Research article

Dynamics and stability for Katugampola random fractional differential equations

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Abstract: This paper deals with some existence of random solutions and the Ulam stability for a class of Katugampola random fractional differential equations in Banach spaces. A random fixed point theorem is used for the existence of random solutions, and we prove that our problem is generalized Ulam-Hyers-Rassias stable. An illustrative example is presented in the last section.

Keywords: differential equation; Katugampola fractional integral; Katugampola fractional derivative; random solution; Banach space; Ulam stability; fixed point

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1. Introduction

The history of fractional calculus dates back to the 17th century. So many mathematicians define the most used fractional derivatives, Riemann-Liouville in 1832, Hadamard in 1891 and Caputo in 1997 [24, 28, 34]. Fractional calculus plays a very important role in several fields such as physics, chemical technology, economics, biology; see [2, 24] and the references therein. In 2011, Katugampola introduced a derivative that is a generalization of the Riemann-Liouville fractional operators and the fractional integral of Hadamard in a single form [21, 22].

There are several articles dealing with different types of fractional operators; see [1, 3, 9–13, 16, 32]. Various results about existence of solutions as well as Ulam stability are provided in [6–8, 14, 15, 17, 19,
In this article we investigate the following class of Katugampola random fractional differential equation
\[
(\iota D_0^\varsigma x)(\xi, w) = f(\xi, x(\xi, w), w); \quad \xi \in I = [0, T], \ w \in \Omega,
\]  
(1.1)  
with the terminal condition
\[
x(T, w) = x_T(w); \quad w \in \Omega.
\]  
(1.2)  
where \(x_T : \Omega \rightarrow E\) is a measurable function, \(\varsigma \in (0, 1], \ T > 0, \ f : I \times E \times \Omega \rightarrow E, \ \iota D_0^\varsigma\) is the Katugampola operator of order \(\varsigma\), and \(\Omega\) is the sample space in a probability space, and \((E, \| \cdot \|)\) is a Banach space.

2. Preliminaries

By \(C(I) := C(I, E)\) we denote the Banach space of all continuous functions \(x : I \rightarrow E\) with the norm
\[
\|x\|_\infty = \sup_{\xi \in I} \|x(\xi)\|,
\]
and \(L^1(I, E)\) denotes the Banach space of measurable function \(x : I \rightarrow E\) with are Bochner integrable, equipped with the norm
\[
\|x\|_{L^1} = \int_I \|x(\xi)\|d\xi.
\]
Let \(C_{\varsigma, \rho}(I)\) be the weighted space of continuous functions defined by
\[
C_{\varsigma, \rho}(I) = \{x : (0, T] \rightarrow E : \xi^{\rho(1-\varsigma)}x(\xi) \in C(I)\},
\]
with the norm
\[
\|x\|_C := \sup_{\xi \in \Omega} \|\xi^{\rho(1-\varsigma)}x(\xi)\|.
\]

**Definition 2.1.** [2]. The Riemann-Liouville fractional integral operator of the function \(h \in L^1(I, E)\) of order \(\varsigma \in \mathbb{R}_+\) is defined by
\[
RL^0_\varsigma h(\xi) = \frac{1}{\Gamma(\varsigma)} \int_0^\xi (\xi - s)^{\varsigma - 1} h(s)ds.
\]

**Definition 2.2.** [2]. The Riemann-Liouville fractional operator of order \(\varsigma \in \mathbb{R}_+\) is defined by
\[
RL^\varsigma_0 h(\xi) = \frac{1}{\Gamma(n - \varsigma)} \left(\frac{d}{d\xi}\right)^n \int_0^\xi (\xi - s)^{n-\varsigma - 1} h(s)ds.
\]

**Definition 2.3.** (Hadamard fractional integral) [4]. The Hadamard fractional integral of order \(r\) is defined as
\[
I_0^\varsigma h(\xi) = \frac{1}{\Gamma(\varsigma)} \int_1^\xi \left(\log \frac{\xi}{s}\right)^{\varsigma - 1} h(s)\frac{ds}{s}, \quad \varsigma > 0,
\]
provided that the left-hand side is well defined for almost every \(\xi \in (0, T)\).
Definition 2.4. (Hadamard fractional derivative) [4]. The Hadamard fractional derivative of order \( r \) is defined as
\[
D^r_0 h(\xi) = \frac{1}{\Gamma(n - \varsigma)} \left( \xi \frac{d}{d\xi} \right)^n \int_1^\xi \left( \log \frac{\xi}{s} \right)^{n-\varsigma-1} h(s) \frac{ds}{s}, \quad \varsigma > 0,
\]
provided that the left-hand side is well defined for almost every \( \xi \in (0, T) \).

Definition 2.5. (Katugampola fractional integral) [21]. The Katugampola fractional integrals of order \( \varsigma > 0 \) is defined by
\[
\nI^\varsigma_0 x(\xi) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^\xi \frac{s^{\rho-1}}{(\xi^\rho - s^\rho)^{1-\varsigma}} x(s) ds \tag{2.1}
\]
for \( \rho > 0 \) and \( \xi \in I \), provided that the left-hand side is well defined for almost every \( \xi \in (0, T) \).

Definition 2.6. (Katugampola fractional derivative) [21]. The Katugampola fractional derivative of order \( \varsigma > 0 \) is defined by:
\[
\nD^\varsigma_0 u(\xi) = \left( \xi^{1-\rho} \frac{d}{d\xi} \right)^n \left( \xi I^{\rho - n} - u(\xi) \right) = \frac{\rho^{n+1}}{\Gamma(n-\varsigma)} \left( \xi^{1-\rho} \frac{d}{d\xi} \right)^n \int_0^\xi \frac{s^{\rho-1}}{(\xi^\rho - s^\rho)^{\varsigma+1}} u(s) ds,
\]
provided that the left-hand side is well defined for almost every \( \xi \in (0, T) \).

We present in the following theorem some properties of Katugampola fractional integrals and derivatives.

Theorem 2.7. [21] Let \( 0 < \text{Re}(\varsigma) < 1 \) and \( 0 < \text{Re}(\eta) < 1 \) and \( \rho > 0 \), for \( a > 0 \):

- **Index property:**
  \[
  (D^\varsigma_a) (D^\eta_a h)(t) = (D^\varsigma+\eta_a h)(t)
  \]
  \[
  (I^\varsigma_a)(I^\eta_a h)(t) = (I^\varsigma+\eta_a h)(t)
  \]

- **Linearity property:**
  \[
  D^\varsigma_a (h + g) = D^\varsigma_a h(t) + D^\varsigma_a g(t)
  \]
  \[
  I^\varsigma_a (h + g) = I^\varsigma_a h(t) + I^\varsigma_a g(t)
  \]

and we have
\[
(t^{1-\rho} \frac{d}{dt}) I^\varsigma_a (I^{1-\tau}_a) u(s) ds.
\]

Theorem 2.8. [21] Let \( r \) be a complex number, \( \text{Re}(r) \geq 0 \), \( n = [\text{Re}(r)] \) and \( \rho > 0 \). Then, for \( t > a \):

1. \( \lim_{r \to -1} (D^r_a I^\varsigma_a h)(t) = \frac{1}{\Gamma(\varsigma+1)} \int_a^t (t - \tau)^{\varsigma-1} h(\tau) d\tau. \)
2. \( \lim_{r \to 0^+} (D^r_a h)(t) = \frac{1}{\Gamma(r)} \int_a^t (\log \frac{\tau}{t})^{r-1} h(\tau) d\tau. \)
3. \( \lim_{r \to 1} (D^r_a h)(t) = \frac{d}{dt} \frac{1}{\Gamma(n-r)} \int_a^t \frac{h(\tau)}{(t-\tau)^{n-r+1}} d\tau. \)
4. \( \lim_{r \to 0} (D^r_a h)(t) = \frac{1}{\Gamma(n-r)} (t \frac{d}{dt})^n \int_a^t (\log \frac{\tau}{t})^{n-r-1} h(\tau) d\tau. \)
Remark 2.9.

(1) \( \lim_{p \to 1} (^pI_0^r h)(t) = (^{RL}I_0^r h)(t). \)
(2) \( \lim_{p \to 0} (^pI_0^r h)(t) = (^H I_0^r h)(t). \)
(3) \( \lim_{p \to 1} (^pD_0^r h)(t) = (^{RL}D_0^r h)(t). \)
(4) \( \lim_{p \to 0} (^pD_0^r h)(t) = (^H D_0^r h)(t). \)

Lemma 2.10. Let \( 0 < r < 1 \). The fractional equation \( (^p D_0^r v)(t) = 0 \), has as solution

\[
v(t) = ct^{p(r-1)}, \tag{2.2}
\]

with \( c \in \mathbb{R} \).

Lemma 2.11. Let \( 0 < r < 1 \). Then

\[
(^p I(^p D_0^r u)(t) = u(t) + ct^{p(r-1)}.
\]

Proof. We have

\[
I_0^r D_0^r u(t) = \left( t^{1-p} \frac{d}{dt} \right) I_0^{r+1} D_0^r u(t)
\]

\[
= \left( t^{1-p} \frac{d}{dt} \right) \frac{\rho^{-r}}{\Gamma(r+1)} \int_0^t (t^p - s^p)^{-r} (^p D_0^r u(s)) ds
\]

\[
= \left( t^{1-p} \frac{d}{dt} \right) \frac{\rho^{-r}}{\Gamma(r+1)} \int_0^t (t^p - s^p)^{-r} \left[ (s^{1-p} \frac{d}{ds} (I_0^{1-r} u(s))) ds \right]
\]

Thus, \( I_0^r D_0^r u(t) = I_1 + I_2 \), with

\[
I_1 = \left( t^{1-p} \frac{d}{dt} \right) \frac{\rho^{-r}}{\Gamma(r+1)} \left[ (t^p - s^p)^r I_0^{1-r} u(s) \right]_0^t,
\]

and

\[
I_2 = \left( t^{1-p} \frac{d}{dt} \right) \frac{\rho^{-r}}{\Gamma(r+1)} \int_0^t (t^p - s^p)^r I_0^{1-r} u(s) ds.
\]

Hence, we get

\[
I_1 = ct^{p(r-1)}
\]

and

\[
I_2 = \left( t^{1-p} \frac{d}{dt} \right) \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{r-1} (t^p - s^p)^r I_0^{1-r} u(s) ds
\]

\[
= \left( t^{1-p} \frac{d}{dt} \right) I_0^r (I_0^{1-r} u(s) ds
\]

\[
= u(t).
\]

Finally we obtain

\[
(I_0^r D_0^r u)(t) = u(t) + ct^{p(r-1)}.
\]
Lemma 2.12. The problem

\[
\begin{aligned}
(\rho D^r_0 u)(t) &= h(t); \quad t \in I := [0, T] \\
u(T) &= u_T
\end{aligned}
\]  

(2.3)

has the following solution

\[
u(t) = \rho^{1-r} \frac{1}{\Gamma(r)} \int_0^t \left( \frac{s^{p-1}}{(T^p - s^p)^{1-r}} h(t) ds - C \rho^{r(t-1)} \right)
\]

(2.4)

where

\[
C = \frac{1}{T^{\rho(r-1)}} \left( \frac{\rho^{1-r}}{\Gamma(r)} \int_0^T \frac{s^{p-1}}{(T^p - s^p)^{1-r}} h(T) ds - u_T \right).
\]

Proof. Solving the equation

\((\rho D^r_0 u)(t) = h(t),\)

we get

\[
u(t) = \rho^{1-r} I_0^\rho h(t) - c \rho^{r(t-1)}.
\]

From the condition, we get

\[
C = \frac{\rho I_0^\rho h(T) - u_T}{T^{\rho(r-1)}}
\]

hence, we obtain (2.4).

Definition 2.13. By a random solution of problem (1.1) and (1.2), we mean a measurable function \(x(w, \cdot) \in C_{c^r}(I)\) which satisfies (1.1) and (1.2).

Lemma 2.14. \(u\) is a random solution of (1.1) and (1.2), if and only if it satisfies

\[
x(\xi, w) = \frac{\rho^{1-c}}{\Gamma(s)} \int_0^\xi \left( \frac{s^{p-1}}{(T^{p} - s^p)^{1-c}} \right) f(\xi, x, w) ds - C(w) \xi^{\rho(r-1)}
\]

(2.5)

where

\[
C(w) = \frac{1}{T^{\rho(s-1)}} \left( \frac{\rho^{1-c}}{\Gamma(s)} \int_0^T \frac{s^{p-1}}{(T^{p} - s^p)^{1-c}} f(T, x, w) ds - x_T(w) \right).
\]

Lemma 2.15. [4, 13] Let \(T : \Omega \times E \to E\) be a mapping such that \(T(\cdot, v)\) is measurable for all \(v \subset E\), and \(T(w, \cdot)\) is continuous for all \(w \subset \Omega\). Then the map \((w, v) \to T(w, v)\) is jointly measurable.

Definition 2.16. A function \(f : I \times E \times \Omega \to E\) is called random Carathéodory if the following conditions are satisfied:

- (i) The map \((s, w) \to f(s, x, w)\) is jointly measurable for all \(x \in E\), and
- (ii) The map \(x \to f(s, x, w)\) is continuous for almost all \(s \in I\) and \(w \subset \Omega\).

Let \(\epsilon > 0\) and \(\Phi : \Omega \times I \to \mathbb{R}_+\) be a jointly measurable function. We consider the following inequality

\[
\|((\rho D^r_0 x)(\xi, w) - f(\xi, u(\xi, w), w)) \| \leq \Phi(\xi, w); \text{ for } \xi \in I, \text{ and } w \in \Omega.
\]

(2.6)
Definition 2.17. [5] The problem (1.1) and (1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\Phi$ if there exists $c_{f,\phi} > 0$ such that for each solution $(\xi, w) \in C_{\varsigma, \rho}(I)$ of the inequality (2.6), there exists $(\gamma, w) \in C_{\varsigma, \rho}$ such that

$$
\| \gamma^{(1-\varsigma)} x(\xi, w) - \xi^{(1-\varsigma)} y(\xi, w) \| \leq c_{f,\phi}(\xi, w); \quad \xi \in I; \quad w \in \Omega.
$$

Theorem 2.18. [18] Let $X$ be a nonempty, closed convex bounded subset of the separable Banach space $E$ and let $G : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $G(w)u = u$ has a random solution.

3. Existence and Ulam stability results

We shall make use of the following hypotheses:

(H$_1$) $f$ is a random Carathéodory function.

(H$_2$) There exist measurable and essentially bounded functions $l_i : \Omega \to C(I); \quad i = 1, 2$ such that

$$
\| f(t, x, w) \| \leq l_1(t, w) + l_2(t, w)p^{q(1-\gamma)}|x|,
$$

for all $x \in E$ and $t \in I$ with

$$
l_i^*(w) = \sup_{t \in I} l_i(t, w); \quad i = 1, 2, \quad w \in \Omega.
$$

Theorem 3.1. If (H$_1$) and (H$_2$) hold, and

$$
\frac{\rho^{-\varsigma} T_\rho}{\Gamma(1+\varsigma)} l_1^*(w) < 1,
$$

then there exists a random solution for (1.1) and (1.2).

Proof. Let $N : \Omega \times C_{\varsigma, \rho}(I) \to C_{\varsigma, \rho}(I)$ be the operator defined by

$$
(Nx)(t, w) = \frac{\rho^{-\varsigma}}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\varsigma-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} f(s, x(s, w), w) ds - C(w)p^{q(1-\varsigma)},
$$

and set

$$
R(w) = \frac{\| C(w) \| + \frac{\rho^{-\varsigma} T_\rho}{\Gamma(1+\varsigma)} l_1^*(w)}{1 - \frac{\rho^{-\varsigma} T_\rho}{\Gamma(1+\varsigma)} l_1^*(w)}; \quad w \in \Omega,
$$

and define the ball

$$
B_R = B(0, R(w)) := \{ x \in C_{\varsigma, \rho}(I) : \| x \|_C \leq R(w) \}.
$$

For any $w \in \Omega$ and each $t \in I$, we have

$$
\| \gamma^{(1-\varsigma)} (Nx)(t, w) \| \leq \| C(w) \| + \frac{\rho^{-\varsigma} T_\rho(1-\varsigma)}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\varsigma-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} |f(s, x(s, w), w)| ds
$$

$$
\leq \| C(w) \| + \frac{\rho^{-\varsigma} T_\rho(1-\varsigma)}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\varsigma-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} l_1(s, w) ds.
$$
\[
+ \frac{\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\rho-1}}{(p^{\rho} - s^{\rho})^{1-s}} \|s^{\rho(1-s)}l_{2}(s, w)x(s, w)\| ds \\
\leq ||C(w)|| + \frac{\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(\varsigma)} \frac{T^{s\rho}}{s^{\rho}} l_{1}(w) \\
+ \frac{l_{2}(w)\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(r)} \int_{0}^{t} \frac{s^{\rho-1}}{(p^{\rho} - s^{\rho})^{1-s}} \|s^{\rho(1-s)}x(s, w)\| ds \\
\leq ||C(w)|| + \frac{\rho^{-s}T^{p}}{\Gamma(1 + s)} l_{1}(w) + \frac{\rho^{-s}T^{p}}{\Gamma(1 + s)} l_{2}(w)||x||_{C} \\
\leq ||C(w)|| + \frac{\rho^{-s}T^{p}}{\Gamma(1 + s)} l_{1}(w) + \frac{\rho^{-s}T^{p}}{\Gamma(1 + s)} l_{2}(w)R(w) \\
\leq R(w).
\]

Thus

\[||N(w)(u)||_{C} \leq R(w).\]

Hence \(N(w)(B_{R}) \subset B_{R}\). We shall prove that \(N : \Omega \times B_{R} \rightarrow B_{R}\) satisfies the assumptions of Theorem 2.18.

**Step 1.** \(N(w)\) is a random operator.

From \((H_{1})\), the map \(w \mapsto f(t, x, w)\) is measurable and further the integral is a limit of a finite sum of measurable functions therefore the map

\[w \mapsto \frac{\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\rho-1}}{(p^{\rho} - s^{\rho})^{1-s}} f(s, x(s, w), w) ds - C(w)\rho^{p(1-1)},\]

is measurable.

**Step 2.** \(N(w)\) is continuous.

Consider the sequence \((x_{n})_{n}\) such that \(x_{n} \rightarrow u\) in \(C_{\varsigma, p}\).

Set

\[v_{n}(t, w) = \rho^{(1-\varsigma)}(N_{x_{n}})(t, w), \quad and \quad v(t, w) = \rho^{(1-\varsigma)}(N_{x})(t, w).\]

Then

\[||v_{n}(t, w) - v(t, w)||\]

\[
\leq \left\| \frac{\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\rho-1}}{(p^{\rho} - s^{\rho})^{1-s}} (f(s, x_{n}(s, w), w) - f(s, x(s, w), w) ds \right\| \\
\leq \frac{\rho^{1-s}T^{s(1-\varsigma)}}{\Gamma(\varsigma)} \int_{0}^{t} \frac{s^{\rho-1}}{(p^{\rho} - s^{\rho})^{1-s}} \|f(s, x_{n}(s, w), w) - f(s, x(s, w), w)\| ds.
\]

By \((H_{1})\) we obtain

\[||v_{n}(\cdot, w) - v(\cdot, w)||_{C} \rightarrow 0 \text{ as } n \rightarrow \infty,\]

Consequently, \(N(w) : B_{R} \subset B_{R}\) is continuous.

**Step 3.** \(N(w)B_{R}\) is equicontinuous. For \(1 \leq t_{1} \leq t_{2} \leq T\), and \(x \in B_{R}\), we have

\[||\rho^{(1-\varsigma)}(N_{x})(t_{2}, w) - \rho^{(1-\varsigma)}(N_{x})(t_{1}, w)||\]
Remark 3.2. Hypothesis (AIMS Mathematics)  

Hyers-Rassias stable. Hence, from Theorem 2.18, we deduce the existence of random solution to problem (1.1) and (1.2).

Now, we prove a result concerning the generalized Ulam-Hyers-Rassias stability of (1.1) and (1.2). We introduce the following additional hypotheses:

\((H_3)\) For any \(w \in \Omega\), \(\Phi(t, \cdot) \subset L^1[0, \infty)\), and there exists a measurable and essentially bounded function \(q : \Omega \to C(I, [0, \infty))\); such that

\[
(1 + \|x - y\|)\|f(t, x(t, w), w) - f(t, y(t, w), w)\| \leq q(t, w)\Phi(t, w)\|x - y\|.
\]

\(\Phi^r(w) = \sup_{t \in I} \Phi(t, w), \ q^r(w) = \sup_{t \in I} q(t, w); \ w \in \Omega.\)

**Theorem 3.3.** If \((H_1), (H_3), (H_4)\) and

\[
\frac{\rho^{-c}T^\rho}{\Gamma(1 + \varsigma)} \Phi^r(w) q^r(w) < 1,
\]

hold. Then the problem (1.1) and (1.2) has random solutions defined on I, and it is generalized Ulam-Hyers-Rassias stable.
Proof. From (H1), (H3) and Remark 3.2; the problem (1.1) and (1.2) has at least one random solution \( y \). Then, we have

\[
y(t, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t s^{\rho-1}(p^{\rho} - s^{\rho})^{1-\varsigma} f(s, y(s, w), w) ds - C(w)T^{\rho(1-\varsigma)}.
\]

Assume \( x \) be a random solution of (2.6). We obtain

\[
\|x(t, w) - y(t, w)\|
\]

\[
\leq \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t s^{\rho-1}(p^{\rho} - s^{\rho})^{1-\varsigma} f(s, x(s, w), w) ds + C(w)
\]

\[
+ \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t s^{\rho-1}(p^{\rho} - s^{\rho})^{1-\varsigma} f(s, y(s, w), w) ds - C(w)
\]

\[
\leq T^{\rho(1-\varsigma)}(\mathcal{I}_0)\Phi(t, w)
\]

\[
+ \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t s^{\rho-1}(p^{\rho} - s^{\rho})^{1-\varsigma} \|f(s, x(s, w), w) - f(s, y(s, w), w)\| ds
\]

\[
\leq T^{\rho(1-\varsigma)}(\mathcal{I}_0)\Phi(t, w)
\]

Thus, we get

\[
\|x(t, w) - y(t, w)\| \leq \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t s^{\rho-1}(p^{\rho} - s^{\rho})^{1-\varsigma} q^*(w)\Phi(s, w) s^{\rho(1-\varsigma)} \frac{\|x - y\|}{1 + \|x - y\|} ds
\]

\[
\leq T^{\rho(1-\varsigma)}(\mathcal{I}_0)\Phi(t, w) + T^{2\rho(1-\varsigma)}(\mathcal{I}_0)\Phi(t, w)q^*(w).
\]

Hence, problem (1.1) and (1.2) is generalized Ulam-Hyers-Rassias stable.

4. An example

Let \( \Omega = (-\infty, 0) \) be equipped with the usual \( \sigma \)-algebra consisting of Lebesgue measurable subsets of \((-\infty, 0)\), and let

\[
\ell^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots), \sum_{n=1}^{\infty} |x_n| < \infty \right\}
\]
be the Banach space with the norm
$$\|x\| = \sum_{n=1}^{\infty} |x_n|.$$ 

Consider the Katugampola random fractional differential equation
$$\left(^\rho D_0^r x\right)(t, w) = f_n(t, x(t, w), w); \quad t \in [0, 1], \ w \in \Omega, \quad (4.1)$$

with the terminal condition
$$x(T, w) = ((1 + w^2)^{-1}, 0, 0, \ldots); \ w \in \Omega, \quad (4.2)$$

with $x = (x_1, x_2, \ldots, x_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$,
$$^\rho D_0^r x = (^\rho D_0^r x_1, \ldots, ^\rho D_0^r x_n, \ldots),$$

and
$$f_n(t, x(t, w), w) = \frac{w^2 \rho^{(1-r)}(2^{-n} + x_n(t, w))}{2(1 + w^2)(1 + \|x\|)} \left( e^{-7-w^2} + \frac{1}{e^{t+5}} \right); \quad t \in [0, 1], \ w \in \Omega.$$ 

We have
$$\|f(t, x, w) - f(t, y, w)\| \leq (e^{-7-w^2} + e^{-t-5}) \frac{w^2 \rho^{(1-r)} \|x - y\|}{1 + \|x - y\|}.$$ 

Hence, hypotheses $(H_3)$ and $(H_4)$ are satisfied with
$$q(t, w) = e^{-7-w^2} + e^{-t-5}, \quad \Phi(t, w) = w^2.$$ 

Hence by theorems 3.1 and 3.3, problem (4.1) and (4.2) admits a random solution, and is generalized Ulam-Hyers-Rassias stable.

5. Conclusions

In this paper, we provided some sufficient conditions ensuring the existence of random solutions and the Ulam stability for a class of fractional differential equations involving the Katugampola fractional derivative in Banach spaces. The techniques used are the random fixed point theory and the notion of Ulam-Hyers-Rassias stability.

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Conflict of interest

The authors declare no conflict of interests.

References


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