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Research article

Dynamics and stability for Katugampola random fractional differential equations

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Abstract: This paper deals with some existence of random solutions and the Ulam stability for a class of Katugampola random fractional differential equations in Banach spaces. A random fixed point theorem is used for the existence of random solutions, and we prove that our problem is generalized Ulam-Hyers-Rassias stable. An illustrative example is presented in the last section.

Keywords: differential equation; Katugampola fractional integral; Katugampola fractional derivative; random solution; Banach space; Ulam stability; fixed point
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1. Introduction

The history of fractional calculus dates back to the 17th century. So many mathematicians define the most used fractional derivatives, Riemann-Liouville in 1832, Hadamard in 1891 and Caputo in 1997 [24, 28, 34]. Fractional calculus plays a very important role in several fields such as physics, chemical technology, economics, biology; see [2,24] and the references therein. In 2011, Katugampola introduced a derivative that is a generalization of the Riemann-Liouville fractional operators and the fractional integral of Hadamard in a single form [21, 22].

There are several articles dealing with different types of fractional operators; see [1, 3, 9–13, 16, 32]. Various results about existence of solutions as well as Ulam stability are provided in [6–8, 14, 15, 17, 19, 20, 23, 25–31, 33]. In this article we investigate the following class of Katugampola random fractional differential equation

$$({}^{\rho}D_0^{\varsigma}x)(\xi, w) = f(\xi, x(\xi, w), w);\ \xi \in I = [0, T],\ w \in \Omega, \tag{1.1}$$

with the terminal condition

$$x(T, w) = x_T(w);\ w \in \Omega, \tag{1.2}$$

where $x_T : \Omega \to E$ is a measurable function, $\varsigma \in (0, 1]$, $T > 0$, $f : I \times E \times \Omega \to E$, ${}^{\rho}D_0^{\varsigma}$ is the Katugampola operator of order ς, and Ω is the sample space in a probability space, and $(E, \|\cdot\|)$ is a Banach space.

2. Preliminaries

By $C(I) := C(I, E)$ we denote the Banach space of all continuous functions $x : I \to E$ with the norm

$$\|x\|_\infty = \sup_{t \in I} \|x(\xi)\|,$$

and $L^1(I, E)$ denotes the Banach space of measurable function $x : I \to E$ with are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_I \|x(\xi)\| d\xi.$$

Let $C_{\varsigma,\rho}(I)$ be the weighted space of continuous functions defined by

$$C_{\varsigma,\rho}(I) = \{x : (0, T] \to E : \xi^{\rho(1-\varsigma)}x(\xi) \in C(I)\},$$

with the norm

$$\|x\|_C := \sup_{\xi \in I} \|\xi^{\rho(1-\varsigma)}x(\xi)\|.$$

Definition 2.1. *[2]. The Riemann-Liouville fractional integral operator of the function $h \in L^1(I, E)$ of order $\varsigma \in \mathbb{R}_+$ is defined by*

$${}^{RL}I_0^{\varsigma}h(\xi) = \frac{1}{\Gamma(r)} \int_0^{\xi} (\xi - s)^{r-1} h(s) ds.$$

Definition 2.2. *[2]. The Riemann-Liouville fractional operator of order $\varsigma \in \mathbb{R}_+$ is defined by*

$${}^{RL}D_0^{\varsigma}h(\xi) = \frac{1}{\Gamma(n-\varsigma)} \left(\frac{d}{d\varsigma}\right)^n \int_0^{\xi} (\xi - s)^{n-\varsigma-1} h(s) ds.$$

Definition 2.3. *(Hadamard fractional integral) [4]. The Hadamard fractional integral of order r is defined as*

$$I_0^{\varsigma}h(\xi) = \frac{1}{\Gamma(\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{s}\right)^{\varsigma-1} h(s) \frac{ds}{s},\ \ \varsigma > 0,$$

provided that the left-hand side is well defined for almost every $\xi \in (0, T)$.

Definition 2.4. *(Hadamard fractional derivative) [4]. The Hadamard fractional derivative of order r is defined as*

$$D_0^{\varsigma}h(\xi) = \frac{1}{\Gamma(n-\varsigma)}\left(\xi\frac{d}{d\xi}\right)^n\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{n-\varsigma-1}h(s)\frac{ds}{s},\ \ \varsigma > 0,$$

provided that the left-hand side is well defined for almost every $\xi \in (0, T)$.

Definition 2.5. *(Katugampola fractional integral) [21]. The Katugampola fractional integrals of order* $(\varsigma > 0)$ *is defined by*

$$^{\rho}I_0^{\varsigma}x(\xi) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^{\xi}\frac{s^{\rho-1}}{(\xi^{\rho}-s^{\rho})^{1-\varsigma}}x(s)ds \tag{2.1}$$

for $\rho > 0$ *and* $\xi \in I$, *provided that the left-hand side is well defined for almost every* $\xi \in (0, T)$.

Definition 2.6. *(Katugampola fractional derivative) [21]. The Katugampola fractional derivative of order* $\varsigma > 0$ *is defined by:*

$$\begin{aligned} ^{\rho}D_0^{r}u(\xi) &= \left(\xi^{1-\rho}\frac{d}{d\xi}\right)^n({}^{\rho}I_0^{n-r}u)(\xi) \\ &= \frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^n\int_0^{\xi}\frac{s^{\rho-1}}{(\xi^{\rho}-s^{\rho})^{r-n+1}}u(s)ds, \end{aligned}$$

provided that the left-hand side is well defined for almost every $\xi \in (0, T)$.

We present in the following theorem some properties of Katugampola fractional integrals and derivatives.

Theorem 2.7. *[21] Let* $0 < Re(\varsigma) < 1$ *and* $0 < Re(\eta) < 1$ *and* $\rho > 0$, *for* $a > 0$*:*

- *Index property:*

$$\begin{aligned} ({}^{\rho}D_a^{\varsigma})({}^{\rho}D_a^{\eta}h)(t) &= {}^{\rho}D_a^{\varsigma+\eta}h(t) \\ ({}^{\rho}I_a^{r})({}^{\rho}I_a^{\eta}h)(t) &= {}^{\rho}I_a^{r+\eta}h(t) \end{aligned}$$

- *Linearity property:*

$$\begin{aligned} ^{\rho}D_a^{r}(h+g) &= {}^{\rho}D_a^{r}h(t) +^{\rho} D_a^{r}g(t) \\ ^{\rho}I_a^{r}(h+g) &= {}^{\rho}I_a^{r}h(t) +^{\rho} I_a^{r}g(t) \end{aligned}$$

and we have

$$(t^{1-\rho}\frac{d}{dt})I_0^{r}(I_0^{1-r})u(s)ds.$$

Theorem 2.8. *[21] Let r be a complex number,* $Re(r) \geq 0$, $n = [Re(r)]$ *and* $\rho > 0$. *Then, for* $t > a$;

(1) $\lim_{\rho\to 1}({}^{\rho}I_a^{r}h)(t) = \frac{1}{\Gamma(r)}\int_a^t(t-\tau)^{r-1}h(\tau)d\tau.$

(2) $\lim_{\rho\to 0^+}({}^{\rho}I_a^{r}h)(t) = \frac{1}{\Gamma(r)}\int_a^t(\log\frac{t}{\tau})^{r-1}h(\tau)\frac{d\tau}{\tau}.$

(3) $\lim_{\rho\to 1}({}^{\rho}D_a^{r}h)(t) = (\frac{d}{dt})^n\frac{1}{\Gamma(n-r)}\int_a^t\frac{h(\tau)}{(t-\tau)^{r-n+1}}d\tau.$

(4) $\lim_{\rho\to 0^+}({}^{\rho}D_a^{r}h)(t) = \frac{1}{\Gamma(n-r)}(t\frac{d}{dt})^n\int_a^t(\log\frac{t}{\tau})^{n-r-1}h(\tau)\frac{d\tau}{\tau}.$

Remark 2.9.

(1) $\lim_{\rho\to 1}({}^{\rho}I_a^r h)(t) = ({}^{RL}I_a^r h)(t)$.
(2) $\lim_{\rho\to 0^+}({}^{\rho}I_a^r h)(t) = ({}^{H}I_a^r h)(t)$.
(3) $\lim_{\rho\to 1}({}^{\rho}D_a^r h)(t) = ({}^{RL}D_a^r h)(t)$.
(4) $\lim_{\rho\to 0^+}({}^{\rho}D_a^r h)(t) = ({}^{H}D_a^r h)(t)$.

Lemma 2.10. *Let* $0 < r < 1$. *The fractional equation* $({}^{\rho}D_0^r v)(t) = 0$, *has as solution*

$$v(t) = ct^{\rho(r-1)}, \tag{2.2}$$

with $c \in \mathbb{R}$.

Lemma 2.11. *Let* $0 < r < 1$. *Then*

$${}^{\rho}I^r({}^{\rho}D_0^r u)(t) = u(t) + ct^{\rho(r-1)}.$$

Proof. We have

$$\begin{aligned}
I_0^r D_0^r u(t) &= \left(t^{1-p}\frac{d}{dt}\right) I_0^{r+1} D_0^r u(t) \\
&= (t^{1-\rho}\frac{d}{dt})\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-r}}({}^{\rho}D_0^r u(s))ds\right) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-r}}\left[\left(s^{1-\rho}\frac{d}{ds}\right)(I_0^{1-r}u)(s)\right]ds\right) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t (t^\rho - s^\rho)^r\left[\frac{d}{ds}(I_0^{1-r}u)(s)\right]ds\right).
\end{aligned}$$

Thus, $I_0^r D_0^r u(t) = I_1 + I_2$, with

$$I_1 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{-r}}{\Gamma(r+1)}\left(\left[(t^\rho - s^\rho)^r I_0^{1-r}u(s)\right]_0^t\right),$$

and

$$I_2 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t r\rho s^{\rho-1}(t^\rho - s^\rho)^{r-1} I_0^{1-r}u(s)ds.$$

Hence, we get

$$I_1 = ct^{\rho(r-1)}$$

and

$$\begin{aligned}
I_2 &= \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} I_0^{1-r}u(s)ds \\
&= \left(t^{1-\rho}\frac{d}{dt}\right) I_0^r(I_0^{1-r})u(s)ds \\
&= u(t).
\end{aligned}$$

Finally we obtain

$$(I_0^r)(D_0^r u)(t) = u(t) + ct^{\rho(r-1)}.$$

Lemma 2.12. *The problem*

$$\begin{cases} ({}^\rho D_0^r u)(t) = h(t); \quad t \in I := [0,T] \\ u(T) = u_T \end{cases} \tag{2.3}$$

has the following solution

$$u(t) = \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-r}} h(t) ds - C t^{\rho(r-1)} \tag{2.4}$$

where

$$C = \frac{1}{T^{\rho(r-1)}} \left(\frac{\rho^{1-r}}{\Gamma(r)} \int_0^T \frac{s^{\rho-1}}{(T^\rho - s^\rho)^{1-r}} h(T) ds - u_T \right).$$

Proof. Solving the equation

$$({}^\rho D_0^r u)(t) = h(t),$$

we get

$$u(t) ={}^\rho I_0^r h(t) - c t^{\rho(r-1)}.$$

From the condition, we get

$$C = \frac{{}^\rho I_0^r h(T) - u_T}{T^{\rho(r-1)}}$$

hence, we obtain (2.4).

Definition 2.13. *By a random solution of problem (1.1) and (1.2), we mean a measurable function $x(w,\cdot) \in C_{\varsigma,\rho}(I)$ which satisfies (1.1) and (1.2).*

Lemma 2.14. *u is a random solution of (1.1) and (1.2), if and only if it satisfies*

$$x(\xi, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^\xi \frac{s^{\rho-1}}{(\xi^\rho - s^\rho)^{1-\varsigma}} f(\xi, x, w) ds - C(w) \xi^{\rho(\varsigma-1)} \tag{2.5}$$

where

$$C(w) = \frac{1}{T^{\rho(\varsigma-1)}} \left(\frac{\rho^{1-\varsigma}}{\Gamma(r)} \int_0^T \frac{s^{\rho-1}}{(T^\rho - s^\rho)^{1-\varsigma}} f(T, x, w) ds - x_T(w) \right).$$

Lemma 2.15. *[4, 13] Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \subset E$, and $T(w, \cdot)$ is continuous for all $w \subset \Omega$. Then the map $(w, v) \to T(w, v)$ is jointly measurable.*

Definition 2.16. *A function $f : I \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

- *(i) The map $(s, w) \to f(s, x, w)$ is jointly measurable for all $x \in E$, and*
- *(ii) The map $x \to f(s, x, w)$ is continuous for almost all $s \in I$ and $w \subset \Omega$.*

Let $\epsilon > 0$ and $\Phi : \Omega \times I \to \mathbb{R}_+$ be a jointly measurable function. We consider the following inequality

$$\|({}^\rho D_0^r x)(\xi, w) - f(\xi, u(\xi, w), w)\| \le \Phi(\xi, w); \ for\ \xi \in I, \ \ and\ w \in \Omega. \tag{2.6}$$

Definition 2.17. *[5] The problem (1.1) and (1.2) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists $c_{f,\phi} > 0$ such that for each solution $x(\cdot, w) \in C_{\varsigma,\rho}(I)$ of the inequality (2.6), there exists $y(\cdot, w) \in C_{\varsigma,\rho}(I)$ satisfies (1.1) and (1.2) with*

$$\|\xi^{\rho(1-\varsigma)}x(\xi, w) - \xi^{\rho(1-\varsigma)}y(\xi, w)\| \leq c_{f,\phi}\phi(\xi, w);\ \xi \in I;\ w \in \Omega.$$

Theorem 2.18. *[18] Let X be a nonempty, closed convex bounded subset of the separable Banach space E and let $G : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $G(w)u = u$ has a random solution.*

3. Existence and Ulam stability results

We shall make use of the following hypotheses:

(H_1) f is a random Carathéodory function.

(H_2) There exist measurable and essentially bounded functions $l_i : \Omega \to C(I);\ i = 1, 2$ such that

$$\|f(t, x, w)\| \leq l_1(t, w) + l_2(t, w)t^{\rho(1-r)}\|x\|,$$

for all $x \in E$ and $t \in I$ with

$$l_i^*(w) = \sup_{t\in I} l_i(t, w);\ i = 1, 2,\ w \in \Omega.$$

Theorem 3.1. *If (H_1) and (H_2) hold, and*

$$\frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w) < 1, \tag{3.1}$$

then there exists a random solution for (1.1) and (1.2).

Proof. Let $N : \Omega \times C_{\varsigma,\rho}(I) \to C_{\varsigma,\rho}(I)$ be the operator defined by

$$(Nx)(t, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} f(s, x(s, w), w)ds - C(w)t^{\rho(\varsigma-1)}, \tag{3.2}$$

and set

$$R(w) > \frac{\|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w)}{1 - \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)};\quad w \in \Omega, \tag{3.3}$$

and define the ball

$$B_R = B(0, R(w)) := \{x \in C_{\varsigma,\rho}(I) : \|x\|_C \leq R(w)\}.$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned} \|t^{\rho(1-\varsigma)}(Nx)(t, w)\| &\leq \|C(w)\| + \|\frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} f(s, x(s, w), w)ds\| \\ &\leq \|C(w)\| + \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}}\|l_1(s, w)\|ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|s^{\rho(1-\varsigma)}l_2(s,w)x(s,w)\|ds \\
&\leq \quad \|C(w)\| + \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\frac{T^{\varsigma\rho}}{\varsigma\rho}l_1^*(w) \\
&+ \quad \frac{l_2^*(w)\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(r)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|s^{\rho(1-\varsigma)}x(s,w)\|ds \\
&\leq \quad \|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w) + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)\|x\|_C \\
&\leq \quad \|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w) + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)R(w) \\
&\leq \quad R(w).
\end{aligned}
$$

Thus

$$\|N(w)(u\|_C \leq R(w).$$

Hence $N(w)(B_R) \subset B_R$. We shall prove that $N : \Omega \times B_R \to B_R$ satisfies the assumptions of Theorem 2.18.

Step 1. $N(w)$ is a random operator.

From (H_1), the map $w \longrightarrow f(t,x,w)$ is measurable and further the integral is a limit of a finite sum of measurable functions therefore the map

$$w \quad \mapsto \quad \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}f(s,x(s,w),w)ds - C(w)t^{\rho(r-1)},$$

is measurable.

Step 2. $N(w)$ is continuous.

Consider the sequence $(x_n)_n$ such that $x_n \to u$ in $C_{\varsigma,\rho}$.

Set

$$v_n(t,w) = t^{\rho(1-\varsigma)}(Nx_n)(t,w), \quad and \quad v(t,w) = t^{\rho(1-\varsigma)}(Nx)(t,w).$$

Then

$$\|v_n(t,w) - v(t,w)\|$$

$$
\begin{aligned}
&\leq \quad \left\|\frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}(f(s,x_n(s,w),w) - f(s,x(s,w),w))ds\right\| \\
&\leq \quad \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|f(s,x_n(s,w),w) - f(s,x(s,w),w))\|ds.
\end{aligned}
$$

By (H_1) we obtain

$$\|v_n(\cdot,w) - v(\cdot,w)\|_C \to 0 \; as \; n \to \infty,$$

Consequently, $N(w) : B_R \subset B_R$ is continuous.

Step 3. $N(w)B_R$ is equicontinuous. For $1 \leq t_1 \leq t_2 \leq T$, and $x \in B_R$, we have

$$\|t_2^{\rho(1-\varsigma)}(Nx)(t_2,w) - t_1^{\rho(1-\varsigma)}(Nx)(t_1,w)\|$$

$$
\begin{aligned}
&\leq \left\| \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right. \\
&- \left. \frac{\rho^{1-\varsigma} t_1^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right\| \\
&\leq \left\| \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(r)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right. \\
&- \frac{\rho^{1-\varsigma} t_1^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \\
&+ \left. \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right\| \\
&\leq \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&+ \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&+ \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&\leq \frac{t_2^{\varsigma\rho} + t_1^{\varsigma\rho} + 2(t_2^\rho - t_1^\rho)^\varsigma}{\rho^\varsigma \Gamma(1+\varsigma)} T^{\rho(1-\varsigma)} (l_1^*(w) + l_2^*(w) R(w)) \\
&\to \ 0; \ \ as \ \ t_2 \to t_1.
\end{aligned}
$$

Arzelá-Ascoli theorem implies that $N : \Omega \times B_R \to B_R$ is continuous and compact. Hence; from Theorem 2.18, we deduce the existence of random solution to problem (1.1) and (1.2).

Now, we prove a result concerning the generalized Ulam-Hyers-Rassias stability of (1.1) and (1.2).

We introduce the following additional hypotheses:

(H_3) For any $w \in \Omega$, $\Phi(t, \cdot) \subset L^1[0, \infty)$, and there exists a measurable and essentially bounded function $q : \Omega \to C(I, [0, \infty))$; such that

$$
(1 + \|x - y\|)\|f(t, x(t,w), w) - f(t, y(t,w), w)\| \leq q(t,w)\Phi(t,w)t^{\rho(1-\varsigma)}\|x - y\|.
$$

(H_4) There exists $\lambda_\Phi > 0$ such that

$$
{}^\rho I_0^\varsigma \Phi(t,w) \leq \lambda_\Phi \Phi(t,w).
$$

Remark 3.2. *Hypothesis (H_3) implies (H_2) with*

$$
l_1(w,t) = f(t, 0, w), \ l_2(w) = q(t,w)\Phi(t,w).
$$

Set

$$
\Phi^*(w) = \sup_{t \in I} \Phi(t,w), \ q^*(w) = \sup_{t \in I} q(t,w); \ w \in \Omega.
$$

Theorem 3.3. *If (H_1), (H_3), (H_4) and*

$$
\frac{\rho^{-\varsigma} T^\rho}{\Gamma(1+\varsigma)} \Phi^*(w) q^*(w) < 1, \tag{3.4}
$$

hold. Then the problem (1.1) and (1.2) has random solutions defined on I, and it is generalized Ulam-Hyers-Rassias stable.

Proof. From (H_1), (H_3) and Remark 3.2; the problem (1.1) and (1.2) has at least one random solution y. Then, we have

$$y(t,w)=\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}f(s,y(s,w),w)ds-C(w)t^{\rho(\varsigma-1)}.$$

Assume x be a random solution of (2.6). We obtain

$$\begin{aligned}\|t^{\rho(1-\varsigma)}x(t,w)&-\frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}f(s,v(s,w),w)ds+C(w)\|\\&\leq T^{\rho(1-\varsigma)}({}^\rho I_0^\varsigma\Phi)(t,w).\end{aligned}$$

From hypotheses (H_3) and (H_4), we have

$$\|t^{\rho(1-\varsigma)}x(t,w)-t^{\rho(1-\varsigma)}y(t,w)\|$$

$$\begin{aligned}&\leq\|t^{\rho(1-\varsigma)}x(t,w)-\frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}f(s,x(s,w),w)ds+C(w)\|\\&+\|\frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}f(s,x(s,w),w)ds-C(w)\\&-\frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}f(s,y(s,w),w)ds+C(w)\|\\&\leq T^{\rho(1-\varsigma)}({}^\rho I_0^\varsigma\Phi)(t,w)\\&+\frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}\|f(s,x(s,w),w)-f(s,y(s,w),w)\|ds\\&\leq T^{\rho(1-\varsigma)}({}^\rho I_0^\varsigma\Phi)(t,w)\\&+\frac{\rho^{1-\varsigma}T^{\rho(1-r)}}{\Gamma(\varsigma)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\varsigma}}q^*(w)\Phi(s,w)s^{\rho(1-\varsigma)}\frac{\|x-y\|}{1+\|x-y\|}ds\\&\leq T^{\rho(1-\varsigma)}\lambda_\Phi\Phi(t,w)+T^{2\rho(1-\varsigma)}\lambda_\Phi\Phi(t,w)q^*(w).\end{aligned}$$

Thus, we get

$$\begin{aligned}\|t^{\rho(1-\varsigma)}x(t,w)-t^{\rho(1-\varsigma)}y(t,w)\|&\leq(1+T^{\rho(1-\varsigma)}q^*(w))T^{\rho(1-\varsigma)}\lambda_\Phi\Phi(t,w)\\&:=c_{f,\Phi}\Phi(t,w).\end{aligned}$$

Hence, problem (1.1) and (1.2) is generalized Ulam-Hyers-Rassias stable.

4. An example

Let $\Omega=(-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$, and let

$$l^1=\left\{x=(x_1,x_2,\dots,x_n,\dots),\sum_{n=1}^\infty|x_n|<\infty\right\}$$

be the Banach space with the norm

$$\|x\| = \sum_{n=1}^{\infty} |x_n|.$$

Consider the Katugampola random fractional differential equation

$$({}^{\rho}D^{r}_{0^+} x_n)(t, w) = f_n(t, x(t, w), w);\ t \in [0, 1],\ w \in \Omega, \tag{4.1}$$

with the terminal condition

$$x(T, w) = ((1 + w^2)^{-1}, 0, 0, \cdots);\ w \in \Omega, \tag{4.2}$$

with $x = (x_1, x_2, \ldots, x_n, \ldots),\ f = (f_1, f_2, \ldots, f_n, \ldots),$

$${}^{\rho}D^{r}_{0^+} x = ({}^{\rho}D^{r}_{0^+} x_1, \ldots, {}^{\rho}D^{r}_{0^+} x_n, \ldots),$$

and

$$f_n(t, x(t, w), w) = \frac{w^2 t^{\rho(1-r)}(2^{-n} + x_n(t, w))}{2(1 + w^2)(1 + \|x\|)} \left(e^{-7-w^2} + \frac{1}{e^{t+5}} \right);\ t \in [0, 1],\ w \in \Omega.$$

We have

$$\|f(t, x, w) - f(t, y, w)\| \le (e^{-7-w^2} + e^{-t-5}) \frac{w^2 t^{\rho(1-r)} \|x - y\|}{1 + \|x - y\|}.$$

Hence, hypotheses (H_3) and (H_4) are satisfied with

$$q(t, w) = e^{-7-w^2} + e^{-t-5}, \quad \Phi(t, w) = w^2.$$

Hence by theorems 3.1 and 3.3, problem (4.1) and (4.2) admits a random solution, and is generalized Ulam-Hyers-Rassias stable.

5. Conclusions

In this paper, we provided some sufficient conditions ensuring the existence of random solutions and the Ulam stability for a class of fractional differential equations involving the Katugampola fractional derivative in Banach spaces. The techniques used are the random fixed point theory and the notion of Ulam-Hyers-Rassias stability.

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Conflict of interest

The authors declare no conflict of interests.

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