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Research article

Gauss-Bonnet theorem in Lorentzian Sasakian space forms

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Abstract: In this paper, we use a Lorentzian approximation scheme to compute the sub-Lorentzian limit of curvatures for curves and Lorentzian surfaces in the Lorentzian Bianci-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian space forms. Based on these results, we get a Gauss-Bonnet theorem in the Lorentzian Sasakian space forms.

Keywords: Lorentzian Sasakian space forms; Gauss-Bonnet theorem; Lorentzian approximation scheme; sub-Riemannian geometry; curvature
Mathematics Subject Classification: 53C40, 53C42

1. Introduction

The Lorentzian Bianci-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian space forms was constructed in [2] according to Theorem 6.5 in [3] which states that there is a one-to-one correspondence between homogeneous contact Riemannian three-manifolds and homogeneous contact Lorentzian three-manifolds. Let c be a real number, set $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 | 1 + \frac{c}{2}(x^2 + y^2) > 0\}$ equipped with the Lorentzian metric $g_c = \frac{dx^2+dy^2}{[1+\frac{c}{2}(x^2+y^2)]^2} - [dz + \frac{ydx-xdy}{1+\frac{c}{2}(x^2+y^2)}]^2$. $(\mathcal{D}, g_c)$ was called the Lorentzian Bianci-Cartan-Vranceanu model of a Lorentzian Sasakian space form and was denoted by $\mathcal{M}^3_1(\mathcal{H})$, where $\mathcal{H} = 2c + 3$ is the constant holomorphic sectional curvature. In [1, 2], Lee described some geometric properties of biharmonic curves and pseudo-Hermitian pseudo-helices in $\mathcal{M}^3_1(\mathcal{H})$. In [5, 6], Balogh et al. used a Riemannian approximation scheme to prove a Heisenberg version of the Gauss-Bonnet theorem. In [7–10], Wang, Wei and Wu investigated sub-Riemannian geometries of some typical spaces such as the affine group, the group of rigid motions of the Minkowski plane, the BCV spaces and the Lorentzian Heisenberg group, and proved Gauss-Bonnet theorems in these groups. Recently, we get a Gauss-Bonnet theorem in the rototranslation group [11].

In this paper, we focus on the case of 3-dimensional Lorentzian Sasakian space forms. After presenting the technical apparatus needed for further investigations, we use a Lorentzian approximation scheme to compute the sub-Lorentzian limit of curvatures for curves and Lorentzian surfaces and prove a Gauss-Bonnet theorem in 3-dimensional Lorentzian Sasakian space forms. The paper is organized in the following way. Basic notions on $\mathcal{M}_1^3(\mathcal{H})$ and the Lorentzian approximants $(\mathcal{D}, g_{cL})$ of $\mathcal{M}_1^3(\mathcal{H})$ are given in Section 2. The sub-Lorentzian limit of curvature of curves in $(\mathcal{D}, g_{cL})$ will be computed. In Sections 4 and 5, we compute sub-Lorentzian limits of geodesic curvature of curves on Lorentzian surfaces and the Riemannian Gaussian curvature of Lorentzian surfaces in $(\mathcal{D}, g_{cL})$. In Section 6, we prove the Gauss-Bonnet theorem in 3-dimensional Lorentzian Sasakian space forms. In Section 7, we summarize this paper as conclusions.

2. Lorentzian approximants of 3-dimensional Lorentzian Sasakian space forms

We recall the Lorentzian Bianci-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian space forms in [1]. Let c be a real number and set

$$\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 | 1 + \frac{c}{2}(x^2 + y^2) > 0\}.$$

Let

$$X_1 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ X_2 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}, \tag{2.1}$$

with brackets

$$[X_1, X_2] = -cyX_1 + cxX_2 + 2X_3,\ [X_2, X_3] = [X_1, X_3] = 0. \tag{2.2}$$

Then

$$\frac{\partial}{\partial x} = \frac{1}{1 + \frac{c}{2}(x^2 + y^2)}(X_1 + yX_3),\ \frac{\partial}{\partial y} = \frac{1}{1 + \frac{c}{2}(x^2 + y^2)}(X_2 - xX_3),\ \frac{\partial}{\partial z} = X_3 \tag{2.3}$$

and $span\{X_1, X_2, X_3\} = T(\mathcal{D})$. Let $H = span\{X_1, X_2\}$ be the horizontal distribution on $\mathcal{D}$. Since $[X_1, X_2] = -cyX_1 + cxX_2 + 2X_3$ it follows that $\{X_1, X_2, [X_1, X_2]\}$ are linearly independent at every point $(x, y, z) \in \mathcal{D}$ and hence it will form a basis at each point. Therefore

$$H_p^2 = H_p + \left[H_p, H_p\right] = span\{X_1(p), X_2(p), X_3(p)\} = T_p(\mathcal{D})$$

and hence the horizontal distribution is step 2 everywhere. Let

$$\omega_1 = \frac{dx}{1 + \frac{c}{2}(x^2 + y^2)}, \omega_2 = \frac{dy}{1 + \frac{c}{2}(x^2 + y^2)}, \omega = dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}.$$

Then $H = \ker\omega$. We equip the Lorentzian metric as following

$$g_c = \frac{dx^2 + dy^2}{[1 + \frac{c}{2}(x^2 + y^2)]^2} - [dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}]^2,$$

and we call $(\mathcal{D}, g_c)$ the Lorentzian Bianci-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian space forms and denote it by $\mathcal{M}_1^3(\mathcal{H})$, where $\mathcal{H} = 2c + 3$ is the constant holomorphic sectional curvature. To describe the Lorentzian approximants to $\mathcal{M}_1^3(\mathcal{H})$, let $L > 0$ and define a metric

$$g_{cL} = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 - L\omega \otimes \omega,$$

so that $X_1, X_2, \widetilde{X_3} := L^{-\frac{1}{2}}X_3$ are orthonormal basis on $T(\mathcal{D})$ with respect to g_{cL}. Hereafter, we denote the Lorentzian approximants to the model $\mathcal{M}_1^3(\mathcal{H})$ of a Lorentzian Sasakian space form by $(\mathcal{D}, g_{cL})$. Note that $g_c = g_{c1}$ be the Lorentzian metric on $\mathcal{D}$. We say that a non-zero vector $\mathbf{x} \in (\mathcal{D}, g_{cL})$ is spacelike, null or timelike if $g_{cL}(\mathbf{x}, \mathbf{x}) > 0$, $g_{cL}(\mathbf{x}, \mathbf{x}) = 0$, or $g_{cL}(\mathbf{x}, \mathbf{x}) < 0$ respectively. The norm of the vector $\mathbf{x}$ is defined by $\|\mathbf{x}\| = \sqrt{|g_{cL}(\mathbf{x}, \mathbf{x})|}$. A straightforward calculation shows the following proposition on the Levi-Civita connection ∇^L on $(\mathcal{D}, g_{cL})$.

Proposition 2.1. *The Levi-civita connection on $(\mathcal{D}, g_{cL})$ relative to the coordinate frame $X_1, X_2, \widetilde{X_3}$ is given by*

$$\begin{aligned} &\nabla^L_{X_1} X_1 = cyX_2,\ \nabla^L_{X_2} X_2 = cxX_1,\ \nabla^L_{X_3} X_3 = 0, \\ &\nabla^L_{X_1} X_2 = -cyX_1 + X_3,\ \nabla^L_{X_2} X_1 = -cxX_2 - X_3,\ \nabla^L_{X_1} X_3 = LX_2, \\ &\nabla^L_{X_3} X_1 = LX_2,\ \nabla^L_{X_2} X_3 = -LX_1,\ \nabla^L_{X_3} X_2 = -LX_1. \end{aligned} \tag{2.4}$$

Proof. It follows from a direct application of the Koszul identity, which here simplifies

$$2\left\langle \nabla^L_{X_i} X_j, X_k \right\rangle_L = \left\langle \left[X_i, X_j\right], X_k \right\rangle_L - \left\langle \left[X_j, X_k\right], X_i \right\rangle_L + \left\langle [X_k X_i], X_j \right\rangle_L, \tag{2.5}$$

where $i, j, k = 1, 2, 3$. □

Define the curvature of the connection ∇^L by

$$R^L(X, Y)Z = \nabla^L_X \nabla^L_Y Z - \nabla^L_Y \nabla^L_X Z - \nabla^L_{[X,Y]} Z.$$

We get the following proposition.

Proposition 2.2. *The curvature tensor of $(\mathcal{D}, g_{cL})$ is given by*

$$\begin{aligned} &R^L(X_1, X_2) X_1 = (-3L - 2c)X_2,\ R^L(X_1, X_2) X_2 = (3L + 2c)X_1 + 2cyX_3,\ R^L(X_1, X_2) X_3 = 0, \\ &R^L(X_1, X_3) X_1 = LX_3,\ R^L(X_1, X_3) X_2 = 0,\ R^L(X_1, X_3) X_3 = L^2 X_1, \\ &R^L(X_2, X_3) X_1 = 0,\ R^L(X_2, X_3) X_2 = LX_3,\ R^L(X_2, X_3) X_3 = L^2 X_2. \end{aligned} \tag{2.6}$$

Proof. It is a direct computation using

$$R^L(X, Y)Z = \nabla^L_X \nabla^L_Y Z - \nabla^L_Y \nabla^L_X Z - \nabla^L_{[X,Y]} Z.$$

Taking

$$R^L(X_1, X_2) X_1 = \left(\nabla^L_{X_1} \nabla^L_{X_2} - \nabla^L_{X_2} \nabla^L_{X_1} - \nabla^L_{[X_1, X_2]}\right) X_1$$

for example, we compute

$$\nabla^L_{X_1}\left(\nabla^L_{X_2} X_1\right) = c^2 xyX_1 - [L + c + \frac{c^2}{2}(x^2 + y^2)]X_2 - cxX_3,$$

$$\nabla^L_{X_2}\left(\nabla^L_{X_1} X_1\right) = cyX_2,\ \nabla^L_{[X_1, X_2]} X_1 = \nabla^L_{X_3} X_1 = [2L - c^2(x^2 + y^2)]X_2 - cxX_3.$$

Hence,

$$R^L(X_1, X_2) X_1 = (-3L - 2c)X_2.$$

□

3. The sub-Lorentzian limit of curvature of curves in $(\mathcal{D}, g_{cL})$

In this section, we will compute the sub-Lorentzian limit of curvature of curves in $(\mathcal{D}, g_{cL})$. Our approach is to define sub-Lorentzian objects as limits of horizontal objects in $(\mathcal{D}, g_{cL})$, where a family of metrics g_{cL} is essentially obtained as an anisotropic blow-up of the Lorentzian metric g_{c1}. At the heart of this approach is the fact that the intrinsic horizontal geometry does not change with L. Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a regular curve, where I is an open interval in $\mathbb{R}$. The regular curve γ is called a spacelike curve, timelike curve or null curve if $\dot{\gamma}(t)$ is a spacelike vector, timelike vector or null vector at any $t \in I$ respectively.

Definition 3.1. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a C^1 smooth curve, we say that γ is regular if $\dot{\gamma} \neq 0$ for every $t \in I$. Moreover we say that $\gamma(t)$ is a horizontal point of γ if*

$$\omega(\dot{\gamma}(t)) = \frac{1}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)} (\dot{\gamma}_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_2(t)) + \dot{\gamma}_3(t) = 0$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$.

As is well know, if γ is a curve with arc length parametrization, then the standard definition of curvature for γ in Riemannian geometry is $\kappa_\gamma^L := \left\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\right\|_L$. If γ is a curve with an arbitrary parametrization, then we give the definitions as follows:

Definition 3.2. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a C^2-smooth regular curve.*

(1) If $\nabla_{\dot{\gamma}}^L \dot{\gamma}$ is a spacelike vector, we define the curvature κ_γ^L of γ at $\gamma(t)$ by

$$\kappa_\gamma^L := \sqrt{\frac{\left\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\right\|_L^2}{\|\dot{\gamma}\|_L^4} - \frac{\left\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma}\right\rangle_L^2}{\langle \dot{\gamma}, \dot{\gamma}\rangle_L^3}}. \tag{3.1}$$

(2) If $\nabla_{\dot{\gamma}}^L \dot{\gamma}$ is a timelike vector, we define the curvature κ_γ^L of γ at $\gamma(t)$ by

$$\kappa_\gamma^L := \sqrt{\frac{\left\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\right\|_L^2}{\|\dot{\gamma}\|_L^4} + \frac{\left\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma}\right\rangle_L^2}{\langle \dot{\gamma}, \dot{\gamma}\rangle_L^3}}. \tag{3.2}$$

Proposition 3.3. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$be a C^2-smooth regular curve and $A = 1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)$.*

(1) If $\nabla_{\dot{\gamma}}^L \dot{\gamma}$ is a spacelike vector, then

$$\begin{aligned}
\kappa_\gamma^L =& \{\{[\frac{\ddot{\gamma}_1}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2} - \frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}]^2 \\
&+ [\frac{\ddot{\gamma}_2}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2} + \frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}]^2 - L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\} \\
&\times [\frac{\dot{\gamma}_1^2}{A^2} + \frac{\dot{\gamma}_2^2}{A^2} - L(\omega(\dot{\gamma}(t)))^2]^{-2} \\
&- \{\frac{\dot{\gamma}_1}{A}[\frac{\ddot{\gamma}_1}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2} - \frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}] \\
&+ \frac{\dot{\gamma}_2}{A}[\frac{\ddot{\gamma}_2}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2} + \frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}] \\
&- L\omega(\dot{\gamma}(t))\frac{d\omega(\dot{\gamma}(t))}{dt}\}^2 \times [\frac{\dot{\gamma}_1^2}{A^2} + \frac{\dot{\gamma}_2^2}{A^2} - L(\omega(\dot{\gamma}(t)))^2]^{-3}\}^{\frac{1}{2}}.
\end{aligned} \tag{3.3}$$

In particular, if $\gamma(t)$ is a horizontal point of γ,

$$\begin{aligned}\kappa_\gamma^L =&\{\{[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}]^2+[\frac{\ddot{\gamma}_2}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}]^2\\&-L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\}\times(\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2})^{-2}-\{\frac{\dot{\gamma}_1}{A}[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}]\\&+\frac{\dot{\gamma}_2}{A}[\frac{\ddot{\gamma}_2(t)}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}]\}^2(\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2})^{-3}\}^{\frac{1}{2}}.\end{aligned}\tag{3.4}$$

(2) If $\nabla^L_{\dot{\gamma}}\dot{\gamma}$ is a timelike vector, then

$$\begin{aligned}\kappa_\gamma^L =&\{-\{[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}-\frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}]^2\\&+[\frac{\ddot{\gamma}_2}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}+\frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}]^2-L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\}\\&\times[\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2}-L(\omega(\dot{\gamma}(t)))^2]^{-2}\\&-\{\frac{\dot{\gamma}_1}{A}[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}-\frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}]\\&+\frac{\dot{\gamma}_2}{A}[\frac{\ddot{\gamma}_2}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}+\frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}]\\&-L\omega(\dot{\gamma}(t))\frac{d\omega(\dot{\gamma}(t))}{dt}\}^2\times[\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2}-L(\omega(\dot{\gamma}(t)))^2]^{-3}\}^{\frac{1}{2}}.\end{aligned}\tag{3.5}$$

In particular, if $\gamma(t)$ is a horizontal point of γ,

$$\begin{aligned}\kappa_\gamma^L =&\{-\{[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}]^2+[\frac{\ddot{\gamma}_2}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}]^2\\&-L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\}\times(\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2})^{-2}-\{\frac{\dot{\gamma}_1}{A}[\frac{\ddot{\gamma}_1}{A}+\frac{c(\gamma_1\dot{\gamma}_2^2-\gamma_1\dot{\gamma}_1^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}]\\&+\frac{\dot{\gamma}_2}{A}[\frac{\ddot{\gamma}_2}{A}+\frac{c(\dot{\gamma}_1^2\gamma_2-\gamma_2\dot{\gamma}_2^2-2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}]\}^2(\frac{\dot{\gamma}_1^2}{A^2}+\frac{\dot{\gamma}_2^2}{A^2})^{-3}\}^{\frac{1}{2}}.\end{aligned}\tag{3.6}$$

Proof. By (2.3), we have

$$\dot{\gamma}(t)=\left(\frac{\dot{\gamma}_1(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}\right)X_1+\left(\frac{\dot{\gamma}_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}\right)X_2+\omega\left(\dot{\gamma}(t)\right)X_3.\tag{3.7}$$

By Proposition 2.1 and (3.7), we have

$$\begin{aligned}\nabla^L_{\dot{\gamma}}X_1&=\left[\frac{c(\dot{\gamma}_1\gamma_2-\gamma_1\dot{\gamma}_2)}{A}+L\omega(\dot{\gamma}(t))\right]X_2-\frac{L\dot{\gamma}_2}{A}X_3,\\\nabla^L_{\dot{\gamma}}X_2&=\left[\frac{c(\gamma_1\dot{\gamma}_2-\dot{\gamma}_1\gamma_2)}{A}-L\omega(\dot{\gamma}(t))\right]X_1+\frac{L\dot{\gamma}_1}{A}X_3,\\\nabla^L_{\dot{\gamma}}X_3&=\frac{-L\dot{\gamma}_2}{A}X_1+\frac{L\dot{\gamma}_1}{A}X_2.\end{aligned}\tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned}\nabla^L_{\dot\gamma}\dot\gamma=&\left[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}-\frac{2L\dot\gamma_2\omega(\dot\gamma(t))}{A}\right]X_1\\&+\left[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}+\frac{2L\dot\gamma_1\omega(\dot\gamma(t))}{A}\right]X_2+\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]X_3.\end{aligned}\tag{3.9}$$

By (3.7), (3.9), and the definition of κ^L_γ, we get Proposition 3.3. □

Definition 3.4. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a C^2-smooth regular curve. We define the intrinsic curvature κ^∞_γ of γ at $\gamma(t)$ to be*

$$\kappa^\infty_\gamma := \lim_{L\to\infty}\kappa^L_\gamma$$

if the limit exists.

We introduce the following notation: For continuous functions $f_1, f_2 : (0,+\infty)\to\mathbb{R}$,

$$f_1(L)\sim f_2(L)\,,\ \text{as } L\to+\infty \Leftrightarrow \lim_{L\to\infty}\frac{f_1(L)}{f_2(L)}=1.$$

Proposition 3.5. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a C^2-smooth regular curve and $A = 1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)$.*

(1) If $\nabla^L_{\dot\gamma}\dot\gamma$ is a spacelike vector, then

$$\kappa^\infty_\gamma=\frac{2\sqrt{\dot\gamma_1^2+\dot\gamma_2^2}}{|\omega(\dot\gamma(t))|},\ \textit{if } \omega(\dot\gamma(t))\neq 0.$$

$$\begin{aligned}\kappa^\infty_\gamma=&\{\{[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]^2+[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]^2\}\\&\times[\frac{\dot\gamma_1^2}{A^2}+\frac{\dot\gamma_2^2}{A^2}]^{-2}-\{\frac{\dot\gamma_1}{A}[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]\\&+\frac{\dot\gamma_2}{A}[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]\}\times[\frac{\dot\gamma_1^2}{A^2}+\frac{\dot\gamma_2^2}{A^2}]^{-3}\}^{\frac{1}{2}},\end{aligned}\tag{3.10}$$

if $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))=0$.

$\lim_{L\to\infty}\frac{\kappa^L_\gamma}{\sqrt{L}}$ *does not exist, if $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0$.*

(2) If $\nabla^L_{\dot\gamma}\dot\gamma$ is a timelike vector, then κ^∞_γ does not exist, if $\omega(\dot\gamma(t))\neq 0$.

$$\begin{aligned}\kappa^\infty_\gamma=&\{-\{[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]^2+[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]^2\}\\&\times[\frac{\dot\gamma_1^2}{A^2}+\frac{\dot\gamma_2^2}{A^2}]^{-2}-\{\frac{\dot\gamma_1}{A}[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]\\&+\frac{\dot\gamma_2}{A}[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]\}\times[\frac{\dot\gamma_1^2}{A^2}+\frac{\dot\gamma_2^2}{A^2}]^{-3}\}^{\frac{1}{2}}.\end{aligned}\tag{3.11}$$

if $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))=0$.

$$\lim_{L\to\infty}\frac{\kappa^L_\gamma}{\sqrt{L}}=\frac{A^2|\frac{d\omega(\dot\gamma(t))}{dt}|}{\dot\gamma_1^2+\dot\gamma_2^2},$$

if $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0$.

Proof. (1) If $\nabla^L_{\dot\gamma}\dot\gamma$ is a spacelike vector, when $\omega(\dot\gamma(t)) \neq 0$, we have

$$\langle \nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma \rangle_L \sim \frac{4L^2\omega^2(\gamma(t))(\dot\gamma_1^2+\dot\gamma_2^2)}{A^2} \text{ as } L \to +\infty,$$

$$\langle \dot\gamma, \dot\gamma \rangle_L \sim -L\omega(\dot\gamma(t))^2,\ \left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \right\rangle_L^2 \sim O\left(L^2\right) \text{ as } L \to +\infty.$$

Therefore

$$\frac{\langle \nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma \rangle_L}{\|\dot\gamma\|_L^4} \to \frac{4(\dot\gamma_1^2+\dot\gamma_2^2)}{\omega(\dot\gamma(t))^2} \text{ as } L \to +\infty,$$

$$\frac{\left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \right\rangle_L^2}{\langle \dot\gamma, \dot\gamma \rangle_L^3} \to 0 \text{ as } L \to +\infty.$$

If $\omega(\dot\gamma(t)) \neq 0$, by (3.1), we have

$$\kappa_\gamma^\infty = \frac{2\sqrt{\dot\gamma_1^2+\dot\gamma_2^2}}{|\omega(\dot\gamma(t))|}.$$

By (3.3) and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, we have (3.10). When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\langle \nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma \rangle_L \sim -L\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]^2 \text{ as } L \to +\infty,$$

$$\langle \dot\gamma, \dot\gamma \rangle_L = \frac{\dot\gamma_1^2+\dot\gamma_2^2}{A^2},$$

$$\frac{\left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \right\rangle_L^2}{\langle \dot\gamma, \dot\gamma \rangle_L^3} = O(1) \text{ as } L \to +\infty.$$

If $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, by (3.1), we get

$$\lim_{L\to\infty} \frac{\kappa_\gamma^L}{\sqrt{L}} = \frac{A^2\sqrt{-(\frac{d\omega(\dot\gamma(t)}{dt})^2}}{\dot\gamma_1^2+\dot\gamma_2^2}.$$

Therefore, this situation does not exist.

(2) If $\nabla^L_{\dot\gamma}\dot\gamma$ is a timelike vector, when $\omega(\dot\gamma(t)) \neq 0$, we have

$$\langle \nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma \rangle_L \sim -\frac{4L^2\omega^2(\gamma(t))(\dot\gamma_1^2+\dot\gamma_2^2)}{A^2} \text{ as } L \to +\infty,$$

$$\langle \dot\gamma, \dot\gamma \rangle_L \sim -L\omega(\dot\gamma(t))^2,\ \left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \right\rangle_L^2 \sim O\left(L^2\right) \text{ as } L \to +\infty.$$

Therefore

$$\frac{\langle \nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma \rangle_L}{\|\dot\gamma\|_L^4} \to -\frac{4(\dot\gamma_1^2+\dot\gamma_2^2)}{\omega(\dot\gamma(t))^2} \text{ as } L \to +\infty,$$

$$\frac{\left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma\right\rangle_L^2}{\langle\dot\gamma,\dot\gamma\rangle_L^3} \to 0 \text{ as } L \to +\infty.$$

If $\omega(\dot\gamma(t)) \neq 0$, by (3.1), we have

$$\kappa_\gamma^\infty = \frac{2\sqrt{-(\dot\gamma_1^2+\dot\gamma_2^2)}}{|\omega(\dot\gamma(t))|},$$

therefore, this situation does not exist. By (3.3) and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, we have (3.11). When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\langle\nabla^L_{\dot\gamma}\gamma, \nabla^L_{\dot\gamma}\gamma\rangle_L \sim L\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]^2 \text{ as } L \to +\infty,$$

$$\langle\dot\gamma,\dot\gamma\rangle_L = \frac{\dot\gamma_1^2+\dot\gamma_2^2}{A^2},$$

$$\frac{\left\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma\right\rangle_L^2}{\langle\dot\gamma,\dot\gamma\rangle_L^3} = O(1) \text{ as } L \to +\infty.$$

If $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, by (3.1), we get

$$\lim_{L\to\infty}\frac{\kappa_\gamma^L}{\sqrt{L}} = \frac{A^2\sqrt{(\frac{d\omega(\dot\gamma(t)}{dt})^2}}{\dot\gamma_1^2+\dot\gamma_2^2}.$$

□

4. The sub-Lorentzian limit of geodesic curvature of curves on Lorentzian surfaces in $(\mathcal{D}, g_{cL})$

We will say that a surface $(\mathcal{D}, g_{cL})$ is regular if Σ is a C^2-smooth compact and oriented surface. In particular we will assume that there exists a C^2-smooth function $u : \mathcal{D} \to \mathbb{R}$ such that

$$\Sigma = \{(x_1, x_2, x_3) \in \mathcal{D} : u(x_1, x_2, x_3) = 0\}$$

and $u_{x_1}\partial_{x_1} + u_{x_2}\partial_{x_2} + u_{x_3}\partial_{x_3} \neq 0$. Let $\nabla_H u = X_1(u)X_1 + X_2(u)X_2$. A point $x \in \Sigma$ is called characteristic if $\nabla_H u(x) = 0$. Our computations will be local and away from characteristic points of Σ. Let us define first $p := X_1 u, q := X_2 u,$ and $r := \widetilde{X_3}u$. Since $p^2+q^2 > 0$, we say $\Sigma \subset \mathcal{D}$ is horizontal spacelike surface. When $L \to \infty$, then $p^2 + q^2 - r^2 > 0$. Then we define

$$\begin{aligned} &l := \sqrt{p^2+q^2}, l_L := \sqrt{p^2+q^2-r^2}, \bar p := \frac{p}{l},\\ &\bar q := \frac{q}{l}, \bar{p_L} := \frac{p}{l_L}, \bar{q_L} := \frac{q}{l_L}, \bar{r_L} := \frac{r}{l_L}. \end{aligned} \tag{4.1}$$

In particular, $\bar p^2 + \bar q^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$v_L = \bar{p_L}X_1 + \bar{q_L}X_2 - \bar{r_L}\widetilde{X_3}, e_1 = \bar q X_1 - \bar p X_2, e_2 = \bar{r_L}\bar p X_1 + \bar{r_L}\bar q X_2 - \frac{l}{l_L}\widetilde{X_3}, \tag{4.2}$$

then v_L is the unit spacelike normal vector to Σ and e_1 is a unit spacelike vector, e_2 is a unit timelike vector of Σ. $\{e_1, e_2\}$ is the orthonormal basis of Σ. Let $\dot{\gamma} = \lambda_1 e_1 + \lambda_2 e_2$, we define $J_L(\dot{\gamma}) = -\lambda_1 e_2 - \lambda_2 e_1$, if γ is a C^2-smooth timelike curve, we define $J_L(\dot{\gamma}) = \lambda_1 e_2 + \lambda_2 e_1$, if γ is a C^2-smooth spacelike curve. For every $U, V \in T\Sigma$, we define $\nabla^{\Sigma,L}_U V = \pi \nabla^L_U V$ where $\pi : TG \to T\Sigma$ is the projection. Then $\nabla^{\Sigma,L}$ is the Levi-Civita connection on Σ with respect to the metric g_{cL}. By (3.9),(4.2) and

$$\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma} = \langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, e_1\rangle_L e_1 - \langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, e_2\rangle_L e_2 \tag{4.3}$$

we have

$$\begin{aligned}
\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma} =& \{\bar{q}[\frac{\ddot{\gamma}_1(t)}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2} - \frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}] \\
& - \bar{p}[\frac{\ddot{\gamma}_2(t)}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2} + \frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}]\}e_1 \\
& + \{\overline{r_L}\bar{p}[\frac{\ddot{\gamma}_1(t)}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2} - \frac{2L\dot{\gamma}_2\omega(\dot{\gamma}(t))}{A}] \\
& + \overline{r_L}\bar{q}[\frac{\ddot{\gamma}_2(t)}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2} + \frac{2L\dot{\gamma}_1\omega(\dot{\gamma}(t))}{A}] + \frac{l}{l_L}L^{\frac{1}{2}}\frac{d}{dt}(\omega(\dot{\gamma}(t)))\}e_2.
\end{aligned} \tag{4.4}$$

Moreover if $\omega(\dot{\gamma}(t)) = 0$, then

$$\begin{aligned}
\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma} =& \{\bar{q}[\frac{\ddot{\gamma}_1(t)}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}] - \bar{p}[\frac{\ddot{\gamma}_2(t)}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}]\}e_1 \\
& + \{\overline{r_L}\bar{p}[\frac{\ddot{\gamma}_1(t)}{A} + \frac{c(\gamma_1\dot{\gamma}_2^2 - \gamma_1\dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_2)}{A^2}] + \overline{r_L}\bar{q}[\frac{\ddot{\gamma}_2(t)}{A} + \frac{c(\dot{\gamma}_1^2\gamma_2 - \gamma_2\dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2\gamma_1)}{A^2}] \\
& + \frac{l}{l_L}L^{\frac{1}{2}}\frac{d}{dt}(\omega(\dot{\gamma}(t)))\}e_2.
\end{aligned} \tag{4.5}$$

Definition 4.1. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface, $\gamma : I \to \Sigma$ be a C^2-smooth regular curve.*

(1) If $\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}$ is a spacelike vector, the geodesic curvature $\kappa^L_{\gamma,\Sigma}$ of γ at $\gamma(t)$ is defined as

$$\kappa^L_{\gamma,\Sigma} := \sqrt{\frac{\|\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\|^2_{\Sigma,L}}{\|\dot{\gamma}\|^4_{\Sigma,L}} - \frac{\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle^2_{\Sigma,L}}{\langle \dot{\gamma}, \dot{\gamma}\rangle^3_{\Sigma,L}}}. \tag{4.6}$$

(2) If $\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}$ is a timelike vector, the geodesic curvature $\kappa^L_{\gamma,\Sigma}$ of γ at $\gamma(t)$ is defined as

$$\kappa^L_{\gamma,\Sigma} := \sqrt{\frac{\|\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\|^2_{\Sigma,L}}{\|\dot{\gamma}\|^4_{\Sigma,L}} + \frac{\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle^2_{\Sigma,L}}{\langle \dot{\gamma}, \dot{\gamma}\rangle^3_{\Sigma,L}}}. \tag{4.7}$$

Definition 4.2. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface, $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $\kappa^\infty_{\gamma,\Sigma}$ of γ at $\gamma(t)$ to be*

$$\kappa^\infty_{\gamma,\Sigma} := \lim_{L\to+\infty} \kappa^L_{\gamma,\Sigma}$$

if the limit exists.

Proposition 4.3. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface, $\gamma : I \to \Sigma$ be a C^2-smooth regular curve.*

(1) If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is a spacelike vector, then

$$\kappa^{\infty}_{\gamma,\Sigma} = \frac{2|\bar p\dot\gamma_1 + \bar q\dot\gamma_2|}{A|\omega(\dot\gamma(t))|},\ \text{if } \omega(\dot\gamma(t)) \neq 0, \tag{4.8}$$

$$\kappa^{\infty}_{\gamma,\Sigma} = 0,\ \text{if } \omega(\dot\gamma(t)) = 0,\ \text{and } \frac{d}{dt}(\omega(\dot\gamma(t))) = 0,$$

If $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, $\lim_{L\to+\infty} \frac{\kappa^L_{\gamma,\Sigma}}{\sqrt{L}}$ does not exist.

(2) If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is a timelike vector, then $\kappa^{\infty}_{\gamma,\Sigma}$ does not exist, if $\omega(\dot\gamma(t)) \neq 0$.

$$\kappa^{\infty}_{\gamma,\Sigma} = 0,\ \text{if } \omega(\dot\gamma(t)) = 0,\ \text{and } \frac{d}{dt}(\omega(\dot\gamma(t))) = 0,$$

If $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$,

$$\lim_{L\to+\infty} \frac{\kappa^L_{\gamma,\Sigma}}{\sqrt{L}} = \frac{A^2|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{(\bar q\dot\gamma_1 - \bar p\dot\gamma_2)^2}.$$

Proof. (1) If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is a spacelike vector, by (3.7) and $\dot\gamma \in T\Sigma$, we have

$$\dot\gamma(t) = \left(\frac{\dot\gamma_1(t)}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)}\right)X_1 + \left(\frac{\dot\gamma_2(t)}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)}\right)X_2 + \omega(\dot\gamma(t))X_3,$$

By (4.4), we have

$$\begin{aligned}\dot\gamma(t) &= ae_1 + be_2\\ &= (a\bar q + b\overline{r_L}\bar p)X_1 + (-a\bar p + b\overline{r_L}\bar q)X_2 - \frac{bl}{l_L}L^{-\frac{1}{2}}X_3.\end{aligned}$$

Thus

$$\begin{cases} a\bar q + b\overline{r_L}\bar p = \frac{\dot\gamma_1(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)},\\ -a\bar p + b\overline{r_L}\bar q = \frac{\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)},\\ -\frac{bl}{l_L}L^{-\frac{1}{2}} = \omega(\dot\gamma(t)),\end{cases}$$

we have

$$\begin{cases} a = \frac{\bar q\dot\gamma_1(t) - \bar p\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)},\\ b = -L^{\frac{1}{2}}\frac{l_L}{l}\omega(\dot\gamma(t)).\end{cases}$$

Thus

$$\dot\gamma = [\frac{\bar q\dot\gamma_1(t) - \bar p\dot\gamma_2(t)}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)}]e_1 - \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot\gamma(t))e_2. \tag{4.9}$$

By (4.4), we have

$$\begin{aligned}\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma\rangle_{\Sigma,L} =&\{\bar q[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}-\frac{2L\dot\gamma_2\omega(\dot\gamma(t))}{A}]\\ &-\bar p[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}+\frac{2L\dot\gamma_1\omega(\dot\gamma(t))}{A}]\}^2\\ &-\{\overline{r_L}\bar p[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}-\frac{2L\dot\gamma_2\omega(\dot\gamma(t))}{A}]\\ &+\overline{r_L}\bar q[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}+\frac{2L\dot\gamma_1\omega(\dot\gamma(t))}{A}]+\frac{l}{l_L}L^{\frac12}\frac{d}{dt}(\omega(\dot\gamma(t)))\}^2.\end{aligned} \tag{4.10}$$

Similarly, we have that when $\omega(\dot\gamma(t)) \neq 0$,

$$\begin{aligned}\langle\dot\gamma,\dot\gamma\rangle_{\Sigma,L} &= [\frac{\bar q\dot\gamma_1(t)-\bar p\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}]^2-(\frac{l_L}{l})^2L\omega^2(\dot\gamma(t))\\ &\sim -L\omega^2(\dot\gamma(t)) \text{ as } L\to+\infty.\end{aligned} \tag{4.11}$$

By (4.4) and (4.9), we have

$$\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_{\Sigma,L} \sim M_0L, \tag{4.12}$$

where M_0 does not depend on L. By Definition 4.1 and (4.10)–(4.12), we get (4.8).

When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, we have

$$\begin{aligned}\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma\rangle_{\Sigma,L} \sim&\{\bar q[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]\\ &-\bar p[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]\}^2 \text{ as } L\to+\infty\end{aligned} \tag{4.13}$$

and

$$\langle\dot\gamma,\dot\gamma\rangle_{\Sigma,L} = [\frac{\bar q\dot\gamma_1(t)-\bar p\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}]^2, \tag{4.14}$$

$$\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_{\Sigma,L} = \bar A B, \tag{4.15}$$

where $\bar A = \bar q[\frac{\ddot\gamma_1(t)}{A}+\frac{c(\gamma_1\dot\gamma_2^2-\gamma_1\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_2\gamma_2)}{A^2}]-\bar p[\frac{\ddot\gamma_2(t)}{A}+\frac{c(\dot\gamma_1^2\gamma_2-\gamma_2\dot\gamma_2^2-2\dot\gamma_1\dot\gamma_2\gamma_1)}{A^2}]$ and $B = \frac{\bar q\dot\gamma_1(t)-\bar p\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}$. By (4.13)–(4.15) and (4.7), we get

$$\kappa^\infty_{\gamma,\Sigma} = \sqrt{\frac{\bar A^2}{B^4}-\frac{\bar A^2B^2}{B^6}} = 0.$$

When $\omega(\dot\gamma(t)) = 0$, and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma\rangle_{\Sigma,L} \sim -L[\frac{d}{dt}(\omega(\dot\gamma(t)))]^2,$$

$$\langle \nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_{\Sigma,L} = O(1),$$

$$\lim_{L\to+\infty}\frac{\kappa^L_{\gamma,\Sigma}}{\sqrt L} = \lim_{L\to+\infty}\frac{1}{\sqrt L}\sqrt{\frac{-L[\frac{d}{dt}(\omega(\dot\gamma(t)))]^2}{(\frac{\bar q\dot\gamma_1(t)-\bar p\dot\gamma_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)})^4}}.$$

Therefore, this situation does not exist.

(2) If $\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}$ is a timelike vector, by similar calculations, we get claim (2). □

Definition 4.4. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface, $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. The signed geodesic curvature $\kappa^{L,s}_{\gamma,\Sigma}$ of γ at $\gamma(t)$ is defined as*

$$\kappa^{L,s}_{\gamma,\Sigma} := \frac{\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\rangle_{\Sigma,L}}{\|\dot{\gamma}\|^3_{\Sigma,L}}.$$

Definition 4.5. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface, $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $\kappa^{\infty}_{\gamma,\Sigma}$ of γ at the non-characteristic point $\gamma(t)$ to be*

$$\kappa^{\infty,s}_{\gamma,\Sigma} := \lim_{L\to+\infty} \kappa^{L,s}_{\gamma,\Sigma},$$

if the limit exists.

Proposition 4.6. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a Lorentzian regular surface.*

(1) If $\gamma : I \to \Sigma$ is a C^2-smooth regular spacelike curve, then $\kappa^{\infty,s}_{\gamma,\Sigma}$ does not exist if $\omega(\dot{\gamma}(t)) \neq 0$.

$$\kappa^{\infty,s}_{\gamma,\Sigma} = 0, \ \text{if } \omega(\dot{\gamma}(t)) = 0, \ \text{and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L\to+\infty} \frac{k^{L,s}_{\gamma,\Sigma}}{\sqrt{L}} = \frac{\left(\frac{\bar{q}\dot{\gamma}_1(t)-\bar{p}\dot{\gamma}_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}\right)\frac{d}{dt}(\omega(\dot{\gamma}(t)))}{\left|\frac{\bar{q}\dot{\gamma}_1(t)-\bar{p}\dot{\gamma}_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}\right|^3},$$

if $\omega(\dot{\gamma}(t)) = 0$, and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$.

(2) If $\gamma : I \to \Sigma$ is a C^2-smooth regular timelike curve, then

$$\kappa^{\infty,s}_{\gamma,\Sigma} = \frac{\bar{q}\dot{\gamma}_2(t) + \bar{p}\dot{\gamma}_1(t)}{(1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2))|\omega(\dot{\gamma}(t))|},$$

if $\omega(\dot{\gamma}(t)) \neq 0$; $\kappa^{\infty,s}_{\gamma,\Sigma}$ does not exist, if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$; $\lim_{L\to+\infty} \frac{k^{L,s}_{\gamma,\Sigma}}{\sqrt{L}}$ does not exist if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$.

Proof. By (4.9), we get

$$J_L(\dot{\gamma}) = -\frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))e_1 + \left[\frac{\bar{q}\dot{\gamma}_1(t) - \bar{p}\dot{\gamma}_2(t)}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)}\right]e_2. \tag{4.16}$$

By (4.4) and (4.16), we have

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\rangle \sim L^{\frac{3}{2}}\omega^2(\dot{\gamma}(t))\left[\frac{\bar{q}\dot{\gamma}_2(t) + \bar{p}\dot{\gamma}_1(t)}{(1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2))}\right] \text{ as } L \to +\infty.$$

When $\omega(\dot{\gamma}(t)) = 0$, and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we get

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\rangle_{L,\Sigma} \sim O(L^{-\frac{1}{2}}) \text{ as } L \to +\infty.$$

So $\kappa^{\infty,s}_{\gamma,\Sigma} = 0$. When $\omega(\dot{\gamma}(t)) = 0$, and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma})\rangle_{L,\Sigma} \sim L^{\frac{1}{2}}[\frac{\bar{q}\dot{\gamma}_1(t) - \bar{p}\dot{\gamma}_2(t)}{1 + \frac{c}{2}(\gamma_1^2 + \gamma_2^2)}]\frac{d}{dt}(\omega(\dot{\gamma}(t))) \text{ as } L \to +\infty.$$

We get

$$\lim_{L\to+\infty} \frac{k^{L,s}_{\gamma,\Sigma}}{\sqrt{L}} = \frac{(\frac{\bar{q}\dot{\gamma}_1(t)-\bar{p}\dot{\gamma}_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)})\frac{d}{dt}(\omega(\dot{\gamma}(t)))}{|\frac{\bar{q}\dot{\gamma}_1(t)-\bar{p}\dot{\gamma}_2(t)}{1+\frac{c}{2}(\gamma_1^2+\gamma_2^2)}|^3}.$$

Similarly, we have (2). □

5. The sub-Lorentzian limit of the Riemannian Gaussian curvature of Lorentzian surfaces in $(\mathcal{D}, g_{cL})$

In the following, we compute the sub-Lorentzian limit of the Riemannian Gaussian curvature of surfaces in $(\mathcal{D}, g_{cL})$. We define the second fundamental form II^L of the embedding of Σ into $(\mathcal{D}, g_{cL})$:

$$II^L = \begin{pmatrix} \langle \nabla^L_{e_1} v_L, e_1\rangle_L & \langle \nabla^L_{e_1} v_L, e_2\rangle_L \\ \langle \nabla^L_{e_2} v_L, e_1\rangle_L & \langle \nabla^L_{e_2} v_L, e_2\rangle_L \end{pmatrix}.$$

We have the following theorem.

Theorem 5.1. *The second fundamental form II_L of the embedding of Σ into $(\mathcal{D}, g_{cL})$ is given by*

$$II^L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

where

$$h_{11} = \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) - c(\bar{p}_L x + \bar{q}_L y),$$

$$h_{12} = h_{21} = -\frac{l_L}{l}\langle e_1, \nabla_H \bar{r_L}\rangle_L + \sqrt{L},$$

$$h_{22} = -\frac{l^2}{l_L^2}\langle e_2, \nabla^L_H(\frac{r}{l})\rangle_L + \widetilde{X_3}(\bar{r_L}) - \frac{l}{l_L}\sqrt{L}\bar{p}_L\bar{q}_L\bar{r}_L.$$

Proof. Since $\langle e_1, v_L\rangle_L = 0, \langle e_2, v_L\rangle_L = 0$, we have

$$\langle \nabla^L_{e_1} v_L, e_1\rangle_L = -\langle \nabla^L_{e_1} e_1, v_L\rangle_L, \langle \nabla^L_{e_2} v_L, e_2\rangle_L = -\langle \nabla^L_{e_2} e_2, v_L\rangle_L.$$

Using the definition of the connection, the identities in (2.4) and grouping terms, we have

$$\begin{aligned}\nabla^L_{e_1} e_1 =& \nabla^L_{\bar{q}X_1 - \bar{p}X_2} \bar{q}X_1 - \bar{p}X_2 \\ =& [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{q})]X_1 - [\bar{q}X_1(\bar{p}) - \bar{p}X_2(\bar{p})]X_2 + [(\bar{p}\bar{q}cy + \bar{p}^2cx)X_1 + (\bar{p}\bar{q}cx + \bar{p}^2cy)X_2].\end{aligned}$$

Since $\bar{p}^2 + \bar{q}^2 = 1$, we have $\bar{p}X_i\bar{p} + \bar{q}X_i\bar{q} = 0, i = 1, 2, 3$. Thus $\bar{q}X_1\bar{q} = -\bar{p}X_1\bar{p}$, $\bar{q}X_2\bar{q} = -\bar{p}X_2\bar{p}$, we have

$$\nabla^L_{e_1} e_1 = -\bar{p}(X_1(\bar{p}) + X_2(\bar{q}))X_1 - \bar{q}(X_1(\bar{p}) + X_2(\bar{q}))X_2 + [(\bar{p}\bar{q}cy + \bar{p}^2cx)X_1 + (\bar{p}\bar{q}cx + \bar{p}^2cy)X_2].$$

Next, we compute the inner product of this with v_L, we obtain

$$\begin{aligned} h_{11} =& -\langle \nabla^L_{e_1} e_1, v_L\rangle_L \\ =& \bar{p}\bar{p}_L(X_1(\bar{p}) + X_2(\bar{q})) + \bar{q}\bar{q}_L(X_1(\bar{p}) + X_2(\bar{q})) - [(\bar{p}_L\bar{p}^2 + \bar{q}_L\bar{p}\bar{q})cx + (\bar{q}_L\bar{q}^2 + \bar{p}_L\bar{p}\bar{q})cy] \\ =& \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) - c(\bar{p}_L x + \bar{q}_L y). \end{aligned}$$

To compute h_{12} and h_{21}, using the definition of the connection, we obtain

$$\begin{aligned} \nabla^L_{e_1} e_2 =& \nabla^L_{\bar{q}X_1 - \bar{p}X_2} \bar{r_L}\bar{p}X_1 + \bar{r_L}\bar{q}X_2 - \frac{l}{l_L}L^{-\frac{1}{2}}X_3 \\ =& (\bar{q}X_1(\bar{r_L}\bar{p}) - \bar{p}X_2(\bar{r_L}\bar{p}))X_1 + (\bar{q}X_1(\bar{r_L}\bar{q}) - \bar{p}X_2(\bar{r_L}\bar{q}))X_2 + (\frac{-\bar{q}}{\sqrt{L}}X_1(\frac{l}{l_L}) + \frac{\bar{p}}{\sqrt{L}}X_2(\frac{l}{l_L}))X_3 \\ =& (-\bar{r}_L\bar{q}^2cy - \bar{r}_L\bar{p}\bar{q}cx - \bar{p}\frac{l\sqrt{L}}{l_L})X_1 + (\bar{r}_L\bar{p}\bar{q}cy + \bar{r}_L\bar{p}^2cx - \bar{q}\frac{l\sqrt{L}}{l_L})X_2 + (\bar{r}_L)X_3. \end{aligned}$$

Next, we compute the inner product of this with v_L. We get

$$\langle \nabla_{e_1} e_2, v_L\rangle_L = \frac{l_L}{l}\langle e_1, \nabla_H \bar{r_L}\rangle_L - \sqrt{L}.$$

Therefore

$$\begin{aligned} h_{12} = h_{21} =& -\langle \nabla_{e_1} e_2, v_L\rangle_L \\ =& -\frac{l_L}{l}\langle e_1, \nabla_H \bar{r_L}\rangle_L + \sqrt{L}. \end{aligned}$$

Since $\langle \nabla_{e_2} v_L, e_2\rangle_L = -\langle \nabla_{e_2} e_2, v_L\rangle_L$, using the definition of connection, the identities in (2.4) and grouping terms, we have

$$\begin{aligned} \nabla^L_{e_2} e_2 =& [\bar{r_L}\bar{p}X_1\bar{r_L}\bar{p} + \bar{r_L}\bar{q}X_2\bar{r_L}\bar{p} - \frac{l}{\sqrt{L}l_L}X_3\bar{r_L}\bar{p}]X_1 + [\bar{r_L}\bar{p}X_1\bar{r_L}\bar{q} + \bar{r_L}\bar{q}X_2\bar{r_L}\bar{q}]X_2 \\ &+(-\frac{\bar{r}_L\bar{p}}{\sqrt{L}}X_1\frac{l}{l_L} - \frac{\bar{r_L}\bar{q}}{\sqrt{L}}X_2\frac{l}{l_L} + \frac{l}{l_L L}X_3\frac{l}{l_L})X_3 + (-\bar{r}_L^2\bar{p}\bar{q}cy + \bar{r}_L^2\bar{q}^2cx + 2\bar{r}_L\bar{q}\frac{l}{l_L}\sqrt{L})X_1 \\ &+(\bar{r}_L^2\bar{p}^2cy - \bar{r}_L^2\bar{p}\bar{q}cx - \bar{r}_L\bar{p}\frac{l}{l_L}\sqrt{L})X_2. \end{aligned}$$

Taking the inner product with v_L and under some similar simplifications to Theorem 4.3 in [4], we get

$$\begin{aligned} h_{22} =& -\langle \nabla^L_{e_2} e_2, v_L\rangle_L \\ =& -\frac{l^2}{l_L^2}\langle e_2, \nabla^L_H(\frac{r}{l})\rangle_L + \widetilde{X_3}(\bar{r_L}) - \frac{l}{l_L}\sqrt{L}\bar{p}\bar{q}\bar{r}_L. \end{aligned}$$

□

Let

$$\mathcal{K}^{\Sigma,L}(e_1, e_2) = \langle -R^{\Sigma,L}(e_1, e_2)\, e_1, e_2\rangle_{\Sigma,L},\ \mathcal{K}^L(e_1, e_2) = -\langle R^L(e_1, e_2)\, e_1, e_2\rangle_L.$$

By the Gauss equation, we have

$$\mathcal{K}^{\Sigma,L}(e_1, e_2) = \mathcal{K}^L(e_1, e_2) + det(II^L). \tag{5.1}$$

Proposition 5.2. *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma,\infty}(e_1, e_2) = 2\langle e_1, \nabla_H(\frac{X_3 u}{|\nabla_H u|})\rangle, \text{as } L \to +\infty.$$

Proof. We compute

$$\begin{aligned}
R^L(e_1, e_2)\, e_1 &= R^L\left(\bar{q}X_1 - \bar{p}X_2, \bar{r}_L\bar{p}X_1 + \bar{r}_L\bar{q}X_2 - \frac{l}{l_L\sqrt{L}}X_3\right)(\bar{q}X_1 - \bar{p}X_2)\\
&= \bar{r}_L\bar{p}\bar{q}^2R^L(X_1, X_1)\, X_1 + \bar{r}_L\bar{q}^3R^L(X_1, X_2)\, X_1 - \frac{l\bar{q}^2}{l_L\sqrt{L}}R^L(X_1, X_3)\, X_1\\
&\quad - \bar{r}_L\bar{p}^2\bar{q}R^L(X_2, X_1)\, X_1 - \bar{r}_L\bar{p}\bar{q}^2R^L(X_2, X_2)\, X_1 + \frac{l\bar{p}\bar{q}}{l_L\sqrt{L}}R^L(X_2, X_3)\, X_1\\
&\quad - \bar{r}_L\bar{p}^2\bar{q}R^L(X_1, X_1)\, X_2 - \bar{r}_L\bar{p}\bar{q}^2R^L(X_1, X_2)\, X_2 + \frac{l\bar{p}\bar{q}}{l_L\sqrt{L}}R^L(X_1, X_3)\, X_2\\
&\quad + \bar{r}_L\bar{p}^3R^L(X_2, X_1)\, X_2 + \bar{r}_L\bar{p}^2\bar{q}R^L(X_2, X_2)\, X_2 - \frac{l\bar{p}^2}{l_L\sqrt{L}}R^L(X_2, X_3)\, X_2\\
&= -\bar{r}_L\bar{p}(3L + 2c)X_1 - \bar{r}_L\bar{q}(3L + 2c)X_2 - (\frac{l\sqrt{L}}{l_L} + 2\bar{r}_L\bar{p}cy)X_3
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}^L(e_1, e_2) &= -\left\langle R^L(e_1, e_2)\, e_1, e_2\right\rangle_L\\
&= \bar{r}_L^2(3L + 2c) + (\frac{l}{l_L})^2L + 2\sqrt{L}\frac{l}{l_L}\bar{r}_L\bar{p}cy.
\end{aligned}$$

To simplify this, we find $L\bar{r}_L^2 \sim \frac{(X_3u)^2}{l^2}$ as $L \to \infty$. Finally, we get

$$\mathcal{K}^L(e_1, e_2) \sim \frac{(X_3u)^2}{l^2} + (\frac{l}{l_L})^2L \text{ as } L \to \infty. \tag{5.2}$$

By Theorem 5.1 and $\nabla_H(\bar{r}_L) = L^{-\frac{1}{2}}\nabla_H\left(\frac{X_3u}{|\nabla_H u|}\right) + O\left(L^{-1}\right)$ as $L \to +\infty$, we get

$$\begin{aligned}
\det\left(II^L\right) &= h_{11}h_{22} - h_{12}^2 \\
&= -L + 2\langle e_1, \nabla_H(\frac{X_3u}{|\nabla_H u|})\rangle + O\left(L^{-1}\right)
\end{aligned} \tag{5.3}$$

as $L \to +\infty$. Note that $L(\frac{l}{l_L})^2 - L = -L\frac{l_L^2 - l^2}{l_L^2} = -L\frac{r^2}{l_L^2} = -\frac{L}{l_L^2}\frac{(X_3u)^2}{L} = -\frac{(X_3u)^2}{l_L^2} \sim -\frac{(X_3u)^2}{l^2}$ as $L \to +\infty$. By (5.1)–(5.3), we get the desired equation. □

6. A Gauss-Bonnet theorem in $(\mathcal{D}, g_{cL})$

Let us first consider the case of a regular timelike curve $\gamma : I \to (\mathcal{D}, g_{cL})$, we define the Riemannian length measure $ds_L = \|\dot{\gamma}\|_L dt$.

Lemma 6.1. *Let $\gamma : I \to (\mathcal{D}, g_{cL})$ be a C^2-smooth timelike curve. Let*

$$ds := |\omega(\dot{\gamma}(t))|\, dt,\ d\bar{s} := \frac{1}{2}\frac{1}{|\omega(\dot{\gamma}(t))|}\left(-\dot{\gamma}_1^2 - \dot{\gamma}_2^2\right) dt.$$

Then

$$\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\gamma ds_L = \int_a^b ds.$$

When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = ds + d\bar{s}L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty,$$

The situation of $\omega(\dot{\gamma}(t)) = 0$ does not exist.

Proof. We know that

$$\|\dot{\gamma}(t)\|_L = \sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2 + L(\omega(\dot{\gamma}(t)))^2},$$

similar to the proof of Lemma 6.1 in [5], we can prove

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\gamma \|\dot{\gamma}(t)\|_L dt &= \int_a^b \lim_{L\to\infty}\frac{1}{\sqrt{L}}\|\dot{\gamma}(t)\|_L dt\\ &= \int_a^b \lim_{L\to\infty}\frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2 + L(\omega(\dot{\gamma}(t)))^2} dt\\ &= \int_a^b |\omega(\dot{\gamma}(t))|\, dt\\ &= \int_a^b ds\end{aligned}$$

is desired. When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = \sqrt{L^{-1}\left(-\dot{\gamma}_1^2 - \dot{\gamma}_2^2\right) + \omega(\dot{\gamma}(t))^2} dt.$$

Using the Taylor expansion, we can prove

$$\frac{1}{\sqrt{L}} ds_L = ds + d\bar{s}L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty.$$

From the definition of ds_L and $\omega(\dot{\gamma}(t)) = 0$, we get

$$\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2} dt,$$

therefore, this situation does not exist. □

Proposition 6.2. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a regular Lorentzian C^2-smooth surface and $\Sigma = \{u = 0\}$ and $d\sigma_{\Sigma,L}$ denote the surface measure on Σ with respect to the metric g_{cL}. Let*

$$d\sigma_\Sigma := (\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega,\ d\bar{\sigma}_\Sigma := \frac{X_3 u}{l}\omega_1 \wedge \omega_2 + \frac{(X_3 u)^2}{2l^2}(\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega.$$

Then

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_\Sigma + d\bar{\sigma}_\Sigma L^{-1} + O\left(L^{-2}\right),\ \text{as } L \to +\infty. \tag{6.1}$$

If $\Sigma = f(D)$ with $f = f(u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to (\mathcal{D}, g_{cL})$ then

$$\lim_{L\to\infty} \frac{1}{\sqrt{L}} \int_\Sigma d\sigma_{\Sigma,L} = \int_D \frac{1}{C^2}\{[f_2((f_1)_{u_2}(f_2)_{u_1} - (f_1)_{u_1}(f_2)_{u_2}) + C((f_2)_{u_1}(f_3)_{u_2} - (f_2)_{u_2}(f_3)_{u_1})]^2$$
$$+ [f_1((f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}) + C((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})]^2\}^{\frac{1}{2}} du_1 du_2,$$

where $C = 1 + \frac{c}{2}(f_1^2 + f_2^2)$.

Proof. It is well known that

$$g_L(X_1, \cdot) = \omega_1,\ g_L(X_2, \cdot) = \omega_2,\ g_L(X_3, \cdot) = -L\omega.$$

We define $e_1{}^* := g_L(e_1, \cdot),\ e_2{}^* := g_L(e_2, \cdot)$, then

$$e_1^* = \bar{q}\omega_1 - \bar{p}\omega_2,\ e_2^* = \bar{r}_L\bar{p}\omega_1 + \bar{r}_L\bar{q}\omega_2 - \frac{l}{l_L}L^{\frac{1}{2}}\omega.$$

Therefore

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^* \wedge e_2^* = \frac{l}{l_L}(\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega + \frac{1}{\sqrt{L}}\bar{r}_L\omega_1 \wedge \omega_2.$$

Recalling

$$\bar{r}_L = \frac{(X_3 u) L^{-\frac{1}{2}}}{\sqrt{p^2 + q^2 - L^{-1}(X_3 u)^2}}$$

and the Taylor expansion

$$\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3}(X_3 u)^2 L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty,$$

we get (6.1). By (2.3), we have

$$\begin{aligned} f_{u_1} &= (f_1)_{u_1} \partial x_1 + (f_2)_{u_1} \partial x_2 + (f_3)_{u_1} \partial x_3 \\ &= \frac{1}{C}(f_1)_{u_1} X_1 + \frac{1}{C}(f_2)_{u_1} X_2 + \sqrt{L}\left(\frac{f_2}{C}(f_1)_{u_1} - \frac{f_1}{C}(f_2)_{u_1} + (f_3)_{u_1}\right)\widetilde{X_3}, \end{aligned}$$

$$\begin{aligned} f_{u_2} &= (f_1)_{u_2}\partial x_1 + (f_2)_{u_2}\partial x_2 + (f_3)_{u_2}\partial\theta \\ &= \frac{1}{C}(f_1)_{u_2} X_1 + \frac{1}{C}(f_2)_{u_2} X_2 + \sqrt{L}\left(\frac{f_2}{C}(f_1)_{u_2} - \frac{f_1}{C}(f_2)_{u_2} + (f_3)_{u_2}\right)\widetilde{X_3}. \end{aligned}$$

Let

$$\bar{v}_L = \begin{vmatrix} X_1 & X_2 & -\widetilde{X_3} \\ \frac{1}{C}(f_1)_{u_1} & \frac{1}{C}(f_2)_{u_1} & \sqrt{L}\left(\frac{f_2}{C}(f_1)_{u_1} - \frac{f_1}{C}(f_2)_{u_1} + (f_3)_{u_1}\right) \\ \frac{1}{C}(f_1)_{u_2} & \frac{1}{C}(f_2)_{u_2} & \sqrt{L}\left(\frac{f_2}{C}(f_1)_{u_2} - \frac{f_1}{C}(f_2)_{u_2} + (f_3)_{u_2}\right) \end{vmatrix} \tag{6.2}$$

$$\begin{aligned} =&\frac{\sqrt{L}}{C^2}\left[f_2((f_1)_{u_2}(f_2)_{u_1} - (f_1)_{u_1}(f_2)_{u_2}) + C((f_2)_{u_1}(f_3)_{u_2} - (f_2)_{u_2}(f_3)_{u_1})\right] X_1 \\ &+ \frac{\sqrt{L}}{C^2}\left[f_1((f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}) + C((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})\right] X_2 \\ &- \frac{1}{C^2}\left[(f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}\right]\widetilde{X_3}. \end{aligned}$$

We know that $d\sigma_{\Sigma,L} = \sqrt{-\det\left(g_{ij}\right)}du_1du_2,\ g_{ij} = g_L\left(f_{u_i}, f_{u_j}\right)$, and

$$\begin{aligned} \det\left(g_{ij}\right) &= -\langle \bar{v}_L, \bar{v}_L\rangle_L \\ &= -\frac{L}{C^4}\left[f_2((f_1)_{u_2}(f_2)_{u_1} - (f_1)_{u_1}(f_2)_{u_2}) + C((f_2)_{u_1}(f_3)_{u_2} - (f_2)_{u_2}(f_3)_{u_1})\right]^2 \\ &- \frac{L}{C^4}\left[f_1((f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}) + C((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})\right]^2 \\ &+ \frac{1}{C^4}\left[(f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}\right]^2, \end{aligned}$$

so by the dominated convergence theorem, we get

$$\begin{aligned} \lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\Sigma d\sigma_{\Sigma,L} = \int_D \frac{1}{C^2}\{&[f_2((f_1)_{u_2}(f_2)_{u_1} - (f_1)_{u_1}(f_2)_{u_2}) + C((f_2)_{u_1}(f_3)_{u_2} - (f_2)_{u_2}(f_3)_{u_1})]^2 \\ &+ [f_1((f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}) + C((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})]^2\}^{\frac{1}{2}}du_1du_2. \end{aligned}$$

□

Similar to the proof of Theorem 4.3 in [7], we get a Gauss-Bonnet theorem in $(\mathcal{D}, g_{cL})$ as following:

Theorem 6.3. *Let $\Sigma \subset (\mathcal{D}, g_{cL})$ be a regular Lorentzian surface with finitely many boundary components $(\partial\Sigma)_i, i \in \{1, \cdots, n\}$, given by Euclidean C^2-smooth regular and closed timelike curves $\gamma_i : [0, 2\pi] \to (\partial\Sigma)_i$. Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}^1(C(\Sigma)) = 0$ and that $\| \nabla_H u \|_H^{-1}$ is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$, then*

$$\int_\Sigma \mathcal{K}^{\Sigma,\infty}d\sigma_\Sigma + \sum_{i=1}^n \int_{\gamma_i} \kappa^{\infty,s}_{\gamma_i,\Sigma}ds = 0.$$

Proof. Using the similar discussions in [5, 6], we assume that all points satisfy $\omega(\dot{\gamma}_i(t)) \neq 0$ on the curve γ_i. Recalling the result in Lemma 3.6 indicates

$$\kappa^{L,s}_{\gamma_i,\Sigma} = \kappa^{\infty,s}_{\gamma_i,\Sigma} + O\left(L^{-1}\right). \tag{6.3}$$

According to the Gauss-Bonnet theorem, we get

$$\int_\Sigma \mathcal{K}^{\Sigma,L}\frac{1}{\sqrt{L}}d\sigma_{\Sigma,L} + \sum_{i=1}^n \int_{\gamma_i} \kappa^{L,s}_{\gamma_i,\Sigma}\frac{1}{\sqrt{L}}ds_L = 2\pi\frac{\chi(\Sigma)}{\sqrt{L}}. \tag{6.4}$$

Let L go to the infinity, and use the dominated convergence theorem, we get the desired result. □

7. Conclusions

This paper dealt with an interesting question of the Gauss-Bonnet theorem in Lorentzian Sasakian space forms from the Lorentzian approximation scheme. The main result of this paper is Theorem 6.3, which is the Gauss-Bonnet-type theorem in the Lorentzian Sasakian space forms. To prove Theorem 6.3, we obtained the sub-Lorentzian limit of the curvature of curves, sub-Lorentzian limits of the geodesic curvature of curves on Lorentzian surfaces, and the sub-Lorentzian limit of the Riemannian Gaussian curvature of Lorentzian surfaces in the Lorentzian Bianci-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian space forms.

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Conflict of interest

The authors declare that there is no conflicts of interests in this work.

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