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Research article

On 2-variable q-Hermite polynomials

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Abstract: The quantum calculus has emerged as a connection between mathematics and physics. It has wide applications, particularly in quantum mechanics, analytic number theory, combinatorial analysis, operation theory etc. The q-calculus, which serves as a powerful tool to model quantum computing, has drawn attention of many researchers in the field of special functions and as a result the q-analogues of certain special functions, especially hypergeometric function, 1-variable Hermite polynomials, Appell polynomials etc., are introduced and studied. In this paper, we introduce the 2-variable q-Hermite polynomials by means of generating function. Also, its certain properties like series definition, recurrence relations, q-differential equation and summation formulas are established. The operational definition and some integral representations of these polynomials are obtained. Some examples are also considerd to show the efficacy of the proposed method. Some concluding remarks are given. At the end of this paper, the graphical representations of these polynomials of certain degrees with specified values of q are given.

Keywords: quantum calculus; post quantum calculus; q-Hermite polynomials; shift operator
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1. Introduction

The quantum calculus, briefly called q-calculus, is infact the investigation of calculus without limits. The q-calculus is a generalization of ordinary calculus as for $q \to 1^-$, the quantum calculus reduces to the ordinary calculus. Recently, it has got attention of many researchers in the field of applied mathematics due to the fact that it provides a connection between mathematics and physics. It has been proved very helpful in the mathematical modelling of the problems arising in quantum computing. This motivates the study of certain special functions in the context of q-calculus. The q-analogues of some special functions like q-Gamma and q-Beta functions, 1-variable q-Hermite polynomials and q-Hermite polynomials-Appell polynomials etc., are introduced and studied [5, 6, 15, 17, 19].

In this paper, we introduce the 2-variable q-Hermite polynomials and investigate some of its properties like series definition, recurrence relations, q-differential equation, summation formulas, operational definition and some integral representations.

We review briefly some definitions and notations of the quantum calculus.

The q-analogue of a complex number α is defined as [2, 3, 11, 12]:

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} \qquad (0<q<1). \tag{1.1}$$

The first five few q-numbers are as follows:

$$[0]_q = 0,\ [1]_q = 1,\ [2]_q = 1+q,\ [3]_q = 1+q+q^2,\ [4]_q = 1+q+q^2+q^3.$$

The q-factorial is defined as:

$$[n]_q! = \begin{cases} \prod_{k=1}^{n}[k]_q!, & 0<q<1 \quad n \geq 1 \\ 1, & n=0, \end{cases}$$

which satisfies

$$[n+1]_q! = [n+1]_q[n]_q!. \tag{1.2}$$

The Gauss q-binomial coefficient is defined as [2, 3, 11, 12]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q\,!}{[n-k]_q![k]_q!} = \frac{(1;q)_n}{(1;q)_k(1;q)_{n-k}}. \qquad k=0,1,\ldots,n. \tag{1.3}$$

The raising and lowering q-powers are defined as [2, 3, 11, 12]:

$$(u \pm a)_q^n = \begin{cases} (u\pm a)(u\pm aq)...(u\pm aq^{n-2})(u\pm aq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n=0, \end{cases} \tag{1.4}$$

or, equivalently

$$(u \pm a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} u^k (\pm a)^{n-k}, \tag{1.5}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is given by Eq (1.3). For $n=1$, it is obvious that $(u\pm a)_q^1 = (u \pm a)$.

In particular, for $a=0$, Eq (1.4) or (1.5), gives

$$(u)_q^n = u^n. \tag{1.6}$$

The two q-exponential functions, denoted by $e_q(u)$ and $E_q(u)$, are defined as [2, 3, 11, 12]:

$$e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} = \frac{1}{(u;q)_\infty}, \qquad |u| < 1, \qquad |q| < 1 \tag{1.7}$$

and

$$E_q(u) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{u^n}{[n]_q!} = (-u;q)_\infty \qquad |q| < 1, \tag{1.8}$$

respectively. The relation between both q-exponential functions is as follows:

$$e_q(u)E_q(-u) = 1. \tag{1.9}$$

Next, the q-derivative of a function f with respect to u, denoted by $D_{q,u}f(u)$, is defined as [13]:

$$D_{q,u}f(u) = \frac{f(qu) - f(u)}{qu - u}, \qquad 0 < q < 1, \ u \neq 0. \tag{1.10}$$

Also, for two arbitrary functions $g(u)$ and $h(u)$, we have

$$D_{q,u}\big(g(u)h(u)\big) = g(u)D_{q,u}h(u) + h(qu)D_{q,u}g(u). \tag{1.11}$$

In particular, we have

$$D_{q,u}\, u^n = [n]_q u^{n-1}, \tag{1.12}$$

$$D_{q,u}e_q(\alpha u) = \alpha e_q(\alpha u) \tag{1.13}$$

and

$$D_{q,u}E_q(\alpha u) = \alpha E_q(\alpha q u).$$

The m^{th} order q-derivative of the q-exponential functions are as follows:

$$D^m_{q,u}e_q(\alpha u) = \alpha^m e_q(\alpha u) \qquad (m \geq 1) \tag{1.14}$$

and

$$D^m_{q,u}E_q(\alpha u) = \alpha^m q^{\binom{m}{2}} E_q(\alpha q^m u) \qquad (m \geq 1),$$

where $D^m_{q,u}$ denotes the m^{th} order q-derivative with respect to u.

Also, the q-definite integral of a function g is defined as [11, 12, 14]:

$$\int_0^a g(u)d_q u = (1-q)a \sum_{n=0}^{\infty} \frac{1}{q^{n+1}}\, g\left(\frac{a}{q^{n+1}}\right), \qquad 0 < \ | \, q \, | \ < 1, a \in \mathbb{R} \tag{1.15}$$

and

$$\int_a^b g(u)d_q u = \int_0^b g(u)d_q u - \int_0^a g(u)d_q u. \tag{1.16}$$

From Eqs (1.15) and (1.16), it is clear that

$$\int_a^b \Big(g(u) + h(u)\Big) d_q u = \int_a^b g(u)d_q u + \int_a^b h(u)d_q u. \tag{1.17}$$

The q-definite integral of the q-derivative of a function $g(u)$ is given as:

$$\int_a^b D_q g(u) d_q u = g(b) - g(a). \tag{1.18}$$

We note that, for $q \to 1^-$, all the results in the q-calculus reduce to the corresponding results in ordinary calculus.

The Hermite polynomials are one of the most applicable classical orthogonal special functions. They are the solutions of the differential equations, which are equivalent to the Schrdinger equation for a harmonic oscillator in quantum mechanics and thus they appear as eigen functions. Moreover, these polynomials have significant role in the study of classical boundary-value problems in parabolic regions, through the use of parabolic coordinates, or in quantum mechanics as well as in other application areas.

Now, we recall the generating function and series definition of the classical Hermite polynomials $H_n(u)$ and the 2-variable Hermite polynomials $H_n(u,v)$. The classical Hermite polynomials $H_n(u)$ are defined by means of the following generating function [1]:

$$e^{2ut-t^2} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \tag{1.19}$$

and the series definition [1]:

$$H_n(u) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2u)^{n-2k}. \tag{1.20}$$

The 2-variable Hermite polynomials (2VHP) $H_n(u,v)$ are defined by means of the following generating function [4]:

$$e^{(ut+vt^2)} = \sum_{n=0}^{\infty} H_n(u,v) \frac{t^n}{n!} \tag{1.21}$$

and the series definition [4]:

$$H_n(u,v) = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} v^k u^{n-2k}. \tag{1.22}$$

The pure and the differential recurrence relations for $H_n(u,v)$ are defined as [4]:

$$H_{n+1}(u,v) = uH_n(u,v) + 2nvH_{n-1}(u,v) \qquad (n \geq 1), \tag{1.23}$$

$$H_{n+1}(u,v) = uH_n(u,v) + 2vD_u H_n(u,v), \tag{1.24}$$

$$nH_n(u,v) = uD_u H_n(u,v) + 2nvD_u H_{n-1}(u,v) \qquad (n \geq 1), \tag{1.25}$$

$$nH_n(u,v) = nuH_{n-1}(u,v) + 2vD_v H_n(u,v) \qquad (n \geq 1), \tag{1.26}$$

and

$$n(n-1)H_{n-1}(u,v) = uD_v H_n(u,v) + 2nuD_v H_{n-1}(u,v) \quad (n \geq 1). \tag{1.27}$$

The 2VHP $H_n(u,v)$ satisfies the following differential equation [4]:

$$2v\frac{\partial^2}{\partial u^2}H_n(u,v)+u\frac{\partial}{\partial u}H_n(u,v)=nH_n(u,v). \tag{1.28}$$

From Eqs (1.19) and (1.21), it is clear that the 2VHP $H_n(u,v)$ is related with the classical Hermite polynomials $H_n(u)$ as:

$$H_n(2u,-1)=H_n(u). \tag{1.29}$$

The q-Hermite polynomials of 1-variable, which is defined in several ways, has many literatures due to its wide applications in various fields of mathematics and physics. For instance, Berg and Ismail [6], have shown that 1-variable q-Hermite polynomials can be used to build the classical q-orthogonal polynomials systemetically.

Recently, Nalci and Pashaev [17] defined the q-Hermite polynomials by means of the following generating function :

$$e_q\big([2]_q ut\big)\ e_q(-t^2)=\sum_{n=0}^{\infty}H_{n,q}(u)\frac{t^n}{[n]_q!} \tag{1.30}$$

and series definition

$$H_{n,q}(u)=[n]_q!\sum_{k=0}^{[n/2]}\frac{(-1)^k([2]_q u)^{n-2k}}{[n-2k]_q![k]_q!}, \tag{1.31}$$

in the context of q-analog of shock soliton solution.

In this paper, we introduce the 2-variable q-Hermite polynomials and study certain properties of these polynomials. An operational definition and some integral representations of these polynomials are obtained. Some examples are also considerd to show the efficacy of the proposed method. Some concluding remarks are given. At the end of this paper the graphical representations of these polynomials of certain degrees with specified values of q are given.

In the next section, we introduce 2-variable q-Hermite polynomials with the help of generating function and we obtain their series definition, recurrence relations, operational differential equations of 2VqHP.

2. The 2-variable q-Hermite polynomials

In this section, we introduce the 2-variable q-Hermite polynomials and obtain their series definition, recurrence relations and differential equation.

In view of Eqs (1.21), (1.29) and (1.30), we define the 2-variable q-Hermite polynomials (2VqHP) $H_{n,q}(u,v)$ by means of the following generating function:

$$e_q(ut)\ e_q(vt^2)=\sum_{n=0}^{\infty}H_{n,q}(u,v)\frac{t^n}{[n]_q\,!}. \tag{2.1}$$

The series on the right hand side of the above equation converges in the finite complex plane.

Expanding the left hand side of Eq (2.1) by using Eq (1.7), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{u^n v^k\ t^{n+2k}}{[n]_q![k]_q!}=\sum_{n=0}^{\infty}H_{n,q}(u,v)\frac{t^n}{[n]_q\,!},$$

which on using the following series rearrangement technique [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]}A(m,n-2m), \tag{2.2}$$

gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{u^{n-2k}v^k\,t^n}{[n-2k]_q![k]_q!}=\sum_{n=0}^{\infty}H_{n,q}(u,v)\frac{t^n}{[n]_q\,!}.$$

Equating the coefficients of equal powers of t from both sides of the above equation, we have the following series definition of 2VqHP $H_{n,q}(u,v)$:

$$H_{n,q}(u,v)=[n]_q!\sum_{k=0}^{[n/2]}\frac{u^{n-2k}v^k}{[n-2k]_q![k]_q!}, \tag{2.3}$$

where [.] denotes the greatest integer function. Being a finite power series, the series on the right hand side of the above equation converges in the finite complex plane.

Taking $v=0$ and $u=0$ one by one in Eq (2.3), we have the following boundary conditions:

$$H_{n,q}(u,0)=(u)_q^n,\qquad H_{n,q}(0,v)=\frac{[n]_q!(v)_q^{[n/2]}}{[[n/2]]_q!},\quad n=0,1,2\dots\ . \tag{2.4}$$

Further, from Eq (2.3), it can be easily verified that

$$H_{n,q}(au,a^2v)=a^nH_{n,q}(u,v), \tag{2.5}$$

where a is a constant.

Now, taking q-partial derivative of both the sides of Eq (2.1) with respect to u and then using Eq (1.13) in the resultant equation, we have

$$te_q(ut)e_q(vt^2)=\sum_{n=0}^{\infty}D_{q,u}H_{n,q}(u,v)\frac{t^n}{[n]_q!}. \tag{2.6}$$

Using Eq (2.1) in the left hand side of Eq (2.6), we have

$$\sum_{n=0}^{\infty}H_{n,q}(u,v)\frac{t^{n+1}}{[n]_q!}=\sum_{n=0}^{\infty}D_{q,u}H_{n,q}(u,v)\frac{t^n}{[n]_q\,!},$$

or, equivalently

$$\sum_{n=1}^{\infty}H_{n-1,q}(u,v)\frac{t^n}{[n-1]_q!}=\sum_{n=0}^{\infty}D_{q,u}H_{n,q}(u,v)\frac{t^n}{[n]_q!}.$$

Comparing the equal powers of t from both the sides of the above equation, we have

$$D_{q,u}H_{n,q}(u,v)=[n]_qH_{n-1,q}(u,v)\qquad(n\geq 1)\ . \tag{2.7}$$

Again, taking 2^{nd} order q-partial derivative of both the sides of Eq (2.1) with respect to u, then using Eq (1.13) for $m=2$ and following the steps involved in the proof of Eq (2.7), we have

$$D_{q,u}^2H_{n,q}(u,v)=[n]_q[n-1]_qH_{n-2,q}(u,v)\qquad(n\geq 2). \tag{2.8}$$

Similarly, taking m^{th} order q-partial derivative of both sides of Eq (2.1) with respect to u, then using Eq (1.13) and following the same steps, we get the following m^{th} order q-partial derivative of $H_{n,q}(u,v)$ with respect to u:

$$D^m_{q,u}\Big(H_{n,q}(u,v)\Big) = \frac{[n]_q!}{[n-m]_q!}H_{n-m,q}(u,v) \qquad (0 \leq m \leq n). \tag{2.9}$$

Next, taking q-partial derivative of both sides of Eq (2.1) with respect to v and following the steps involved in the proof of Eq (2.7), we have

$$D_{q,v}\Big(H_{n,q}(u,v)\Big) = [n]_q[n-1]_qH_{n-2,q}(u,v) \qquad (n \geq 2). \tag{2.10}$$

Similarly, taking m^{th} order q-partial derivative of both sides of Eq (2.1) with respect to v and following the same steps, we have

$$D^m_{q,v}\Big(H_{n,q}(u,v)\Big) = \frac{[n]_q!}{[n-2m]_q!}H_{n-2m,q}(u,v) \qquad (m \geq 1). \tag{2.11}$$

Again, in view of Eqs (2.7), (2.10) and (2.5), it is easy to verify that the 2VqHP $H_{n,q}(u,v)$ satisfy the following q-partial differential equation:

$$D^2_{q,u}H_{n,q}(u,v) = D_{q,v}H_{n,q}(u,v). \tag{2.12}$$

Next, we proceed to establish the recurrence relations for the 2VqHP $H_{n,q}(u,v)$. For this, we need to prove the following lemma:

Lemma 2.1. *If $D_{q,t}$ denotes the q-partial derivative with respect to t, then*

$$D_{q,t}\; e_q(vt^2) = vt\; e_q(vt^2) + qvt\; e_q(qvt^2). \tag{2.13}$$

Proof. Using Eqs (1.7) and (1.13), we have

$$D_{q,t}\; e_q(vt^2) = \sum_{n=0}^{\infty} \frac{v^n[2n]_q t^{2n-1}}{[n]_q!}. \tag{2.14}$$

The series on the right hand side of the above equation converges in the finite complex plane.

Since, from Eq (1.1), we have $\frac{[2n]_q}{[n]_q}$ $=1+q^n$, therefore using Eq (1.2) in the right hand side of the above equation, we have

$$D_{q,t}\; e_q(vt^2) = \sum_{n=1}^{\infty} \frac{v^n(1+q^n)t^{2n-1}}{[n-1]_q!},$$

which on simplifying the right hand side, gives

$$\begin{aligned} D_{q,t}\; e_q(vt^2) &= \sum_{n=1}^{\infty} \frac{v^n t^{2n-1}}{[n-1]_q!} + \sum_{n=1}^{\infty} \frac{(qv)^n t^{2n-1}}{[n-1]_q!} \\ &= vt\sum_{n=0}^{\infty} \frac{(vt^2)^n}{[n]_q!} + qvt\sum_{n=0}^{\infty} \frac{(qvt^2)^n}{[n]_q!}. \end{aligned} \tag{2.15}$$

Using Eq (1.7) in the right hand side of above equation, we get the assertion (2.13). □

Remark 2.1 Since, for $q \to 1^-$, we have $D_{q,t} \to D_t$ and $e_q(vt^2) \to e^{vt^2}$, therefore for $q \to 1^-$, Lemma 2.2, gives the following well known result of ordinary calculus:

$$D_t e^{vt^2} = 2vte^{vt^2},$$

where D_t denotes the ordinary derivative with respect to t.

Now, we establish the pure and q-differential recurrence relations for $H_{n,q}(u,v)$ in the form of following theorem:

Theorem 2.2. *The 2VqHP $H_{n,q}(u,v)$ satisfy the following recurrence relations:*

$$H_{n+1,q}(u,v) = uH_{n,q}(u,v) + [n]_q v\Big(H_{n-1,q}(qu,v) + qH_{n-1,q}(qu,qv)\Big) = 0 \ (n \geq 1), \tag{2.16}$$

$$qH_{n+1,q}(u,v) = quH_{n,q}(u,v) + vD_{q,u}\Big(H_{n,q}(qu,v) + qH_{n,q}(qu,qv)\Big), \tag{2.17}$$

$$q[n]_q H_{n,q}(u,v) = quD_{q,u}H_{n,q}(u,v) + [n]_q v\, D_{q,v}\Big(H_{n-1,q}(qu,v) + qH_{n-1,q}(qu,qv)\Big)(n \geq 1), \tag{2.18}$$

$$[n]_q H_{n,q}(u,v) = [n]_q uH_{n-1,q}(u,v) + vD_{q,v}\Big(H_{n,q}(qu,v) + H_{n,q}(qu,qv)\Big)(n \geq 1) \tag{2.19}$$

and

$$[n]_q[n-1]_q H_{n-1,q}(u,v) = D_{q,v}uH_{n,q}(u,v) + [n]_q vD_{q,v}\Big(H_{n-1,q}(qu,v) + H_{n-1,q}(qu,qv)\Big)(n \geq 1). \tag{2.20}$$

Proof. Taking q-derivative of both sides of Eq (2.1) with respect to t and then using Eq (1.11) in the right hand side of the resultant equation, we have

$$\sum_{n=0}^{\infty} D_{q,t}H_{n,q}(u,v)\frac{t^n}{[n]_q\,!} = e_q(vt^2)D_{q,t}e_q(ut) + e_q(qut)D_{q,t}\ e_q(vt^2),$$

which on using Eqs (1.12) and (1.2) in the left hand side and using Eqs (1.13) and (2.13) in the right hand side, gives

$$\sum_{n=1}^{\infty} H_{n,q}(u,v)\frac{t^{n-1}}{[n-1]_q\,!} = ue_q(ut)e_q(vt^2) + vt\ e_q(qut)e_q(vt^2) + qvt\ e_q(qut)e_q(qvt^2).$$

Using Eq (2.1) in both sides of the above equation and then comparing the equal powers of t from both sides of the resultant equation, we get the assertion (2.16).

If a is an arbitrary constant, then in view of Eq (2.5), we have

$$H_{n-1,q}(qu,av) = q^{n-1}H_{n-1,q}\Big(u, \frac{1}{q^2}av\Big),$$

which on using Eq (2.7), gives

$$H_{n-1,q}(qu,av) = q^{n-1}\frac{1}{[n]_q}D_{q,u}H_{n,q}\Big(u, \frac{1}{q^2}av\Big). \tag{2.21}$$

Again, using Eq (2.5) in the right hand side of Eq (2.21), we have

$$H_{n-1,q}(qu,av) = \frac{1}{q}\frac{1}{[n]_q} D_{q,u}H_{n,q}(qu,av). \tag{2.22}$$

Using Eq (2.22) for $a = 1$ and $a = q$ in the right hand side of Eq (2.16), we have the assertion (2.17).

Now, replacing n by $n-1$ in Eq (2.17), we have

$$qH_{n,q}(u,v) = quH_{n-1,q}(u,v) + vD_{q,u}\Big(H_{n-1,q}(qu,v) + qH_{n-1,q}(qu,qv)\Big) \qquad (n \geq 1),$$

which on using Eq (2.7) in the right hand side, gives the assertion (2.18).

Next, replacing n by $n-1$ in Eq (2.16), we have

$$H_{n,q}(u,v) = xH_{n-1,q}(u,v) + [n-1]_q v\Big(H_{n-2,q}(qu,v) + qH_{n-2,q}(qu,qv)\Big) = 0 \quad (n \geq 2). \tag{2.23}$$

In view of Eq (2.5), for any constant a, we have

$$H_{n-2,q}(qu,av) = \left(\sqrt{a}\right)^{n-2} H_{n-2,q}\left(\frac{q}{\sqrt{a}}u, v\right),$$

which on using Eq (2.10), gives

$$H_{n-2,q}(qu,av) = \left(\sqrt{a}\right)^{n-2} \frac{1}{[n]_q[n-1]_q} D_{q,v}H_{n,q}\left(\frac{q}{\sqrt{a}}u, v\right). \tag{2.24}$$

Again, using Eq (2.5) in the right hand side of Eq (2.24), we have

$$H_{n-2,q}(qu,av) = \frac{1}{a}\frac{1}{[n]_q[n-1]_q} D_{q,v}H_{n,q}(qu,av). \tag{2.25}$$

Using Eq (2.25) for $a = 1$ and $a = q$, in the right hand side of Eq (2.23), we have the assertion (2.19).

Again, replacing n by $n-1$ in Eq (2.19), we have

$$[n-1]_q H_{n-1,q}(u,v) = [n-1]_q uH_{n-2,q}(u,v) + vD_{q,v}\Big(H_{n-1,q}(qu,v) + H_{n-1,q}(qu,qv)\Big),\ n \geq 1, \tag{2.26}$$

which on using Eq (2.10) in the right hand side of Eq (2.26), gives the assertion (2.20). □

In order, to establish the differential equation of 2VqHP $H_{n,q}(u,v)$, we define the following operators:

Let $f(u,v)$ be a q-function of two variables, then we define the shift operators $L_{a,u}$ and $L_{a,v}$ as:

$$L_{a,u}f(u,v) = f(au,v) \tag{2.27}$$

and

$$L_{a,v}f(u,v) = f(u,av), \tag{2.28}$$

where a is a constant. In view of (2.27), the shift operator $L_{a,u}$ satisfy the following properties:

$$L_{a,u}L_{b,u}f(u,v) = f(abu,v) = L_{ab,u}f(u,v). \tag{2.29}$$

In particular, for a=b, we have

$$L_{a^2,u}f(u,v) = f(a^2uv,) = L_{a,u}L_{a,u}f(v = L^2_{a,u}f(u,v). \tag{2.30}$$

If $L^{-1}_{a,u}$ is the inverse of the operator $L_{a,u}$, that is $L^{-1}_{a,u}\ L_{a,u} = I$, where I is an identity operator such that $If(u,v) = f(u,v)$.

Then, from Eq (2.27), we have

$$L^{-1}_{a,u}f(au,v) = f(u,v).$$

Replacing au by u in the above equation, we have

$$L^{-1}_{a,u}\ f(u,v) = f\left(\frac{1}{a}u,v\right) = f(a^{-1}u,v),$$

which on using Eq (2.27), gives

$$L^{-1}_{a,u}\ f(u,v) = L_{a^{-1},u}\ f(u,v). \tag{2.31}$$

Using induction method, Eqs (2.29) and (2.30), gives

$$L_{a^r,u}f(u,v) = L^r_{a,u}f(u,v), \tag{2.32}$$

where r is any integer.

Similarly, it can be shown that $L_{a,v}$ satisfies the following properties:

$$L_{ab,v}f(u,v) = L_{a,v}L_{b,v}f(u,v), \tag{2.33}$$

$$L_{a^r,v}f(u,v) = L^r_{a,v}f(u,v), \tag{2.34}$$

where r is any integer.

Now, we prove the following result:

Theorem 2.3. *The 2-variable q-Hermite polynomials $H_{n,q}(u,v)$ satisfy the following q-differential equation:*

$$\left(vL_{q,u}\left(1 + q^{\frac{n}{2}}L_{\sqrt{\frac{1}{q}},u}\right)D^2_{q,u} + uD_{q,u} - [n]_q\right)H_{n,q}(u,v) = 0, \tag{2.35}$$

where $L_{.,u}$ is the shift operator defined in Eq (2.27).

Proof. Replacing n by $(n-1)$ in Eq (2.16) and then using (2.7) in resultant equation, we have

$$H_{n,q}(u,v) - uH_{n-1,q}(u,v) - [n-1]_qv\left(H_{n-2,q}(qu,v) + q^{n-1}H_{n-2,q}\left(u,\frac{1}{q}v\right)\right) = 0,\ n \geq 2. \tag{2.36}$$

In view of Eq (2.27), we have

$$H_{n-2,q}(qu,v) = L_{q,u}H_{n-2,q}(u,v). \tag{2.37}$$

Also, in view of Eqs (2.5) and (2.27), we have

$$H_{n-2,q}\Big(u,\frac{1}{q}v\Big)=\left(\sqrt{\frac{1}{q}}\right)^{n-2}L_{\sqrt{q},u}H_{n-2,q}(u,v). \tag{2.38}$$

Using Eqs (2.27), (2.37) and (2.38) in Eq (2.36), we have

$$H_{n,q}(u,v)-uH_{n-1,q}(u,v)-[n-1]_q v\left(L_{q,u}+q^{n-1}\left(\sqrt{\frac{1}{q}}\right)^{n-2}L_{\sqrt{q},u}\right)H_{n-2,q}(u,v)=0,\ n\geq 2. \tag{2.39}$$

Now, using Eqs (2.7) and (2.8) in Eq (2.39), we have

$$H_{n,q}(u,v)-\frac{u}{[n]_q}D_{q,u}H_{n,q}(u,v)-\frac{v}{[n]_q}\Big(L_{q,u}+q^n\left(\sqrt{\frac{1}{q}}\right)^n L_{q^{\frac{-1}{2}},u}\Big)D^2_{q,u}H_{n,q}(u,v)=0,$$

or, equivalently

$$v\Big(L_{q,u}+q^n\Big(\sqrt{\frac{1}{q}}\Big)^n L_{\sqrt{q},u}\Big)D^2_{q,u}H_{n,q}(u,v)+uD_{q,u}H_{n,q}(u,v)-[n]_qH_{n,q}(u,v)=0, \tag{2.40}$$

which, on further simplification yields assertion (2.35). □

Remark 2.2 It can be easily verified that

$$L_{a,u}D^m_{q,u}H_{n,q}(u,v)=\frac{1}{a^n}D^m_{q,u}L_{a,u}H_{n,q}(u,v). \tag{2.41}$$

Using Eq (2.41), for $m=1$ and $m=2$, in Eq (2.40), we have the following equivalent form of differential equation:

$$\left(\frac{1}{q^2}D^2_{q,u}L_{q,u}+q^{n+1}\left(\sqrt{\frac{1}{q}}\right)^n D^2_{q,u}L_{q^{\frac{n-1}{2}},u}\right)H_{n,q}(u,v)+uD_{q,u}H_{n,q}(u,v)-[n]_qH_{n,q}(u,v)=0 \tag{2.42}$$

Another, forms of the differential Eq (2.35) are as follows

$$\Big(q^{n-2}v\,L^{-1}_{q,v}(L_{1/q,v}+q)D^2_{q,u}+uD_{q,u}-[n]_q\Big)H_{n,q}(u,v)=0 \tag{2.43}$$

and

$$\Big(vL_{q,u}(1+qL_{q,v})D^2_{q,u}+quD_{q,u}-q[n]_q\Big)H_{n,q}(u,v)=0, \tag{2.44}$$

where $L_{q,v}$ is the shift operator defined by Eq (2.28).

Remark 2.3 Since, for $q\to 1^-$, the q-calculus reduces to the ordinary calculus. Therefore, for $q\to 1^-$, Eq (2.1) gives the generating function of 2VHP $H_n(u,v)$, given by Eq (1.21). Also, the series definition, recurrence relations and differential equation for 2VHP $H_n(u,v)$, given by the Eqs (1.21), (1.22) and (1.23)–(1.28), can be obtained by taking $q\to 1^-$ in Eqs (2.3), (2.16)–(2.20) and (2.35), respectively.

In the next section, we establish certain summation formulas involving the 2-variables q-Hermite polynomials $H_{n,q}(u,v)$ and its q-derivatives.

3. Summation formulas

In this section, we obtain certain summation formulas for 2VqHP by exploiting the identities (1.7) and (1.9) and using the generating function of $H_{n,q}(u,v)$. The following summation formulas for the 2VqHP $H_{n,q}(u,v)$ hold:

Theorem 3.1. *The following summation formulas hold in the finite complex plane:*

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-v)^r H_{n-2r,q}(u,v)}{[r]_q![n-2r]_q!} = \frac{(u)_q^n}{[n]_q!}. \tag{3.1}$$

If n is even i.e. $n = 2m$ ($m \in \mathbb{N}$), then

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-u)^r H_{2m-r,q}(u,v)}{[r]_q![2m-r]_q!} = \frac{(v)_q^m}{[m]_q!}, \tag{3.2}$$

if n is odd i.e. $n = 2m+1$ ($m \in \mathbb{N} \cup \{0\}$), then

$$\sum_{r=0}^{2m+1} \frac{q^{\binom{r}{2}}(-u)^r H_{2m+1-r,q}(u,v)}{[r]_q![2m+1-r]_q!} = 0\ . \tag{3.3}$$

Proof. In view of Eq (1.9), we have

$$e_q(ut)\ e_q(vt^2)E_q(-vt^2) = e_q(ut), \tag{3.4}$$

which on using Eqs (1.7), (1.8) and (2.1), gives

$$\sum_{n=0}^{\infty} H_{n,q}(u,v)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}(-v)^r t^{2r}}{[r]_q!} = \sum_{n=0}^{\infty}\frac{u^n t^n}{[n]_q!},$$

or, equivalently

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{H_{n,q}(u,v)q^{\binom{r}{2}}(-v)^r\ t^{n+2r}}{[n]_q![r]_q!} = \sum_{n=0}^{\infty}\frac{u^n t^n}{[n]_q!}.$$

Using Eq (2.2) in the left hand side of above equation, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{H_{n-2r,q}(u,v)q^{\binom{r}{2}}(-v)^r\ t^{n}}{[n-2r]_q![r]_q!} = \sum_{n=0}^{\infty}\frac{u^n t^n}{[n]_q!},$$

which on equating the coefficients of equal powers of t from both sides, gives the assertion (3.1).

Again, using Eq (1.9), we have

$$E_q(-ut)\ e_q(ut)e_q(vt^2) = e_q(vt^2). \tag{3.5}$$

Using Eqs (1.7), (1.8) and (2.1) in the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}(-u)^r H_{n,q}(u,v)\ t^{n+r}}{[n]_q![r]_q!} = \sum_{n=0}^{\infty}\frac{v^n t^{2n}}{[n]_q!},$$

which on using the following series rearrangement technique [1]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{3.6}$$

gives

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{q^{\binom{r}{2}}(-u)^{r}H_{n-r,q}(u,v)t^{n}}{[r]_q![n-r]_q!}=\sum_{n=0}^{\infty}\frac{v^{n}t^{2n}}{[n]_q!}. \tag{3.7}$$

Comparing the even and odd powers of t from both sides of Eq (3.7), we get the assertions (3.2) and (3.3), respectively. □

Remark 3.1 The summation formula, given by Eq (3.3), has the following equivalent form:

$$\sum_{r=0}^{2m}\frac{q^{\binom{r}{2}}(-u)^{r}H_{2m+1-r,q}(u,v)}{[r]_q![2m+1-r]_q!}=\frac{q^{\binom{2m+1}{2}}u^{2m+1}}{[2m+1]_q!}. \tag{3.8}$$

From Theorem 3.1 and Remark 3.1, we deduce the following summation formulas for q-derivative of 2VqHP $H_{n,q}(u,v)$ with respect to u:

Corollary 3.2. *The following summation formulas hold in the finite complex plane:*

$$\sum_{r=0}^{[n/2]}\frac{q^{\binom{r}{2}}(-v)^{r}D_{q,u}H_{n+1-2r,q}(u,v)}{[r]_q![n+1-2r]_q!}=\frac{(u)_q^{n}}{[n]_q!}, \tag{3.9}$$

$$\sum_{r=0}^{2m}\frac{q^{\binom{r}{2}}(-u)^{r}D_{q,u}H_{2m+1-r,q}(u,v)}{[r]_q![2m+1-r]_q!}=\frac{(v)_q^{m}}{[m]_q!} \tag{3.10}$$

and

$$\sum_{r=0}^{2m}\frac{q^{\binom{r}{2}}(-u)^{r}D_{q,u}H_{2m+2-r,q}(u,v)}{[r]_q![2m+2-r]_q!}=\frac{q^{\binom{2m+1}{2}}u^{2m+1}}{[2m+1]_q!}. \tag{3.11}$$

Proof. Using Eq (2.7), we have

$$D_{q,u}H_{n+1-2r,q}(u,v)=[n+1-2r]_qH_{n-2r,q}(u,v), \tag{3.12}$$

which on simplifying, gives

$$H_{n-2r,q}(u,v)=\frac{D_{q,u}H_{n+1-2r,q}(u,v)}{[n+1-2r]_q}. \tag{3.13}$$

Using Eqs (3.13) and (1.6) in Eq (3.1), we have the assertion (3.9).

Similarly, using Eq (2.7), we have

$$H_{2m-r,q}(u,v)=\frac{D_{q,u}H_{2m+1-r,q}(u,v)}{[2m+1-r]_q}. \tag{3.14}$$

Using Eqs (3.14) and (1.6) in Eq (3.2), we have the assertion (3.10).

From Eq (3.14), we have

$$H_{2m+1-r,q}(u,v)=\frac{D_{q,u}H_{2m+2-r,q}(u,v)}{[2m+2-r]_q}. \tag{3.15}$$

Using Eq (3.15) in Eq (3.8), we have the assertion (3.11). □

Remark 3.2 Using Eq (2.10) in the right hand sides of Eqs (3.1), (3.2) and (3.8), we have the following summation formulas for q-derivative of 2VqHP $H_{n,q}(u,v)$ with respect to v:

Corollary 3.3. *The following summation formulas hold in the finite complex plane:*

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-v)^r D_{q,v} H_{n-2r+2,q}(u,v)}{[r]_q![n-2r+2]_q!} = \frac{(u)_q^n}{[n]_q!}, \tag{3.16}$$

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-u)^r D_{q,v} H_{2m+2-r,q}(u,v)}{[r]_q![2m+2-r]_q!} = \frac{(v)_q^m}{[m]_q!} \tag{3.17}$$

and

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-u)^r D_{q,v} H_{2m+3-r,q}(u,v)}{[r]_q![2m+3-r]_q!} = \frac{q^{\binom{2m+1}{2}}(u)^{2m+1}}{[2m+1]_q!}. \tag{3.18}$$

Remark 3.3 For $q \to 1^-$, Eqs (3.1)–(3.3), (3.8)–(3.11), (3.16)–(3.18) give summation formulas for 2VHP $H_n(u,v)$ and its derivatives.

Now, we list some examples in Table 1 to show the efficacy of our results established in sections 2 and 3.

In the next section, we obtain the operational and integral representations for the 2VqHP $H_{n,q}(u,v)$.

4. Operational and integral representations

It has been realized that the use of operational identities have simplified the study of special polynomials. In this section, we obtain the operational and integral representations of 2VqHP $H_{n,q}(u,v)$.

First, we establish the following result:

Theorem 4.1. *The operational representation of 2-variable q-Hermite polynomials $H_{n,q}(u,v)$ is as follows:*

$$H_{n,q}(u,v) = e_q(vD_{q,u}^2)(u)_q^n. \tag{4.1}$$

Proof. In view of Eq (1.12), we have

$$D_{q,u}^{2k}\,(u)_q^n = \frac{[n]_q!}{[n-2k]_q!}(u)_q^{n-2k}. \tag{4.2}$$

Using the above equation in the right hand side of Eq (2.3), we have

$$H_{n,q}(u,v) = \sum_{k=0}^{[n/2]} \frac{(v)_q^k D_{q,u}^{2k}\,(u)_q^n}{[k]_q!},$$

which on using Eq (1.6), gives

$$H_{n,q}(u,v) = \sum_{k=0}^{[n/2]} \frac{\left(vD_{q,u}^2\right)^k}{[k]_q!}(u)_q^n.$$

Using Eq (1.7) in the right hand side of above equation, we have the assertion (4.1). □

Table 1. Properties of 2-variable q-Hermite polynomials $H_{4,q}(u,v)$ and $H_{7,q}(u,v)$.

S. No.	Polynomials	Name of the properties	Results
I	$H_{4,q}(u,v)$	Series definition	$H_{4,q}(u,v)=u^4+[4]_q[3]_qu^2v+[4]_q[3]_qv^2$
		Recurrence relation 1	$H_{5,q}(u,v)=uH_{4,q}(u,v)+[4]_qv\Big(H_{3,q}(qu,v)+qH_{3,q}(qu,qv)\Big)=0$
		Recurrence relation 2	$H_{7,q}(u,v)=uH_{6,q}(u,v)+[7]_qv\Big(H_{6,q}(qu,v)+qH_{6,q}(qu,qv)\Big)=0$
		Recurrence relation 3	$qH_{5,q}(u,v)=quH_{4,q}(u,v)+vD_{q,u}\Big(H_{4,q}(qu,v)+qH_{4,q}(qu,qv)\Big)$
		Recurrence relation 4	$[4]_qH_{4,q}(u,v)=\ [4]_quH_{3,q}(u,v)+vD_{q,v}\Big(H_{4,q}(qu,v)+H_{4,q}(qu,qv)\Big)$
		Recurrence relation 5	$[4]_q[3]_qH_{3,q}(u,v)=D_{q,v}uH_{4,q}(u,v)+[4]_qvD_{q,v}\Big(H_{3,q}(qu,v)+H_{3,q}(qu,qv)\Big)$
		q-Differential equation	$\Big(vL_{q,u}\Big(1+q^2L_{\sqrt{\frac{1}{q}},u}\Big)D^2_{q,u}+uD_{q,u}-[4]_q\Big)H_{4,q}(u,v)=0$
		Summation Formula 1	$\frac{H_{4,q}(u,v)}{[4]_q!}-\frac{vH_{2,q}(u,v)}{[2]_q!}+\frac{qv^2}{[2]_q!}=\frac{(u)^4_q}{[4]_q!}$
		Summation Formula 2	$\frac{H_{4,q}(u,v)}{[4]_q!}-\frac{uH_{3,q}(u,v)}{[3]_q!}+\frac{qu^2H_{2,q}(u,v)}{[2]_q!}-\frac{q^3u^3H_{1,q}(u,v)}{[3]_q!}+\frac{q^6u^4}{[2]_q!}=\frac{(v)^2_q}{[4]_q!}$
		Summation Formula 3	$\frac{D_{q,u}H_{5,q}(u,v)}{[5]_q!}-\frac{vD_{q,u}H_{3,q}(u,v)}{[3]_q!}+\frac{qv^2D_{q,u}H_{1,q}(u,v)}{[2]_q!}=\frac{(u)^4_q}{[4]_q!}$
		Summation Formula 4	$\frac{D_{q,u}H_{5,q}(u,v)}{[5]_q!}-\frac{uD_{q,u}H_{4,q}(u,v)}{[4]_q!}+\frac{qu^2D_{q,u}H_{3,q}(u,v)}{[2]_q![3]_q!}-\frac{q^3u^3D_{q,u}H_{2,q}(u,v)}{[2]_q![3]_q!}+\frac{q^6u^4D_{q,u}H_{1,q}(u,v)}{[2]_q!}=\frac{(v)^2_q}{[4]_q!}$
		Summation Formula 5	$\frac{D_{q,v}H_{6,q}(u,v)}{[6]_q!}-\frac{vD_{q,v}H_{4,q}(u,v)}{[4]_q!}+\frac{qv^2D_{q,v}H_{2,q}(u,v)}{[2]_q![2]_q!}=\frac{(u)^4_q}{[4]_q!}$
		Summation Formula 6	$\frac{D_{q,v}H_{6,q}(u,v)}{[6]_q!}-\frac{uD_{q,v}H_{5,q}(u,v)}{[5]_q!}+\frac{qu^2D_{q,v}H_{4,q}(u,v)}{[4]_q![2]_q!}-\frac{q^3u^3D_{q,v}H_{2,q}(u,v)}{[3]_q![3]_q!}+\frac{q^6u^4D_{q,v}H_{2,q}(u,v)}{[4]_q![2]_q!}=\frac{(v)^2_q}{[2]_q!}$
II	$H_{7,q}(u,v)$	Series definition	$H_{7,q}(u,v)=u^7+[7]_q[6]_qu^5v+[7]_q[6]_q[5]_q(1+q^2)u^3v^2+[7]_q[6]_q[5]_q[4]_quv^3$
		Recurrence relation 1	$H_{7,q}(u,v)=uH_{6,q}(u,v)+[7]_qv\Big(H_{6,q}(qu,v)+qH_{6,q}(qu,qv)\Big)=0$
		Recurrence relation 2	$qH_{8,q}(u,v)=quH_{7,q}(u,v)+vD_{q,u}\Big(H_{7,q}(qu,v)+qH_{7,q}(qu,qv)\Big)$
		Recurrence relation 3	$q[7]_qH_{7,q}(u,v)=quD_{q,u}H_{7,q}(u,v)+[7]_qvD_{q,v}\Big(H_{6,q}(qu,v)+qH_{6,q}(qu,qv)\Big)$
		Recurrence relation 4	$[7]_qH_{7,q}(u,v)=\ [7]_quH_{6,q}(u,v)+vD_{q,v}\Big(H_{7,q}(qu,v)+H_{7,q}(qu,qv)\Big)$
		Recurrence relation 5	$[7]_q[6]_qH_{6,q}(u,v)=D_{q,v}uH_{6,q}(u,v)+[7]_qvD_{q,v}\Big(H_{6,q}(qu,v)+H_{6,q}(qu,qv)\Big)$
		q-Differential equation	$\Big(vL_{q,u}\Big(1+q^2L_{\sqrt{\frac{1}{q}},u}\Big)D^2_{q,u}+uD_{q,u}-[7]_q\Big)H_{7,q}(u,v)=0$
		Summation Formula 1	$\frac{H_{7,q}(u,v)}{[7]_q!}-\frac{vH_{5,q}(u,v)}{[5]_q!}+\frac{qvH_{3,q}(u,v)}{[2]_q![3]_q!}-\frac{q^3v^3H_{1,q}(u,v)}{[3]_q!}=\frac{(u)^7_q}{[7]_q!}$
		Summation Formula 2	$\frac{H_{7,q}(u,v)}{[7]_q!}-\frac{uH_{6,q}(u,v)}{[6]_q!}+\frac{qu^2H_{5,q}(u,v)}{[2]_q![5]_q!}-\frac{q^3u^3H_{4,q}(u,v)}{[3]_q![4]_q!}+\frac{q^6u^4H_{3,q}(u,v)}{[3]_q![4]_q!}-\frac{q^{10}u^5H_{2,q}(u,v)}{[5]_q![2]_q!}+\frac{q^{15}u^6H_{1,q}(u,v)}{[6]_q!}-\frac{q^{21}u^7}{[7]_q!}=0$
		Summation Formula 3	$\frac{D_{q,u}H_{8,q}(u,v)}{[8]_q!}-\frac{vD_{q,u}H_{6,q}(u,v)}{[6]_q!}+\frac{qvD_{q,u}H_{4,q}(u,v)}{[2]_q![4]_q!}-\frac{q^3v^3D_{q,u}H_{2,q}(u,v)}{[2]_q![3]_q!}=\frac{(u)^7_q}{[7]_q!}$
		Summation Formula 4	$\frac{D_{q,u}H_{8,q}(u,v)}{[8]_q!}-\frac{uD_{q,u}H_{7,q}(u,v)}{[7]_q!}+\frac{qu^2D_{q,u}H_{6,q}(u,v)}{[2]_q![6]_q!}-\frac{q^3u^3D_{q,u}H_{5,q}(u,v)}{[3]_q![5]_q!}+\frac{q^6u^4D_{q,u}H_{4,q}(u,v)}{[4]_q![4]_q!}-\frac{q^{10}u^5D_{q,u}H_{3,q}(u,v)}{[5]_q![3]_q!}+\frac{q^{15}u^6D_{q,u}H_{2,q}(u,v)}{[2]_q![6]_q!}=\frac{q^{21}u^7}{[7]_q!}$
		Summation Formula 5	$\frac{D_{q,v}H_{9,q}(u,v)}{[9]_q!}-\frac{vD_{q,v}H_{7,q}(u,v)}{[7]_q!}+\frac{qvD_{q,7}H_{5,q}(u,v)}{[2]_q![5]_q!}-\frac{q^3v^3D_{q,v}H_{3,q}(u,v)}{[3]_q![3]_q!}=\frac{(u)^7_q}{[7]_q!}$
		Summation Formula 6	$\frac{D_{q,v}H_{9,q}(u,v)}{[9]_q!}-\frac{uD_{q,v}H_{8,q}(u,v)}{[8]_q!}+\frac{qu^2D_{q,v}H_{7,q}(u,v)}{[2]_q![7]_q!}-\frac{q^3u^3D_{q,v}H_{6,q}(u,v)}{[3]_q![6]_q!}+\frac{q^6u^4D_{q,v}H_{5,q}(u,v)}{[4]_q![5]_q!}-\frac{q^{10}u^5D_{q,v}H_{4,q}(u,v)}{[5]_q![4]_q!}+\frac{q^{15}u^6D_{q,v}H_{6,q}(u,v)}{[3]_q![6]_q!}=\frac{q^{21}u^7}{[7]_q!}$

Remark 4.1 For $q \to 1^-$, Eq (4.1) becomes [4]:

$$H_n(u,v) = e^{\left(v\frac{\partial^2}{\partial u^2}\right)} u^n \ . \tag{4.3}$$

Now, we obtain the following integral representations for the 2VqHP $H_{n,q}(u,v)$:

Theorem 4.2. *The definite q-integral of 2VqHP $H_{n,q}(u,v)$ with respect to u is as follows:*

$$\int_a^b H_{n,q}(u,v)d_qu = \frac{H_{n+1,q}(b,v) - H_{n+1,q}(a,v)}{[n+1]_q}. \tag{4.4}$$

Proof. Using Eq (2.7), we have

$$\int_a^b H_{n,q}(u,v)d_qu = \frac{1}{[n+1]_q}\int_a^b D_{q,u}H_{n+1,q}(u,v)d_qu,$$

which on using Eq (1.18) in the right hand side, gives the assertion (4.4). □

Next, we establish the following result:

Theorem 4.3. *The definite q-integral of 2VqHP $H_{n,q}(u,v)$ with respect to v is as follows:*

$$\int_c^d H_{n,q}(u,v)d_qv = \frac{H_{n+2,q}(u,d) - H_{n+2,q}(u,c)}{[n+1]_q[n+2]_q}. \tag{4.5}$$

Proof. Using Eq (2.10), we have

$$\int_a^b H_{n,q}(u,v)d_qv = \frac{1}{[n+1]_q[n+2]_q}\int_a^b D_{q,y}H_{n+2,q}(u,v)d_qv,$$

which on using Eq (1.18) in the right hand side, gives assertion (4.5). □

In view of Theorems 4.2 and 4.3, we have the following result:

Corollary 4.4. *The double q-integration of 2VqHP $H_{n,q}(u,v)$ is as follows:*

$$\begin{aligned}
\int_c^d \left(\int_a^b H_{n,q}(u,v)d_qu\right)d_qv &= \int_a^b \left(\int_c^d H_{n,q}(u,v)d_qv\right)d_qu \\
&= \int_{u=a}^b \int_{v=c}^d H_{n,q}(u,v)d_qv\, d_qu \\
&= \frac{[n]_q!}{[n+3]_q!}\left(H_{n+3,q}(b,d) + H_{n+3,q}(a,c) - H_{n+3,q}(b,c) - H_{n+3,q}(a,d)\right).
\end{aligned} \tag{4.6}$$

Proof. Integrating Eq (4.4) with respect to v by taking limit from c to d and using Eq (1.18), we have

$$\int_c^d \left(\int_a^b H_{n,q}(u,v) d_q u \right) d_q v = \frac{1}{[n+1]_q} \left(\int_c^d H_{n+1,q}(b,v) d_q v - \int_c^d H_{n+1,q}(a,v) d_q v \right),$$

which in view of Eq (4.5), gives

$$\int_c^d \left(\int_a^b H_{n,q}(u,v) d_q u \right) d_q v = \frac{1}{[n+1]_q [n+2]_q [n+3]_q} \Bigg(H_{n+3,q}(b,d) + H_{n+3,q}(a,c) - H_{n+3,q}(b,c) - H_{n+3,q}(a,d) \Bigg). \quad (4.7)$$

Similarly, integrating Eq (4.5) with respect to u by taking limit from a to b and using Eq (1.18), we have

$$\int_a^b \left(\int_c^d H_{n,q}(u,v) d_q v \right) d_q u = \frac{1}{[n+1]_q [n+2]_q} \left(\int_a^b H_{n+2,q}(u,d) d_q u - \int_a^b H_{n+2,q}(u,c) d_q u \right), \quad (4.8)$$

which in view of Eq (4.4), gives

$$\int_a^b \left(\int_c^d H_{n,q}(u,v) d_q v \right) d_q u = \frac{1}{[n+1]_q [n+2]_q [n+3]_q} \Bigg(H_{n+3,q}(b,d) + H_{n+3,q}(a,c) - H_{n+3,q}(b,c) - H_{n+3,q}(a,d) \Bigg). \quad (4.9)$$

In view of Eqs (4.7) and (4.9), we have the assertion (4.6). □

Remark 4.2 For $q \to 1^-$, Eqs (4.4) and (4.5) reduce to the integral representations for 2VHP $H_n(u,v)$ [4]. Also, for $q \to 1^-$, Eq (4.6) gives:

$$\begin{aligned} \int_c^d \left(\int_a^b H_n(u,v) du \right) dv &= \int_a^b \left(\int_c^d H_n(u,v) dy \right) du \\ &= \int_{u=a}^b \int_{v=c}^d H_n(u,v) dv\, du \\ &= \frac{n!}{(n+3)!} \Bigg(H_{n+3}(b,d) + H_{n+3}(a,c) - H_{n+3}(b,c) - H_{n+3}(a,d) \Bigg). \end{aligned}$$

Examples :
Now, we consider some examples to show the efficacy of the results obtained, in this section.
I. First, we take $n = 6$ in Eqs (4.1), (4.4)–(4.6) to obtain the corresponding results for $H_{6,q}(u,v)$.

For $n = 6$, Eq (4.1), the following operational representation of $H_{6,q}(u,v)$:

$$H_{6,q}(u,v) = e_q(vD_{q,u}^2)(u)_q^6. \quad (4.10)$$

Also, taking $n=6, a=1, b=2, c=2, d=3$ in Eqs (4.4)–(4.6) and then simplifying by using Eq (2.3), we have the following integral representations of $H_{6,q}(u,v)$:

$$\begin{aligned}\int_1^2 H_{6,q}(u,v)d_qu &= \frac{1}{[7]_q}\Big((2^7-1)+[7]_q[6]_q(2^5-1)v+[7]_q[6]_q[5]_q(1+q^2)(2^3-1)v^2 \\ &\quad +[7]_q[6]_q[5]_q[4]_q((2-1))v^3\Big) \\ &= \frac{127}{[7]_q}+[6]_q31v+[6]_q[5]_q(1+q^2)7v^2+[6]_q[5]_q[4]_qv^3, \end{aligned} \tag{4.11}$$

$$\begin{aligned}\int_2^3 H_{6,q}(u,v)d_qv = \frac{1}{[7]_q[8]_q}u^8+[6]_qu^6+\frac{5[6]_q[5]_q}{[8]_q}(1+q^2+q^4)u^4+\frac{19[6]_q[5]_q[4]_q}{[8]_q}(1+q^2+q^4)u^2 \\ +65[6]_q[5]_q \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\int_2^3\int_1^2 H_{7,q}(u,v)\ d_qu\ d_qv = \frac{255}{[8]_q}+\frac{315[7]_q}{[8]_q}(1+q^2+q^4)+\frac{304[7]_q[6]_q[5]_q}{[8]_q[3]_q}(1+q^2+q^4) \\ +195[7]_q[6]_q[5]_q(1+q^2+q^4). \end{aligned} \tag{4.13}$$

II. Next, we take $n=3$ in Eqs (4.1), (4.4)–(4.6) to obtain the corresponding results for $H_{3,q}(u,v)$.

For $n=3$, Eq (4.1), the following operational representation of $H_{3,q}(u,v)$:

$$H_{3,q}(u,v)=e_q(vD^2_{q,u})(u)^3_q. \tag{4.14}$$

Also, taking $n=3, a=0, b=2, c=3, d=4$ in Eqs (4.4)–(4.6) and then simplifying by using Eq (2.3), we have the following integral representations of $H_{3,q}(u,v)$:

$$\begin{aligned}\int_0^2 H_{3,q}(u,v)d_qu &= \frac{1}{[4]_q}\Big((2^4-0)+[4]_q[3]_q(2^2-0)v+[4]_q[3]_qv^2\Big) \\ &= \frac{16}{[4]_q}+[3]_q4v+[3]_qv^2, \end{aligned} \tag{4.15}$$

$$\int_3^4 H_{3,q}(u,v)d_qv = \frac{1}{[4]_q[5]_q}u^5+u^3+7[3]_qu \tag{4.16}$$

and

$$\int_3^4\int_0^2 H_{3,q}(u,v)\ d_qu\ d_qv = \frac{16}{[4]_q}+\frac{28[3]_q}{[4]_q}(1+q^2). \tag{4.17}$$

5. Concluding remarks

The post quantum calculus, briefly called (p,q)-calculus, is considered as a generalization of the q-calculus by some mathematicians , as for $p=1$, the (p,q)-calculus reduces to the q-calculus. The (p,q)-analogues of certain ordinary special functions and q-special functions as well as polynomials like Beta function, Gamma function, Euler polynomials and Bernoulli polynomials have been introduced and studied [5, 9, 16, 18]. We conclude this paper with the introduction and study of (p,q)-analogue of 2VqHP $H_{n,q}(u,v)$.

We recall the (p,q)-number $[n]_{p,q}$, which is defined as [8]:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} \qquad (0 < q < p \leq 1), \tag{5.1}$$

for any positive integer $n \in \mathbb{N}$. The (p,q)-factorial is defined as [8]:

$$[n]_{p,q}! = \begin{cases} \prod_{k=1}^{n} [k]_{p,q}! = [1]_{p,q}[2]_{p,q}\ldots[n]_{p,q}, & 0 < q < p \leq 1 \quad n \geq 1 \\ 1, & n = 0. \end{cases}$$

Also, we recall that

$$(u)_{p,q}^{n} = p^{\binom{n}{2}} u^n \ . \tag{5.2}$$

The (p,q)-exponential function, denoted by $e_{p,q}(u)$, is defined as [8]:

$$e_{p,q}(u) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{u^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \frac{(u)_{p,q}^{n}}{[n]_{p,q}!}. \tag{5.3}$$

The (p,q)-derivative of a function f with respect to u, denoted by $\mathrm{D}_{p,q,u}\ f(u)$, is defined as [7]:

$$D_{p,q,u} f(u) = \frac{f(pu) - f(qu)}{(p-q)u} \qquad (x \neq 0). \tag{5.4}$$

Recently, Duran et al. defined the 1-variable (p,q)-Hermite polynomials as [10]:

$$e_{p,q}\big([2]_{p,q} ut\big)\ e_{p,q}(-t^2) = \sum_{n=0}^{\infty} H_{n,p,q}(u) \frac{t^n}{[n]_{p,q}\ !}. \tag{5.5}$$

In view of Eqs (2.1) and (5.5), we define the 2-variable (p,q)-Hermite polynomials 2V(p,q)HP $H_{n,p,q}(u,v)$ by means of the following generating function:

$$e_{p,q}(ut)\ e_{p,q}(vt^2) = \sum_{n=0}^{\infty} H_{n,p,q}(u,v) \frac{t^n}{[n]_{p,q}\ !}. \tag{5.6}$$

The series on the right hand side of the above equation converges in the finite complex plane.

Simplifying and equating the coefficients of equal powers of t from both sides of Eq (5.6), gives the following series definition of 2V(p,q)HP $H_{n,p,q}(u,v)$:

$$H_{n,p,q}(u,v) = [n]_{p,q}! \sum_{k=0}^{[n/2]} \frac{(u)_{p,q}^{n-2k} (v)_{p,q}^{k}}{[n-2k]_{p,q}! [k]_{p,q}!}. \tag{5.7}$$

Being a finite power series, the series on the right hand side of the above equation converges in the finite complex plane.

Similar to the Lemma 2.1, we find the following (p,q)-partial derivative of the function $e_{p,q}(vt^2)$:

$$D_{p,q,t}\ e_{p,q}(vt^2) = pvt\ e_{p,q}(p^2vt^2) + qvt\ e_{p,q}(pqvt^2), \tag{5.8}$$

which is used to obtain the following (p,q)-pure and differential recurrence relations for 2V(p,q)HP $H_{n,p,q}(u,v)$:

$$H_{n+1,p,q}(u,v) = uH_{n,p,q}(pu,p^2v) + [n]_{p,q}v\Big(pH_{n-1,p,q}(qu,p^2v) + qH_{n-1,p,q}(qu,pqv)\Big) = 0\ (n \geq 1), \tag{5.9}$$

$$qH_{n+1,p,q}(u,v) = quH_{n,p,q}(pu,p^2v) + puD_{p,q,u}\Big(pH_{n,p,q}\Big(\frac{q}{p}u,p^2v\Big) + qH_{n,p,q}\Big(\frac{q}{p}u,pqv\Big)\Big), \tag{5.10}$$

$$q[n]_{p,q}H_{n,p,q}(u,v) = quD_{p,q,u}H_{n,p,q}(u,p^2v) + [n]_{p,q}pv\ D_{p,q,u}\Big(pH_{n-1,p,q}\Big(\frac{q}{p}u,p^2v\Big) + qH_{n-1,p,q}\Big(\frac{q}{p}u,pqv\Big)\Big)(n \geq 1), \tag{5.11}$$

$$[n]_{p,q}H_{n,p,q}(u,v) = [n]_{p,q}uH_{n-1,p,q}(pu,p^2v) + vD_{p,q,v}\Big(H_{n,p,q}(qu,pv\Big) + H_{n,p,q}(qu,qv)\Big)(n \geq 1) \tag{5.12}$$

and

$$p[n]_{p,q}[n-1]_{p,q}H_{n-1,p,q}(u,v) = D_{p,q,v}uH_{n,p,q}(pu,pv) + [n]_{p,q}pvD_{p,q,v}\Big(H_{n-1,p,q}(qu,pv\Big) + H_{n-1,p,q}(qu,qv)\Big)(n \geq 1). \tag{5.13}$$

Now, in order to establish the operational representation for the 2V(p,q)HP $H_{n,p,q}(u,v)$, we define a (p,q)-operator $X_{p,q,u}$ as:

$$\Big(X_{p,q,u}\Big)^r_{p,q} := L^{-r}_{p,u}\ D^r_{p,q,u} \qquad (r \geq 1), \tag{5.14}$$

which, in view of Eqs (2.31), (2.32) and (5.2), acts on a (p,q)-function in the following manner:

$$X^r_{p,q,u}f_{p,q}(u) = p^{-\binom{r}{2}}\Big(L^{-1}_{p,u}\Big)^r\Big(D_{p,q,u}\Big)^r f_{p,q}(u) \qquad (r \geq 1), \tag{5.15}$$

where $L^{-1}_{p,u}$ is the inverse shift operator, defined by Eq (2.31). It is easy to verify that the shift-differential operator $X_{p,q,u}$ satisfies the following:

$$X^{2k}_{p,q,u}\ (u)^n_{p,q} = \frac{[n]_{p,q}!}{[n-2k]_{p,q}!}(u)^{n-2k}_{p,q}. \tag{5.16}$$

Using the above equation in the right hand side of Eq (5.7), we have

$$H_{n,p,q}(u,v) = \sum_{k=0}^{[n/2]} \frac{(v)^k_{p,q}X^{2k}_{p,q,u}\ (u)^n_{p,q}}{[k]_{p,q}!},$$

which on using Eq (5.2), gives

$$H_{n,p,q}(u,v)=\sum_{k=0}^{[n/2]}\frac{\left(vX^{2}_{p,q,u}\right)^{k}_{p,q}}{[k]_{p,q}!}(u)^{n}_{p,q}.$$

Using Eq (5.3) in the right hand side of above equation, we have the following operational identity of the 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(u,v)$:

$$H_{n,p,q}(u,v)=e_{p,q}(vX^{2}_{p,q,u})(u)^{n}_{p,q}, \tag{5.17}$$

where the (p,q)-operator $X^{2}_{p,q,u}:=p^{-1}L^{-2}_{p,u}D^{2}_{p,q,u}$.

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Conflict of interest

The authors declare that they have no competing interest.

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Supplementary

In order to plot the graphs of the 2-variable q-Hermite and (p,q)-Hermite polynomials for certain values of p and q, we obtain the following first few q-Hermite and (p,q)-Hermite polynomials by using Eqs (2.3) and (5.7) respectively:

$H_{0,q}(u,v) = 1,$
$H_{1,q}(x,y) = u,$
$H_{2,q}(u,v) = u^2 + [2]_q v,$
$H_{3,q}(u,v) = u^3 + [3]_q[2]_q uv,$
$H_{4,q}(u,v) = u^4 + [4]_q[3]_q u^2 v + [4]_q[3]_q v^2$
and
$H_{0,p,q}(u,v) = 1,$
$H_{1,p,q}(u,v) = u,$
$H_{2,p,q}(u,v) = pu^2 + [2]_{p,q} v,$
$H_{3,p,q}(u,v) = p^3 u^3 + [3]_{p,q}[2]_{p,q} uv,$
$H_{4,p,q}(u,v) = p^6 u^4 + [4]_{p,q}[3]_{p,q} p^2 u^2 v + [4]_{p,q}[3]_{p,q} p v^2.$

The following graphical representations of the first few 2-variable q-Hermite polynomials $H_{n,q}(u,v)$ and 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(u,v)$ for certain values of p and q, are obtained by using the above suitable expression in the software MATLAB:

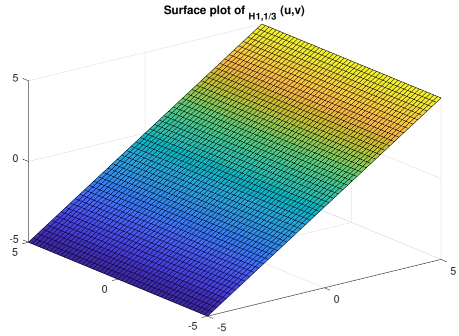

Figure S1. The Surface plot for 2VqHP $H_{1,1/3}(u,v)$.

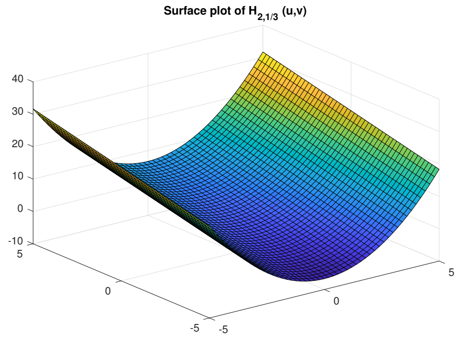

Figure S2. The Surface plot for 2VqHP $H_{2,1/3}(u,v)$.

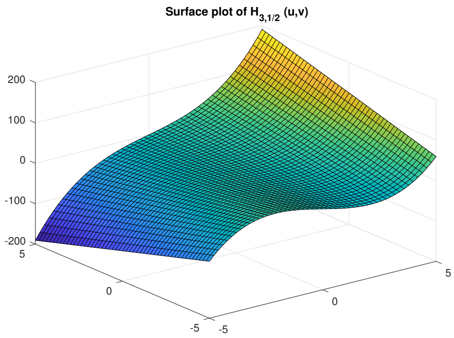

Figure S3. The Surface plot for 2VqHP $H_{3,1/2}(u,v)$.

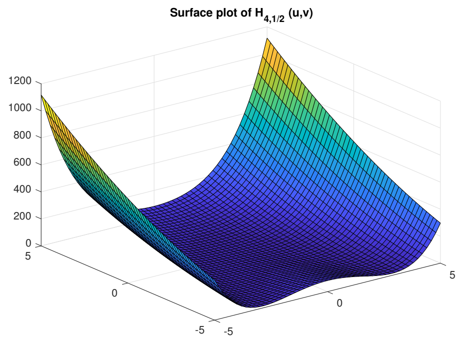

Figure S4. The Surface plot for 2VqHP $H_{4,1/2}(u,v)$.

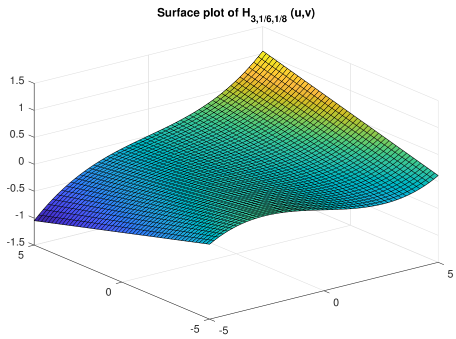

Figure S5. The Surface plot for 2V(p,q)HP $H_{3,1/6,1/8}(u,v)$.

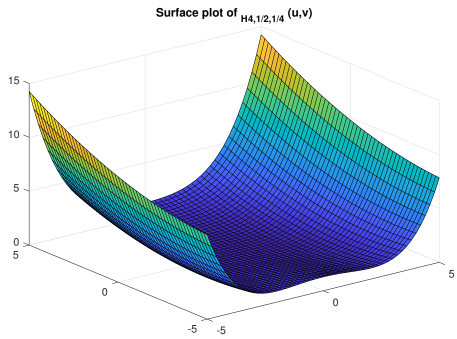

Figure S6. The Surface plot for 2V(p,q)HP $H_{4,1/2,1/4}(u,v)$.

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