s)-\lambda_5(s)} e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta}\Big|_{s=t_0}^{s=t} - \varepsilon \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s)\, z_i(s)}{\lambda_i(s)-\lambda_5(s)}\right) e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta} ds\right]$$

$$= \varepsilon \left[\frac{K(t,t)\, z_i(t)}{\lambda_i(t)-\lambda_5(t)} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_i(\theta)d\theta} - \frac{K(t,t_0)\, z_i(t_0)}{\lambda_i(t_0)-\lambda_5(t_0)} e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta}\right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_5(\theta)d\theta} \int_{t_0}^{t} \left(\frac{\partial}{\partial s} \frac{K(t,s)\, z_i(s)}{\lambda_i(s)-\lambda_5(s)}\right) e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(\lambda_i(\theta)-\lambda_5(\theta))d\theta} ds$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(t,s)\,z_i(s)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_i(\theta)d\theta} - \left(I_i^{\nu}\left(K(t,s)\,z_i(s)\right)\right)_{s=t_0} e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\right],$$

$$I_i^0 = \frac{1}{\lambda_i(s)-\lambda_5(s)}, I_i^{\nu} = \frac{1}{\lambda_i(s)-\lambda_5(s)}\frac{\partial}{\partial s}I_i^{\nu-1}\,(\nu \geq 1), i = \overline{1,4};$$

$$J_m(t,\varepsilon) = e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\int_{t_0}^{t}K(t,s)\,z^m(s)\,e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta}ds$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\int_{t_0}^{t}\frac{K(t,s)\,z^m(s)}{(m-e_5,\lambda(s))}de^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta} \quad = \varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\left[\frac{K(t,s)\,z^m(s)}{(m-e_5,\lambda(s))}e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta}\right|_{s=t_0}^{s=t}$$

$$\left. - \varepsilon\int_{t_0}^{t}\left(\frac{\partial}{\partial s}\frac{K(t,s)\,z^m(s)}{(m-e_5,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta}ds\right]$$

$$= \varepsilon\left[\frac{K(t,t)\,z^m(t)}{(m-e_5,\lambda(t))}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta} - \frac{K(t,t_0)\,z^m(t_0)}{(m-e_5,\lambda(t_0))}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\int_{t_0}^{t}\left(\frac{\partial}{\partial s}\frac{K(t,s)\,z^m(s)}{(m-e_5,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}(m-e_5,\lambda(\theta))d\theta}$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_{5,m}^{\nu}\left(K(t,s)\,z^m(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta} - \left(I_{5,m}^{\nu}\left(K(t,s)\,z^m(s)\right)\right)_{s=t_0}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\right],$$

$$I_{5,m}^0 = \frac{1}{(m-e_5,\lambda(s))}, I_{5,m}^{\nu} = \frac{1}{(m-e_5,\lambda(s))}\frac{\partial}{\partial s}I_{5,m}^{\nu-1}\,(\nu \geq 1), 2 \leq |m| \leq N_z.$$

Here it is taken into account that $(m-e_5,\lambda(s)) \neq 0$, since by definition of the space U, multi-indices $m \notin \Gamma_5$. This means that the image of the operator J on the element (2.4) of the space U is represented as a series

$$Jz(t,\tau) = e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta))d\theta}\int_{t_0}^{t}K(t,s)\,z_5(s)\,ds + \sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_0^{\nu}\left(K(t,s)\,z_0(s)\right)\right)_{s=t_0}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta))d\theta}\right.$$

$$\left. - \left(I_0^{\nu}\left(K(t,s)\,z_0(s)\right)\right)_{s=t}\right] + \sum_{i=1,i\neq 5}^{5}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(t,s)\,z_i(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_i(\theta))d\theta}\right.$$

$$\left. - \left(I_i^{\nu}\left(K(t,s)\,z_i(s)\right)\right)_{s=t_0}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\right]$$

$$+ \sum_{2\leq|m|\leq N_z}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_{5,m}^{\nu}\left(K(t,s)\,z^m(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta} - \left(I_{5,m}^{\nu}\left(K(t,s)\,z^m(s)\right)\right)_{s=t_0}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_5(\theta)d\theta}\right].$$

It is easy to show (see, for example, [22], pages 291–294) that this series converges asymptotically as $\varepsilon \to +0$ (uniformly in $t \in [t_0, T]$). This means that the class M_ε is asymptotically invariant (as $\varepsilon \to +0$) with respect to the operator J.

We introduce the operators $R_\nu : U \to U$, acting on each element $z(t,\tau) \in U$ of the form (2.4) according to the law:

$$R_0 z(t,\tau) = e^{\tau_5}\int_{t_0}^{t}K(t,s)\,z_5(s)\,ds, \qquad (2.5_0)$$

$$R_1 z(t,\tau) = [\left(I_0^0(K(t,s)\,z_0(s))\right)_{s=t_0} e^{\tau_5} - \left(I_0^0(K(t,s)\,z_0(s))\right)_{s=t}]$$
$$+\sum_{i=1}^{4}[\left(I_i^0(K(t,s)\,z_i(s))\right)_{s=t} e^{\tau_i} - \left(I_i^0(K(t,s)\,z_i(s))\right)_{s=t_0} e^{\tau_5}] \qquad (2.5_1)$$
$$+\sum_{2\le|m|\le N_z}^{*}[\left(I_{5,m}^0(K(t,s)\,z^m(s))\right)_{s=t} e^{(m,\tau)} - \left(I_{5,m}^0(K(t,s)\,z^m(s))\right)_{s=t_0} e^{\tau_5}],$$

$$R_{\nu+1} z(t,\tau) = \sum_{\nu=0}^{\infty}(-1)^{\nu}[(I_0^{\nu}(K(t,s)\,z_0(s)))_{s=t_0} e^{\tau_5} - (I_0^{\nu}(K(t,s)\,z_0(s)))_{s=t}]$$
$$+\sum_{i=1}^{4}\sum_{\nu=0}^{\infty}(-1)^{\nu}[(I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t} e^{\tau_i} - (I_i^{\nu}(K(t,s)\,z_i(s)))_{s=t_0} e^{\tau_5}] \qquad (2.5_{\nu+1})$$
$$+\sum_{2\le|m|\le N_z}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}[\left(I_{5,m}^{\nu}(K(t,s)\,z^m(s))\right)_{s=t} e^{(m,\tau)} - \left(I_{5,m}^{\nu}(K(t,s)\,z^m(s))\right)_{s=t_0} e^{\tau_5}].$$

Let now $\tilde{z}(t,\tau,\varepsilon)$ be an arbitrary continuous function in $(t,\tau) \in [t_0,T] \times \left\{\tau : \mathrm{Re}\tau_j \le 0,\, j=\overline{1,5}\right\}$ with asymptotic expansion

$$\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau),\, z_k(t,\tau) \in U \qquad (2.6)$$

converging as $\varepsilon \to +0$ (uniformly in $(t,\tau) \in [t_0,T] \times \left\{\tau : \mathrm{Re}\tau_j \le 0,\, j=\overline{1,5}\right\}$). Then the image $J\tilde{z}(t,\tau,\varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k J z_k(t,\tau) = \sum_{r=0}^{\infty}\varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau)\,|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of the operator J on series of the form (2.6):

$$\tilde{J}\tilde{z}(t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau)\right) \overset{def}{=} \sum_{r=0}^{\infty}\varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau). \qquad (2.7)$$

Although the operator $\tilde{J}$ is formally defined, its usefulness is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (2.1), in which only N-th partial sums of the series (2.6) will take part, which have not formal, but true meaning. Now we can write down a problem that is completely regularized with respect to the original problem (2.1):

$$L_\varepsilon \tilde{z}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{z}}{\partial t} + \sum_{j=1}^{5}\lambda_j(t)\frac{\partial\tilde{z}}{\partial\tau_j} - A(t)\tilde{z} - \varepsilon\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B\tilde{z} -$$
$$-\tilde{J}\tilde{z} = h(t), \quad \tilde{z}(t,\tau,\varepsilon)|_{t=t_0,\tau=0} = z^0, \quad t \in [t_0,T], \qquad (2.8)$$

were the operator $\tilde{J}$ has the form (2.7).

3. Iterative problems and their solvability in the space *U*

Substituting the series (2.6) into (2.8) and equating the coefficients for the same powers of ε, we obtain the following iterative problems:

$$Lz_0(t,\tau) \equiv \sum_{j=1}^{5} \lambda_j(t) \frac{\partial z_0}{\partial \tau_j} - A(t)z_0 - R_0 z_0 = h(t), \quad z_0(t_0,0) = z^0; \tag{3.1$_0$}$$

$$Lz_1(t,\tau) = -\frac{\partial z_0}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right) B(t) z_0 + R_1 z_0, \quad z_1(t_0,0) = 0; \tag{3.1$_1$}$$

$$Lz_2(t,\tau) = -\frac{\partial z_1}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right) B(t) z_1 + R_1 z_1 + R_2 z_0, \quad z_2(t_0,0) = 0; \tag{3.1$_2$}$$

$$\cdots$$

$$Lz_k(t,\tau) = -\frac{\partial z_{k-1}}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right) B(t) z_{k-1} + R_k z_0 + ... + R_1 z_{k-1}, \quad z_k(t_0,0) = 0, \ k \geq 1. \tag{3.1$_k$}$$

Each of the iterative problem (3.1$_k$) can be written as

$$Lz(t,\tau) \equiv \sum_{j=1}^{5} \lambda_j(t) \frac{\partial z}{\partial \tau_j} - A(t)z - R_0 z = H(t,\tau), \quad z(t_0,0) = z^*, \tag{3.2}$$

where $H(t,\tau) = H_0(t) + \sum_{j=1}^{5} H_j(t)\, e^{\tau_j} + \sum_{2\leq|m|\leq N_H}^{*} H^m(t)\, e^{(m,\tau)}$ is a well-known vector-function of the space U, z^* is a well-known constant vector of a complex space $\mathbb{C}^2$, and the operator R_0 has the form (see (2.5$_0$))

$$R_0 z \equiv R_0\left(z_0(t) + \sum_{j=1}^{5} z_j(t)\, e^{\tau_j} + \sum_{2\leq|m|\leq N_z}^{*} z^m(t)\, e^{(m,\tau)}\right) = e^{\tau_5}\int_{t_0}^{t} K(t,s)\, z_5(s)\, ds.$$

In the future we need the $\lambda_j(t)$-eigenvectors of the matrix $A(t)$:

$$\varphi_1(t) = \begin{pmatrix} 1 \\ -i\omega(t) \end{pmatrix}, \varphi_2(t) = \begin{pmatrix} 1 \\ +i\omega(t) \end{pmatrix},$$

as well as $\bar{\lambda}_j(t)$-eigenvectors of the matrix $A^*(t)$:

$$\chi_1(t) = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{i}{\omega(t)} \end{pmatrix}, \chi_2(t) = \frac{1}{2}\begin{pmatrix} 1 \\ +\frac{i}{\omega(t)} \end{pmatrix}.$$

These vectors form a biorthogonal system, i.e.

$$\left(\varphi_k(t), \chi_j(t)\right) = \begin{cases} 1, k = j, \\ 0, k \neq j \end{cases} \ (k, j = 1, 2).$$

We introduce the scalar product (for each $t \in [t_0, T]$) in the space U :

$$< z, w > \equiv < z_0(t) + \sum_{j=1}^{5} z_j(t)\, e^{\tau_j} + \sum_{2\leq|m|\leq N_z}^{*} z^m(t)\, e^{(m,\tau)}, \quad w_0(t) + \sum_{j=1}^{5} w_j(t)\, e^{\tau_j}$$

$$+\sum_{2\le|m|\le N_w}^{*} w^m(t)\, e^{(m,\tau)} > \stackrel{def}{=} (z_0(t), w_0(t)) + \sum_{j=1}^{5}\Big(z_j(t), w_j(t)\Big) + \sum_{2\le|m|\le \min(N_z,N_w)}^{*} (z^m(t), w^m(t)),$$

where we denote by $(*,*)$ the ordinary scalar product in a complex space $\mathbb{C}^2$. We prove the following statement.

Theorem 1. *Let conditions 1) and 2) are satisfied and the right-hand side* $H(t,\tau) = H_0(t) + \sum_{j=1}^{5} H_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} H^m(t)\, e^{(m,\tau)}$ *of the system (3.2) belongs to the space U. Then for the solvability of the system (3.2) in U it is necessary and sufficient that the identities*

$$< H(t,\tau), \chi_k(t)\, e^{\tau_k} > \equiv 0, k = 1,2, \forall t \in [t_0, T], \tag{3.3}$$

are fulfilled.

Proof. We will determine the solution to the system (3.2) in the form of an element (2.4) of the space U:

$$z(t,\tau) = z_0(t) + \sum_{j=1}^{5} z_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)\, e^{(m,\tau)}. \tag{3.4}$$

Substituting (3.4) into the system (3.2), we have

$$\sum_{j=1}^{5}\Big[\lambda_j(t) I - A(t)\Big] z_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} [(m,\lambda(t)) I - A(t)]\, z^m(t)\, e^{(m,\tau)}$$

$$-A(t)\, z_0(t) - e^{\tau_5}\int_{t_0}^{t} K(t,s)\, z_5(s)\, ds = H_0(t) + \sum_{j=1}^{5} H_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} H^m(t)\, e^{(m,\tau)}.$$

Equating here separately the free terms and coefficients at the same exponents, we obtain the following systems of equations:

$$-A(t)\, z_0(t) = H_0(t), \tag{3.5_0}$$

$$\Big[\lambda_j(t) I - A(t)\Big] z_j(t) = H_j(t), \; j = \overline{1,4}, \tag{3.5_j}$$

$$[\lambda_5(t) I - A(t)]\, z_5(t) - \int_{t_0}^{t} K(t,s)\, z_5(s)\, ds = H_5(t), \tag{3.5_5}$$

$$[(m,\lambda(t)) I - A(t)]\, z^m(t) = H^m(t), \;\; 2 \le |m| \le N_z, \;\; m \notin \bigcup_{j=0}^{5} \Gamma_j. \tag{3.5_m}$$

Due to the invertibility of the matrix $A(t)$, the system (3.5_0) has the solution $-A^{-1}(t)\, H_0(t)$. Since $\lambda_5(t) = \mu(t)$ is a real function, and the eigenvalues of the matrix $A(t)$ are purely imaginary, the matrix $\lambda_5(t) I - A(t)$ is invertible and therefore the system (3.5_5) can be written as

$$z_5(t) = \int_{t_0}^{t}\Big([\lambda_5(t) I - A(t)]^{-1} K(t,s)\Big) z_5(s)\, ds + [\lambda_5(t) I - A(t)]^{-1} H_5(t). \tag{3.6}$$

Due to the smoothness of the kernel $\left([\lambda_5(t)I - A(t)]^{-1} K(t,s)\right)$ and the heterogeneity $[\lambda_5(t)I - A(t)]^{-1} H_5(t)$, this Volterra integral system has a unique solution $z_5(t) \in C^\infty\left([t_0,T],\mathbb{C}^2\right)$. Systems (3.5_3) and (3.5_4) also have unique solutions

$$z_j(t) = \left[\lambda_j(t)I - A(t)\right]^{-1} H_j(t) \in C^\infty\left([t_0,T],\mathbb{C}^2\right), j = 3,4, \tag{3.7}$$

since $\lambda_3(t)$, $\lambda_4(t)$ do not belong to the spectrum of the matrix $A(t)$. Systems (3.5_1) and (3.5_2) are solvable in the space $C^\infty\left([t_0,T],\mathbb{C}^2\right)$ if and only if the identities $\left(H_j(t),\chi_j(t)\right) \equiv 0\ \forall t \in [t_0,T], j = 1,2$ hold.

It is easy to see that these identities coincide with the identities (3.3).

Further, since multi-indices $m \notin \bigcup_{j=0}^{5}\Gamma_j$ in systems (3.5_m), then these systems are uniquely solvable in the space $C^\infty\left([t_0,T],\mathbb{C}^2\right)$ in the form of functions

$$z^m(t) = [(m,\lambda(t))I - A(t)]^{-1} H^m(t),\ 0 \le |m| \le N_H. \tag{3.8}$$

Thus, condition (3.3) is necessary and sufficient for the solvability of the system (3.2) in the space U. The theorem 1 is proved.

Remark 1. If identity (3.3) holds, then under conditions 1) and 2) the system (3.2) has (see (3.6) – (3.8)) the following solution in the space U :

$$z(t,\tau) = z_0(t) + \sum_{j=1}^{5} z_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)} \equiv z_0(t) + \sum_{k=1}^{2}\alpha_k(t)\varphi_k(t)e^{\tau_k}$$

$$+h_{12}(t)\varphi_2(t)e^{\tau_1} + +h_{21}(t)\varphi_1(t)e^{\tau_2} + z_5(t)e^{\tau_5} + \sum_{j=3}^{4}P_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} P^m(t)e^{(m,\tau)}, \tag{3.9}$$

where $\alpha_k(t) \in C^\infty\left([t_0,T],\mathbb{C}^1\right)$ are arbitrary functions, $k = 1,2$, $z_0(t) = -A^{-1}H_0(t)$, $z_5(t)$ is the solution of the integral system (3.6) and the notations are introduced:

$$h_{12}(t) \equiv \frac{(H_1(t),\chi_2(t))}{\lambda_1(t) - \lambda_2(t)}, h_{21}(t) \equiv \frac{(H_2(t),\chi_1(t))}{\lambda_2(t) - \lambda_1(t)}, P_j(t) \equiv \left[\lambda_j(t)I - A(t)\right]^{-1} H_j(t),$$

$$P^m(t) \equiv [(m,\lambda(t))I - A(t)]^{-1} H^m(t).$$

4. Unique solvability of the general iterative problem in the space U. The remainder term theorem

We proceed to the description of the conditions for the unique solvability of the system (3.2) in the space U. Along with the problem (3.2), we consider the system

$$Lw(t,\tau) = -\frac{\partial z}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)z + R_1 z + Q(t,\tau), \tag{4.1}$$

where $z = z(t,\tau)$ is the solution (3.9) of the system (3.2), $Q(t,\tau) \in U$ is the known function of the space U. The right-hand side of this system:

$$G(t,\tau) \equiv -\frac{\partial z}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right)B(t)z + R_1 z + Q(t,\tau)$$

$$= -\frac{\partial}{\partial t}[z_0(t) + \sum_{j=1}^{5} z_j(t) e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t) e^{(m,\tau)}]$$

$$+\frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2) B(t)\left[z_0(t) + \sum_{j=1}^{5} z_j(t) e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t) e^{(m,\tau)}\right] + R_1 z + Q(t,\tau),$$

may not belong to the space U, if $z = z(t,\tau) \in U$. Since $-\frac{\partial z}{\partial t}$, $R_1 z$, $Q(t,\tau) \in U$, then this fact needs to be checked for the function

$$Z(t,\tau) \equiv \frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2) B(t)[z_0(t) + \sum_{j=1}^{5} z_j(t) e^{\tau_j}$$

$$+ \sum_{2\le|m|\le N_H}^{*} z^m(t) e^{(m,\tau)}] = \frac{g(t)}{2} B(t) z_0(t)(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)$$

$$+ \sum_{j=1}^{5} \frac{g(t)}{2} B(t) z_j(t)(e^{\tau_j+\tau_3}\sigma_1 + e^{\tau_j+\tau_4}\sigma_2) + \frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2) B(t) \sum_{2\le|m|\le N_H}^{*} z^m(t) e^{(m,\tau)}.$$

Function $Z(t,\tau) \notin U$, since it has resonant exponents

$$e^{\tau_3+\tau_4} = e^{(m,\tau)}|_{m=(0,0,1,1,0)}, e^{\tau_3+(m,\tau)}(m_3 + 1 = m_4, m_1 = m_2 = m_5 = 0),$$

$$e^{\tau_4+(m,\tau)}(m_4 + 1 = m_3, m_1 = m_2 = m_5 = 0),$$

therefore, the right-hand side $G(t,\tau) = Z(t,\tau) - \frac{\partial z}{\partial t} + R_1 z + Q(t,\tau)$ of the system (19) also does not belong to the space U. Then, according to the well-known theory (see [18], p. 234), it is necessary to embed $\wedge: G(t,\tau) \to \hat{G}(t,\tau)$ the right-hand side $G(t,\tau)$ of the system (4.1) in the space U. This operation is defined as follows.

Let the function $G(t,\tau) = \sum_{|m|=0}^{N} w^m(t) e^{(m,\tau)}$ contain resonant exponentials, i.e. $G(t,\tau)$ has the form

$$G(t,\tau) = w_0(t) + \sum_{j=1}^{5} w_j(t) e^{\tau_j} + \sum_{j=0}^{5} \sum_{|m^j|=2:\, m^j\in\Gamma_j}^{N} w^{m^j}(t) e^{(m^j,\tau)} + \sum_{|m|=2, m\ne m^j, j=\overline{0,5}}^{N} w^m(t) e^{(m,\tau)}.$$

Then

$$\hat{G}(t,\tau) = w_0(t) + \sum_{j=1}^{5} w_j(t) e^{\tau_j} + \sum_{j=0}^{5} \sum_{|m^j|=2:\, m^j\in\Gamma_j}^{N} w^{m^j}(t) e^{\tau_j} + \sum_{|m|=2, m\ne m^j, j=\overline{0,5}}^{N} w^m(t) e^{(m,\tau)}.$$

Therefore, the embedding operation acts only on the resonant exponentials and replaces them with a unit or exponents e^{τ_j} of the first dimension according to the rule:

$$\left(e^{(m,\tau)}|_{m\in\Gamma_0}\right)^{\wedge} = e^0 = 1,\ \left(e^{(m,\tau)}|_{m\in\Gamma_j}\right)^{\wedge} = e^{\tau_j},\ j = \overline{1,5}.$$

We now turn to the proof of the following statement.

Theorem 2. *Suppose that conditions 1) and 2) are satisfied and the right-hand side* $H(t,\tau) = H_0(t) + \sum_{j=1}^{5} H_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} H^m(t)\, e^{(m,\tau)} \in U$ *of the system (3.2) satisfies condition (3.3). Then the problem (3.2) under additional conditions*

$$< \widehat{G}(t,\tau), \chi_k(t)\, e^{\tau_k} > \equiv 0 \forall t \in [t_0, T], k = 1,2, \tag{4.2}$$

where $Q(t,\tau) = Q_0(t) + \sum_{k=1}^{5} Q_k(t)\, e^{\tau_k} + \sum_{2\le|m|\le N_Q}^{*} Q^m(t)\, e^{(m,\tau)}$ *is the well-known vector function of the space* U, *is uniquely solvable in* U.

Proof. Since the right-hand side of the system (3.2) satisfies condition (3.3), this system has a solution in the space U in the form (3.9), where $\alpha_k(t) \in C^\infty\left([t_0, T], \mathbb{C}^1\right)$, $k = 1,2$ are arbitrary functions so far. Subordinate (3.9) to the initial condition $z(t_0, 0) = z^*$. We obtain $\sum_{k=1}^{2} \alpha_k(t_0)\, \varphi_k(t_0) = z_*$, where is indicated

$$z_* = z^* + A^{-1}(t_0)\, H_0(t_0) - [\lambda_5(t_0)\, I - A(t_0)]^{-1} H_5(t_0) - \sum_{j=3}^{4} \left[\lambda_j(t_0)\, I - A(t_0)\right]^{-1} H_j(t_0)$$

$$-\frac{(H_1(t_0), \chi_2(t_0))}{\lambda_1(t_0) - \lambda_2(t_0)} \varphi_2(t_0) - \frac{(H_2(t_0), \chi_1(t_0))}{\lambda_2(t_0) - \lambda_1(t_0)} \varphi_1(t_0) - \sum_{2\le|m|\le N_H}^{*} P^m(t_0).$$

Multiplying scalarly the equality $\sum_{k=1}^{2} \alpha_k(t_0)\, \varphi_k(t_0) = z_*$ by $\chi_j(t_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\left\{\chi_j(t)\right\}$, we find the values $\alpha_k(t_0) = (z_*, \chi_k(t_0)), k = 1,2$. Now we subordinate the solution (3.9) to the orthogonality condition (4.2). We write in more detail the right-hand side $G(t,\tau)$ of the system (4.1):

$$G(t,\tau) \equiv -\frac{\partial}{\partial t}[z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\, \varphi_k(t)\, e^{\tau_k} + h_{12}(t)\, \varphi_2(t)\, e^{\tau_1}$$

$$+h_{21}(t)\, \varphi_1(t)\, e^{\tau_2} + z_5(t)\, e^{\tau_5} + \sum_{j=3}^{4} P_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} P^m(t)\, e^{(m,\tau)}]$$

$$+\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2\right) B(t)\, [z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\, \varphi_k(t)\, e^{\tau_k} + h_{12}(t)\, \varphi_2(t)\, e^{\tau_1}$$

$$+h_{21}(t)\, \varphi_1(t)\, e^{\tau_2} + z_5(t)\, e^{\tau_5} + \sum_{j=3}^{4} P_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} P^m(t)\, e^{(m,\tau)}]$$

$$+R_1[z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\, \varphi_k(t)\, e^{\tau_k} + h_{12}(t)\, \varphi_2(t)\, e^{\tau_1} + h_{21}(t)\, \varphi_1(t)\, e^{\tau_2}$$

$$+z_5(t)\, e^{\tau_5} + \sum_{j=3}^{4} P_j(t)\, e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} P^m(t)\, e^{(m,\tau)}] + Q(t,\tau).$$

Putting this function into the space U, we will have

$$\hat{G}(t,\tau) \equiv -\frac{\partial}{\partial t}[z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\, \varphi_k(t)\, e^{\tau_k} + h_{12}(t)\, \varphi_2(t)\, e^{\tau_1}$$

$$+h_{21}(t)\varphi_1(t)e^{\tau_2}+z_5(t)e^{\tau_5}+\sum_{j=3}^{4}P_j(t)e^{\tau_j}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}]$$

$$+\{\frac{g(t)}{2}(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2)B(t)(z_0(t)+\sum_{k=1}^{2}\alpha_k(t)\varphi_k(t)e^{\tau_k}+h_{12}(t)\varphi_2(t)e^{\tau_1}$$

$$+h_{21}(t)\varphi_1(t)e^{\tau_2}+z_5(t)e^{\tau_5}+\sum_{j=3}^{4}P_j(t)e^{\tau_j}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\}^{\wedge}$$

$$+R_1[z_0(t)+\sum_{k=1}^{2}\alpha_k(t)\varphi_k(t)e^{\tau_k}+h_{12}(t)\varphi_2(t)e^{\tau_1}+h_{21}(t)\varphi_1(t)e^{\tau_2}$$

$$+z_5(t)e^{\tau_5}+\sum_{j=3}^{4}P_j(t)e^{\tau_j}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}]+Q(t,\tau)$$

$$=-\frac{\partial}{\partial t}[z_0(t)+\sum_{k=1}^{2}\alpha_k(t)\varphi_k(t)e^{\tau_k}+h_{12}(t)\varphi_2(t)e^{\tau_1}+h_{21}(t)\varphi_1(t)e^{\tau_2}$$

$$+z_5(t)e^{\tau_5}+\sum_{j=3}^{4}P_j(t)e^{\tau_j}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}] \qquad (**)$$

$$+\{\frac{1}{2}g(t)B(t)(e^{\tau_3}\sigma_1z_0(t)+e^{\tau_3+\tau_1}\sigma_1\alpha_1(t)\varphi_1(t)+e^{\tau_3+\tau_2}\sigma_1\alpha_2(t)\varphi_2(t)$$

$$+e^{\tau_3+\tau_1}\sigma_1h_{12}(t)\varphi_2(t)+e^{\tau_3+\tau_2}\sigma_1h_{21}(t)\varphi_1(t)+e^{\tau_3+\tau_5}\sigma_1z_5(t)$$

$$+e^{2\tau_3}\sigma_1P_3(t)+e^{\tau_3+\tau_4}\sigma_1P_4(t)+e^{\tau_4}\sigma_2z_0(t)+e^{\tau_4+\tau_1}\sigma_2\alpha_1(t)\varphi_1(t)$$

$$+e^{\tau_4+\tau_2}\sigma_2\alpha_2(t)\varphi_2(t)+e^{\tau_4+\tau_1}\sigma_2h_{12}(t)\varphi_2(t)+e^{\tau_4+\tau_2}\sigma_2h_{21}(t)\varphi_1(t)$$

$$+e^{\tau_4+\tau_5}\sigma_2z_5(t)+e^{\tau_3+\tau_4}\sigma_2P_3(t)+e^{2\tau_4}\sigma_2P_4(t)$$

$$+\frac{1}{2}g(t)B(t)\sum_{2\le|m|\le N_H}^{*}P^m(t)(e^{m\tau+\tau_3}\sigma_1+e^{m\tau+\tau_4}\sigma_2)\}^{\wedge}$$

$$+R_1[z_0(t)+\sum_{k=1}^{2}\alpha_k(t)\varphi_k(t)e^{\tau_k}+h_{12}(t)\varphi_2(t)e^{\tau_1}+h_{21}(t)\varphi_1(t)e^{\tau_2}$$

$$+z_5(t)e^{\tau_5}+\sum_{i=3}^{4}P_i(t)e^{\tau_i}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}]+Q(t,\tau).$$

Given that the expression $R_1(z_0(t,\tau))$ linearly depends on $\alpha_1(t)$ and $\alpha_2(t)$ (see the formula (2.5_1)) :

$$R_1(z_0(t,\tau)) \equiv R_1[z_0(t) + \sum_{k=1}^{2} \alpha_k(t)\varphi_k(t)e^{\tau_k} + h_{12}(t)\varphi_2(t)e^{\tau_1} + h_{21}(t)\varphi_1(t)e^{\tau_2}$$

$$+z_5(t)e^{\tau_5} + \sum_{j=3}^{4} P_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} P^m(t)e^{(m,\tau)}] \equiv \sum_{j=1}^{2} F_j(\alpha_1(t),\alpha_2(t),t)e^{\tau_j} + \tilde{R}_1(z_0(t,\tau)),$$

(here $F_j(\alpha_1(t),\alpha_2(t),t)$ are linear functions of $\alpha_1(t),\alpha_2(t)$, and the expression $\tilde{R}_1(z_0(t,\tau))$ does not contain linear terms of $\alpha_1(t),\alpha_2(t)$), we conclude that, after the embedding operation, the function $\hat{G}(t,\tau)$ will linearly depend on scalar functions $\alpha_1(t)$ and $\alpha_2(t)$.

Taking into account that under conditions (4.2), scalar multiplication by vector functions $\chi_k(t)e^{\tau_k}$, containing only exponentials e^{τ_k}, $k = 1,2$, it is necessary to keep in the expression $\widehat{G}(t,\tau)$ only terms with exponents e^{τ_1} and e^{τ_2}. Then it follows from (**) that conditions (4.2) are written in the form

$$< -\frac{\partial}{\partial t}\left(\sum_{k=1}^{2} \alpha_k(t)\varphi_k(t)e^{\tau_k} + h_{12}(t)\varphi_2(t)e^{\tau_1} + h_{21}(t)\varphi_1(t)e^{\tau_2}\right)$$

$$+\left(F_1(\alpha_1(t),\alpha_2(t),t) + \textstyle\sum_{|m^1|=2:\,m^1\in\Gamma_1}^{N} w^{m^1}(\alpha_1(t),\alpha_2(t),t)\right)e^{\tau_1}$$
$$+\left(F_2(\alpha_1(t),\alpha_2(t),t)\textstyle\sum_{|m^2|=2:\,m^2\in\Gamma_2}^{N} w^{m^2}(\alpha_1(t),\alpha_2(t),t)\right)e^{\tau_2}$$
$$+Q_1(t)e^{\tau_1} + Q_2(t)e^{\tau_2}, \chi_k(t)e^{\tau_k} >\equiv 0, \forall t \in [t_0,T], k = 1,2,$$

where the functions $w^{m^j}(\alpha_1(t),\alpha_2(t),t)$, $j = 1,2$, depend on $\alpha_1(t)$ and $\alpha_2(t)$ in a linear way. Performing scalar multiplication here, we obtain linear ordinary differential equations with respect to the functions $\alpha_k(t)$, $k = 1,2$, involved in the solution (3.9) of the system (3.2). Attaching the initial conditions $\alpha_k(t_0) = (z_*, \chi_k(t_0))$, $k = 1,2$, calculated earlier to them, we find uniquely functions $\alpha_k(t)$, and, therefore, construct a solution (3.9) to the problem (3.2) in the space U in a unique way. The theorem 2 is proved.

As mentioned above, the right-hand sides of iterative problems (3.1_k) (if them solve sequentially) may not belong to the space U. Then, according to [18] (p. 234), the right-hand sides of these problems must be embedded into the U, according to the above rule. As a result, we obtain the following problems:

$$Lz_0(t,\tau) \equiv \sum_{j=1}^{5} \lambda_j(t)\frac{\partial z_0}{\partial \tau_j} - A(t)z_0 - R_0 z_0 = h(t), \quad z_0(t_0,0) = z^0; \qquad (\overline{3.1}_0)$$

$$Lz_1(t,\tau) = -\frac{\partial z_0}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)B(t)z_0\right]^{\wedge} + R_1 z_0, \quad z_1(t_0,0) = 0; \qquad (\overline{3.1}_1)$$

$$Lz_2(t,\tau) = -\tfrac{\partial z_1}{\partial t} + \left[\tfrac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)B(t)z_1\right]^{\wedge} + R_1 z_1 + R_2 z_0, z_2(t_0,0) = 0; \qquad (\overline{3.1}_2)$$

$$\cdots$$

$$Lz_k(t,\tau) = -\tfrac{\partial z_{k-1}}{\partial t} + \left[\tfrac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2)B(t)z_{k-1}\right]^{\wedge} + R_k z_0 + ... + R_1 z_{k-1},\ z_k(t_0,0) = 0,\ k \ge 1, \quad (\overline{3.1}_k)$$

(images of linear operators $\frac{\partial}{\partial t}$ and R_ν do not need to be embedded in the space U, since these operators act from U to U). Such a replacement will not affect the construction of an asymptotic solution to the original problem (1.1) (or its equivalent problem (2.1)), so on the narrowing $\tau = \frac{\psi(t)}{\varepsilon}$ the series of problems (3.1_k) will coincide with the series of problems $\left(\overline{3.1_k}\right)$ (see [18], pp. 234–235].

Applying Theorems 1 and 2 to iterative problems $\left(\overline{3.1_k}\right)$, we find their solutions uniquely in the space U and construct series (2.6). As in [18] (pp. 63-69), we prove the following statement.

Theorem 3. *Let conditions 1)–2) be satisfied for the system (2.1). Then*, for $\varepsilon \in (0, \varepsilon_0](\varepsilon_0 > 0$ *is sufficiently small*) *system (2.1) has a unique solution* $z(t, \varepsilon) \in C^1([t_0, T], \mathbb{C}^2)$; at the same time there is the estimate

$$\|z(t, \varepsilon) - z_{\varepsilon N}(t)\|_{C[t_0,T]} \leq c_N \varepsilon^{N+1},\ N = 0, 1, 2, ...,$$

where $z_{\varepsilon N}(t)$ *is the restriction on* $\tau = \frac{\psi(t)}{\varepsilon}$ *of the N -th partial sum of the series (2.6) (with coefficients* $z_k(t, \tau) \in U$, *satisfying the iterative problems* $(\overline{3.1_k})$*) and the constant* $c_N > 0$ *does not depend on* ε *at* $\varepsilon \in (0, \varepsilon_0]$.

5. Construction of a solution of the first iterative problem

Using Theorem 1, we try to find a solution to the first iterative problem $\left(\overline{3.1_0}\right)$. Since the right-hand side $h(t)$ of the system $\left(\overline{3.1_0}\right)$ satisfies condition (3.3), this system (according to (3.9)) has a solution in the space U in the form

$$z_0(t, \tau) = z_0^{(0)}(t) + \sum_{k=1}^{2} \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau_k}, \tag{5.1}$$

where $\alpha_k^{(0)}(t) \in C^\infty\left([t_0, T], \mathbb{C}^1\right)$ are arbitrary functions, $k = 1, 2,\ z_0^{(0)}(t) = -A^{-1}(t) h(t)$. Subordinating (4.2) to the initial condition $z_0(t_0, 0) = z^0$, we have

$$\sum_{k=1}^{2} \alpha_k^{(0)}(t_0) \varphi_k(t_0) = z^0 + A^{-1}(t_0) h(t_0).$$

Multiplying this equality scalarly $\chi_j(t_0)$ and taking into account biorthogonality property of the systems $\{\varphi_k(t)\}$ and $\left\{\chi_j(t)\right\}$, find the values

$$\alpha_k^{(0)}(t_0) = \left(z^0 + A^{-1}(t_0) h(t_0), \chi_k(t_0)\right), k = 1, 2. \tag{5.2}$$

For a complete calculation of the functions $\alpha_k^{(0)}(t)$, we proceed to the next iterative problem $\left(\overline{3.1_1}\right)$. Substituting the solution (5.1) of the system $\left(\overline{3.1_0}\right)$ into it, we arrive at the following system:

$$Lz_1(t, \tau) = -\frac{d}{dt} z_0^{(0)}(t) - \sum_{k=1}^{2} \frac{d}{dt}(\alpha_k^{(0)}(t) \varphi_k(t)) e^{\tau_k} + \frac{K(t, t) z_0^{(0)}(t)}{\lambda_5(t)} e^{\tau_5} - \frac{K(t, t_0) z_0^{(0)}(t_0)}{\lambda_5(t_0)}$$

$$+ \left[\frac{g(t)}{2}(e^{\tau_3}\sigma_1 + e^{\tau_4}\sigma_2) B(t) \left(z_0^{(0)}(t) + \sum_{k=1}^{2} \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau_k}\right)\right]^{\wedge} \tag{5.3}$$

$$+\sum_{j=1}^{2}\left[\frac{\left(K\left(t,t\right)\alpha_{j}^{(0)}\left(t\right)\varphi_{j}\left(t\right)\right)}{\lambda_{j}\left(t\right)}e^{\tau_{j}}-\frac{\left(K\left(t,t_{0}\right)\alpha_{j}^{(0)}\left(t_{0}\right)\varphi_{j}\left(t_{0}\right)\right)}{\lambda_{j}\left(t_{0}\right)}\right],$$

(here we used the expression (2.5_1) for $R_1 z(t,\tau)$ and took into account that when $z(t,\tau)=z_0(t,\tau)$ in the sum (2.5_1) only terms with e^{τ_1}, e^{τ_2} and e^{τ_5} remain). We calculate

$$M=\left[\frac{g(t)}{2}\left(e^{\tau_3}\sigma_1+e^{\tau_4}\sigma_2\right)B\left(t\right)\left(z_0^{(0)}\left(t\right)+\sum_{k=1}^{2}\alpha_k^{(0)}\left(t\right)\varphi_k\left(t\right)e^{\tau_k}\right)\right]^{\wedge}$$

$$=\frac{1}{2}g\left(t\right)B\left(t\right)[\alpha_1^{(0)}\sigma_1\varphi_1\left(t\right)e^{\tau_3+\tau_1}+\sigma_2\alpha_2^{(0)}\left(t\right)\varphi_1\left(t\right)e^{\tau_4+\tau_1}$$

$$+\sigma_1\alpha_2^{(0)}\left(t\right)\varphi_2\left(t\right)e^{\tau_3+\tau_2}+\sigma_2\alpha_2^{(0)}\left(t\right)\varphi_2\left(t\right)e^{\tau_4+\tau_2}+e^{\tau_3}\sigma_1 z_0\left(t\right)+e^{\tau_4}\sigma_2 z_0\left(t\right))]^{\wedge}.$$

Let us analyze the exponents of the second dimension included here for their resonance:

$$e^{\tau_3+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(-i\beta'-i\omega)d\theta},$$

$$-i\beta'-i\omega=\left[\begin{array}{l}0,\\-i\omega,\\+i\omega,\end{array}\right.\Leftrightarrow\emptyset,\ -i\beta'-i\omega=\left[\begin{array}{l}-i\beta',\\+i\beta',\\\mu\end{array}\right.\Leftrightarrow\emptyset;$$

$$e^{\tau_4+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(+i\beta'-i\omega)d\theta},$$

$$+i\beta'-i\omega=\left[\begin{array}{l}(0),\\-i\omega,\\(+i\omega),\end{array}\right.\Leftrightarrow\left[\begin{array}{l}\beta'=\omega,\\\beta'=2\omega,\end{array}\right.$$

$$+i\beta'-i\omega=\left[\begin{array}{l}(-i\beta'),\\+i\beta',\\\mu,\end{array}\right.\Leftrightarrow 2\beta'=\omega\Rightarrow$$

$$\Rightarrow\left[\begin{array}{l}\widehat{e^{\tau_4+\tau_1}}=e^0=1\ (\beta'=\omega),\\\widehat{e^{\tau_4+\tau_1}}=e^{\tau_2}\ (\beta'=2\omega),\\\widehat{e^{\tau_4+\tau_1}}=e^{\tau_3}\ (2\beta'=\omega);\end{array}\right.$$

$$e^{\tau_3+\tau_2}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(-i\beta'+i\omega)d\theta},$$

$$-i\beta'+i\omega=\left[\begin{array}{l}(0),\\(-i\omega),\\+i\omega,\end{array}\right.\Leftrightarrow\left[\begin{array}{l}\beta'=\omega,\\\beta'=2\omega;\end{array}\right.$$

$$-i\beta'+i\omega=\left[\begin{array}{l}-i\beta',\\(+i\beta'),\\\mu,\end{array}\right.\Leftrightarrow 2\beta'=\omega$$

$$\Rightarrow\left[\begin{array}{l}\widehat{e^{\tau_3+\tau_2}}=e^0=1\ (\beta'=\omega),\\\widehat{e^{\tau_3+\tau_2}}=e^{\tau_1}\ (\beta'=2\omega),\\\widehat{e^{\tau_3+\tau_2}}=e^{\tau_4}\ (2\beta'=\omega);\end{array}\right.$$

$$e^{\tau_4+\tau_2}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(+i\beta'+i\omega)d\theta},$$

$$+i\beta'+i\omega=\left[\begin{array}{l}0,\\-i\omega,\\+i\omega,\end{array}\right.\Leftrightarrow\emptyset,\ +i\beta'+i\omega=\left[\begin{array}{l}-i\beta',\\+i\beta',\\\mu,\end{array}\right.\Leftrightarrow\emptyset.$$

Thus, the exponents $e^{\tau_3+\tau_1}$and $e^{\tau_4+\tau_2}$ are not resonant, and the exponents $e^{\tau_4+\tau_1}$ and $e^{\tau_3+\tau_2}$ are resonant at certain ratios between frequencies $\beta'(t)$, and $\omega(t)$, moreover, their embeddings are carried out as follows:

$$\left[\begin{array}{l} \widehat{e^{\tau_4+\tau_1}} = e^0 = 1\ (\beta' = \omega), \\ \widehat{e^{\tau_4+\tau_1}} = e^{\tau_2}\ (\beta' = 2\omega), \\ \widehat{e^{\tau_4+\tau_1}} = e^{\tau_3}\,(2\beta' = \omega), \end{array}\right. \quad \left[\begin{array}{l} \widehat{e^{\tau_3+\tau_2}} = e^0 = 1\ (\beta' = \omega), \\ \widehat{e^{\tau_3+\tau_2}} = e^{\tau_1}\ (\beta' = 2\omega), \\ \widehat{e^{\tau_3+\tau_2}} = e^{\tau_4}\ (2\beta' = \omega). \end{array}\right.$$

So, resonances are possible only in the following cases of relations between frequencies: a) $\beta' = 2\omega$, b) $\beta' = \omega$, c) $2\beta' = \omega$. Case b) is not considered (see condition (1)). We consider cases a) and c).

a) $\beta' = 2\omega$. In this case, the system (5.3) after embedding takes the form

$$Lz_1(t,\tau) = -\frac{d}{dt}z_0^{(0)}(t) - \sum_{k=1}^{2}\frac{d}{dt}(\alpha_k^{(0)}(t)\varphi_k(t))e^{\tau_k} + \frac{K(t,t)\,z_0^{(0)}(t)}{\lambda_5(t)}e^{\tau_5}$$

$$-\frac{K(t,t_0)\,z_0^{(0)}(t_0)}{\lambda_5(t_0)} + \frac{1}{2}g(t)\,B(t)\,[\sigma_1\alpha_1^{(0)}(t)\,\varphi_1(t)\,e^{\tau_3+\tau_1} + \sigma_2\alpha_1^{(0)}(t)\,\varphi_1(t)\,e^{\tau_2}$$

$$+\sigma_1\alpha_2^{(0)}(t)\,\varphi_2(t)\,e^{\tau_1} + \sigma_2\alpha_2^{(0)}(t)\,\varphi_2(t)\,e^{\tau_4+\tau_2} + e^{\tau_3}\sigma_1 z_0(t) + e^{\tau_4}\sigma_2 z_0(t))]$$

$$+\sum_{j=1}^{2}\left[\frac{\left(K(t,t)\,\alpha_j^{(0)}(t)\,\varphi_j(t)\right)}{\lambda_j(t)}e^{\tau_j} - \frac{\left(K(t,t_0)\,\alpha_j^{(0)}(t_0)\,\varphi_j(t_0)\right)}{\lambda_j(t_0)}\right].$$

This system is solvable in the space U if and only if the conditions of orthogonality are satisfied:

$$\langle -\sum_{k=1}^{2}\frac{d}{dt}(\alpha_k^{(0)}(t)\,\varphi_k(t))e^{\tau_k} + \frac{1}{2}g(t)\,B(t)\,[\sigma_1\alpha_2^{(0)}(t)\,\varphi_2(t)\,e^{\tau_1}$$

$$+\sigma_2\alpha_1^{(0)}(t)\,\varphi_1(t)\,e^{\tau_2}] + \sum_{i=1}^{2}\frac{\left(K(t,t)\,\alpha_i^{(0)}(t)\,\varphi_i(t)\right)}{\lambda_i(t)}e^{\tau_i}, \chi_j(t)\,e^{\tau_j}\rangle \equiv 0,\ j = 1,2.$$

Performing scalar multiplication here, we obtain a system of ordinary differential equations:

$$\begin{array}{l} -\frac{d\alpha_1^{(0)}(t)}{dt} - (\dot{\varphi}_1(t), \chi_1(t))\,\alpha_1^{(0)}(t) + \frac{1}{2}g(t)\,\sigma_1\,(B(t)\,\varphi_2(t), \chi_1(t))\,\alpha_2^{(0)}(t) + \frac{(K(t,t),\chi_1(t))}{\lambda_1(t)}\alpha_1^{(0)}(t) \equiv 0, \\ -\frac{d\alpha_2^{(0)}(t)}{dt} - (\dot{\varphi}_2(t), \chi_2(t))\,\alpha_2^{(0)}(t) + +\frac{1}{2}g(t)\,\sigma_2\,(B(t)\,\varphi_1(t), \chi_2(t))\,\alpha_1^{(0)}(t) + \frac{(K(t,t),\chi_2(t)(t))}{\lambda_2(t)}\alpha_2^{(0)}(t) \equiv 0. \end{array} \tag{5.4}$$

Adding the initial conditions (5.2) to this system, we find uniquely functions $\alpha_k^{(0)}(t)$, $k = 1,2$, and, therefore, uniquely calculate the solution (5.1) of the problem $\left(\overline{3.1}_0\right)$ in the space U. Moreover, the main term of the asymptotic solution of the problem (2.1) has the form

$$z_{\varepsilon 0}(t) = z_0^{(0)}(t) + \sum_{k=1}^{2}\alpha_k^{(0)}(t)\,\varphi_k(t)\,e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_k(\theta)d\theta}, \tag{5.5}$$

where the functions $\alpha_k^{(0)}(t_0)$ satisfy the problem (5.2), (5.4), $z_0^{(0)}(t) = -A^{-1}(t)h(t)$. We draw attention to the fact that the system of equations (5.4) does not decompose into separate differential equations (as was the case in ordinary integro-differential equations). The presence of a rapidly oscillating coefficient in the problem (1.1) leads to more complex differential systems of type (5.4), the solution of which, although they exist on the interval $[t_0, T]$, is not always possible to find them explicitly. However, in third case this it manages to be done.

c) $2\beta' = \omega$. In this case, the system (5.3) after embedding takes the form (take into account that $\widehat{e^{\tau_4+\tau_1}} = e^{\tau_3}$, $\widehat{e^{\tau_3+\tau_2}} = e^{\tau_4}$)

$$Lz_1(t,\tau) = -\frac{d}{dt}z_0^{(0)}(t) - \sum_{k=1}^{2}\frac{d}{dt}(\alpha_k^{(0)}(t)\varphi_k(t))e^{\tau_k} + \frac{K(t,t)z_0^{(0)}(t)}{\lambda_5(t)}e^{\tau_5} - \frac{K(t,t_0)z_0^{(0)}(t_0)}{\lambda_5(t_0)}$$

$$+\frac{1}{2}g(t)B(t)[\sigma_1\alpha_1^{(0)}(t)\varphi_1(t)e^{\tau_3+\tau_1} + \sigma_2\alpha_1^{(0)}(t)\varphi_1(t)e^{\tau_2}$$

$$+\sigma_1\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_1} + \sigma_2\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_4+\tau_2} + e^{\tau_3}\sigma_1 z_0(t) + e^{\tau_4}\sigma_2 z_0(t))]$$

$$+\sum_{j=1}^{2}\left[\frac{\left(K(t,t)\alpha_j^{(0)}(t)\varphi_j(t)\right)}{\lambda_j(t)}e^{\tau_j} - \frac{\left(K(t,t_0)\alpha_j^{(0)}(t_0)\varphi_j(t_0)\right)}{\lambda_j(t_0)}\right].$$

This system is solvable in the space U if and only if the conditions of orthogonality

$$\langle -\sum_{k=1}^{2}\frac{d}{dt}(\alpha_k^{(0)}(t)\varphi_k(t))e^{\tau_k} + \sum_{i=1}^{2}\frac{\left(K(t,t)\alpha_i^{(0)}(t)\varphi_i(t)\right)}{\lambda_i(t)}e^{\tau_i}, \chi_j(t)e^{\tau_j}\rangle \equiv 0,$$

$j = 1, 2$, are satisfied. Performing scalar multiplication here, we obtain a system of diverging ordinary differential equations

$$-\frac{d\alpha_1^{(0)}(t)}{dt} - (\dot{\varphi}_1(t), \chi_1(t))\alpha_1^{(0)}(t) + \frac{(K(t,t),\chi_1(t))}{\lambda_1(t)}\alpha_1^{(0)}(t) \equiv 0,$$

$$-\frac{d\alpha_2^{(0)}(t)}{dt} - (\dot{\varphi}_2(t), \chi_2(t))\alpha_2^{(0)}(t) + \frac{(K(t,t),\chi_2(t)(t))}{\lambda_2(t)}\alpha_2^{(0)}(t) \equiv 0.$$

Together with the initial conditions (5.2), it has a unique solution

$$\alpha_k^{(0)}(t) = \left(z^0 + A^{-1}(t_0)h(t_0), \chi_k(t_0)\right)exp\left\{\int_{t_0}^{t}\frac{(K(\theta,\theta) - \dot{\varphi}_k(\theta), \chi_k(\theta))}{\lambda_k(\theta)}d\theta\right\},$$

$k = 1, 2$, and therefore, the solution (5.1) of the problem $\left(\overline{3.1_0}\right)$ will be found uniquely in the space U. In this case the leading term of the asymptotics has the form (5.5), but with functions $\alpha_k^{(0)}(t)$, explicitly calculated. Its influence is revealed when constructing the asymptotics of the first and higher orders.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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