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*Research article*

# The Górnicki -Proinov type contraction on quasi-metric spaces

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**Abstract:** In this manuscript, we look for the answer of the question: Under which conditions the Górnicki-Proinov type contractions possesses a fixed point in the framework of quasi-metric spaces. The observed results are not only generalize but also uniform several existing fixed point theorem in this direction. We also present an example to demonstrate the validity of the obtained main result.

**Keywords:** the Górnicki-Proinov type contraction; fixed point theorems; quasi-metric space
**Mathematics Subject Classification:** 47H09, 47H10, 54H25

## 1. Introduction and preliminaries

Metric fixed point theory connects three main branches of Mathematics: Functional Analysis, Topology and Applied Mathematics. One of the initial fixed point results was used by Picard [27] in the solution of a certain differential equations. Banach [4] abstracted the idea of the fixed point in the paper of Picard in the framework of normed spaces. Later, Cacciopoli [9] restated the Banach fixed point results in the setting of standard metric space. After Cacciopoli [9], a number of different contractions were defined to extend and generalize the renowned Banach contraction mapping principle in the setting of the complete metric spaces.

Especially in the last two decades, numerous articles have been published on metric fixed point theory. On the one hand, this is pleasing for the advances of the theory, on the other hand, such intense interest has caused a lot of trouble. For instance, we have seen that the generalizations obtained in some publications are either equivalent to an existing theorem or simply a consequence of it. As a clearer example, we may consider the publications of the fixed point theorem in the setting of the cone metric spaces that emerged after 2007. It was noticed that published fixed point results, in the context of cone metric spaces, can be converted into their standard versions by using the scalarization function (see e.g. [11] and related references there in). Likewise, it was proved that the published fixed point results in the setting of the G-metric proved are equivalent to the corresponding version of fixed point theorem in the framework of the quasi-metrics, see e.g [32]. In this case, it has become a very necessary need to examine the newly obtained results and to classify the equivalent ones. One of the best examples of this is Proinov's [29] article. He observed that several recently published fixed point results were the consequence of Skof Theorem and he generalized the main result of Skof [31].

In this paper, we combine the outstanding results of Proinov [29] and Górnicki [12] in a more general setting, in quasi-metric spaces. Consequently, the obtained results cover several existing results in this direction see e.g. [2, 3, 5, 8, 10, 15, 18–25, 30]. First, we recall basic notations and fundamental results.

**Definition 1.** *[28] Let $\mathtt{X}$ be a non-empty set and $\alpha : \mathtt{X} \times \mathtt{X} \to [0, \infty)$ be a function. We say that a mapping $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ is*

- $\alpha$-orbital admissible *($\alpha_{o.a}$) if*

$$\alpha(z, \mathtt{f}z) \geq 1 \Rightarrow \alpha(\mathtt{f}z, \mathtt{f}^2 z) \geq 1, \tag{1.1}$$

  *for any $z, v \in \mathtt{X}$,*
- triangular $\alpha$-orbital admissible *($\alpha_{t.o.a}$) if it is $\alpha$-orbital admissible and*

$$\alpha(z, v) \geq 1 \text{ and } \alpha(v, \mathtt{f}v) \geq 1 \Rightarrow \alpha(z, \mathtt{f}v) \geq 1, \tag{1.2}$$

  *for any $z, v \in \mathtt{X}$.*

**Lemma 1.** *[28] Suppose that $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ is an triangular $\alpha$-orbital admissible function and $z_m = \mathtt{f}z_{m-1}, m \in \mathbb{N}$. If there exists $z_0 \in \mathtt{X}$ such that $\alpha(z_0, \mathtt{f}z_0) \geq 1$, then we have*

$$\alpha(z_m, z_n) \geq 1 \text{ for any } m, n \in \mathbb{N}, m < n.$$

Let $\mathtt{P} = \{\psi \mid \psi : (0, \infty) \to \mathbb{R}\}$. For two functions $\psi, \varphi \in \mathtt{P}$ we consider the following axioms:

($A_1$) $\varphi(t) < \psi(t)$ for any $t > 0$;
($A_2$) $\psi$ is nondecreasing;
($A_3$) $\limsup_{t\to e+} \varphi(t) < \psi(e+)$ for any $e > 0$;
($A_4$) $\inf_{t>e} \psi(t) > -\infty$ for any $e$;
($A_5$) $\limsup_{t\to e+} \varphi(t) < \liminf_{t\to e} \psi(t)$ or $\limsup_{t\to e} \varphi(t) < \liminf_{t\to e+} \psi(t)$ for any $e > 0$;
($A_6$) $\limsup_{t\to 0+} \varphi(t) < \liminf_{t\to e} \psi(t)$ for any $e > 0$;
($A_7$) if the sequences $(\psi(t_n))$ and $(\varphi(t_n))$ are convergent with the same limit and $(\psi(t_n))$ is strictly decreasing then $t_n \to 0$ as $n \to \infty$.

**Theorem 1.** *[Theorem 3.6., [29]] Let $(\mathtt{X}, \mathtt{d})$ be a complete metric space, $\psi, \varphi \in \mathtt{P}$ and $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ be a mapping such that*

$$\psi(\mathtt{d}(\mathtt{f}z, \mathtt{f}v)) \leq \varphi(\mathtt{d}(z, v)),$$

*for all $z, v \in \mathtt{X}$ with $\mathtt{d}(\mathtt{f}z, \mathtt{f}v) > 0$. If the assumptions ($A_1$), ($A_2$), ($A_3$) are satisfied then $\mathtt{f}$ admits a unique fixed point.*

**Theorem 2.** *[Theorem 3.7., [29]] Let $\psi, \varphi \in \mathtt{P}$ be two functions satisfying ($A_1$), ($A_4$), ($A_5$), ($A_6$), ($A_7$). On a complete metric space $(\mathtt{X}, \mathtt{d})$ a mapping $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ has a unique fixed point provided that*

$$\psi(\mathtt{d}(\mathtt{f}z, \mathtt{f}v)) \leq \varphi(\mathtt{d}(z, v)),$$

*for all $z, v \in \mathtt{X}$ with $q(\mathtt{f}z, \mathtt{f}v) > 0$.*

Let $\mathtt{f}$ be a self-mapping on a metric space $(\mathtt{X}, \mathtt{d})$, $z \in \mathtt{X}$, $\{\mathtt{f}^n z\}$ be the Picard sequence and the set $O(\mathtt{f}, z) := \{\mathtt{f}^n z : n = 0, 1, 2, ...\}$.

**Definition 2.** *[12] The mapping $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ is said to be* asymptotically regular at a point $z \in \mathtt{X}$ *if*

$$\lim_{m\to\infty} \mathtt{d}(\mathtt{f}^m z, \mathtt{f}^{m+1} z) = 0. \tag{1.3}$$

*Moreover, if $\mathtt{f}$ is asymptotically regular at each point of $\mathtt{X}$ it is called* asymptotically regular.

**Theorem 3.** *[Theorem 2.6., [12]] Let $(\mathtt{X}, \mathtt{d})$ be a complete metric space and $\mathtt{f}$: $\mathtt{X} \to \mathtt{X}$ be a continuous asymptotically regular mapping. Then $\mathtt{f}$ has a unique fixed point if there exist $0 \leq c < 1$ and $0 \leq \mathcal{K} < +\infty$ such that*

$$\mathtt{d}(\mathtt{f}z, \mathtt{f}v) \leq c \cdot \mathtt{d}(z, v) + \mathcal{K} \cdot \{\mathtt{d}(z, \mathtt{f}z) + \mathtt{d}(v, \mathtt{f}v)\}, \tag{1.4}$$

*for all $z, v \in \mathtt{X}$.*

**Definition 3.** *[7], [26] We say that the mapping $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ is:*

(c.o.) orbitally continuous at a point $w \in \mathtt{X}$ *if for any sequence $\{z_n\}$ in $O(\mathtt{f}, z)$ for some $z \in \mathtt{X}$, $\lim_{n\to\infty} \mathtt{d}(z_n, w) = 0$ implies $\lim_{n\to\infty} \mathtt{d}(\mathtt{f}z_n, \mathtt{f}w) = 0$.*

(c.r.) $r$-continuous at a point $w \in \mathtt{X}$ *($r = 1, 2, 3, ...$) if for any sequence $\{z_n\}$ in $\mathtt{X}$ $\lim_{n\to\infty} \mathtt{d}(\mathtt{f}^{r-1} z_n, w) = 0$ implies $\lim_{n\to\infty} \mathtt{d}(\mathtt{f}^r z_n, \mathtt{f}w) = 0$.*

**Remark 1.** *As it was shown in [26], for the case $r > 1$, the continuity of $\mathtt{f}^r$ and the r-continuity of $\mathtt{f}$ are independent conditions.*

**Theorem 4.** *[Theorem 2.1., [6]] On a complete metric space $(\mathtt{X}, \mathtt{d})$ let $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ a mapping such that there exist $0 \leq c < 1$ and $0 \leq \mathcal{K} < +\infty$ such that*

$$\mathtt{d}(\mathtt{f}z, \mathtt{f}v) \leq c \cdot \mathtt{d}(z, v) + \mathcal{K} \cdot \{\mathtt{d}(z, \mathtt{f}z) + \mathtt{d}(v, \mathtt{f}v)\}, \tag{1.5}$$

*for all $z, v \in \mathtt{X}$. Then, $\mathtt{f}$ possesses a unique fixed point if $\mathtt{f}$ is either r-continuous for $r \geq 1$, or orbitally continuous.*

**Theorem 5.** *[Theorem 2.2., [13]] Let $(\mathtt{X}, \mathtt{d})$ be a complete metric space and $\mathtt{f} : \mathtt{X} \to \mathtt{X}$ be an asymptotically regular mapping. Suppose that there exist $\psi : [0, \infty) \to [0, \infty)$ and $0 \leq \mathcal{K} < \infty$ such that*

$$\mathtt{d}(\mathtt{f}z, \mathtt{f}v) \leq \varphi(\mathtt{d}(z, v)) + \mathcal{K} \cdot \{\mathtt{d}(z, \mathtt{f}z) + \mathtt{d}(v, \mathtt{f}v)\},$$

*for all $z, v \in \mathtt{X}$. Suppose also that:*

*(i) $\varphi(t) < t$ for all $t > 0$ and $\varphi$ is upper semi-continuous;*
*(ii) either $\mathtt{f}$ is orbitally continuous or $\mathtt{f}$ is r-continuous for $r \geq 1$.*

*Then, $\mathtt{f}$ possesses a unique fixed point $z_* \in X$ and for each $z \in X$, $\mathtt{f}^n z \to z_*$ as $n \to \infty$.*

**Theorem 6.** *[Theorem 2.1., [13]] Let $(X, d)$ be a complete metric space and $\mathtt{f} : X \to X$ be an $(\mathtt{a.r.})$ mapping such that there exist $\varsigma : [0, \infty) \to [0, 1)$ and $0 \le \mathcal{K} < \infty$ such that*

$$d(\mathtt{f}z, \mathtt{f}v) \le \varsigma(d(z, v)) \cdot d(z, v) + \mathcal{K} \cdot \{d(z, \mathtt{f}z) + d(v, \mathtt{f}v)\},$$

*for all $z, v \in X$. Suppose that:*

*(i)* $\varsigma(t_n) \to 1 \Rightarrow t_n \to 0$;
*(ii)* *either $\mathtt{f}$ is orbitally continuous or $\mathtt{f}$ is r-continuous for $r \ge 1$.*

*Then, $\mathtt{f}$ has a unique fixed point $z_* \in X$ and for each $z \in X$, $\mathtt{f}^n z \to z_*$ as $n \to \infty$.*

A different, but interesting extension of the contraction mapping was given by Istrăţescu. We recall here this interesting result.

**Theorem 7.** *[14] (Istrăţescu's fixed point theorem) On a complete metric space $(X, d)$, a map $\mathtt{f} : X \to X$ is a Picard operator provided that there exists $c_1, c_2 \in (0, 1)$ such that*

$$d(\mathtt{f}^2 z, \mathtt{f}^2 v) \le c_1 \cdot d(\mathtt{f}z, \mathtt{f}v) + c_2 \cdot d(z, v),$$

*for all $z, v \in X$.*

**Definition 4.** *Let $X$ be a non-empty set and $q : X \times X \to \mathbb{R}_0^+$ be a function. We say that $q$ is a* quasi-metric *if the followings are held:*

$(q_1)$ $q(z, v) = q(v, z) = 0 \Leftrightarrow z = v$ ;
$(q_2)$ $q(z, v) \le q(z, y) + q(y, v)$, *for all $z, v, y \in X$.*

*In this case, the pair $(X, q)$ forms a quasi-metric space.*

Of course, each metric is a quasi-metric, but the converse is not necessarily true. For example, the functions $q_l, q_r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$, where $q_l(z, v) = \max\{z - v, 0\}$ and $q_r(z, v) = \max\{v - z, 0\}$ define quasi-metrics but not metrics. However, starting with $q_l, q_r$, a standard metric can be defined as follows $d(z, v) := \max\{q_l(z, v), q_r(v, z)\}$.

**Definition 5.** *A sequence $\{z_m\}$ in a quasi-metric space $(X, q)$ is*

*(i)* convergent *to $z \in X$ if*

$$\lim_{m\to\infty} q(z_m, z) = \lim_{m\to\infty} q(z, z_m) = 0; \tag{1.6}$$

$$\lim_{m\to\infty} q(z_m, y) = q(z, y) \text{ and } \lim_{m\to\infty} q(y, z_m) = q(y, z).$$

*(ii)* left-Cauchy *if for every $\epsilon > 0$ there exists a positive integer $p = p(\epsilon)$ such that $q(z_n, z_m) < \epsilon$ for all $n \ge m > p$;*
*(iii)* right-Cauchy *if for every $\epsilon > 0$ there exists a positive integer $p = p(\epsilon)$ such that $q(z_n, z_m) < \epsilon$ for all $m \ge n > p$.*
*(iv)* Cauchy *if for every $\epsilon > 0$ there is a positive integer $p = p(\epsilon)$ such that $q(z_n, z_m) < \epsilon$ for all $m, n > p$.*

**Remark 2.** *1. The limit for a convergent sequence is unique; if $z_n \to z$, we have for all $z \in X$.*
*2. The sequence $\{z_m\}$ is a Cauchy sequence if and only if it is left-Cauchy and right-Cauchy.*

A quasi-metric space $(X, q)$ is said to be *left-complete* (respectively, *right-complete*, *complete*) if each left-Cauchy sequence (respectively, right-Cauchy sequence, Cauchy sequence) in $X$ is convergent.

**Definition 6.** *Suppose that $(X, q)$ is a quasi-metric space, $f$ is a self-mapping on $X$. Let $\{z_m\}$ be a sequence in $X$ and $z \in X$. We say that $f$ is*

*(i)* left-continuous *if $q(fz, fz_m) \to 0$ whenever $q(z, z_m) \to 0$;*
*(ii)* right-continuous *if $q(fz_m, fz) \to 0$ whenever $q(z_m, z) \to 0$;*
*(iii)* continuous *if $\{fz_m\} \to fz$ whenever $\{z_m\} \to z$.*

**Definition 7.** *We say that a quasi-metric space $(X, q)$ is $\delta$-*symmetric *if there exists a positive real number $\delta > 0$ such that*

$$q(v, z) \leq \delta \cdot q(z, v) \text{ for all } z, v \in X. \tag{1.7}$$

**Remark 3.** *When $\delta = 1$, the $\delta$-symmetric quasi-metric space $(X, q)$ is a metric space.*

**Example 1.** *Let $X = \mathbb{R}$ be a non-empty set and $d : X \times X \to \mathbb{R}_0^+$ be a distance on $X$. Let $q : X \times X \to \mathbb{R}_0^+$ be the quasi-metric, defined by*

$$q(z, v) = \begin{cases} 2 \cdot d(z, v) & \text{if } z \geq v \\ d(z, v) & \text{otherwise} \end{cases}$$

*The space $(X, q)$ is not a metric space, but it is a 2-symmetric quasi-metric space.*

The main properties of $\delta$-symmetric quasi-metric spaces are recall in what follows.

**Lemma 2.** *(See e.g. [18]) Let $\{z_m\}$ be a sequence on a $\delta$-symmetric quasi-metric space $(X, q)$. It holds:*

*(i) $\{z_m\}$ right-converges to $z \in X \Leftrightarrow \{z_m\}$ left-converges to $z \Leftrightarrow \{z_m\}$ converges to $z$.*
*(ii) $\{z_m\}$ is right-Cauchy $\Leftrightarrow \{z_m\}$ is left-Cauchy $\Leftrightarrow \{z_m\}$ is Cauchy.*
*(iii) If $\{v_m\}$ is a sequence in $X$ and $q(z_m, v_m) \to 0$ then $q(v_m, z_m) \to 0$.*

We conclude this section by proving the following crucial Lemma.

**Lemma 3.** *Let $(z_m)$ be a sequence on a $\delta$-symmetric quasi-metric space $(X, q)$ such that $\lim_{m\to\infty} q(z_m, z_{m+1}) = 0$. If the sequence $(z_m)$ is not right-Cauchy (R-C), then there exist $e > 0$ and two subsequences $\{z_{m_l}\}$, $\left\{z_{p_l}\right\}$ of $\{z_m\}$ such that*

$$\begin{aligned} \lim_{l\to\infty} q(z_{m_l}, z_{p_l}) &= \lim_{l\to\infty} q(z_{m_l+1}, z_{p_l}) = \lim_{l\to\infty} q(z_{m_l+1}, z_{p_l+1}) \\ &= \lim_{l\to\infty} q(z_{m_l}, z_{p_l+1}) = e \end{aligned} \tag{1.8}$$

*Proof.* First of all, since the space is $\delta$-symmetric quasi-metric, there exists $\delta > 0$ such that $0 \leq q(z_{m+1}, z_m) \leq \delta \cdot q(z_m, z_{m+1})$. Therefore, $\lim_{m\to\infty} q(z_{m+1}, z_m) = 0$. Moreover, since the sequence $\{z_m\}$ is not right-Cauchy, we can find $e > 0$ and build the subsequences $\{z_{m_l}\}$, $\{z_{n_l}\}$ of $\{z_m\}$ such that

$$e < q(z_{m_l+1}, z_{p_l+1}) \text{ and } q(z_{m_l+1}, z_{p_l}) \leq e.$$

Thus, by using $(q_2)$, we have

$$e < q(z_{m_l+1}, z_{p_l+1}) \leq q(z_{m_l+1}, z_{p_l}) + q(z_{p_l}, z_{p_l+1}) \leq e + q(z_{p_l}, z_{p_l+1}).$$

Letting $l \to \infty$ and keeping in mind $\lim_{m\to\infty} q(z_m, z_{m+1}) = 0$ we get

$$\lim_{m\to\infty} q(z_{m_l+1}, z_{p_l+1}) = e.$$

Moreover,

$$q(z_{m_l+1}, z_{p_l+1}) - q(z_{p_l}, z_{p_l+1}) \leq q(z_{m+1}, z_{p_l}) \leq e$$

and letting $l \to \infty$ we get $\lim_{l\to\infty} q(z_{m_l+1}, z_{p_l}) = e$. On the other hand, since

$$q(z_{m_l+1}, z_{p_l}) - q(z_{m_l+1}, z_{m_l}) \leq q(z_{m_l}, z_{p_l}) \leq q(z_{m_l+1}, z_{p_l}) + q(z_{m_l+1}, z_{m_l})$$

we have $\lim_{l\to\infty} q(z_{m_l}, z_{p_l}) = e$ and from the inequality

$$q(z_{m_l+1}, z_{p_l+1}) - q(z_{m_l+1}, z_{m_l}) \leq q(z_{m_l}, z_{p_l+1}) \leq q(z_{m_l}, z_{m_l+1}) + q(z_{m_l+1}, z_{p_l+1})$$

it follows that, also, $\lim_{l\to\infty} q(z_{m_l}, z_{p_l+1}) = e$. □

**Remark 4.** *Let $(X, q)$ be a δ-symetric quasi-metric space and $f : X \to X$ be an asymptotically regular mapping. Thus, from (1.3) together with (1.7) we have*

$$\lim_{m\to\infty} q(f^n z, f^{n+1} z) = \lim_{m\to\infty} q(f^{n+1} z, f^n z) = 0, \tag{1.9}$$

*for any $z \in X$.*

## 2. Main results

Let $(X, q)$ be a $\delta$-symmetric quasi-metric space and the function $\alpha : X \times X \to [0, \infty)$. Regarding to the $\alpha$ function, we denote by $(I)$ the following statement,

$(I)$ $\alpha(z, v) \geq 1$, for all $z, v \in Fix_X f = \{z \in X : fz = z\}$

**Definition 8.** *Let $(X, q)$ be a δ-symmetric quasi-metric space, the functions $\psi, \varphi : (0, \infty) \to \mathbb{R}$ and $\alpha : X \times X \to [0, \infty)$. We say that an asymptotically regular mapping $f : X \to X$ is a* generalized $(\alpha$-$\psi$-$\varphi)$-contraction *if we can find $0 \leq C < \infty$ such that*

$$\alpha(z, v)\psi(q(z, v)) \leq \varphi\left(\max\left\{q(z, v), \frac{q(z, fv) + q(fz, v)}{2}\right\}\right) + C \cdot \left\{q(z, fz) + q(v, fv)\right\}, \tag{2.1}$$

*for each $z, v \in X$ with $q(fz, fv) > 0$.*

**Theorem 8.** *On a complete δ-symmetric quasi-metric space $(X, q)$ a generalized $(\alpha$-$\psi$-$\varphi)$-contraction $f$: $X \to X$ admits a fixed point provided that*

*(1) $f$ is triangular α-orbital admissible and there exists $z_0 \in X$ such that $\alpha(z_0, fz_0) \geq 1$;*

*(2) $\psi, \varphi \in P$ satisfy $(A_1)$ and $(A_5)$;*
*(3) either $\mathtt{f}$ is orbitally continuous or $\mathtt{f}$ is r-continuous for some $r \geq 1$.*

*If we supplementary add the assumption $(I)$, the uniqueness of the fixed point is ensured.*

*Proof.* By assumption, there exists a point $z_0 \in \mathtt{X}$ such that $\alpha(z_0, \mathtt{f}z_0) \geq 1$ and $\alpha(\mathtt{f}z_0, z_0) \geq 1$. Let $\{x_n\} \subset X$ be the sequence defined as

$$z_1 = \mathtt{f}z_0,\ ...,\ z_m = \mathtt{f}z_{m-1} = \mathtt{f}^m z_0, \text{ for } m \in \mathbb{N}, \tag{2.2}$$

where $z_m \neq z_{m+1}$, for any $m \in \mathbb{N}$ (since on the contrary, if we can find $n_0 \in \mathbb{N}$ such that $\mathtt{f}z_{n_0} = z_{n_0+1} = z_{n_0}$ then $z_{n_0}$ is a fixed point of $\mathtt{f}$). Thus, taking into account the fact that $\mathtt{f}$ is $\alpha_{t.o.a}$ admissible and the Lemma 1 we have

$$\alpha(z_m, z_n) \geq 1, \text{ for all } m, n \in \mathbb{N}, m < n. \tag{2.3}$$

We shall prove that the sequence $\{z_m\}$ is Cauchy. Supposing the contrary, by Lemma 3, there exist a strictly positive number $e$ and two subsequences $\{z_{m_l}\}$ and $\left\{z_{p_l}\right\}$ such that (1.8) hold. Replacing $z = z_{m_l}$ and $v = z_{p_l}$ in (2.1), and keeping (2.3) in mind we get

$$\begin{aligned}\psi(q(z_{m_l+1}, z_{p_l+1})) &\leq \alpha(z_{m_l}, z_{p_l})\psi(q(\mathtt{f}z_{m_l}, \mathtt{f}z_{p_l}))\\ &\leq \varphi\left(\max\left\{q(z_{m_l}, z_{p_l}), \tfrac{q(z_{m_l}, \mathtt{f}z_{p_l}) + q(\mathtt{f}z_{m_l}, z_{p_l})}{2}\right\}\right) +\\ &\quad + C \cdot \left\{q(z_{m_l}, \mathtt{f}z_{m_l}) + q(z_{p_l}, \mathtt{f}z_{p_l})\right\}.\end{aligned}$$

Letting $s_l = q(z_{m_l+1}, z_{p_l+1})$ and $t_l = \max\left\{q(z_{m_l}, z_{p_l}), \frac{q(z_{m_l}, z_{p_l+1}) + q(z_{m_l+1}, z_{p_l})}{2}\right\}$ and taking into account $(a_1)$, the above inequality becomes

$$\psi(s_l) \leq \varphi(t_l) + C \cdot \left\{q(z_{m_l}, z_{m_l+1}) + q(z_{p_l}, z_{p_l+1})\right\} < \psi(t_l) + C \cdot \left\{q(z_{m_l}, z_{m_l+1}) + q(z_{p_l}, z_{p_l+1})\right\}. \tag{2.4}$$

But, on the one hand, by Lemma 1,

$$\begin{aligned}\lim_{l\to\infty} s_l &= \lim_{l\to\infty} q(z_{m_l+1}, z_{p_l+1}) = e\\ \lim_{l\to\infty} t_l &= \lim_{l\to\infty} \max\left\{q(z_{m_l+1}, z_{p_l+1}), \frac{q(z_{m_l}, z_{p_l+1}) + q(z_{m_l+1}, z_{p_l})}{2}\right\} = e\end{aligned}$$

and on the other hand, since the mapping $\mathtt{f}$ is supposed to be asymptotically regular, letting $l \to \infty$ in the inequality (2.4) we get

$$\liminf_{t\to e} \psi(t) \leq \liminf_{l\to\infty} \psi(s_l) \leq \limsup_{l\to\infty} \varphi(t_m) \leq \limsup_{t\to e} \varphi(t),$$

which contradicts the assumption $(A_5)$. Thus, $\lim\limits_{l\to\infty} q(z_{m_l}, z_{p_l}) = 0$, for any $n \geq 1$, that is, the sequence $\{z_l\}$ is right-Cauchy and moreover, from Lemma 2, it is a Cauchy sequence.

Thus, since the space $(\mathtt{X}, q)$ is complete, we can find $z^*$ such that

$$\lim_{m\to\infty} z_m = z^* \tag{2.5}$$

and we shall show that $z_*$ is in fact a fixed point of $\mathtt{f}$. Using the first part of the assumption (3), that is, $\mathtt{f}$ is orbitally continuous, since $\{z_n\} \in O(\mathtt{f}, z)$ and $z_n \to z_*$ we have $z_{n+1} = \mathtt{f}z_n \to \mathtt{f}z_*$ as $n \to \infty$. The uniqueness of the limit gives $\mathtt{f}z_* = z_*$.

If we assume that the second assumption of (3) holds, that is $\mathtt{f}$ is $r$-continuous, by (2.5) for some $r \geq 1$, we have $\lim_{n\to\infty} \mathtt{f}^{r-1} z_n = z^*$ which implies $\lim_{n\to\infty} \mathtt{f}^{r} z_n = \mathtt{f}z^*$ (because $\mathtt{f}$ is $r$-continuous). Therefore, by uniqueness of the limit we have $\mathtt{f}z_* = z_*$.

Now, if we can find another point $v_* \in \mathtt{X}$ with $z^* \neq v^*$ such that $\mathtt{f}v^* = v^*$, $z^* = \mathtt{f}z^*$, from (2.1), keeping in mind $(A_1)$ and the condition $(\mathcal{I})$ we get

$$\begin{aligned}\psi(q(z^*, v^*)) &\leq \alpha(z^*, v^*)\psi(q(\mathtt{f}z^*, \mathtt{f}v^*))\\ &\leq \varphi\left(\max\left\{q(z^*, v^*), \tfrac{q(z^*, \mathtt{f}v^*)+q(\mathtt{f}z^*, v^*)}{2}\right\}\right) + C\cdot\left\{q(z^*, \mathtt{f}z^*) + q(v^*, \mathtt{f}v^*)\right\}\\ &= \varphi(q(z^*, v^*)) < \psi(q(z^*, v^*)).\end{aligned}$$

This is a contradiction. Thus, $z^* = v^*$. □

**Corollary 1.** *On a complete $\delta$-symmetric quasi-metric space $(\mathtt{X}, q)$, let $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ be an asymptotically regular mapping. Suppose that there exists $0 \leq C < \infty$ such that*

$$\psi(q(z, v)) \leq \varphi\left(\max\left\{q(z, v), \frac{q(z, \mathtt{f}v) + q(\mathtt{f}z, v)}{2}\right\}\right) + C\cdot\left\{q(z, \mathtt{f}z) + q(v, \mathtt{f}v)\right\}, \tag{2.6}$$

*for each $z, v \in \mathtt{X}$ with $q(\mathtt{f}z, \mathtt{f}v) > 0$. Suppose also that:*

*(1) $\psi, \varphi \in P$ satisfy $(A_1)$ and $(A_5)$;*
*(2) either $\mathtt{f}$ is orbitally continuous or $\mathtt{f}$ is $r$-continuous for some $r \geq 1$.*

*Then $\mathtt{f}$ admits a unique fixed point.*

*Proof.* Put $\alpha(z, v) = 0$ in Theorem 8. □

**Corollary 2.** *On a complete $\delta$-symmetric quasi-metric space $(\mathtt{X}, q)$, let $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ be an asymptotically regular mapping such that*

$$\psi(q(z, v)) \leq \varphi\left(\max\left\{q(z, v), \frac{q(z, \mathtt{f}v) + q(\mathtt{f}z, v)}{2}\right\}\right), \tag{2.7}$$

*for each $z, v \in \mathtt{X}$ with $q(\mathtt{f}z, \mathtt{f}v) > 0$. Suppose also that:*

*(1) $\psi, \varphi \in P$ satisfy $(A_1)$ and $(A_5)$;*
*(2) either $\mathtt{f}$ is orbitally continuous or $\mathtt{f}$ is $r$-continuous for some $r \geq 1$.*

*Then $\mathtt{f}$ admits a unique fixed point.*

*Proof.* Put $C = 0$ in Corollary 1. □

**Example 2.** *Let $\mathtt{X} = \mathbb{R}$ and the 2 symmetric-quasi-metric $q : \mathtt{X}\times\mathtt{X} \to [0, \infty)$ defined by*

$$q(z, v) = \begin{cases} 2(z - v), & \text{for } z \geq v\\ v - z, & \text{otherwise}\end{cases}.$$

*Let the mapping $\mathtt{f} : \mathtt{X} \to \mathtt{X}$, where*

$$\mathtt{f}z = \begin{cases} \frac{z}{z+1}, & \text{for } z \geq 0\\ -\frac{z^2}{2}, & \text{for } z \in [-1, 0)\\ 0, & \text{for } z < -1\end{cases}.$$

*Let* $\alpha : X \times X \to [0, \infty)$ *defined by*

$$\alpha(z, v) = \begin{cases} 1, & \text{for } z, v \geq 0, z \geq v \\ 3, & \text{for } z, v < -1 \\ 2, & \text{for } z = -1, v = 0 \\ 0, & \text{otherwise} \end{cases}$$

*and we choose the functions* $\psi, \varphi \in P$, *with* $\psi(t) = t$ *and* $\varphi(t) = \frac{t}{1+\frac{t}{2}}$ *for each* $t > 0$.
*Because it is easily to see that the assumptions (1)-(3) from the Theorem 8 are satisfied, we shall show that the mapping* $f$ *satisfy the inequality (2.1). The next cases should be check.*

$(c_1)$ *If* $z, v \geq 0$, $z > v$ *then* $fz = \frac{z}{1+z}$, $fv = \frac{v}{1+v}$ *and* $q(z, v) = 2|z - v| = 2(z - v)$, $q(fz, fv) = \frac{2(z-v)}{(z+1)(v+1)}$.
*Thus,*

$$\alpha(z, v)\psi(q(fz, fv)) = \frac{2(z - v)}{(z + 1)(v + 1)} \leq \frac{2(z - v)}{1 + (z - v)} = \varphi(q(z, v)).$$

$(c_2)$ *If* $z, v < -1$, *then* $fz = fv = 0$, *so* $q(fz, fv) = 0$.

$(c_3)$ *If* $z = -1$ *and* $v = 0$ *then* $q(-1, 0) = 1$, $q(f(-1), f0) = q(-\frac{1}{2}, 0) = \frac{1}{2}$, $q(-1, f(-1)) = q(-1, -\frac{1}{2}) = \frac{1}{2}$, $q(0, f0) = 0$. *Therefore,*

$$\alpha(-1, 0)\psi(q(f(-1), f0)) = 2 \cdot \frac{1}{2} \leq \frac{1}{3} + 4 \cdot \frac{1}{2} = \varphi(q(-1, 0)) + C \cdot (q(-1, f(-1)) + q(0, f0)),$$

*where we put* $C = 4$.

*(All other cases are non-interesting, since* $\alpha(z, v) = 0$.*)*
*Thus,* $z = 0$ *is the unique fixed point of* $f$.

In the following, inspired by Istrăţescu's results, we consider a new type of generalized ($\alpha$-$\psi$-$\varphi$)-contractions and list some useful consequences.

**Definition 9.** *Let* $(X, q)$ *be a* $\delta$*-symmetric quasi-metric space, the functions* $\psi, \varphi : (0, \infty) \to \mathbb{R}$ *and* $\alpha : X \times X \to [0, \infty)$. *A mapping* $f : X \to X$ *is said to be generalized-Istrăţescu* ($\alpha$-$\psi$-$\varphi$)*-contraction if it is asymptotically regular and there is* $0 \leq C < \infty$ *such that*

$$\begin{aligned} \alpha(z, v)\psi(q(f^2 z, f^2 v)) \leq \varphi\left(\max\left\{q(z, v), q(fz, fv), \tfrac{q(z,fv)+q(fz,v)}{2}, \tfrac{q(fz,f^2 v)+q(f^2 z,fv)}{2}\right\}\right) \\ +C \cdot \left\{q(z, fz) + q(v, fv) + q(fz, f^2 z) + q(fv, f^2 v)\right\}, \end{aligned} \tag{2.8}$$

*for each* $z, v \in X$ *with* $q(f^2 z, f^2 v) > 0$.

**Theorem 9.** *On a complete* $\delta$*-symmetric quasi metric space* $(X, q)$ *a generalized-Istrăţescu* ($\alpha$-$\psi$-$\varphi$)*-contraction* $f$: $X \to X$ *admits a fixed point provided that*

*(1)* $f$ *is triangular* $\alpha$*-orbital admissible and there exists* $z_0 \in X$ *such that* $\alpha(z_0, fz_0) \geq 1$;
*(2)* $\psi, \varphi \in P$ *satisfy* $(A_1)$, $(A_2)$ *and* $(A_3)$;
*(3) either* $f$ *is orbitally continuous or* $f$ *is* $r$*-continuous for some* $r \geq 1$.

*Besides, if property* $(I)$ *is added, the fixed point of* $f$ *is unique.*

*Proof.* Let $z_0 \in \mathtt{X}$ and the sequence $\{z_m\}$ be defined by (2.2). Supposing that it is not a (left) Cauchy sequence, by Lemma 3 we can find two subsequences $\{z_{m_l}\}$, respectively $\{z_{m_l}\}$ and a strictly positive real number $e$ such that the equalities (1.8) hold. On the other hand, taking Lemma 1 into account, from (2.8) we have

$$\begin{aligned}\psi(q(z_{m_l+2}, z_{p_l+2})) &\leq \alpha(z_{m_l}, z_{p_l})\psi(q(\mathtt{f}^2 z_{m_l}, \mathtt{f} z_{p_l}))\\ &< \varphi\left(H(z_{m_l}, z_{p_l})\right) + C\left(q(z_{m_l}, \mathtt{f} z_{m_l}) + q(z_{p_l}, \mathtt{f} z_{p_l}) +\right.\\ &\quad \left. + q(\mathtt{f} z_{m_l}, \mathtt{f}^2 z_{m_l}) + q(\mathtt{f} z_{p_l}, \mathtt{f}^2 z_{p_l})\right),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned} H(z_{m_l}, z_{p_l}) &= \max\left\{\begin{array}{c} q(z_{m_l}, z_{p_l}), q(\mathtt{f} z_{m_l}, \mathtt{f} z_{p_l}), \frac{q(z_{m_l}, \mathtt{f} z_{p_l}) + q(\mathtt{f} z_{m_l}, z_{p_l})}{2}, \\ \frac{q(z_{m_l}, \mathtt{f}^2 z_{p_l}) + q(\mathtt{f}^2 z_{m_l}, \mathtt{f} z_{p_l})}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} q(z_{m_l}, z_{p_l}), q(z_{m_l+1}, z_{p_l+1}), \frac{q(z_{m_l}, z_{p_l+1}) + q(z_{m_l+1}, z_{p_l})}{2}, \\ \frac{q(z_{m_l+1}, z_{p_l+2}) + q(z_{m_l+2}, z_{p_l+1})}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} q(z_{m_l}, z_{p_l}), q(z_{m_l+1}, z_{p_l+1}), \frac{q(z_{m_l}, z_{p_l+1}) + q(z_{m_l+1}, z_{p_l})}{2}, \\ \frac{q(z_{m_l+1}, z_{p_l+1}) + q(z_{p_l+1}, z_{p_l+2}) + q(z_{m_l+2}, z_{m_l+1}) + q(z_{m_l+1}, z_{p_l+1})}{2} \end{array}\right\}. \end{aligned}$$

Also, by $(q_2)$ we have

$$q(z_{m_l+1}, z_{p_l+1}) - q(z_{m_l+1}, z_{m_l+2}) - q(z_{p_l+2}, z_{p_l+1}) \leq q(z_{m_l+2}, z_{p_l+2}).$$

Letting $l \to \infty$ and keeping in mind (1.9) and (1.8), we get

$$\begin{aligned}\lim_{l\to\infty} s_l &= e\\ \lim_{l\to\infty} t_l &= e.\end{aligned} \tag{2.10}$$

Taking into account $(A_1)$, by (2.9) and from the monotonicity of $\psi$ it follows

$$\begin{aligned}\psi(s_l) &\leq \psi(q(z_{m_l+2}, z_{p_l+2})) < \varphi(H(z_{m_l}, z_{p_l})) +\\ &\quad + C\cdot(q(z_{m_l}, \mathtt{f} z_{m_l}) + q(z_{p_l}, \mathtt{f} z_{p_l}) + q(\mathtt{f} z_{m_l}, \mathtt{f}^2 z_{m_l}) + q(\mathtt{f} z_{p_l}, \mathtt{f}^2 z_{p_l}))\\ &= \varphi(t_l) + C\cdot(q(z_{m_l}, z_{m_l+1}) + q(z_{p_l}, z_{p_l+1}) + q(z_{m_l+1}, z_{m_l+2}) + q(z_{p_l+1}, z_{p_l+2})).\end{aligned}$$

Taking the limit in the above inequality,

$$\begin{aligned}\psi(e) &= \lim_{l\to\infty}\psi(s_l) \leq \limsup_{l\to\infty}\varphi(t_l) +\\ &\quad + C\limsup_{l\to\infty}\left(q(z_{m_l}, \mathtt{f} z_{m_l}) + q(z_{p_l}, \mathtt{f} z_{p_l}) + q(\mathtt{f} z_{m_l}, \mathtt{f}^2 z_{m_l}) + q(\mathtt{f} z_{p_l}, \mathtt{f}^2 z_{p_l})\right)\\ &\leq \limsup_{t\to e}\varphi(t).\end{aligned} \tag{2.11}$$

This contradicts the assumption $(A_3)$. Thus, $\lim_{m,p\to\infty} q(z_m, z_p) = 0$, for all $m, p \in \mathbb{N}$, $m < p$ and the sequence $\{z_m\}$ is right-Cauchy, so that it is Cauchy sequence (by Lemma 1) on a complete $\delta$-symmetric quasi-metric space. So, there exists a point $z^* \in \mathtt{X}$ such that $z_m \to z^*$, as $m \to \infty$. Using the third assumption, as in Theorem 8, we easily can see that $\mathtt{f}z^* = z^*$. To prove the uniqueness, we can assume that there exists another point $v^*$, different by $z^*$, such that $\mathtt{f}v^* = v^*$. Taking into account the condition $U$, that is $\alpha(z^*, v^*) \leq 1$, form (2.8) it follows

$$\begin{aligned}
\psi(q(z^*, v^*)) &\leq \alpha(z^*, v^*)\psi(q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}(\mathtt{f}v^*))) \\
&\leq \varphi\left(\max\left\{\begin{array}{c} q(z^*, v^*), q(\mathtt{f}z^*, \mathtt{f}v^*), \frac{q(z^*,\mathtt{f}v^*)+q(\mathtt{f}z^*,v^*)}{2}, \\ \frac{q(\mathtt{f}z^*,\mathtt{f}(\mathtt{f}v^*))+q(\mathtt{f}(\mathtt{f}z^*),\mathtt{f}v^*)}{2} \end{array}\right\}\right)+ \\
&\quad +C \cdot \max\left\{q(z^*, \mathtt{f}z^*) + q(v^*, \mathtt{f}v^*)\right\} \\
&= \varphi\left(q(z^*, v^*)\right) + C \cdot \max\left\{q(z^*, z^*) + q(v^*, v^*)\right\} \\
&< \psi(q(z^*, v^*)),
\end{aligned}$$

which is a contradiction. Therefore, $z^*$ is the unique fixed point of $\mathtt{f}$. □

**Corollary 3.** *On a complete δ-symmetric quasi metric space* $(\mathtt{X}, q)$ *let* $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ *be an* (a.r.) *mapping such that there exists* $0 \leq C < \infty$ *such that*

$$\begin{aligned}
\psi(q(\mathtt{f}^2 z, \mathtt{f}^2 v)) \leq \varphi\Big(\max\Big\{q(z, v), q(\mathtt{f}z, \mathtt{f}v), \tfrac{q(z,\mathtt{f}v)+q(\mathtt{f}z,v)}{2}, \tfrac{q(\mathtt{f}z,\mathtt{f}^2 v)+q(\mathtt{f}^2 z,\mathtt{f}v)}{2}\Big\}\Big) \\
+C \cdot \left\{q(z, \mathtt{f}z) + q(v, \mathtt{f}v) + q(\mathtt{f}z, \mathtt{f}^2 z) + q(\mathtt{f}v, \mathtt{f}^2 v)\right\},
\end{aligned} \tag{2.12}$$

*for each* $z, v \in \mathtt{X}$ *with* $q(\mathtt{f}^2 z, \mathtt{f}^2 v) > 0$. *The mapping* $\mathtt{f}$ *has a unique fixed point provided that*

*(1)* $\psi, \varphi \in P$ *satisfy* $(A_1)$, $(A_2)$ *and* $(A_3)$;
*(2) either* $\mathtt{f}$ *is orbitally continuous or* $\mathtt{f}$ *is r-continuous for some* $r \geq 1$.

*Proof.* Put $\alpha(z, v) = 1$ in Theorem 9. □

**Corollary 4.** *On a complete δ-symmetric quasi metric space* $(\mathtt{X}, q)$ *let* $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ *be an asymptotically regular mapping such that there exists* $0 \leq C < \infty$ *such that*

$$\psi(q(\mathtt{f}^2 z, \mathtt{f}^2 v)) \leq \varphi\Big(\max\Big\{q(z, v), q(\mathtt{f}z, \mathtt{f}v), \tfrac{q(z,\mathtt{f}v)+q(\mathtt{f}z,v)}{2}, \tfrac{q(\mathtt{f}z,\mathtt{f}^2 v)+q(\mathtt{f}^2 z,\mathtt{f}v)}{2}\Big\}\Big) \tag{2.13}$$

*for each* $z, v \in \mathtt{X}$ *with* $q(\mathtt{f}^2 z, \mathtt{f}^2 v) > 0$. *The mapping* $\mathtt{f}$ *has a unique fixed point provided that*

*(1)* $\psi, \varphi \in P$ *satisfy* $(A_1)$, $(A_2)$ *and* $(A_3)$;
*(2) either* $\mathtt{f}$ *is orbitally continuous or* $\mathtt{f}$ *is r-continuous for some* $r \geq 1$..

*Proof.* Put $C = 0$ in Corollary 3. □

**Theorem 10.** *On a complete δ-symmetric quasi metric space* $(\mathtt{X}, q)$ *let* $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ *be a mapping such that*

$$\alpha(z, v)\psi\left(S(z, v)\right) \leq \varphi(H(z, v)) + C \cdot G(z, v) \tag{2.14}$$

*for all $z, v \in X$ with $q(fz, fv) > 0$ and $q(f^2 z, f^2 v) > 0$, where*

$$\begin{aligned}
S(z, v) &= \min\left\{q(fz, fv), q(f^2 z, f^2 v)\right\};\\
H(z, v) &= \max\left\{q(fz, f^2 z), q(v, fv), q(v, f^2 z)\right\};\\
G(z, v) &= \max\left\{q(z, fz)\cdot q(fv, f^2 v)\cdot q(fv, f^2 z),\ q(v, fv)\cdot q(fz, f^2 z)\cdot q(v, fz)\right\},
\end{aligned}$$

*$\psi, \varphi \in P$, $\alpha : X \times X \to [0, \infty)$ and $0 \le C < \infty$. Suppose that:*

*(1) $f$ is triangular $\alpha$-orbital admissible and there exists $z_0 \in X$ such that $\alpha(z_0, fz_0) \ge 1$;*
*(2) the functions $\psi, \varphi$ satisfy $(A_1)$, $(A_2)$ and $(A_3)$;*
*(3) either $f$ is continuous, or*
*(4) $f^2$ is continuous and $\alpha(fz, z) \ge 1$, $\alpha(z, fz) \ge 1$ for any $z \in Fix_{f^2}(X)$*

*Then $f$ has a fixed point.*
*Besides, if property $(I)$ is added, the fixed point of $f$ is unique.*

*Proof.* Let $z_0 \in X$, such that $\alpha(z_0, fz_0) \ge 1$ and $\{z_m\}$ be a sequence in $X$, where $z_m = fz_{m-1} = f^m z_0$, for $m \in \mathbb{N}$. Since $f$ is triangular $\alpha$-orbital admissible and, by Lemma 3, $\alpha(z_m, z_{m-1}) \ge 1$ and letting $z = z_{m-1}$, $v = z_m$ in (2.14) we have

$$\psi(S(z_{m-1}, z_m)) \le \alpha(z_{m-1}, z_m)\psi(S(z_{m-1}, z_m)) \le \varphi(H(z_{m-1}, z_m)) + C\cdot P(z_{m-1}, z_m) \tag{2.15}$$

for any $m \in \mathbb{N}$, where

$$\begin{aligned}
S(z_{m-1}, z_m) &= \min\left\{q(fz_{m-1}, fz_m), q(f^2 z_{m-1}, f^2 z_m)\right\},\\
&= \min\left\{q(z_m, z_{m+1}), q(z_{m+1}, z_{m+2})\right\}\\
H(z_{m-1}, z_m) &= \max\left\{q(fz_{m-1}, f^2 z_{m-1}), q(z_m, fz_m), q(z_m, f^2 z_{m-1})\right\}\\
&= \max\left\{q(z_m, z_{m+1}), q(z_m, z_{m+1}), q(z_m, z_{m+1})\right\}\\
&= q(z_m, z_{m+1}),\\
G(z_{m-1}, z_m) &= \max\left\{\begin{array}{c} q(z_{m-1}, fz_{m-1})\cdot q(fz_m, f^2 z_m)\cdot q(fz_m, f^2 z_{m-1}),\\ q(z_m, fz_m)\cdot q(fz_{m-1}, f^2 z_{m-1})\cdot q(z_m, fz_{m-1}) \end{array}\right\}\\
&= \max\left\{\begin{array}{c} q(z_{m-1}, z_m)\cdot q(z_{m+1}, z_{m+2})\cdot q(z_{m+1}, z_{m+1}),\\ q(z_m, z_{m+1})\cdot q(z_m, z_{m+1})\cdot q(z_m, z_m) \end{array}\right\}\\
&= 0.
\end{aligned}$$

Taking into account $(A_1)$, the inequality (2.15) becomes

$$\psi(S(z_{m-1}, z_m)) \le \varphi(H(z_{m-1}, z_m)) < \psi(H(z_{m-1}, z_m)) \tag{2.16}$$

and from $(A_2)$ we get

$$\begin{aligned}
0 &< \min\left\{q(z_m, z_{m+1}), q(z_{m+1}, z_{m+2})\right\} = S(z_{m-1}, z_m)\\
&< H(z_{m-1}, z_m) = q(z_m, z_{m+1}),
\end{aligned} \tag{2.17}$$

which implies $q(z_{m+1}, z_{m+2}) < q(z_m, z_{m+1})$, for $m \in \mathbb{N}$. Therefore, the sequence $\left\{q(z_m, z_{m+1})\right\}$ is strictly decreasing and bounded, that is convergent to a point $d_0 \leq 0$. If we suppose that $d_0 > 0$, letting $m \to \infty$ in the first part of (2.20), we have

$$\begin{aligned}
\psi(d_0) &= \lim_{m\to\infty} \psi(q(z_m, z_{m+1})) = \lim_{m\to\infty} \psi(S(z_{m-1}, z_m)) \\
&\leq \limsup_{m\to\infty} \varphi(H(z_{m-1}, z_m)) = \limsup_{m\to\infty} \varphi(q(z_m, z_{m+1})) \\
&\leq \limsup_{t\to d_0} \varphi(t).
\end{aligned}$$

This contradicts the assumption $(A_3)$. Therefore, $0 = d_0 = \lim_{m\to\infty} q(z_m, z_{m+1})$ and since the space $(\mathtt{X}, q)$ is $\delta$-symmetric, there exists $\delta > 0$ such that

$$0 \leq q(z_{m+1}, z_m) \leq \delta \cdot q(z_m, z_{m+1}).$$

Taking the limit as $m \to \infty$ in the above inequality and using the Sandwich Lemma, it follows

$$\lim_{m\to\infty} q(z_m, z_{m+1}) = 0 = \lim_{m\to\infty} q(z_{m+1}, z_m). \tag{2.18}$$

Assume now, that the sequence $\{z_m\}$ is not Cauchy. Thus, by Lemma 3, there exists $\mathit{e} > 0$ such that (1.8) hold, where $\{z_{m_l}\}$, $\left\{z_{p_l}\right\}$ are two subsequences of $\{z_m\}$. Now, since

$$q(z_{m_l+1}, z_{p_l+1}) - q(z_{m_l+1}, z_{m_l+2}) - q(z_{p_l+2}, z_{p_l+1}) \leq q(z_{m_l+2}, z_{p_l+2}),$$

we have

$$\begin{aligned}
s_l &= \min\left\{q(z_{m_l+1}, z_{p_l+1}), q(z_{m_l+1}, z_{p_l+1}) - q(z_{m_l+1}, z_{m_l+2}) - q(z_{p_l+2}, z_{p_l+1})\right\} \\
&\leq \min\left\{q(z_{m_l+1}, z_{p_l+1}), q(z_{m_l+2}, z_{p_l+2})\right\} \\
&= \min\left\{q(\mathtt{f}z_{m_l}, \mathtt{f}z_{p_l}), q(\mathtt{f}^2 z_{m_l}, \mathtt{f}^2 z_{p_l})\right\} = S(z_{m_l}, z_{p_l}).
\end{aligned}$$

On the other hand, for $z = z_{m_l}$ and $v = z_{p_l}$ we have

$$\begin{aligned}
& q(z_{p_l}, z_{m_l+1}) - q(z_{m_l+1}, z_{m_l+2}) \leq q(z_{p_l}, z_{m_l+2}) \leq H(z_{m_l}, z_{p_l}) \\
&\quad = \max\left\{q(z_{m_l}, z_{m_l+1}), q(z_{p_l}, z_{p_l+1}), q(z_{p_l}, z_{m_l+2})\right\} \\
&\quad \leq \max\left\{q(z_{m_l}, z_{m_l+1}), q(z_{p_l}, z_{p_l+1}), q(z_{p_l}, z_{m_l+1}) + q(z_{m_l+1}, z_{m_l+2})\right\},
\end{aligned}$$

$$\begin{aligned}
G(z_{m_l}, z_{p_l}) &= \max\left\{\begin{array}{c} q(z_{m_l}, \mathtt{f}z_{m_l}) \cdot q(\mathtt{f}z_{p_l}, \mathtt{f}^2 z_{p_l}) \cdot q(\mathtt{f}z_{p_l}, \mathtt{f}^2 z_{m_l}), \\ q(z_{p_l}, \mathtt{f}z_{p_l}) \cdot q(\mathtt{f}z_{m_l}, \mathtt{f}^2 z_{m_l}) \cdot q(z_{p_l}, \mathtt{f}z_{m_l}) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} q(z_{m_l}, z_{m_l+1}) \cdot q(z_{p_l+1}, z_{p_l+2}) \cdot q(z_{p_l+1}, z_{m_l+2}), \\ q(z_{p_l}, z_{p_l+1}) \cdot q(z_{m_l+1}, z_{m_l+2}) \cdot q(z_{p_l}, z_{m_l+1}) \end{array}\right\}
\end{aligned}$$

and taking into account (1.8) and (2.18), we get

$$H(z_{m_l}, z_{p_l}) \to \mathit{e},\ s_l \to \mathit{e} \text{ and } P(z_{m_l}, z_{p_l}) \to 0, \text{ as } l \to \infty.$$

Moreover, applying (2.14) for $z = z_{m_l}$, $v = z_{p_l}$ and taking Lemma 1 into account, together with $(A_2)$, we have

$$\psi(s_l) \le \psi(S(z_{m_l}, z_{p_l})) \le \alpha(z_{m_l}, z_{p_l})\psi(S(z_{m_l}, z_{p_l})) \le \varphi(H(z_{m_l}, z_{p_l})) + C \cdot P(z_{m_l}, z_{p_l}).$$

Taking the limit as $l \to \infty$ in the above inequality, we get

$$\psi(e) = \liminf_{l\to\infty} \psi(s_l) \le \liminf_{l\to\infty} \psi(S(z_{m_l}, z_{p_l}) \le \limsup_{l\to\infty} \varphi(H(z_{m_l}, z_{p_l})) \le \limsup_{t\to e} \varphi(t)$$

which contradicts $(A_3)$. Consequently, $e = 0$, so that the sequence $\{z_m\}$ is right Cauchy and from Lemma 2 it is a Cauchy sequence on a complete quasi-metric space. So, there exists $z^* \in \mathtt{X}$ such that $\lim_{m\to\infty} z_m = z^*$.

Further, assuming that (3) holds, it follows that

$$\lim_{m\to\infty} q(z_m, \mathtt{f}z) = \lim_{m\to\infty} q(\mathtt{f}z_{m-1}, \mathtt{f}z) = 0$$

and then $z$ is a fixed point of $\mathtt{f}$.

Assuming that (4) holds, we derive $\lim_{m\to\infty} q(z_m, \mathtt{f}^2 z) = \lim_{m\to\infty} q(\mathtt{f}^2 z_{m-2}, \mathtt{f}^2 z) = 0$ and due to the uniqueness of the limit for a convergent sequence, we conclude that $z^*$ is a fixed point of $\mathtt{f}^2$. Using the method of Reductio ad absurdum, we shall show that $z^*$ is a fixed point of $\mathtt{f}$. Presuming that $\mathtt{f}z^* \neq z^*$, from (2.14) and keeping in mind the second part of (4),

$$\psi(S(\mathtt{f}z^*, z^*)) \le \alpha(\mathtt{f}z^*, z^*)\psi(S(\mathtt{f}z^*, z^*)) \le \varphi(H(\mathtt{f}z^*, z^*)) + C \cdot P(\mathtt{f}z^*, z^*), \tag{2.19}$$

where

$$\begin{aligned}
S(\mathtt{f}z^*, z^*) &= \min\left\{q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}z^*), q(\mathtt{f}^2(\mathtt{f}z^*), \mathtt{f}^2 z^*)\right\} = \min\left\{q(\mathtt{f}^2 z^*, \mathtt{f}z^*), q(\mathtt{f}(\mathtt{f}^2 z^*), \mathtt{f}^2 z^*)\right\} \\
&= \min\left\{q(z^*, \mathtt{f}z^*), q(\mathtt{f}z^*, z^*)\right\}, \\
H(\mathtt{f}z^*, z^*) &= \max\left\{q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}^2(\mathtt{f}z^*)), q(z^*, \mathtt{f}z^*), q(z^*, \mathtt{f}^2(\mathtt{f}z^*)\right\} \\
&= \max\left\{q(\mathtt{f}^2 z^*, \mathtt{f}(\mathtt{f}^2 z^*)), q(z^*, \mathtt{f}z^*), q(z^*, \mathtt{f}(\mathtt{f}^2 z^*)\right\} \\
&= \max\left\{q(z^*, \mathtt{f}z^*), q(z^*, \mathtt{f}z^*), q(z^*, \mathtt{f}z^*)\right\} \\
&= q(z^*, \mathtt{f}z^*), \\
G(\mathtt{f}z^*, z^*) &= \max\left\{\begin{array}{c} q(\mathtt{f}z^*, \mathtt{f}(\mathtt{f}z^*)) \cdot q(\mathtt{f}z^*, \mathtt{f}^2 z^*) \cdot q(\mathtt{f}z^*, \mathtt{f}^2(\mathtt{f}z^*), \\ q(z^*, \mathtt{f}z^*) \cdot q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}^2(\mathtt{f}z^*)) \cdot q(z^*, \mathtt{f}(\mathtt{f}z^*)) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} q(\mathtt{f}z^*, z^*) \cdot q(\mathtt{f}z^*, z^*) \cdot q(\mathtt{f}z^*, \mathtt{f}z^*), \\ q(z^*, \mathtt{f}z^*) \cdot q(z^*, \mathtt{f}z^*) \cdot q(z^*, z^*) \end{array}\right\} \\
&= 0.
\end{aligned}$$

Thereupon, by (2.19) together with (2) we have

$$\min\left\{q(\mathtt{f}^2 z^*, \mathtt{f}z^*), q(\mathtt{f}(\mathtt{f}^2 z^*), \mathtt{f}^2 z^*)\right\} < q(z^*, \mathtt{f}z^*). \tag{2.20}$$

In the same way, we have

$$\psi(S(z^*, \mathtt{f}z^*)) \leq \alpha(z^*, \mathtt{f}z^*)\psi(S(z^*, \mathtt{f}z^*)) \leq \varphi(H(z^*, \mathtt{f}z^*)) + C \cdot P(z^*, \mathtt{f}z^*), \tag{2.21}$$

where

$$\begin{aligned}
S(z^*, \mathtt{f}z^*) &= \min\left\{q(\mathtt{f}z^*, \mathtt{f}(\mathtt{f}z^*)), q(\mathtt{f}^2 z^*, \mathtt{f}^2(\mathtt{f}z^*))\right\} = \min\left\{q(\mathtt{f}z^*, z^*), q(z^*, \mathtt{f}z^*)\right\}, \\
H(z^*, \mathtt{f}z^*) &= \max\left\{q(\mathtt{f}z^*, \mathtt{f}^2(\mathtt{f}z^*)), q(\mathtt{f}z^*, \mathtt{f}(\mathtt{f}z^*)), q(\mathtt{f}z^*, \mathtt{f}^2 z^*)\right\} \\
&= \max\left\{q(\mathtt{f}z^*, \mathtt{f}z^*), q(\mathtt{f}z^*, z^*), q(\mathtt{f}z^*, z^*)\right\} \\
&= q(\mathtt{f}z^*, z^*), \\
G(z^*, \mathtt{f}z^*) &= \max\left\{\begin{array}{c} q(z^*, \mathtt{f}z^*) \cdot q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}^2(\mathtt{f}z^*)) \cdot q(\mathtt{f}(\mathtt{f}z^*), \mathtt{f}^2(z^*), \\ q(\mathtt{f}z^*, \mathtt{f}(\mathtt{f}z^*)) \cdot q(\mathtt{f}(z^*), \mathtt{f}^2(z^*)) \cdot q(\mathtt{f}z^*, \mathtt{f}z^*) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} q(z^*, \mathtt{f}z^*) \cdot q(z^*, \mathtt{f}z^*) \cdot q(z^*, z^*, \\ q(\mathtt{f}z^*, z^*) \cdot q(\mathtt{f}(z^*), z^*) \cdot q(\mathtt{f}z^*, \mathtt{f}z^*) \end{array}\right\} \\
&= 0,
\end{aligned}$$

and so

$$\min\left\{q(\mathtt{f}z^*, z^*), q(z^*, \mathtt{f}z^*)\right\} = S(z^*, \mathtt{f}z^*) < H(z^*, \mathtt{f}z^*) = q(\mathtt{f}z^*, z^*). \tag{2.22}$$

In conclusion, from (2.20) and (2.22) we have

$$\min\left\{q(\mathtt{f}z^*, z^*), q(z^*, \mathtt{f}z^*)\right\} < \min\left\{q(\mathtt{f}z^*, z^*), q(z^*, \mathtt{f}z^*)\right\},$$

which is a contradiction. Therefore, $\mathtt{f}z^* = z^*$, which shows that $z^*$ is a fixed point of $\mathtt{f}$.
We claim that this is the only fixed point of $\mathtt{f}$. Indeed, supposing that there exists another point $v^* \in Fix_{\mathtt{f}}(\mathtt{X})$, such that $v^* \neq z^*$, by using the condition $I$,

$$\psi(S(v^*, z^*)) \leq \alpha(v^*, z^*)\psi(S(v^*, z^*)) \leq \varphi(H(v^*, z^*)) + C \cdot P(v^*, z^*), \tag{2.23}$$

with

$$\begin{aligned}
S(v^*, z^*) &= \min\left\{q(\mathtt{f}^2 v^*, \mathtt{f}^2 z^*), q(\mathtt{f}v^*, \mathtt{f}z^*)\right\} = q(v^*, z^*); \\
H(v^*, z^*) &= \max\left\{q(\mathtt{f}v^*, \mathtt{f}^2 v^*), q(z^*, \mathtt{f}z^*), q(z^*, \mathtt{f}^2 v^*)\right\} = q(z^*, v^*); \\
G(v^*, z^*) &= \max\left\{\begin{array}{c} q(v^*, \mathtt{f}v^*) \cdot q(\mathtt{f}z^*, \mathtt{f}^2 z^*) \cdot q(\mathtt{f}z^*, \mathtt{f}^2 v^*), \\ q(z^*, \mathtt{f}z^*) \cdot q(\mathtt{f}v^*, \mathtt{f}^2 v^*) \cdot q(z^*, \mathtt{f}v^*) \end{array}\right\} \\
&= 0.
\end{aligned}$$

Thus, from (2.23) together with the hypothesis (1), we get

$$q(v^*, z^*) = S(v^*, z^*) < H(v^*, z^*) = q(z^*, v^*). \tag{2.24}$$

Similarly, since

$$\psi(S(z^*, v^*)) \leq \alpha(z^*, v^*)\psi(S(z^*, v^*)) \leq \varphi(H(z^*, v^*)) + C \cdot P(z^*, v^*),$$

we have

$$q(z^*, v^*) = S(z^*, v^*) < H(z^*, v^*) = q(v^*, z^*). \tag{2.25}$$

Combining (2.24) with (2.25) we have

$$q(v^*, z^*) < q(z^*, v^*) < q(v^*, z^*),$$

which is a contradiction. Consequently, the fixed point of $\mathtt{f}$ is unique. □

**Example 3.** *Let* $X = [-1, +\infty)$ *and the* 2*-symmetric quasi-metric* $q : X \times X \to [0, +\infty)$*,* $q(z, v) = \begin{cases} 2(z-v), & \text{for } z \geq v \\ v - z, & \text{otherwise} \end{cases}$*. Let the mapping* $\mathtt{f} : X \to X$*, with*

$$\mathtt{f}z = \begin{cases} z^2, & \text{for } z \in [-1, 0) \\ \frac{\sqrt{6}}{4}, & \text{for } z \in [0, 1] \\ \frac{\sqrt{z^2+1}}{2}, & \text{for } z > 1 \end{cases}.$$

*First of all, we can easily see that* $\mathtt{f}$ *is not continuous, instead*

$$\mathtt{f}^2 z = \begin{cases} \frac{\sqrt{6}}{4}, & \text{for } z \in [-1, 1] \\ \frac{\sqrt{z^2+5}}{4}, & \text{for } z > 1 \end{cases}$$

*it is a continuous mapping. Let the function* $\alpha : X \times X \to [0, +\infty)$*,*

$$\alpha(z, v) = \begin{cases} \ln(z^2 + v^2 + 1), & \text{for } z, v \in [-1, 1) \\ 3, & \text{for } z = 2, v = 1 \\ 0, & \text{otherwise} \end{cases}$$

*and* $\psi, \varphi \in P$*,* $\psi(t) = e^t$*,* $\varphi(t) = t + 1$*, for every* $t > 0$*. With these choices, the assumptions (1), (2) and (4) from Theorem 9 are satisfied and we have to check the that the inequality (2.8) holds for every* $z, v \in X$*, with* $\min\left\{q(\mathtt{f}z, \mathtt{f}v), q(\mathtt{f}^2 z, \mathtt{f}^2 v)\right\} > 0$*. Thus, due to the definition of the function* $\alpha$*, the only interesting case is when* $z = 2$ *and* $v = 1$*. We have:*

$$\begin{aligned} S(2,1) &= \min\left\{q(\mathtt{f}2, \mathtt{f}1), q(\mathtt{f}^2 2, \mathtt{f}^2 1)\right\} = \min\left\{q(\tfrac{\sqrt{5}}{2}, \tfrac{\sqrt{6}}{4}\right\} = 0.275 \\ H(2,1) &= \max\left\{q(\mathtt{f}2, \mathtt{f}^2), q(1, \mathtt{f}1), q(1, \mathtt{f}^2 2)\right\} = q(1, \mathtt{f}1) = q(1, \tfrac{\sqrt{6}}{4}) = 0.775 \\ G(2,1) &= \max\left\{q(2, \mathtt{f}2) \cdot q(\mathtt{f}1, \mathtt{f}^2 1) \cdot q(\mathtt{f}1, \mathtt{f}^2 2), q(1, \mathtt{f}1) \cdot q(\mathtt{f}2, \mathtt{f}^2 2) \cdot q(1, \mathtt{f}2)\right\} \\ &= \max\{0, 0.067\} = 0.067 \end{aligned}$$

*and*

$$\alpha(2,1)\psi(S(2,1)) = 3 \cdot e^{S(2,1)} = 3.950 < 5.125 = (0.775 + 1) + 50 \cdot 0.067 = \varphi(H(2,1)) + C \cdot G(2,1).$$

*Therefore, the assumptions of Theorem 10 hold, so that the mapping* $\mathtt{f}$ *admits a unique fixed point, that is* $z = \frac{\sqrt{6}}{4}$*.*

**Corollary 5.** *On a complete $\delta$-symmetric quasi metric space $(\mathtt{X}, q)$, let $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ be a mapping such that*

$$\psi(S(z, v)) \leq \varphi(H(z, v)) + C \cdot G(z, v) \tag{2.26}$$

*for all $z, v \in \mathtt{X}$ with $q(\mathtt{f}z, \mathtt{f}v) > 0$ and $q(\mathtt{f}^2 z, \mathtt{f}^2 v) > 0$, where*

$$\begin{aligned} S(z, v) &= \min\left\{q(\mathtt{f}z, \mathtt{f}v), q(\mathtt{f}^2 z, \mathtt{f}^2 v)\right\}; \\ H(z, v) &= \max\left\{q(\mathtt{f}z, \mathtt{f}^2 z), q(v, \mathtt{f}v), q(v, \mathtt{f}^2 z)\right\}; \\ G(z, v) &= \max\left\{q(z, \mathtt{f}z) \cdot q(\mathtt{f}v, \mathtt{f}^2 v) \cdot q(\mathtt{f}v, \mathtt{f}^2 z),\ q(v, \mathtt{f}v) \cdot q(\mathtt{f}z, \mathtt{f}^2 z) \cdot q(v, \mathtt{f}z)\right\}, \end{aligned}$$

*with $\psi, \varphi \in P$. Suppose that:*

*(1) the functions $\psi, \varphi$ satisfy ($A_1$), ($A_2$) and ($A_3$);*
*(3) either $\mathtt{f}$ is continuous, or*
*(4) $\mathtt{f}^2$ is continuous.*

*Then $\mathtt{f}$ has a unique fixed point.*

*Proof.* Put $\alpha(z, v) = 1$ in Theorem 10. □

**Corollary 6.** *On a complete $\delta$-symmetric quasi metric space $(\mathtt{X}, q)$, let $\mathtt{f}\colon \mathtt{X} \to \mathtt{X}$ be a mapping such that*

$$\psi(S(z, v)) \leq \varphi(H(z, v)), \tag{2.27}$$

*for all $z, v \in \mathtt{X}$ with $q(\mathtt{f}z, \mathtt{f}v) > 0$ and $q(\mathtt{f}^2 z, \mathtt{f}^2 v) > 0$, where*

$$\begin{aligned} S(z, v) &= \min\left\{q(\mathtt{f}z, \mathtt{f}v), q(\mathtt{f}^2 z, \mathtt{f}^2 v)\right\}; \\ H(z, v) &= \max\left\{q(\mathtt{f}z, \mathtt{f}^2 z), q(v, \mathtt{f}v), q(v, \mathtt{f}^2 z)\right\}, \end{aligned}$$

*with $\psi, \varphi \in P$. Suppose that:*

*(1) the functions $\psi, \varphi$ satisfy ($A_1$), ($A_2$) and ($A_3$);*
*(3) either $\mathtt{f}$ is continuous, or*
*(4) $\mathtt{f}^2$ is continuous.*

*Then $\mathtt{f}$ has a unique fixed point.*

*Proof.* Put $C = 0$ in Corollary 5. □

## 3. Conclusions

In this paper, we presented very general results on the uniqueness and existence of a fixed point of Górnicki-Proinov type contraction in the context of quasi-metric spaces. As one can easily see, several consequences of our obtained results can be stated by simply choosing different expression for functions $\psi$ and $\varphi$. One of the essential details in this work is the inclusion of admissible function as an auxiliary function in the contraction inequality. This auxiliary function which may seem insignificant plays a vital role in combining several existing results that appear far apart in a shrinkage inequality. More precisely, some results in the framework of cyclic contractions and the results in the framework of partially ordered sets as well as the standard results can be formulated via admissible mappings. This leads to unify the distinct trends in research of the metric fixed point. In particular, if the admissible mapping is equal to 1, then the unification formula yields the standard results (more specific details can be found, e.g. [17]). Consequently, the usefulness of the work presented can be seen more easily. For further generalizations, this approach can be used in different abstract spaces.

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