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Research article

Derivative-free method based on DFP updating formula for solving convex constrained nonlinear monotone equations and application

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Abstract: In this paper, a new derivative-free approach for solving nonlinear monotone system of equations with convex constraints is proposed. The search direction of the proposed algorithm is derived based on the modified scaled Davidon–Fletcher–Powell (DFP) updating formula in such a way that it is sufficiently descent. Under some mild assumptions, the search direction is shown to be bounded. Subsequently, the convergence result of the proposed method is established. The performance of the proposed algorithm on a collection of some test problems as well as signal recovery problems is demonstrated in comparison with some existing algorithms with similar characteristics. The results of the numerical experiments confirm the efficiency as well as the robustness of the proposed algorithm by comparing it with some existing methods in the literature.

Keywords: derivative-free method; nonlinear monotone equations; hyperplane projection technique; DFP method; quasi–Newton method; signal reconstruction problem
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1. Introduction

Let $U \subseteq \mathbb{R}^n$ be a nonempty closed convex subset where $\mathbb{R}^n$ is an $n-$dimensional Euclidean space equipped with the Euclidean norm $\|\cdot\|$. Recall that a mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ is said to satisfy monotonicity and Lipschitz continuity conditions if for all $u, v \in \mathbb{R}^n$, the inequalities

$$0 \leq (u - v)^T (P(u) - P(v)), \tag{1.1}$$

$$\|P(u) - P(v)\| \leq L\|u - v\|, \quad L > 0, \tag{1.2}$$

hold, respectively. Consider the following nonlinear system of equations

$$P(u) = 0, \tag{1.3}$$

where $P : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a monotone mapping. The problem (1.3) is of great importance to many researchers as it appears in a wide range of applications such as robotic motion control problems, signal recovery problems, image deblurring problems, varational inequality and so on (see [1–4]).

Quasi–Newton methods are among the very popular iterative methods for solving problem (1.3) as well as the general unconstrained optimization problem, $\min\{p(u) \in \mathbb{R} : u \in \mathbb{R}^n\}$. Note that $P(u)$ can be viewed as $\nabla p(u)$, that is, the gradient of p at u. The quasi-Newton methods define their search directions as $q_k := -W_k P(u_k)$, where the matrix W_k is either an approximation of the Hessian inverse of p at u_k or the approximation of Jacobian of P at u_k. The matrix W_k is usually updated in every iteration via a suitable updating formula such as DFP (Davidon–Fletcher–Powell), BFGS (Broyden–Fletcher–Goldfarb–Shanno), SR1 (Symmetric rank-one), diagonal updating formulas and so on [5–8]. However, one of the shortcomings associated with these methods, i.e, DFP, BFGS and SR1, is the need to compute and store matrices in every iteration. Consequently, a number of researchers have modified some quasi–Newton search directions in such a way that they mimic the behaviour of the popular conjugate gradient methods. Conjugate gradient methods are known for their abilities to handle large-scale problems efficiently due to the fact that they neither require to form nor store matrices throughout the entire iteration processes.

For instance, based on the BFGS updating formula and Polak–Ribière–Polyak conjugate gradient parameter [9, 10], Zhang et al. [11] developed a three-term conjugate gradient method for solving general unconstrained optimization problems. With the aid of the Armijo-type line search, their method was shown to be globally convergent. Also, Andrei [12] proposed another simple three-term conjugate gradient algorithm for unconstrained optimization problem based on the BFGS updating formula. The proposed search direction in [12] satisfies both descent and conjugacy conditions independent of any line search strategy used. In a similar approach, Awwal et al. [13] presented an interesting three-term derivative-free method for solving problem (1.3) with convex constraints. Their algorithm was developed by incorporating the modified Perry conjugate gradient parameter [14] into the modified BFGS updating formula. Their algorithm was also applied to recover some disturbed signals. Very recently, based on a modified scaled SR1 updating formula and the projection technique of Solodov and Svaiter [15], Abubakar et al. [16] proposed another derivative-free method for solving problem (1.3) with convex constraints. The convergence analysis of their method was established based on the monotonicity and Lipschitz continuity assumption on the underlying mapping.

The DFP updating formula is among the earliest quasi–Newton updating formulas that are positive definite. However, it has not received the required attention in recent time, especially with regards to the system of nonlinear equations. In this paper, based on the modified scaled DFP updating formula, we propose a new three-term iterative method for solving problem (1.3) with convex constraints. The new method mimics the conjugate gradient method and also satisfies the sufficient descent property. The followings are some of the contributions of this paper:

- This paper presents a new iterative algorithm for solving convex constrained monotone nonlinear system of equations.
- It also explores the efficiency as well as the numerical performance of the DFP-based iterative method.
- The new search direction is derived in such a way that it satisfies the sufficient descent condition.
- The global convergence of the proposed method is established under mild conditions.
- The proposed method is successfully applied on signal recovery problems.

The remaining part of this paper is organized as follows. The proposed method and its convergence analysis are described in Section 2. The numerical performance of the proposed method is discussed in Section 3 and the application is described in Section 4. Finally, some concluding remarks are given in Section 5.

2. Proposed DFDFP method and its global convergence results

Recall that quasi–Newton methods update their sequence of points $\{u_k\}$ via the recursive formula

$$u_{k+1} := u_k + t_k q_k, \quad k = 0, 1, 2, \ldots,$$

where the step length $t_k > 0$ is usually obtained via a suitable line search technique. The search direction q_k is defined as

$$q_k := -W_k P(u_k), \quad k = 0, 1, 2, \ldots, \tag{2.1}$$

where for $k = 0$, $W_0 = I$ and I is an identity matrix. For $k \geq 1$, W_k is a given matrix updated via a suitable formula such that the following secant equation is satisfied

$$W_k \gamma_{k-1} = s_{k-1},$$

with $\gamma_{k-1} := P(u_k) - P(u_{k-1})$ and $s_{k-1} := u_k - u_{k-1}$.

Consider the DFP updating formula, which was originally proposed by Davidon and later developed by Fletcher and Powell [5], given as

$$W_k := W_{k-1} + \frac{s_{k-1}s_{k-1}^T}{s_{k-1}^T\gamma_{k-1}} - \frac{W_{k-1}\gamma_{k-1}\gamma_{k-1}^T W_{k-1}}{\gamma_{k-1}^T W_{k-1}\gamma_{k-1}}. \tag{2.2}$$

Motivated by the approach in [11, 13], we consider the following DFP–like formula given as

$$W_k := \mu_k W_{k-1} + \frac{s_{k-1}s_{k-1}^T}{s_{k-1}^T\gamma_{k-1}} - \frac{W_{k-1}\gamma_{k-1}\gamma_{k-1}^T W_{k-1}}{\gamma_{k-1}^T W_{k-1}\gamma_{k-1}}, \tag{2.3}$$

where μ_k is a positive parameter to be determined. Note that if $\mu_k = 1$ for all k, then (2.3) reduces to the classical DFP updating formula (2.2). The aim here is to derive a DFP–like search direction, q_k, where the parameter μ_k will be determined in such a way that the search direction q_k in (2.1) satisfies the following sufficient descent condition

$$P(u_k)^T q_k \leq -\widetilde{\alpha}\|P(u_k)\|^2, \quad \widetilde{\alpha} > 0. \tag{2.4}$$

Now, following the approach in [17], setting $W_{k-1} \equiv \tau_k I$ in (2.3), where I is an identity matrix, we have

$$W_k = \mu_k \tau_k I + \frac{s_{k-1}s_{k-1}^T}{s_{k-1}^T \gamma_{k-1}} - \tau_k \frac{\gamma_{k-1}\gamma_{k-1}^T}{\gamma_{k-1}^T \gamma_{k-1}}, \tag{2.5}$$

where $\tau_k \in [z_1, z_2], \;\; z_1, z_2 > 0$, is a scaling parameter. Substituting (2.5) into (2.1) gives

$$q_k = -\mu_k \tau_k P(u_k) - \frac{s_{k-1}^T P(u_k)}{s_{k-1}^T \gamma_{k-1}} s_{k-1} + \tau_k \frac{\gamma_{k-1}^T P(u_k)}{\|\gamma_{k-1}\|^2} \gamma_{k-1}. \tag{2.6}$$

This means that the search direction (2.6) mimics the behaviour of the classical three-term conjugate gradient method. For the search direction q_k in (2.6) to be well-defined, we must ensure that $s_{k-1}^T \gamma_{k-1} \neq 0$ and $\|\gamma_{k-1}\| \neq 0$.

Now, redefining γ_{k-1} as $\widehat{\gamma}_{k-1} := P(u_k) - P(u_{k-1}) + c s_{k-1}$, $c > 0$, $s_{k-1} := u_k - u_{k-1}$ and assume that the mapping P is monotone and Lipschitz continuous, we have

$$\begin{aligned}
\|\widehat{\gamma}_{k-1}\|^2 &= (P(u_k) - P(u_{k-1}) + c(u_k - u_{k-1}))^T (P(u_k) - P(u_{k-1}) + c(u_k - u_{k-1})) \\
&= \|P(u_k) - P(u_{k-1})\|^2 + 2c(P(u_k) - P(u_{k-1}))^T (u_k - u_{k-1}) + c^2\|u_k - u_{k-1}\|^2 \\
&\geq \|P(u_k) - P(u_{k-1})\|^2 + c^2\|u_k - u_{k-1}\|^2 \\
&\geq c^2\|s_{k-1}\|^2 > 0, \quad u_k \neq u_{k-1},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\widehat{\gamma}_{k-1}^T s_{k-1} &= (P(u_k) - P(u_{k-1}) + c(u_k - u_{k-1}))^T (u_k - u_{k-1}) \\
&= (P(u_k) - P(u_{k-1}))^T (u_k - u_{k-1}) + c\|u_k - u_{k-1}\|^2 \\
&\geq c\|s_{k-1}\|^2 > 0, \quad u_k \neq u_{k-1}.
\end{aligned} \tag{2.8}$$

Therefore, it is justifiable to redefine the search direction q_k as follows

$$q_k := -\mu_k \tau_k P(u_k) - \frac{s_{k-1}^T P(u_k)}{s_{k-1}^T \widehat{\gamma}_{k-1}} s_{k-1} + \tau_k \frac{\widehat{\gamma}_{k-1}^T P(u_k)}{\|\widehat{\gamma}_{k-1}\|^2} \widehat{\gamma}_{k-1}, \tag{2.9}$$

where $\tau_k := \frac{\|s_{k-1}\|^2}{\widehat{\gamma}_{k-1}^T s_{k-1}}$, for $k \geq 1$.

Remark 2.1. *The choice of the parameter τ_k is motivated by the work of Andrei [17], i.e, Eq* (32) *of [17], to be specific. The inequality* (2.8) *guarantees the well–definedness of τ_k for all k. Furthermore, we show that $\tau_k \in [z_1, z_2], \;\; z_1, z_2 > 0$ holds as follows*

$$\begin{aligned}
\widehat{\gamma}_{k-1}^T s_{k-1} &= (P(u_k) - P(u_{k-1}) + c(u_k - u_{k-1}))^T (u_k - u_{k-1}) \\
&= (P(u_k) - P(u_{k-1}))^T (u_k - u_{k-1}) + c\|u_k - u_{k-1}\|^2 \\
&\leq \|P(u_k) - P(u_{k-1})\|\|u_k - u_{k-1}\| + c\|u_k - u_{k-1}\|^2 \\
&\leq L\|u_k - u_{k-1}\|^2 + c\|u_k - u_{k-1}\|^2
\end{aligned}$$

$$= (c + L)\|s_{k-1}\|^2, \quad c, L > 0. \tag{2.10}$$

This together with (2.8) *gives*

$$\frac{1}{c} \geq \tau_k \geq \frac{1}{c + L}, \quad c > 0, \tag{2.11}$$

which means that taking $z_1 = 1/(c + L)$ *and* $z_2 = 1/c$ *yields the desired result.*

Next, we determine the value of the spectral parameter μ_k such that (2.4) is fulfilled for q_k given in (2.9). Let $k \geq 1$ and $P(u_k) \neq 0$. Then, we have from the definition of q_k in (2.9)

$$\begin{aligned} P(u_k)^T q_k &= -\mu_k \tau_k \|P(u_k)\|^2 - \frac{(s_{k-1}^T P(u_k))^2}{s_{k-1}^T \widehat{\gamma}_{k-1}} + \tau_k \frac{(\widehat{\gamma}_{k-1}^T P(u_k))^2}{\|\widehat{\gamma}_{k-1}\|^2} \\ &\leq -\mu_k \tau_k \|P(u_k)\|^2 + \tau_k \frac{(\widehat{\gamma}_{k-1}^T P(u_k))^2}{\|\widehat{\gamma}_{k-1}\|^2} \\ &\leq -\mu_k \tau_k \|P(u_k)\|^2 + \tau_k \frac{\|\widehat{\gamma}_{k-1}\|^2 \|P(u_k)\|^2}{\|\widehat{\gamma}_{k-1}\|^2} \\ &\leq -\tau_k (\mu_k - 1)\|P(u_k)\|^2. \end{aligned} \tag{2.12}$$

Since $\tau_k \geq z_1$, this means that by setting

$$\mu_k \geq \alpha + 1, \quad \alpha > 0,$$

we have

$$P(u_k)^T q_k \leq -z_1 \alpha \|P(u_k)\|^2.$$

Now, taking $\widetilde{\alpha} = z_1 \alpha$ means that the sufficient descent condition (2.4) holds. Therefore, without loss of generality, it makes sense to set the value of μ_k as

$$\mu_k := \alpha + 1, \quad \text{for all } k \geq 1, \quad \text{where } \alpha > 0. \tag{2.13}$$

Next, we state the following definition

Definition 2.2. *Let* u *be an arbitrary point in* $\mathbb{R}^n$, *then its projection onto the feasible set* U *is given as* $\Gamma_U(u) := \arg\min\{\|u - v\| : v \in U\}$.

The projection operator $\Gamma_U(u)$ satisfies the following inequality

$$\|\Gamma_U(u) - v\| \leq \|u - v\|, \quad \text{for all } v \in U. \tag{2.14}$$

We now state the steps of the proposed algorithm for solving problem (1.3).

Algorithm 1: Derivative–Free DFP Method (DFDFP).

Input : Given an initial point $u_0 \in U \subset \mathbb{R}^n$ and parameters: $\rho \in (0,1)$, $c, \alpha, \sigma, \kappa > 0$, $0 < \ell < 2$, stopping tolerance $Tol \geq 0$. Set $k = 0$.

Step 1: Compute $P(u_k)$. If $\|P(u_k)\| \leq Tol$, stop. Else, go to Step 2.

Step 2: Compute the search direction $q_0 := -P(u_0)$ and for $k \geq 1$,

$$q_k := -(\alpha+1)\tau_k P(u_k) - \frac{s_{k-1}^T P(u_k)}{s_{k-1}^T \widehat{\gamma}_{k-1}} s_{k-1} + \tau_k \frac{\widehat{\gamma}_{k-1}^T P(u_k)}{\|\widehat{\gamma}_{k-1}\|^2} \widehat{\gamma}_{k-1}, \tag{2.15}$$

where

$$\tau_k := \frac{\|s_{k-1}\|^2}{\widehat{\gamma}_{k-1}^T s_{k-1}}. \tag{2.16}$$

Step 3: Set $v_k := u_k + t_k q_k$, where $t_k := \kappa\rho^i$ such that i is the smallest non-negative integer for which

$$-P(u_k + \kappa\rho^i q_k)^T q_k \geq \sigma\kappa\rho^i \|P(u_k + \kappa\rho^i q_k)\|^{1/h} \|q_k\|^2, \quad h \geq 1. \tag{2.17}$$

Step 4: If $\|P(v_k)\| = 0$, stop. Otherwise, the next iterate is computed as

$$u_{k+1} := \Gamma_U \left[u_k - \ell \frac{P(v_k)^T (u_k - v_k)}{\|P(v_k)\|^2} P(v_k) \right], \quad P(v_k) \neq 0. \tag{2.18}$$

Step 5: Set $k := k+1$ and go to step 1.

Remark 2.3. *The line search used in **Step 3** of Algorithm 1 (DFDFP) is a modification of the popular Solodov and Svaiter [15] and was adopted from [18, 19]. It was shown in [18] that there exists a step length $t_k > 0$ that satisfies the line search* (2.17)*.*

Lemma 2.4. *Let P be a mapping that satisfies the monotonicity condition. If $\widehat{u} \in U$ is such that $P(\widehat{u}) = 0$ and the sequence $\{u_k\}$ is generated by Algorithm 1 (DFDFP), then, the $\lim_{k\to\infty} \|u_k - \widehat{u}\|$ exists.*

Proof. Let $\widehat{u} \in U$ such that $P(\widehat{u}) = 0$, then $P(\widehat{u})^T(u_k - \widehat{u}) = 0$. Then it holds that

$$\begin{aligned} P(v_k)^T(u_k - \widehat{u}) &= P(v_k)^T(u_k - v_k + v_k - \widehat{u}) \\ &= P(v_k)^T(u_k - v_k) + P(v_k)^T(v_k - \widehat{u}) \\ &\geq P(v_k)^T(u_k - v_k) + P(\widehat{u})^T(v_k - \widehat{u}) \\ &= P(v_k)^T(u_k - v_k), \end{aligned} \tag{2.19}$$

where the inequality follows from the monotonicity of P, i.e, $P(v_k)^T(v_k - \widehat{u}) \geq P(\widehat{u})^T(v_k - \widehat{u}) = 0$. Next, since $0 < \ell < 2$, then by (2.14) and (2.18), we have

$$\begin{aligned} \|u_{k+1} - \widehat{u}\|^2 &= \left\| \Gamma_U \left(u_k - \ell \frac{P(v_k)^T(u_k - v_k)}{\|P(v_k)\|^2} P(v_k) \right) - \widehat{u} \right\|^2 \\ &\leq \left\| u_k - \ell \frac{P(v_k)^T(u_k - v_k)}{\|P(v_k)\|^2} P(v_k) - \widehat{u} \right\|^2 \\ &= \left\| (u_k - \widehat{u}) - \ell \frac{P(v_k)^T(u_k - v_k)}{\|P(v_k)\|^2} P(v_k) \right\|^2 \\ &= \|u_k - \widehat{u}\|^2 - 2\ell \frac{P(v_k)^T(u_k - v_k)}{\|P(v_k)\|^2} P(v_k)^T(u_k - \widehat{u}) + \ell^2 \frac{[P(v_k)^T(u_k - v_k)]^2}{\|P(v_k)\|^2} \end{aligned}$$

$$
\begin{aligned}
&\leq \|u_k - \widehat{u}\|^2 - 2\ell \frac{P(v_k)^T(u_k - v_k)}{\|P(v_k)\|^2} P(v_k)^T(u_k - v_k) + \ell^2 \frac{[P(v_k)^T(u_k - v_k)]^2}{\|P(v_k)\|^2} \\
&= \|u_k - \widehat{u}\|^2 - \ell(2-\ell) \frac{[P(v_k)^T(u_k - v_k)]^2}{\|P(v_k)\|^2} \qquad (2.20)\\
&\leq \|u_k - \widehat{u}\|^2. \qquad (2.21)
\end{aligned}
$$

The inequality (2.21) means that the sequence $\{\|u_k - \widehat{u}\|\}$ is a decreasing sequence and therefore the proof is complete. □

Lemma 2.5. *Let P be a mapping that satisfies the Lipschitz continuity. If the sequence of the search direction $\{q_k\}$ is generated by the Algorithm 1 (DFDFP) then*

$$\lim_{k\to\infty} t_k \|q_k\| = 0. \tag{2.22}$$

Proof. From Lemma 2.4, we can have some positive constant, say a_1, such that

$$\|u_k - \widehat{u}\| \leq a_1, \quad a_1 > 0. \tag{2.23}$$

Also,

$$
\begin{aligned}
\|\widehat{\gamma}_{k-1}\| &= \|P(u_k) - P(u_{k-1}) + c(u_k - u_{k-1})\| \\
&\leq \|P(u_k) - P(u_{k-1})\| + c\|u_k - u_{k-1}\| \\
&\leq L\|u_k - u_{k-1}\| + c\|u_k - u_{k-1}\| \\
&\leq (L+c)\|s_{k-1}\|, \quad c, L > 0. \qquad (2.24)
\end{aligned}
$$

Next, from (2.15), we have

$$
\begin{aligned}
\|q_k\| &= (\alpha+1)\tau_k\|P(u_k)\| + \left|\frac{s_{k-1}^T P(u_k)}{s_{k-1}^T \widehat{\gamma}_{k-1}}\right| \|s_{k-1}\| + \tau_k \left|\frac{\widehat{\gamma}_{k-1}^T P(u_k)}{\|\widehat{\gamma}_{k-1}\|^2}\right| \|\widehat{\gamma}_{k-1}\| \\
&\leq (\alpha+1)\tau_k\|P(u_k)\| + \frac{\|s_{k-1}\|\|P(u_k)\|}{c\|s_{k-1}\|^2}\|s_{k-1}\| + \tau_k \frac{\|\widehat{\gamma}_{k-1}\|\|P(u_k)\|}{c^2\|s_{k-1}\|^2}\|\widehat{\gamma}_{k-1}\| \\
&\leq (\alpha+1)\tau_k\|P(u_k)\| + \frac{\|s_{k-1}\|\|P(u_k)\|}{c\|s_{k-1}\|^2}\|s_{k-1}\| + \tau_k \frac{(L+c)^2\|s_{k-1}\|^2\|P(u_k)\|}{c^2\|s_{k-1}\|^2} \\
&\leq (\alpha+1)\frac{1}{c}\|P(u_k)\| + \frac{\|P(u_k)\|}{c} + \frac{1}{c}\frac{(L+c)^2\|P(u_k)\|}{c^2} \\
&= \left[(\alpha+1)\frac{1}{c} + \frac{1}{c} + \frac{(L+c)^2}{c^3}\right]\|P(u_k)\|, \quad \alpha, c, L > 0. \qquad (2.25)
\end{aligned}
$$

The first inequality follows from Cauchy–Schwarz inequality, the (2.7) and (2.8). The second and the last inequalities follow from (2.24) and (2.11), respectively.

Since P is Lipschitz continuous, then it holds that

$$\|P(u_k)\| = \|P(u_k) - P(\widehat{u})\| \leq L\|u_k - \widehat{u}\| \leq La_1, \tag{2.26}$$

where the last inequality follows from (2.23). By combining (2.25) and (2.26), we have

$$\|q_k\| \leq a_2, \tag{2.27}$$

where $a_2 := \left[(\alpha+1)\frac{1}{c}+\frac{1}{c}+\frac{(L+c)^2}{c^3}\right]La_1$.

Moreover, by the boundedness of $\{u_k\}$ and (2.27) as well as the definition of v_k in **Step 3** of Algorithm 1 (DFDFP), it holds that $\{v_k\}$ is also bounded. Therefore, since P is Lipschitz continuous, we have

$$\|P(v_k)\| \leq a_3, \quad a_3 > 0. \tag{2.28}$$

From the line search (2.17) and the inequality (2.20), it follows that

$$\begin{aligned}\sigma^2 t_k^4 \|P(v_k)\|^{2/h}\|q_k\|^4 &\leq t_k^2[P(v_k)^T q_k]^2 \\ &\leq \frac{\|P(v_k)\|^2}{\ell(2-\ell)}\left(\|u_k-\widehat{u}\|^2-\|u_{k+1}-\widehat{u}\|^2\right).\end{aligned} \tag{2.29}$$

By multiplying both sides of the inequality (2.29) by $\|P(v_k)\|^{-2/h}$ and using (2.28) gives

$$\begin{aligned}\sigma^2 t_k^4\|q_k\|^4 &\leq \frac{\|P(v_k)\|^{2-2/h}}{\ell(2-\ell)}\left(\|u_k-\widehat{u}\|^2-\|u_{k+1}-\widehat{u}\|^2\right) \\ &\leq \frac{a_3^{2-2/h}}{\ell(2-\ell)}\left(\|u_k-\widehat{u}\|^2-\|u_{k+1}-\widehat{u}\|^2\right).\end{aligned} \tag{2.30}$$

Using the fact that the $\lim_{k\to\infty}\|u_k-\widehat{u}\|$ exists, (see, Lemma 2.4) and the fact that $0<\ell<2$, we have

$$\sigma^2 \lim_{k\to\infty} t_k^4\|q_k\|^4 = 0. \tag{2.31}$$

Hence,

$$\lim_{k\to\infty} t_k\|q_k\| = 0. \tag{2.32}$$

□

Theorem 2.6. *Suppose that the mapping P is Lipschitz continuous, then the sequence $\{u_k\}$ generated by Algorithm 1 converges to $\widehat{u}\in U$ such that $P(\widehat{u})=0$.*

Proof. We begin with the claim that

$$\liminf_{k\to\infty}\|P(u_k)\| = 0. \tag{2.33}$$

We provide the proof of this claim by contradiction. If (2.33) does not hold, then

$$\|P(u_k)\| \geq b, \quad b>0, \quad k=1,2,3,\ldots. \tag{2.34}$$

For $t_k \neq \kappa$, implies that $t_k' := \rho^{-1}t_k$ will not satisfy the line search (2.17). Therefore by defining $v_k' := u_k + t_k' q_k$, then

$$\sigma t_k'\|P(v_k')\|^{1/h}\|q_k\|^2 + P(v_k')^T q_k > 0. \tag{2.35}$$

This means applying Cauchy-Schwartz inequality on the inequality (2.4) yields

$$\begin{aligned}\widetilde{\alpha}\|P(u_k)\|^2 &\leq -P(u_k)^T q_k \\ &< P(v_k')^T q_k - P(u_k)^T q_k + \sigma t_k'\|P(v_k')\|^{1/h}\|q_k\|^2 \\ &= (P(v_k')-P(u_k))^T q_k + \sigma t_k'\|P(v_k')\|^{1/h}\|q_k\|^2 \\ &\leq L t_k'\|q_k\|^2 + \sigma t_k'\|P(v_k')\|^{1/h}\|q_k\|^2\end{aligned}$$

$$= \rho^{-1} t_k \left(L + \sigma \|P(v_k')\|^{1/h}\right) \|q_k\|^2$$
$$\leq \rho^{-1} t_k \left(L + \sigma a_3^{1/h}\right) a_2^2. \tag{2.36}$$

The last inequality follows from (2.27) and (2.28).

On the other hand, using the inequalities (2.4) and (2.34) it is easy to see that

$$\|q_k\| \geq \widetilde{\alpha} b. \tag{2.37}$$

Now, rearranging (2.36) and multiplying both sides by $\|q_k\|$ yields

$$t_k \|q_k\| > \frac{\rho \widetilde{\alpha} \|P(u_k)\|^2}{\left(L + \sigma a_3^{1/h}\right) a_2^2} \|q_k\| \geq \frac{\rho \widetilde{\alpha} b^2}{\left(L + \sigma a_3^{1/h}\right) a_2^2} \widetilde{\alpha} b. \tag{2.38}$$

Taking the limits on both sides of (2.38) shows a contradiction with (2.22). Hence, (2.33) must hold.

Now, the continuity of P implies that there is some accumulation point $\overline{u}$ of $\{u_k\}$ such that $P(\widehat{u}) = 0$. Since $\lim_{k\to\infty} \|u_k - \widehat{u}\|$ exists and the fact that $\widehat{u}$ is an accumulation point of $\{u_k\}$, it then follows that $\{u_k\}$ converges to $\widehat{u}$. □

3. Numerical experiments on collection of some test problems

Attention now turns to numerical experiments where the performance of the proposed Algorithm 1 (DFDFP) will be demonstrated in comparison with some existing methods. The following two existing methods are considered for the numerical comparison:

(i) A paper titled "Solving nonlinear monotone operator equations via modified SR1 update" by Abubakar et al. [16]. This method shall be referred to as MSR1.

(ii) A paper titled "Modified Hager-Zhang conjugate gradient methods via singular value analysis for solving monotone nonlinear equations with convex constraint" by Sabi'u et al. [20]. This method will be denoted by MHZ1.

Each of the three algorithms is applied to solve eleven (11) test problems (see Supplementary) by varying their dimensions and starting points (SP). Each test problem is solved using five dimensions: $1,000$, $5,000$, $10,000$, $50,000$, $100,000$ as well as six starting points, namely, $u_1 = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10})^T$, $u_2 = (\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n})^T$, $u_3 = (2, 2, 2, \ldots, 2)^T$, $u_4 = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})^T$, $u_5 = (1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}, \ldots, 0)^T$ and $u_6 = \text{rand}(0, 1)$. Therefore, a total of 330 problems were solved in the course of this experiment. The parameters used in implementing MSR1 and MHZ1 are taken from [16] and [20], while Algorithm 1 (DFDFP) is implemented using the parameters: $h = 5, \rho = 0.5$, $\alpha = 0.1$, $c = 0.01$, $\sigma = 0.01, \kappa = 1$, and $\ell = 1.99$. All the three algorithms are coded in MATLAB R2019b and run on a PC with intel Core(TM) i5-8250u processor with 4 GB of RAM and CPU 1.60 GHZ. Each algorithm is terminated whenever $\|P(u_k)\| \leq 10^{-6}$. If this condition is not satisfied after 1000 iterations, failure is declared which is denoted as "–". The performance of each algorithm is assessed by #iter (number of iterations), #fval (number of function evaluations) and #time (CPU time) recorded. In addition, for each problem, $\|P(\widehat{u})\|$ (denoted by Norm) is reported to indicate that an algorithm successfully obtained an approximate solution of a particular problem.

Table 1. Numerical results obtained by each solver for Test Problem S1.

Problem S1		DFDFP				MHZ1				MSR1			
Dim	**SP**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**
1000	u_1	4	9	0.48256	0	45	92	0.24873	5.43E-07	23	48	0.19064	8.31E-07
	u_2	3	7	0.029884	9.93E-16	78	157	0.26647	9.47E-07	60	121	0.18319	9.21E-07
	u_3	5	11	0.006002	0	31	64	0.12431	3.62E-07	36	74	0.10687	5.4E-07
	u_4	5	11	0.006962	0	81	164	0.53498	3.04E-07	71	144	0.19159	9.1E-07
	u_5	1	3	0.029565	0	98	197	0.59459	7.09E-07	83	168	0.39915	2.91E-07
	u_6	1	3	0.060751	0	71	143	0.39624	9.94E-07	82	166	0.26122	1.25E-07
5000	u_1	3	8	0.089361	7.74E-07	18	38	0.45183	1.08E-07	25	52	0.41923	4.33E-07
	u_2	3	7	0.012835	9.93E-16	78	157	1.5444	9.47E-07	60	121	0.85383	9.21E-07
	u_3	4	9	0.027107	4.48E-07	56	114	1.1319	9.32E-07	53	108	0.76293	8.21E-07
	u_4	5	11	0.014045	0	114	230	2.3908	2.62E-07	73	148	1.1159	5.01E-07
	u_5	1	3	0.009025	0	106	214	2.4709	5.87E-07	113	228	2.7638	5.97E-07
	u_6	1	3	0.015162	0	102	206	2.7577	2.22E-07	77	156	1.0927	9.17E-07
10000	u_1	3	7	0.030846	8.56E-07	74	150	3.2215	1.87E-07	28	58	0.72663	5.87E-07
	u_2	3	7	0.025247	9.93E-16	78	157	2.8553	9.47E-07	60	121	1.7425	9.21E-07
	u_3	4	9	0.023782	2.24E-07	56	114	2.0416	5.03E-07	38	78	1.3803	3.76E-07
	u_4	5	11	0.025989	0	101	204	4.0838	2.88E-07	64	130	1.8753	9.86E-07
	u_5	1	3	0.009144	0	96	194	5.0807	3.95E-07	100	201	5.679	5.27E-07
	u_6	1	3	0.008168	0	87	175	4.42	6.27E-07	118	238	3.4093	6.11E-07
50000	u_1	3	7	0.081842	1.71E-07	36	74	4.8787	6.5E-07	34	70	4.4627	9.08E-07
	u_2	3	7	0.050614	9.93E-16	78	157	12.1027	9.47E-07	60	121	5.6989	9.21E-07
	u_3	4	9	0.10314	4.48E-08	97	196	14.3989	8.18E-07	43	87	5.4439	7.7E-07
	u_4	5	11	0.14313	0	78	158	11.1623	5.41E-07	74	150	9.5104	8.63E-07
	u_5	1	3	0.028587	0	114	230	18.5395	7.24E-08	118	238	17.9265	6.8E-07
	u_6	1	3	0.039125	0	140	281	28.5848	9.89E-07	83	168	10.8405	2.93E-07
100000	u_1	3	7	0.13701	8.56E-08	91	184	29.0825	1.86E-07	31	64	8.3281	4.12E-07
	u_2	3	7	0.14199	9.93E-16	78	157	24.4619	9.47E-07	60	121	11.8481	9.21E-07
	u_3	4	9	0.22311	2.24E-08	44	90	10.9536	8.4E-07	50	101	11.5163	3.97E-07
	u_4	5	11	0.39097	0	62	126	17.9314	2.75E-07	66	134	14.2214	1.47E-07
	u_5	1	3	0.093319	0	91	184	32.0206	9.11E-07	117	236	41.765	2.09E-07
	u_6	1	3	0.08865	0	110	222	39.9285	9.77E-07	97	196	20.7738	5.43E-07

Table 2. Numerical results obtained by each solver for Test Problem S2.

Problem S2		DFDFP				MHZ1				MSR1			
Dim	**SP**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**
1000	u_1	2	5	0.031363	0	5	12	0.01137	3.53E-07	3	8	0.009218	3.54E-07
	u_2	1	3	0.001852	0	34	70	0.12219	9.71E-07	27	56	0.084745	5.7E-07
	u_3	1	3	0.004707	0	6	14	0.007615	3.75E-07	6	14	0.006487	6.9E-07
	u_4	1	3	0.005185	0	57	116	0.23817	6.11E-07	46	94	0.13593	7.25E-07
	u_5	1	3	0.004714	0	119	239	0.82436	9.89E-07	61	124	0.18722	5.72E-07
	u_6	1	3	0.00428	0	128	258	0.72424	1.86E-07	54	110	0.22666	7.22E-07
5000	u_1	2	5	0.006958	0	5	12	0.03552	7.89E-07	3	8	0.017727	7.92E-07
	u_2	1	3	0.004865	0	34	70	0.64707	9.71E-07	27	56	0.41949	5.7E-07
	u_3	1	3	0.003816	0	6	14	0.029316	8.39E-07	7	16	0.032238	1.48E-07
	u_4	1	3	0.005479	0	47	96	0.90982	4.95E-07	28	58	0.46709	3.11E-07
	u_5	1	3	0.003977	0	99	200	1.9332	2.59E-07	62	126	1.0511	3.92E-07
	u_6	1	3	0.004061	0	83	168	1.6346	9.36E-07	40	82	0.75104	8.65E-07
10000	u_1	2	5	0.007792	0	6	14	0.066752	7.21E-08	4	10	0.040052	4.61E-11
	u_2	1	3	0.006022	0	34	70	1.0824	9.71E-07	27	56	0.92457	5.7E-07
	u_3	1	3	0.016321	0	7	16	0.1281	7.68E-08	7	16	0.071434	2.1E-07
	u_4	1	3	0.009001	0	90	182	4.123	3.33E-07	38	78	1.2366	4.13E-07
	u_5	1	3	0.010013	0	120	242	5.015	4.98E-07	62	126	2.0712	8.66E-07
	u_6	1	3	0.006726	0	94	190	3.4621	3.35E-07	58	118	2.0485	9.06E-07
50000	u_1	2	5	0.034855	0	6	14	0.18487	1.61E-07	4	10	0.17294	1.03E-10
	u_2	1	3	0.027138	0	34	70	4.1625	9.71E-07	27	56	3.9809	5.7E-07
	u_3	1	3	0.025318	0	7	16	0.31912	1.72E-07	7	16	0.29224	4.69E-07
	u_4	1	3	0.021162	0	87	176	14.3238	4.91E-07	37	76	5.7219	8.79E-07
	u_5	1	3	0.018548	0	128	258	18.4653	2.53E-07	42	86	4.9108	8.95E-07
	u_6	1	3	0.023746	0	115	231	17.8026	9.32E-07	75	152	8.462	4.41E-07
100000	u_1	2	5	0.081906	0	6	14	0.40636	2.28E-07	4	10	0.23069	1.46E-10
	u_2	1	3	0.034853	0	34	70	7.8705	9.71E-07	27	56	6.2263	5.7E-07
	u_3	1	3	0.056144	0	7	16	0.4276	2.43E-07	7	16	0.41684	6.64E-07
	u_4	1	3	0.034081	0	61	124	17.5348	1.39E-07	26	54	5.8445	6.33E-07
	u_5	1	3	0.058074	0	118	237	30.0785	8.23E-07	77	156	15.381	6.16E-07
	u_6	1	3	0.035151	0	139	280	39.1573	2.33E-07	56	114	11.6696	6.55E-07

Table 3. Numerical results obtained by each solver for Test Problem S3.

Problem S3		DFDFP				MHZ1				MSR1			
Dim	**SP**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**
1000	u_1	2	5	0.013923	0	1	3	0.018354	0	1	3	0.001765	0
	u_2	3	7	0.002502	0	57	115	0.14979	9.04E-07	54	109	0.14005	2.25E-07
	u_3	3	7	0.002142	0	1	3	0.002034	0	1	3	0.002192	0
	u_4	4	9	0.001898	2.25E-10	1	3	0.003452	0	1	3	0.001137	0
	u_5	3	7	0.003571	0	1	3	0.030843	0	1	3	0.002175	0
	u_6	3	7	0.002668	0	1	3	0.001811	0	1	3	0.00168	0
5000	u_1	2	5	0.005219	0	1	3	0.002313	0	1	3	0.002724	0
	u_2	3	7	0.007174	0	57	115	0.78153	9.04E-07	54	109	0.72639	2.25E-07
	u_3	3	7	0.016398	0	1	3	0.007756	0	1	3	0.007543	0
	u_4	4	9	0.010441	7.38E-10	1	3	0.004123	0	1	3	0.004676	0
	u_5	3	7	0.009432	0	1	3	0.003395	0	1	3	0.003856	0
	u_6	3	7	0.007108	0	1	3	0.004636	0	1	3	0.003651	0
10000	u_1	2	5	0.006813	0	1	3	0.003936	0	1	3	0.004051	0
	u_2	3	7	0.011481	0	57	115	1.6313	9.04E-07	54	109	1.4313	2.25E-07
	u_3	3	7	0.019902	0	1	3	0.01059	0	1	3	0.014422	0
	u_4	4	9	0.012332	8.25E-10	1	3	0.007651	0	1	3	0.007375	0
	u_5	3	7	0.013883	0	1	3	0.006927	0	1	3	0.006071	0
	u_6	3	7	0.02216	0	1	3	0.009314	0	1	3	0.009548	0
50000	u_1	2	5	0.046352	0	1	3	0.023268	0	1	3	0.018181	0
	u_2	3	7	0.046062	0	57	115	6.1302	9.04E-07	54	109	4.9523	2.25E-07
	u_3	3	7	0.086003	0	1	3	0.048501	0	1	3	0.040369	0
	u_4	4	9	0.053699	8.98E-10	1	3	0.026984	0	1	3	0.023978	0
	u_5	3	7	0.087192	0	1	3	0.021776	0	54	110	5.5365	4.77E-07
	u_6	3	7	0.039373	0	1	3	0.019716	0	70	142	5.7925	2.53E-07
100000	u_1	2	5	0.064413	0	1	3	0.025935	0	1	3	0.019695	0
	u_2	3	7	0.071834	0	57	115	10.2612	9.04E-07	54	109	8.604	2.25E-07
	u_3	3	7	0.10396	0	1	3	0.073225	0	1	3	0.070045	0
	u_4	4	9	0.12226	9.07E-10	1	3	0.036078	0	1	3	0.038975	0
	u_5	3	7	0.074867	0	1	3	0.033273	0	63	128	10.9085	8.98E-07
	u_6	3	7	0.084393	0	1	3	0.033253	0	51	104	8.7145	6.66E-07

Table 4. Numerical results obtained by each solver for Test Problem S4.

Problem S4		DFDFP				MHZ1				MSR1			
Dim	**SP**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**	**#iter**	**#fval**	**#time**	**Norm**
1000	u_1	6	14	0.047845	7.68E-07	59	120	0.66268	7.34E-07	68	138	0.33676	4.72E-07
	u_2	6	14	0.015071	7.97E-07	47	96	0.46936	1.32E-07	43	88	0.21762	8.87E-07
	u_3	6	13	0.005775	4.6E-07	73	148	0.7509	5.77E-07	28	58	0.20173	5.55E-07
	u_4	6	14	0.007564	7.95E-07	51	104	0.50307	9.35E-07	42	86	0.30882	4.34E-07
	u_5	6	14	0.017716	6.56E-07	49	100	0.46112	9.24E-07	59	120	0.39154	5.39E-07
	u_6	6	14	0.008052	6.54E-07	77	156	0.85605	9.72E-07	48	98	0.31011	6.73E-07
5000	u_1	7	15	0.045193	3.17E-07	47	96	1.915	7.08E-07	29	60	0.98394	5.53E-08
	u_2	7	15	0.052406	3.29E-07	45	92	1.8371	5.68E-07	50	102	1.4487	8.31E-07
	u_3	6	14	0.028274	4.74E-07	39	80	1.6605	2.05E-07	40	82	1.2022	3.32E-07
	u_4	7	15	0.052491	3.29E-07	64	130	2.6093	2.83E-07	38	78	1.0623	1.08E-07
	u_5	7	15	0.0407	2.71E-07	69	140	2.3476	8.44E-07	54	110	1.4306	3.78E-07
	u_6	7	15	0.042739	2.72E-07	59	120	1.6883	5.67E-07	44	90	1.1093	9.53E-07
10000	u_1	7	15	0.051501	4.49E-07	55	112	4.2115	8.87E-07	31	64	1.6594	8.36E-07
	u_2	7	15	0.066162	4.66E-07	50	102	3.8228	5.36E-07	20	42	1.2047	9.03E-07
	u_3	6	14	0.075946	6.7E-07	43	88	3.0267	9.34E-07	36	74	1.8432	8.29E-07
	u_4	7	15	0.046467	4.66E-07	48	98	3.1033	2.36E-07	32	66	1.7019	7.03E-07
	u_5	7	15	0.049351	3.84E-07	54	110	3.99	3.7E-07	36	74	1.9736	7.06E-07
	u_6	7	15	0.08133	3.82E-07	75	152	5.411	3.67E-07	54	110	2.8501	7.3E-07
50000	u_1	7	16	0.31424	4.57E-07	41	84	9.4062	1.51E-07	34	70	5.5924	6.59E-07
	u_2	7	16	0.23878	4.75E-07	34	70	10.863	6.07E-07	24	50	3.9577	3.96E-07
	u_3	7	15	0.20746	2.75E-07	23	48	4.7979	9.75E-07	48	98	7.4587	5.43E-07
	u_4	7	16	0.24162	4.75E-07	48	98	11.5802	3.5E-07	36	74	6.5536	2.48E-07
	u_5	7	15	0.17315	8.58E-07	41	84	9.0374	3.82E-07	35	72	5.9663	9.81E-07
	u_6	7	15	0.19535	8.57E-07	53	108	12.9513	5.33E-07	48	98	8.1367	4.6E-07
100000	u_1	7	16	0.71869	6.47E-07	41	84	18.8007	1.54E-07	30	62	10.1722	8.25E-07
	u_2	7	16	0.43497	6.71E-07	36	74	16.3771	4.29E-07	23	48	7.8144	7.35E-08
	u_3	7	15	0.43489	3.89E-07	37	76	15.9985	1.15E-07	22	46	7.6662	8.93E-07
	u_4	7	16	0.6723	6.71E-07	42	86	22.8704	4.95E-07	30	62	9.7449	9.27E-07
	u_5	7	16	0.54883	5.52E-07	40	82	19.1641	9.28E-07	44	90	14.7894	9.81E-07
	u_6	7	16	0.47699	5.53E-07	54	110	25.5087	8.47E-07	40	82	12.9854	5.04E-07

Table 5. Numerical results obtained by each solver for Test Problem S5.

Problem S5		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	8	17	0.094398	8.15E-07	6	14	0.13168	1.79E-07	5	12	0.008132	5.72E-07
	u_2	8	18	0.010861	4.55E-07	64	130	0.30373	3.54E-07	64	130	0.17894	5.52E-07
	u_3	9	20	0.007579	7.27E-07	7	16	0.014495	8.14E-08	5	12	0.007984	9.58E-08
	u_4	8	17	0.00494	9.96E-07	76	154	0.60339	7.67E-07	77	156	0.21348	9.05E-07
	u_5	9	19	0.014671	2.82E-07	86	174	0.58752	6.23E-07	73	148	0.27376	9.27E-07
	u_6	9	19	0.006652	2.65E-07	85	172	0.5166	7.27E-07	87	176	0.36262	6.17E-07
5000	u_1	8	18	0.022152	8.25E-07	6	14	0.058821	4E-07	6	14	0.046998	8.45E-08
	u_2	9	19	0.032439	2E-07	109	220	3.0441	7.74E-07	72	146	1.3163	4.7E-07
	u_3	10	21	0.02748	3.18E-07	7	16	0.075856	1.82E-07	5	12	0.061931	2.14E-07
	u_4	9	19	0.036464	1.99E-07	61	124	1.9923	9.91E-07	85	172	1.6644	9.76E-07
	u_5	9	19	0.041459	6.33E-07	94	190	2.8496	5.33E-07	69	140	1.3369	6.15E-07
	u_6	9	19	0.024546	6.39E-07	105	212	2.8959	4.75E-07	76	154	1.5943	9.07E-07
10000	u_1	9	19	0.04262	2.29E-07	6	14	0.10989	5.66E-07	6	14	0.095868	1.2E-07
	u_2	9	19	0.055835	2.82E-07	88	178	5.3888	7.78E-07	68	138	2.5773	9.94E-07
	u_3	10	21	0.050379	4.5E-07	7	16	0.13778	2.57E-07	5	12	0.090774	3.03E-07
	u_4	9	19	0.042381	2.82E-07	101	204	5.806	8.47E-07	78	158	3.3728	5.75E-07
	u_5	9	19	0.044335	8.95E-07	69	140	3.6074	9.28E-07	63	128	2.4046	6.93E-07
	u_6	9	19	0.043301	8.86E-07	101	204	5.4388	8.3E-07	100	202	3.4124	8.33E-07
50000	u_1	9	19	0.14953	5.11E-07	7	16	0.36223	8.78E-08	6	14	0.25144	2.67E-07
	u_2	9	19	0.14781	6.32E-07	75	152	13.5343	9.75E-07	74	150	8.7364	6.08E-07
	u_3	10	22	0.30637	4.56E-07	7	16	0.34524	5.76E-07	5	12	0.25944	6.77E-07
	u_4	9	19	0.15798	6.31E-07	99	200	16.9497	7.5E-07	78	158	9.3924	6.87E-07
	u_5	9	20	0.16491	9.07E-07	93	188	15.311	7.72E-07	114	230	11.1489	6.07E-07
	u_6	9	20	0.30637	9.22E-07	80	162	14.7116	6.82E-07	76	154	7.7256	5.7E-07
100000	u_1	9	19	0.26344	7.23E-07	7	16	0.72455	1.24E-07	6	14	0.52487	3.78E-07
	u_2	9	19	0.31537	8.93E-07	72	146	21.4883	1.05E-07	323	648	72.4707	9.11E-07
	u_3	10	22	0.43901	6.45E-07	7	16	0.77916	8.14E-07	5	12	0.51208	9.58E-07
	u_4	9	19	0.34277	8.93E-07	122	246	38.6026	4.32E-07	69	140	14.2496	5.49E-07
	u_5	10	21	0.34534	2.51E-07	130	262	40.0755	6.22E-07	91	184	17.2271	4.82E-07
	u_6	10	21	0.41642	2.5E-07	110	222	35.9738	2.59E-07	67	136	14.1726	8.74E-07

Table 6. Numerical results obtained by each solver for Test Problem S6.

Problem S6		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	1	3	0.023502	0	5	12	0.40516	4.02E-08	6	14	0.014702	1.19E-07
	u_2	1	3	0.001891	2.22E-16	-	-	-	-	34	70	0.4993	1.77E-07
	u_3	1	3	0.001371	0	1	3	0.017517	0	1	3	0.002591	0
	u_4	2	5	0.01432	0	79	160	0.77187	3.18E-07	83	168	0.44596	7.01E-08
	u_5	4	9	0.006505	0	104	210	0.86134	5.73E-07	271	544	29.0816	2.89E-07
	u_6	4	9	0.005021	0	129	259	1.0778	9.5E-07	146	294	9.1366	8.21E-08
5000	u_1	1	3	0.009991	0	5	12	0.12233	8.99E-08	6	14	0.059605	2.66E-07
	u_2	1	3	0.011139	2.22E-16	-	-	-	-	34	70	2.8883	1.77E-07
	u_3	1	3	0.008842	0	1	3	0.018605	0	1	3	0.013731	0
	u_4	2	5	0.01217	0	67	136	2.1526	5.91E-07	75	152	6.2191	3.44E-07
	u_5	4	9	0.023737	0	119	240	3.3519	8.6E-07	442	886	79.8121	8E-07
	u_6	4	9	0.012639	0	90	182	2.5788	3.93E-07	88	177	8.3229	9.06E-07
10000	u_1	1	3	0.011103	0	5	12	0.11466	1.27E-07	6	14	0.14169	3.76E-07
	u_2	1	3	0.019253	2.22E-16	-	-	-	-	34	70	5.6562	1.77E-07
	u_3	1	3	0.010877	0	1	3	0.019788	0	1	3	0.026974	0
	u_4	2	5	0.02582	0	67	136	4.1844	1.4E-07	71	144	2.9069	6.21E-07
	u_5	4	9	0.035336	0	102	206	7.0593	3.42E-07	118	237	53.3688	1.34E-11
	u_6	4	9	0.018758	0	89	180	7.9395	3.21E-07	47	96	4.7949	1.29E-07
50000	u_1	1	3	0.031187	0	5	12	1.0266	2.84E-07	6	14	0.47099	8.41E-07
	u_2	1	3	0.031017	2.22E-16	-	-	-	-	34	70	17.3786	1.77E-07
	u_3	1	3	0.052319	0	1	3	0.1182	0	1	3	0.12863	0
	u_4	2	5	0.065554	0	94	190	23.1127	1.77E-07	83	168	10.313	3.32E-07
	u_5	4	9	0.068887	0	112	226	25.8815	3.33E-07	46	93	12.573	0
	u_6	4	9	0.097557	0	106	214	24.5656	8.13E-08	77	156	21.8676	2.46E-07
100000	u_1	1	3	0.067286	0	5	12	1.8042	4.02E-07	7	16	1.032	8.89E-08
	u_2	1	3	0.10303	2.22E-16	-	-	-	-	34	70	30.4462	1.77E-07
	u_3	1	3	0.089003	0	1	3	0.26787	0	8	18	1.2931	3.17E-07
	u_4	2	5	0.15703	0	138	278	108.1459	2.53E-07	73	148	15.3179	8.42E-07
	u_5	4	9	0.19473	0	105	212	69.0383	1.63E-07	66	134	34.8849	1.2E-07
	u_6	4	9	0.15302	0	109	220	47.2701	4.22E-07	161	324	300.1426	1.42E-07

Table 7. Numerical results obtained by each solver for Test Problem S7.

Problem S7		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	12	26	0.028013	2.47E-07	320	642	2.8867	8.99E-07	90	182	1.6379	8.67E-07
	u_2	19	40	0.017335	5.38E-07	67	136	0.28463	8.85E-07	70	142	0.73466	5.11E-07
	u_3	20	42	0.026734	8.06E-07	84	170	0.46202	7.01E-07	97	196	1.43	4.17E-07
	u_4	22	46	0.032051	4.96E-07	94	190	0.63812	9.75E-07	75	152	1.3266	9.83E-07
	u_5	29	60	0.052402	7.86E-07	173	348	1.7682	5.49E-07	96	194	1.4102	4.7E-07
	u_6	34	69	0.038085	4.64E-07	62	126	0.29384	8.78E-07	96	194	1.7679	7.67E-07
5000	u_1	11	24	0.037096	4.12E-07	148	298	5.9632	9.71E-07	76	154	3.7005	6.69E-07
	u_2	19	40	0.21184	5.38E-07	67	136	1.4155	8.85E-07	70	142	2.6085	5.11E-07
	u_3	27	56	0.10082	4.63E-07	83	168	1.7434	8.16E-07	99	200	4.7166	1.56E-07
	u_4	24	50	0.09978	8.31E-07	66	134	1.1454	7.04E-07	84	170	3.0923	4.31E-07
	u_5	30	62	0.08732	7.74E-07	93	187	2.1076	9.68E-07	209	420	23.9859	5.93E-07
	u_6	32	66	0.081649	6.75E-07	66	134	1.1019	7.4E-07	96	194	4.0319	9.46E-07
10000	u_1	11	24	0.075453	6.7E-07	84	170	3.7199	7.05E-07	77	155	6.5857	9.05E-07
	u_2	19	40	0.11085	5.38E-07	67	136	3.8672	8.85E-07	70	142	7.1976	5.11E-07
	u_3	16	34	0.13465	6.58E-07	85	172	4.6587	5.63E-07	93	188	12.1457	3.36E-07
	u_4	25	52	0.13806	7.36E-07	66	134	3.078	9.38E-07	84	170	7.8196	7.54E-07
	u_5	30	62	0.18235	5.34E-07	107	215	7.2431	9.57E-07	100	201	12.6079	8.82E-07
	u_6	30	62	0.16288	8.74E-07	62	126	2.9034	8.67E-07	101	204	9.6561	5.98E-07
50000	u_1	12	26	0.2802	7.86E-07	79	160	11.4706	8.08E-07	85	172	31.8656	4.19E-07
	u_2	19	40	0.43481	5.38E-07	67	136	12.0333	8.85E-07	70	142	24.2639	5.11E-07
	u_3	30	62	0.76631	7.19E-07	116	234	25.2459	9.23E-07	96	194	42.3263	7.54E-07
	u_4	25	52	0.6839	5.19E-07	79	160	14.9076	9.96E-07	77	156	25.1985	8.39E-07
	u_5	33	68	0.83441	6.93E-07	138	277	39.1921	8.72E-07	388	778	399.656	5.7E-07
	u_6	32	66	0.71504	7.78E-07	69	140	10.1216	6.64E-07	88	178	21.8279	9.13E-07
100000	u_1	15	32	0.82211	9.24E-07	81	164	25.2432	7.8E-07	83	168	40.5274	8.23E-07
	u_2	19	40	1.0184	5.38E-07	67	136	26.3368	8.85E-07	70	142	35.8217	5.11E-07
	u_3	29	59	1.925	7.21E-07	127	256	64.7206	5.5E-07	119	240	93.0809	7.68E-07
	u_4	26	54	1.2243	6.32E-07	71	144	21.8526	8.09E-07	82	166	60.4051	8.31E-07
	u_5	31	63	1.3684	8.74E-07	200	402	114.8491	8.06E-07	573	1148	1180.484	8.59E-07
	u_6	34	70	1.78	3.97E-07	72	145	23.2595	9.62E-07	93	188	49.199	7.02E-07

Table 8. Numerical results obtained by each solver for Test Problem S8.

Problem S8		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	32	65	0.067909	8.7E-07	76	154	0.84765	6.19E-07	82	166	0.30626	8.04E-07
	u_2	33	68	0.017936	7.34E-07	80	162	0.41871	9.73E-07	71	144	0.20436	8.09E-07
	u_3	34	70	0.013939	9.12E-07	83	168	0.45197	4.36E-07	69	140	0.21799	6.44E-07
	u_4	37	75	0.016929	7.09E-07	81	164	0.4279	8.7E-07	79	160	0.26593	9.77E-07
	u_5	24	49	0.018076	7.76E-07	66	134	0.34267	7.57E-07	91	184	0.29765	9.01E-07
	u_6	49	99	0.024855	8.34E-07	100	202	0.77655	7.06E-07	85	172	0.23956	9.8E-07
5000	u_1	37	76	0.087844	7.15E-07	69	140	1.7293	9.94E-07	59	120	0.73175	6.48E-07
	u_2	25	52	0.050198	8.37E-07	86	174	2.1444	4.14E-07	97	196	1.7822	8.49E-07
	u_3	34	70	0.066447	6.34E-07	107	216	2.1178	8.43E-07	92	186	1.3894	7.04E-07
	u_4	31	64	0.05287	9.95E-07	90	182	1.9472	5.84E-07	75	152	1.7382	9.88E-07
	u_5	40	82	0.065926	8.78E-07	81	164	1.5774	7.71E-07	81	164	1.1561	5.32E-07
	u_6	49	100	0.068172	9.13E-07	99	200	2.1795	8.49E-07	98	198	2.2511	7.83E-07
10000	u_1	35	72	0.14042	6.43E-07	86	174	5.2774	8.42E-07	77	156	3.8617	8.53E-07
	u_2	31	64	0.13028	6.32E-07	88	178	8.1808	7.05E-07	84	170	4.275	9.14E-07
	u_3	39	80	0.15325	4.23E-07	105	212	7.9398	7.67E-07	93	188	3.9553	8.29E-07
	u_4	32	66	0.12752	7.67E-07	78	158	5.4422	3.8E-07	63	128	2.6708	9.74E-07
	u_5	40	82	0.16646	5.77E-07	80	162	5.8745	5.53E-07	83	168	3.6379	9.18E-07
	u_6	54	110	0.22599	8.84E-07	93	188	6.5827	5.86E-07	99	200	3.8256	9.24E-07
50000	u_1	22	46	0.56066	7.39E-07	74	150	16.5795	6.52E-07	74	150	10.0037	8.17E-07
	u_2	37	75	0.97652	9.88E-07	107	216	26.2616	5.08E-07	70	142	10.0485	4.12E-07
	u_3	29	60	0.39301	9.4E-07	82	166	17.2379	8.62E-07	98	198	13.805	8.31E-07
	u_4	37	76	0.60217	7.57E-07	102	206	23.7085	5.01E-07	79	160	10.4765	9.51E-07
	u_5	36	74	0.62045	6.21E-07	86	174	20.5267	7.99E-07	78	158	11.0269	9.03E-07
	u_6	54	110	1.3132	7.25E-07	117	236	31.2894	9.72E-07	105	212	15.2667	4.96E-07
100000	u_1	37	76	2.4333	7.67E-07	115	232	75.7182	9.2E-07	82	166	22.4979	7.56E-07
	u_2	32	66	1.7346	9.94E-07	69	140	34.8828	6.77E-07	75	152	20.6891	7.52E-07
	u_3	28	58	1.1769	6.26E-07	124	250	72.3798	6.18E-07	100	202	26.4477	6.2E-07
	u_4	30	62	1.2119	7.65E-07	100	202	58.7053	9.54E-07	88	178	24.213	4.56E-07
	u_5	18	38	0.71769	9.71E-07	91	184	47.297	5.02E-07	97	196	26.4774	8.02E-07
	u_6	65	132	3.2073	9.51E-07	122	246	69.6999	1.82E-07	111	224	30.4336	4.24E-07

Table 9. Numerical results obtained by each solver for Test Problem S9.

Problem S9		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	8	17	0.025969	6.12E-07	6	14	0.075705	1.77E-07	6	14	0.042201	7.98E-08
	u_2	17	35	0.014313	7.57E-07	189	380	1.7859	7.62E-07	98	198	0.55824	7.91E-07
	u_3	9	19	0.006941	3.02E-07	6	14	0.022716	7.67E-07	6	14	0.033993	2.43E-07
	u_4	17	36	0.013382	5.81E-07	189	380	1.7149	5.72E-07	101	204	0.96113	6.65E-07
	u_5	11	24	0.015511	9.88E-07	139	280	0.984	9.56E-07	87	176	0.37508	7.64E-07
	u_6	17	36	0.017729	8.42E-07	128	258	0.99134	8.25E-07	119	240	0.62727	9.57E-07
5000	u_1	8	18	0.027665	6.2E-07	6	14	0.080861	3.95E-07	6	14	0.097936	1.78E-07
	u_2	14	30	0.045813	6.71E-07	181	364	6.4779	8.01E-07	102	206	2.2535	6.51E-07
	u_3	9	19	0.025112	6.75E-07	7	16	0.078723	1.19E-07	6	14	0.095586	5.43E-07
	u_4	16	34	0.042592	7.57E-07	208	418	8.3356	8.48E-07	99	200	2.3479	8.9E-07
	u_5	11	23	0.032739	8.84E-07	200	402	7.0807	8.13E-07	119	240	3.2876	8.89E-07
	u_6	16	33	0.052506	9.55E-07	124	250	3.8386	6.13E-07	124	250	3.2627	7.69E-07
10000	u_1	8	18	0.045152	8.76E-07	6	14	0.16242	5.58E-07	6	14	0.14519	2.52E-07
	u_2	15	31	0.080117	9.93E-07	198	398	17.9306	9.59E-07	104	210	4.4272	9.53E-07
	u_3	9	19	0.048073	9.55E-07	7	16	0.20734	1.68E-07	6	14	0.16772	7.68E-07
	u_4	14	30	0.07971	6.43E-07	198	398	15.985	6.07E-07	108	218	6.2596	4.94E-07
	u_5	13	27	0.076531	5.74E-07	120	242	7.7442	8.38E-07	105	212	4.8412	9.53E-07
	u_6	16	34	0.087092	4.34E-07	132	266	8.4628	9.65E-07	114	230	6.0401	7.58E-07
50000	u_1	9	19	0.23374	3.84E-07	7	16	0.557	8.66E-08	6	14	0.49989	5.64E-07
	u_2	16	33	0.37283	9.3E-07	223	448	58.8284	7.18E-07	92	186	11.9885	8.69E-07
	u_3	9	20	0.32426	9.67E-07	7	16	0.5144	3.76E-07	7	16	0.41536	1.09E-07
	u_4	15	32	0.34609	6.95E-07	194	390	47.1427	7.86E-07	117	236	20.5687	6.71E-07
	u_5	14	29	0.30337	2.54E-07	200	402	58.0404	5.32E-07	117	236	19.2491	6.66E-07
	u_6	17	36	0.35947	5.27E-07	152	306	37.2605	6.62E-07	105	212	17.1264	9.48E-07
100000	u_1	9	19	0.54618	5.43E-07	7	16	1.3775	1.22E-07	6	14	0.70926	7.98E-07
	u_2	16	33	0.81285	4.24E-07	200	402	104.2223	7.76E-07	96	194	29.299	7.45E-07
	u_3	10	21	0.46885	2.68E-07	7	16	1.2672	5.32E-07	7	16	0.97489	1.54E-07
	u_4	14	30	0.75746	5.11E-07	187	376	97.8726	7.73E-07	113	228	34.3038	6.33E-07
	u_5	13	27	0.57462	5.55E-07	193	388	92.8696	7.27E-07	126	254	39.2473	4.16E-07
	u_6	18	38	1.2593	9.29E-07	155	312	71.3791	6.39E-07	112	226	32.747	7.75E-07

Table 10. Numerical results obtained by each solver for Test Problem S10.

Problem S10		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	10	22	0.11046	7.88E-07	116	234	0.43475	2.96E-07	108	218	0.56259	6.94E-07
	u_2	10	21	0.005842	2.38E-07	96	194	0.44799	5.86E-07	76	154	0.3433	2.21E-07
	u_3	9	19	0.005554	5.43E-07	90	182	0.41907	3.63E-07	108	218	0.63798	3.56E-07
	u_4	9	20	0.00619	5.71E-07	87	176	0.39902	1.12E-07	59	120	0.22293	8.86E-07
	u_5	9	19	0.008743	6.06E-07	122	246	0.78529	9.77E-07	80	162	0.36608	9.64E-07
	u_6	14	29	0.009908	2.8E-07	110	222	0.61065	2.43E-07	130	262	0.76511	9.44E-07
5000	u_1	12	26	0.032939	8.83E-07	83	168	1.5675	4.8E-07	112	226	2.7047	2.35E-07
	u_2	13	27	0.025855	7.03E-07	117	236	2.9584	3.75E-07	66	134	1.2375	1.52E-07
	u_3	9	20	0.019964	7.92E-07	110	222	2.5519	4.14E-07	122	246	3.4749	5.67E-07
	u_4	13	27	0.024625	3.86E-07	80	162	1.6306	5.74E-07	110	222	2.25	6.93E-07
	u_5	11	23	0.023894	1.98E-07	90	182	3.0141	4.58E-07	113	228	2.609	9.29E-07
	u_6	15	31	0.03049	5.18E-07	135	272	3.9153	5.8E-07	144	290	3.0587	6.04E-07
10000	u_1	13	27	0.051319	2.49E-07	207	416	10.8512	2.98E-07	93	188	4.8252	1.51E-07
	u_2	13	27	0.047583	7.11E-07	103	208	3.3416	8.84E-07	131	264	4.9783	9.49E-07
	u_3	10	21	0.03176	2.83E-07	158	318	8.3841	6.57E-07	148	298	7.9777	9.44E-07
	u_4	14	29	0.04767	4.77E-07	116	234	4.9302	5.4E-07	132	266	7.0218	2.86E-07
	u_5	11	23	0.03825	2.57E-07	90	182	3.8282	2.88E-07	124	250	6.3506	8.79E-07
	u_6	16	33	0.051415	2.45E-07	120	242	6.1758	6.37E-07	142	286	7.7652	7.74E-07
50000	u_1	13	27	0.26275	2.03E-07	96	194	15.6075	5.08E-07	158	318	26.386	5.06E-07
	u_2	13	27	0.19118	4.72E-07	111	224	15.7787	7.18E-07	178	358	40.0501	5.98E-07
	u_3	10	21	0.24425	8.11E-07	112	226	21.0176	4.65E-07	183	368	40.4276	5.04E-07
	u_4	13	27	0.21474	4.18E-07	114	230	24.7249	3.92E-07	140	282	24.2936	2.73E-07
	u_5	12	25	0.1791	3.34E-07	82	166	18.2966	4.87E-07	80	162	9.1164	8.34E-07
	u_6	11	23	0.18327	8.03E-07	157	316	40.5439	8.78E-07	180	362	36.9538	2.3E-08
100000	u_1	13	27	0.32951	2.93E-07	417	836	236.9696	7.85E-07	166	334	59.9467	7.33E-07
	u_2	13	27	0.34883	2.87E-07	124	250	31.0736	8.82E-07	203	408	87.9469	7.69E-07
	u_3	11	23	0.38535	8.22E-07	109	220	36.7679	5.29E-07	188	378	75.6608	5.29E-07
	u_4	13	27	0.32996	6.5E-07	85	172	33.662	9.24E-07	181	364	72.7134	5.94E-07
	u_5	15	31	0.39815	4.26E-07	116	234	26.9585	5.9E-07	127	256	58.1576	2.04E-07
	u_6	11	24	0.31554	6.47E-07	132	266	99.1451	9.48E-07	180	362	70.9571	9.13E-07

Table 11. Numerical results obtained by each solver for Test Problem S11.

Problem S11		DFDFP				MHZ1				MSR1			
Dim	SP	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm	#iter	#fval	#time	Norm
1000	u_1	1	3	0.023903	0	5	12	0.025364	1.86E-07	5	12	0.11039	3.32E-07
	u_2	2	5	0.002771	2.22E-16	69	139	0.36823	6.78E-07	42	85	0.32449	9.74E-07
	u_3	2	5	0.003885	0	7	16	0.008952	1.21E-07	7	16	0.02398	2.97E-07
	u_4	3	7	0.004212	0	74	150	0.4115	7.41E-07	59	120	0.1818	9.96E-07
	u_5	2	5	0.004043	0	55	112	0.36647	3.03E-07	50	101	0.20617	8.91E-07
	u_6	2	5	0.002063	0	89	179	0.50734	8.95E-07	78	157	0.25138	9.55E-07
5000	u_1	1	3	0.003115	0	5	12	0.040685	4.16E-07	5	12	0.032039	7.42E-07
	u_2	2	5	0.006287	2.22E-16	69	139	1.5067	6.78E-07	42	85	0.7069	9.74E-07
	u_3	2	5	0.007322	0	7	16	0.039131	2.71E-07	7	16	0.10269	6.64E-07
	u_4	3	7	0.008981	0	74	150	1.7505	3.57E-07	54	110	0.76827	7.59E-07
	u_5	2	5	0.004747	0	82	166	2.118	2.87E-07	72	146	1.7218	8.98E-07
	u_6	2	5	0.005077	0	94	190	2.0405	7.22E-07	71	143	1.1722	3.24E-07
10000	u_1	1	3	0.005187	0	5	12	0.05858	5.88E-07	6	14	0.06697	7.83E-08
	u_2	2	5	0.009115	2.22E-16	69	139	2.6739	6.78E-07	42	85	1.3654	9.74E-07
	u_3	2	5	0.012953	0	7	16	0.076134	3.83E-07	7	16	0.33193	9.4E-07
	u_4	3	7	0.012575	0	58	118	2.5736	2.32E-07	67	136	2.5239	2.54E-07
	u_5	2	5	0.010119	0	95	192	3.828	1.92E-07	65	132	2.5324	8.38E-07
	u_6	2	5	0.009051	0	85	172	3.4107	7.77E-07	71	144	2.6807	9.33E-07
50000	u_1	1	3	0.014918	0	6	14	0.22534	8.5E-08	6	14	0.37819	1.75E-07
	u_2	2	5	0.04416	2.22E-16	69	139	10.5982	6.78E-07	42	85	4.495	9.74E-07
	u_3	2	5	0.057451	0	7	16	0.23197	8.56E-07	8	18	1.0505	1.89E-07
	u_4	3	7	0.045526	0	85	172	14.4394	9.72E-07	51	104	6.279	3.91E-07
	u_5	2	5	0.038559	0	121	244	18.8559	2.59E-07	64	130	8.3197	6.34E-07
	u_6	2	5	0.038099	0	84	170	13.8624	4.95E-07	88	178	10.5632	8.47E-07
100000	u_1	1	3	0.030776	0	6	14	0.67393	1.2E-07	6	14	0.82034	2.48E-07
	u_2	2	5	0.064372	2.22E-16	69	139	18.6219	6.78E-07	42	85	9.5874	9.74E-07
	u_3	2	5	0.096329	0	8	18	0.55442	7.83E-08	8	18	1.0961	2.67E-07
	u_4	3	7	0.10163	0	68	138	20.9117	3.74E-07	56	114	13.0225	6.88E-07
	u_5	2	5	0.059877	0	94	190	25.1025	4.81E-07	73	148	17.7766	4.43E-07
	u_6	2	5	0.05885	0	78	158	22.5392	2.51E-07	58	117	13.5515	9.91E-07

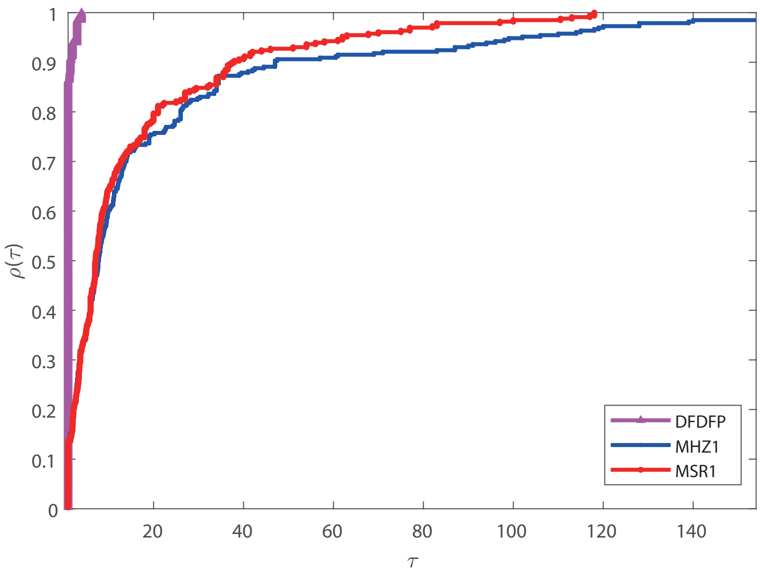

Figure 1. Dolan and Moré performance profile with respect to number of iterations.

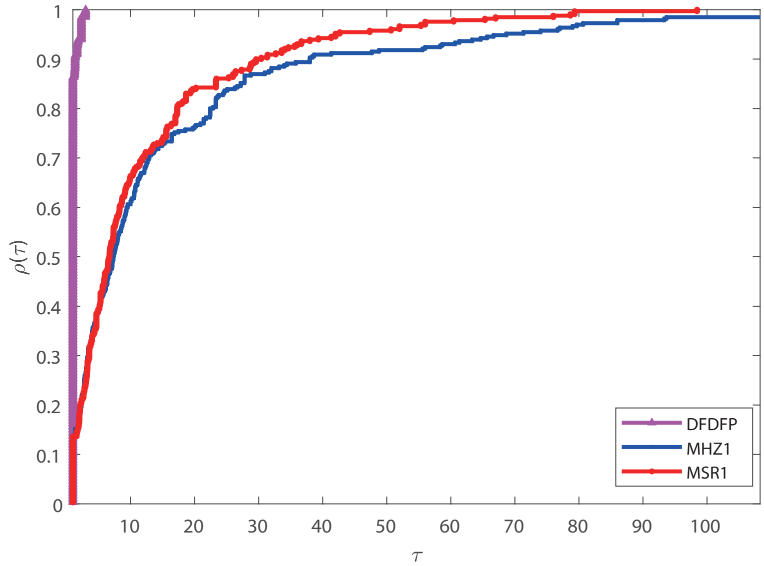

Figure 2. Dolan and Moré performance profile with respect to number of function evaluations.

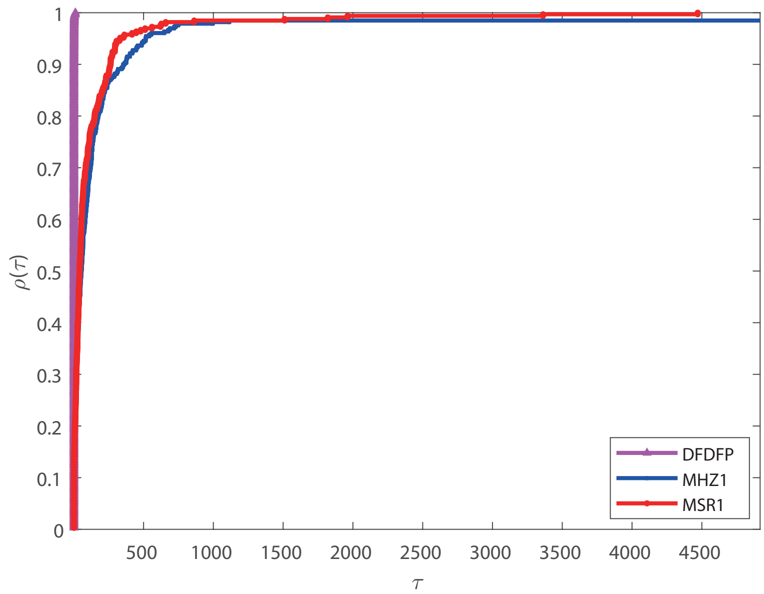

Figure 3. Dolan and Moré performance profile with respect to CPU time.

The results recorded by each algorithm are reported in Tables 1–11. The Norm reported in the tables revealed that all the three algorithms successfully obtained a solution of each of the test problems considered except for Problem S6 with the initial point of u_2, where MHZ1 failed. Moreover, the Norm values reported in Tables 1, 2, 3, 6 and 11 showed that DFDFP obtained the solutions of these problems with very high degree of accuracy. Perusing through Tables 1–11, it can be seen that DFDFP solves the entire 330 test problems with least #iter except for Problem S3. The huge difference between #iter recorded by DFDFP against the two existing methods showed the efficiency of the proposed method. The corresponding #fval values also revealed the superior performance of DFDFP method over the two existing methods considered. Lastly, the #time reported in the tables showed that DFDFP is faster than both MSR1 and MHZ1 in virtually all the cases considered in the experiments. On the other hand, Figures 1–3 present the graphical view of the performance recorded by each algorithm. These figures summarized the #iter, #fval and #time reported in Tables 1–11 which was done with the aid of the popular Dolan and Moré performance profile [21]. It can as well be seen from the Figures 1–3 that DFDFP outperforms MSR1 and MHZ1 with wide margin. This underscores the efficiency as well as the robustness of the proposed Algorithm 1 (DFDFP) over its main competitors, namely, MSR1 and MHZ1.

4. Signal reconstruction problem

Signal processing involves finding some sparse solutions to ill-conditioned system of linear equations. Popular approaches for recovering some disturbed signals include minimizing some objective functions containing a quadratic (ℓ_2) error term and a sparse ℓ_1-regularization term. Now, consider the convex unconstrained minimization problem below

$$\min_{u} p(u), \quad p(u) = \frac{1}{2}\|v - Qu\|_2^2 + \eta\|u\|_1, \tag{4.1}$$

where $\eta > 0$ is a regularization term, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^k$ is an observation, $Q \in \mathbb{R}^{k\times n}$ ($k << n$) is a linear operator, $\|\cdot\|_2$ and $\|\cdot\|_1$ denote the norm 2 and norm 1, respectively. The presence of the $\|\cdot\|_1$ in (4.1) makes iterative algorithms involving gradients unsuitable to handle it in its original form. Recently, different derivative-free iterative algorithms for dealing with signal recovery problem have been proposed, thanks to the reformulation of problem (4.1) into nonlinear system of equations by Xiao et al. [23] which was motivated by the work of Figueiredo et al. [22]. Since Algorithm 1 (DFDFP) is derivative-free, we briefly review the reformulation of problem (4.1) into the form of problem (1.3) and then apply the algorithm to solve it.

Define $(\cdot)_+ = \max\{0, \cdot\}$ and let a and b be two vectors such that $u = a - b$ with $a_i = (u_i)_+$ and $b_i = (-u_i)_+$ for all $i = 1, 2, \ldots, n$. Then Figueiredo et al. [22] have shown that the ℓ_1-regularization problem (4.1) is transformed into the following problem

$$\min_{w\geq 0} \frac{1}{2} w^T Z w + r^T w, \tag{4.2}$$

where $w = [a \ \ b]^T$, $r = \eta e_{2n} + \begin{bmatrix} -Q^T v & Q^T v \end{bmatrix}^T$ and $Z = \begin{bmatrix} Q^T Q & -Q^T Q \\ -Q^T Q & Q^T Q \end{bmatrix}$.

It is easy to see that Z is a positive semi-definite matrix, which means that Eq (4.2) is a convex quadratic problem.

Taking it further, Xiao et al. [23] showed that the above convex quadratic problem (4.2) is equivalent to the following system of nonlinear equation

$$P(w) = \min\{w,\ Zw + r\} = 0, \tag{4.3}$$

where the "min" is interpreted as componentwise minimum. However, recall that the convergence analysis of Algorithm 1 (DFDFP) was established based on the assumptions that the underlying mapping is monotone and Lipschitz continuous. Therefore, the mapping P in (4.3) must satisfy monotonicity and Lipschitzian conditions. Fortunately, Pang [24] has shown that the mapping P in (4.3) is Lipschitzian and in addition, Xiao et al. [23] proved that it is also monotone.

Next, we describe the signal reconstruction experiment as follows. Consider the reconstruction of sparse signal with length n from k observations. In the course of the experiment, the size of the signal is chosen as $n = 2^{11}$, $k = 2^9$ where the original signal contains 2^7 randomly nonzero elements. Furthermore, the measurement v is distributed with some noise, $v = Q\tilde{u} + \bar{\varrho}$, with Q being a randomly generated Gaussian matrix and $\bar{\varrho}$ is the Gaussian noise distributed normally with mean 0 and variance 10^{-4}.

The performance of the proposed Algorithm 1 (DFDFP) on signal reconstruction is assessed by comparing it with DPP algorithm [13]. The two algorithms are coded in MATLAB R2019b and run on a PC with intel Core(TM) i5-8250u processor with 4 GB of RAM and CPU 1.60 GHZ. The parameters used for Algorithm 1 (DFDFP) are the same as in Section 3 except that $\alpha = \frac{1}{\tau_k} - 1$, whereas that of DPP are taken from [13]. Moreover, both algorithms were run from the same initial point $u_0 = Q^T v$ and the same continuation technique on the parameter η. The following termination criteria

$$\left| \frac{p(u_k) - p(u_{k-1})}{p(u_{k-1})} \right| < 10^{-5},$$

is used throughout the experiment.

The numerical performance of each algorithm is assessed by Iter (number of iterations) and Time (CPU time) required to successfully recover the disturbed signal. In addition, the quality of the reconstruction of the disturbed signal is assessed by MSE (mean of squared error) to the original signal $\tilde{u}$, that is, $MSE = \frac{1}{n}\|\tilde{u} - u_*\|^2$, where u_* is the recovered signal. Figures 4-5 show that the DFDFP and DPP successfully reconstruct the disturbed signal. However, the quality of the reconstruction varies as can be seen from the MSEs recorded by each algorithm. The MSE recorded by DFDFP is 0.000926 which is less than that recorded by DPP, that is 0.00179. This means that the quality of the signal recovered by the proposed Algorithm 1 (DFDFP) is better than that of DPP. In addition, Algorithm 1 (DFDFP) successfully reconstructed the disturbed signal in less than 1 second with 89 iterations. Comparing with Iter= 145 and Time= 1.77 seconds recorded by DPP, it shows that DFDFP is faster and very efficient in signal reconstruction problem.

5. Conclusions

Based on the quasi–Newton updating formula, line search strategy and the projection technique, a Davidon–Fletcher–Powell (DFP) like derivative-free method for solving nonlinear monotone system of equations with convex constraints has been proposed. We have shown that the proposed method is globally convergent to a solution of problem (1.3). In addition, the search direction satisfies the sufficient descent condition independent of the line search used. Numerical results reported showed that

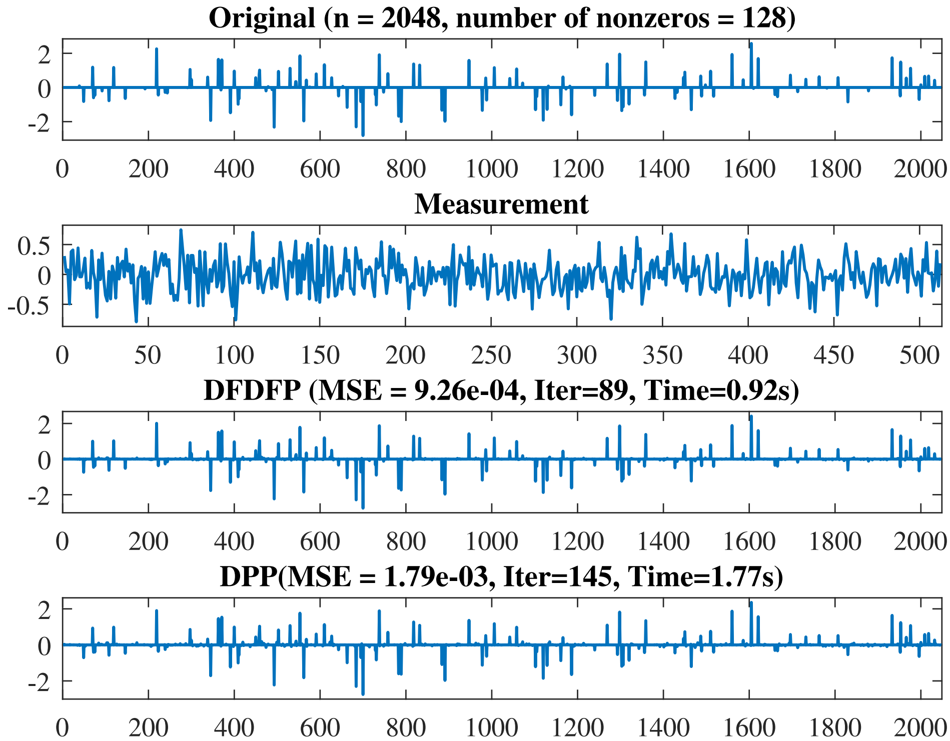

Figure 4. From top to bottom: The original signal, the measurement, the recovered signal by the DFDFP and DPP methods, respectively.

the proposed method outperformed two existing methods [16, 20] on a collection of some test problems and was able to recover some noisy signals in compressive sensing with a better quality recovery than the method in [13]. Future work includes extending the new algorithm to solve robotic motion control.

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Conflict of interest

The authors declare that they have no conflict of interest.

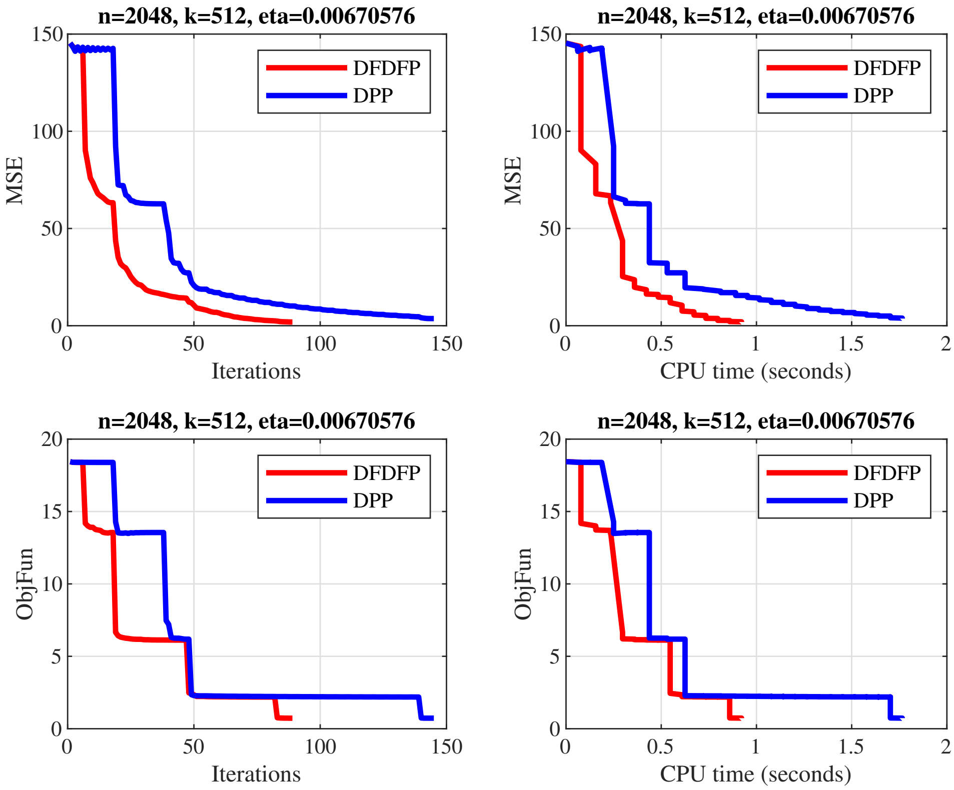

Figure 5. Comparison result of the DFDFP and DPP methods. The x-axis represent the number of Iterations top left and bottom left and CPU time in seconds top right and bottom right. The y-axis represent the MSE top left and top right and the objective function values bottom left and bottom right.

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Supplementary

We use the following monotone nonlinear equation for the experiments in Section 3 where $P(u) = (p_1(u), p_2(u), \dots, p_n(u))^T$, and $u = (u_1, u_2, \dots, u_n)^T$.

Problem S1. [25]

$$\begin{aligned} p_1(u) &= e^{u_1} - 1, \\ p_i(u) &= e^{u_i} + u_{i-1} - 1, \quad i = 1, 2, \dots, n-1. \end{aligned} \quad \text{where } U = \mathbb{R}^n_+,$$

Problem S2. [26]

$$p_i(u) = 2u_i - \sin|u_i|, \quad i = 1, 2, \dots, n, \quad \text{where } U = \mathbb{R}^n_+.$$

Problem S3. [27]

$$p_i(u) = e^{u_i} - 1, \quad i = 1, 2, \dots, n, \quad \text{where } U = \mathbb{R}^n_+.$$

Problem S4. [28]

$$\begin{aligned} p_1(u) &= u_1 - \exp\left(\cos\left(\frac{u_1 + u_2}{n+1}\right)\right), \\ p_i(u) &= u_i - \exp\left(\cos\left(\frac{u_{i-1} + u_i + u_{i+1}}{n+1}\right)\right), \quad 2 \le i \le n-1, \quad \text{where } U = \mathbb{R}^n_+, \\ p_n(u) &= u_n - \exp\left(\cos\left(\frac{u_{n-1} + u_n}{n+1}\right)\right). \end{aligned}$$

Problem S5. [29]

$$p_i(u) = u_i - \sin(|u_i - 1|), \quad i = 1, 2, \dots, n-1,$$

where $U = \{u \in \mathbb{R}^n : \sum_{i=1}^{n} u_i \le n, \ u_i \ge -1, \ i = 1, 2, \cdots, n\}$.

Problem S6. [30]

$$p_i(u) = e^{u_i^2} + \frac{3}{2}\sin(2u_i) - 1, \quad i = 1, 2, \ldots, n, \quad \text{where} \quad U = \mathbb{R}^n_+.$$

Problem S7.

$$\begin{aligned} p_1(u) &= 2u_1 - u_2 + e^{u_1} - 1, \\ p_i(u) &= -u_{i-1} + 2u_i - u_{i+1} + e^{u_i} - 1, \quad i =, 2, \ldots, n-1, \quad \text{where} \quad U = \mathbb{R}^n_+, \\ p_n(u) &= -u_{n-1} + 2u_n + e^{u_n} - 1. \end{aligned}$$

Problem S8. [31]

$$\begin{aligned} p_1(u) &= \frac{5}{2}u_1 + u_2 - 1, \\ p_i(u) &= u_{i-1} + \frac{5}{2}u_i + u_{i+1} - 1, \quad i =, 2, \ldots, n-1, \quad \text{where} \quad U = \mathbb{R}^n_+, \\ p_n(u) &= u_{n-1} + \frac{5}{2}u_n - 1. \end{aligned}$$

Problem S9. [28]

$$\begin{aligned} p_1(u) &= u_1 + \sin(u_1) - 1, \\ p_i(u) &= -u_{i-1} + 2u_i + \sin(u_i) - 1, \quad i = 2, \ldots, n-1, \quad \text{where} \quad U = \mathbb{R}^n_+, \\ p_n(u) &= u_n + \sin(u_n) - 1. \end{aligned}$$

Problem S10. [13]

$$p_i(u) = \frac{i}{n}e^{u_i} - 1, \quad i = 1, 2, \ldots, n, \quad \text{where} \quad U = \mathbb{R}^n_+.$$

Problem S11.

$$p_i(u) = \cos(u_i) + u_i - 1, \quad i = 1, 2, \ldots, n, \quad \text{where} \quad U = \mathbb{R}^n_+.$$