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Research article

Error estimates in L^2 and L^∞ norms of finite volume method for the bilinear elliptic optimal control problem

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Abstract: This paper discusses some a priori error estimates of bilinear elliptic optimal control problems based on the finite volume element approximation. A case-based numerical example serves to discuss with optimal L^2-norm error estimates and L^∞-norm error estimates, and supports two key insights. First, the approximate orders for the state, costate and control variables are $O(h^2)$ in the sense of L^2-norm. Second, the approximate orders for the state, costate and control variables are $O(h^2\sqrt{|lnh|})$ in the sense of L^∞-norm.

Keywords: bilinear elliptic optimal control problem; finite volume method; a priori error estimates; variational discretization
Mathematics Subject Classification: 49J20, 65M08

1. Introduction

Optimal control problems are frequently used in science, engineering and various application areas in the operation of social and physical. Many studies discussed the optimal control problems. The finite element methods seem to be the most popular numerical method used to solve the problems [6, 10]. For optimal control problems governed by linear elliptic equations, there were two pioneering works on finite element approximation by Falk [22] and Geveci [23]. However, it is a challenge that to solve the optimal control problems governed by nonlinear state equations. In [28, 30], the authors established a priori error estimates and a posteriori error estimates for the finite element approximation of a class of nonlinear optimal control problems. In [12–14, 16, 25–27, 31], the authors systematically introduced a finite element method for optimal control problems. Our objective is to investigate a class of bilinear optimal control problems that are often encountered in engineering, physical and others fields. Accordingly, it is necessary to study optimal control problems governed by bilinear state equations. Lu and Chen studied the finite element and mixed finite element approximation of bilinear optimal control problems [15, 32]. However, there are relatively scarce that incorporate the bilinear optimal control problems based on the finite volume method.

The finite volume method is an effective discretization technique for partial differential equations. Due to its local conservative property and other attractive properties, the finite volume method is a promising tool commonly applied in the numerical approximation of some problems for partial differential equations. Since the method was proposed, voluminous studies of the mathematical theory for finite volume method in the literature [2, 3, 9, 11, 17, 18, 20, 21]. Bank and Rose obtained some results for elliptic boundary value problems that the finite volume element approximation was comparable with the finite element approximation in H^1-norm which can be found in [2]. In [21], the authors presented the optimal L^2-error estimate for second-order elliptic boundary value problems under the assumption that $f \in H^1$, they also obtained the H^1-norm and maximum-norm error estimates for those problems. In [11], Chatzipantelidis proposed a nonconforming finite volume method and obtained the L^2-norm and H^1-norm error estimates for elliptic boundary value problems in two dimensions. The authors discussed prior estimates for linear elliptic optimal control problem in [33], they derived the optimal order error estimates in L^2 and L^∞-norm for the state, costate and control variables, and the optimal H^1 and $W^{1,\infty}$-norm error estimates for the state and costate variables.

Finite volume methods lie somewhere between finite difference and finite element methods [34]. It has a flexibility similar to that of finite element methods for handling complicated solution domain geometries and boundary conditions, and simplicity for implementation comparable to finite difference methods with triangulations of a simple structure [35]. The finite volume methods and finite element methods are commonly employed in computational fluid mechanics and computational solid dynamics, where the finite volume method is traditionally associated with computational fluid mechanics and the finite element method associated with computational solid dynamics. In general, two different functional spaces (one for the trial space and one for the test space) are used in the finite volume method. Owing to the two different spaces, the numerical analysis of the finite volume method is more difficult than that of the finite element method and finite difference method.

The goal of our paper is to establish a priori error estimates for the finite volume element approximation of the bilinear optimal control problem. First, we use the finite volume method to discretize the state and adjoint equation of the optimal control problem. Hinze proposed a variational discretization concept for optimal control problems with control constraints [24]. With the variational discretization concept, the control variable is not discretized directly but discretized by a projection of the discrete costate variable. Then we obtain some optimal order error estimates under some reasonable assumptions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|v\|_{m,p}^p$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, and the semi-norm $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) =$

$\{v \in W^{m,p}(\Omega) : v \mid_{\partial\Omega} = 0\}$. When p=2, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ with norm $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. Let $\|\cdot\|_\infty$ be the maximum norm, $\|f\|_\infty = \operatorname{ess}\sup_{x\in\Omega} |f(x)|$. As usual, we denote by $(\cdot,\cdot)$ of the $L^2(\Omega)$-inner product.

We consider the following bilinear elliptic optimal control problem:

$$\min_{u\in U_{ad}} \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}, \tag{1.1}$$

$$-\operatorname{div}(A\nabla y) + uy = f, \quad \text{in } \Omega, \tag{1.2}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygon domain with boundary $\partial\Omega$. f, $y_d \in H^1(\Omega)$, and y, u are unknowns functions, U_{ad} is denoted by

$$U_{ad} = \{u \in L^2(\Omega) : u(x) \geq 0, \ a.e. \text{ in } \Omega\}.$$

Furthermore, we assume that the coefficient matrix $A \in W^{2,\infty}(\Omega)$ is a symmetric positive definite matrix and there exists a constant $c > 0$ satisfies $\mathbf{X}^t A\mathbf{X} \geq c\|\mathbf{X}\|^2_{\mathbb{R}^2}$, if $\forall\, \mathbf{X} \in \mathbb{R}^2$. Actually, it belongs to a class of parameter estimation problems in which through the measured data y, the real parameter u is calculated by the least square formulation.

The paper is organized as follows. In the next Section, we introduce some notations and describe the finite volume method briefly. In Section 3, we apply the piecewise linear finite volume method and variational discretization concept to the optimal control problem (1.1)–(1.3) and obtain the discretized optimal system. In Section 4, we analyze the error estimates between the exact solution and the finite volume element approximation. And in Section 5, a few numerical examples are presented to test the theoretical results. While Second 6 gives a conclusion and some possible future work.

2. Finite volume element approximation

As is shown in [21], the partition $\mathcal{T}_h$ is quasi-uniform, i.e., there exists a positive constant C such that

$$C^{-1}h^2 \leq \operatorname{meas}(V_i) \leq Ch^2, \quad \forall\, V_i \in \mathcal{T}_h.$$

For the convex polygon Ω, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of a closed triangle elements K such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of K as shown in Figure 1.

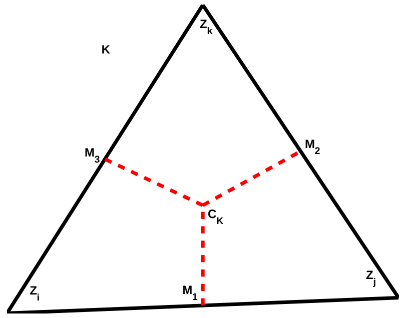

Figure 1. The dual partition of a triangular K.

The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as shown in Figure 2.

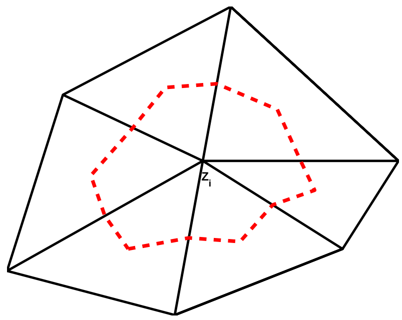

Figure 2. The control volume V_i sharing the same vertex z_i.

The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h = \max\{h_K\}$, where h_K is the diameter of the triangle K. The dual partition $\mathcal{T}_h^*$ is quasi-uniform as well.

We define the finite dimensional space V_h associated with $\mathcal{T}_h$ for the trial functions by

$$V_h = \{v :\ v \in H^1(\Omega),\ v|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h,\ v|_{\partial\Omega} = 0\},$$

and define the finite dimensional space Q_h associated with the dual partition $\mathcal{T}_h^*$ for the test functions by

$$Q_h = \{q \in L^2(\Omega) :\ q|_V \in P_0(V),\ \forall\ V \in \mathcal{T}_h^*;\ q|_{V_z} = 0,\ z \in \partial\Omega\},$$

where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V, respectively.

To connect the trial space and test space, we define a transfer operator $I_h : V_h \to Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i,\quad I_h v_h|_{V_i} = v_h(z_i),\qquad \forall\ V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i.

To address the finite volume method clearly, we consider the following problem

$$-\operatorname{div}(A\nabla\varphi) + u\varphi = f, \quad \text{in } \Omega, \tag{2.1}$$
$$\varphi = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

where A, Ω, $\partial\Omega$ are the same as in (1.2), (1.3), $f \in H^1(\Omega)$.

The finite volume element approximation φ_h of (2.1), (2.2) is defined as the solution of the problem: find $\varphi_h \in V_h$ such that

$$a(\varphi_h, I_h v_h) + (u_h\varphi_h, I_h v_h) = (f, I_h v_h), \quad \forall\, v_h \in V_h, \tag{2.3}$$

where the bilinear form $a(\varphi_h, I_h v_h)$ is defined by

$$a(\varphi_h, I_h v_h) = -\sum_{z_i \in N_h} v_h(z_i) \int_{\partial V_i} A\nabla\varphi_h \cdot \mathbb{n}\, ds, \quad \varphi_h, v_h \in H_0^1(\Omega) \cap V_h,$$

where $\mathbb{n}$ is the unit outward normal vector to ∂V_i.

The bilinear form $a(\cdot,\cdot)$ is not symmetric though the problem is self-adjoint. Then for all $w_h, v_h \in V_h$, there exists positive constants C and $h_0 \geq 0$ such that [18] for all $0 < h < h_0$

$$|a(w_h, I_h v_h) - a(v_h, I_h w_h)| \leq Ch\|w_h\|_1\, \|v_h\|_1. \tag{2.4}$$

It is well known (see, e.g., [28]) that the optimal control problem (1.1)–(1.3) has at least one solution (y, p, u), and that if a triplet (y, p, u) is the solution of (1.1)–(1.3), then there is a co-state $p \in H_0^1(\Omega)$ such that (y, p, u) satisfies the following optimality conditions:

$$(A\nabla y, \nabla w) + (uy, w) = (f, w), \quad \forall\, w \in H_0^1(\Omega), \tag{2.5}$$
$$(A\nabla p, \nabla q) + (up, q) = (y - y_d, q), \quad \forall\, q \in H_0^1(\Omega), \tag{2.6}$$
$$(u - yp, v - u) \geq 0, \quad \forall\, v \in U_{ad}. \tag{2.7}$$

If $y \in H_0^1(\Omega) \cap C^2(\Omega)$ and $p \in H_0^1(\Omega) \cap C^2(\Omega)$, then optimality condition (2.5)–(2.7) (see, e.g., [33]) can be written as

$$-\operatorname{div}(A\nabla y) + uy = f, \ \ \forall\, x \in \Omega, \tag{2.8}$$
$$y(x) = 0, \ \ \forall\, x \in \partial\Omega, \tag{2.9}$$
$$-\operatorname{div}(A\nabla p) + up = y - y_d, \ \ \forall\, x \in \Omega, \tag{2.10}$$
$$p(x) = 0, \ \ \forall\, x \in \partial\Omega, \tag{2.11}$$
$$(u - yp, v - u) \geq 0, \quad \forall\, v \in U_{ad}. \tag{2.12}$$

We use finite volume method to discretize the state and costate equation. Then the optimal control problem (2.5)–(2.7) can be approximated by: find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ such that

$$a(y_h, I_h w_h) + (u_h y_h, I_h w_h) = (f, I_h w_h), \quad \forall\, w_h \in V_h, \tag{2.13}$$
$$a(p_h, I_h q_h) + (u_h p_h, I_h q_h) = (y_h - y_d, I_h q_h), \quad \forall\, q_h \in V_h, \tag{2.14}$$
$$(u_h - y_h p_h, v - u_h) \geq 0, \quad \forall\, v \in U_{ad}. \tag{2.15}$$

Similar to [24], we can find that the variational inequality (2.12) is equivalent to

$$u(x) = \max(0, y(x)p(x)). \tag{2.16}$$

And the variational inequality (2.15) is equivalent to

$$u_h(x) = \max(0, y_h(x)p_h(x)). \tag{2.17}$$

Then the discrete optimality condition can be rewritten by: find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ such that

$$a(y_h, I_h w_h) + (u_h y_h, I_h w_h) = (f, I_h w_h), \quad \forall\, w_h \in V_h, \tag{2.18}$$

$$a(p_h, I_h q_h) + (u_h p_h, I_h q_h) = (y_h - y_d, I_h q_h), \quad \forall\, q_h \in V_h, \tag{2.19}$$

$$u_h(x) = \max(0, y_h p_h). \tag{2.20}$$

3. L^2 error estimates

In this section, we consider the error analysis of the finite volume element approximation. Let $(y_h(u), p_h(u))$ be the solution of

$$a(y_h(u), I_h w_h) + (u y_h(u), I_h w_h) = (f, I_h w_h), \quad \forall\, w_h \in V_h, \tag{3.1}$$

$$a(p_h(u), I_h q_h) + (u p_h(u), I_h q_h) = (y_h(u) - y_d, I_h q_h), \quad \forall\, q_h \in V_h. \tag{3.2}$$

For $y_h(u)$ and $p_h(u)$, noting that $y_h = y_h(u_h)$ and $p_h = p_h(u_h)$.

Firstly, we introduce some intermediate error estimates.

Lemma 3.1. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. Assume that $(y_h(u), p_h(u))$ and (y_h, p_h) be the solutions of (3.1), (3.2) and (2.13)–(2.15), respectively. Then it holds that*

$$\|p_h(u) - p_h\|_1 + \|y_h(u) - y_h\|_1 \le C\|u - u_h\|. \tag{3.3}$$

Proof. Subtracting (2.13), (2.14) from (3.1), (3.2), we have

$$a(y_h(u) - y_h, I_h w_h) + (u y_h(u) - u_h y_h, I_h w_h) = 0, \quad \forall\, w_h \in V_h,$$

$$a(p_h(u) - p_h, I_h q_h) + (u p_h(u) - u_h p_h, I_h q_h) = (y_h(u) - y_h, I_h q_h), \quad \forall\, q_h \in V_h.$$

Then we have

$$a(y_h(u) - y_h, I_h w_h) + (u(y_h(u) - y_h), I_h w_h) = (y_h(u_h - u), I_h w_h), \quad \forall\, w_h \in V_h, \tag{3.4}$$

$$\begin{aligned} &a(p_h(u) - p_h, I_h q_h) + (u(p_h(u) - p_h), I_h q_h) \\ &\qquad = (y_h(u) - y_h, I_h q_h) + (p_h(u_h - u), I_h q_h), \quad \forall\, q_h \in V_h. \end{aligned} \tag{3.5}$$

Let $w_h = y_h(u) - y_h$ and $q_h = p_h(u) - p_h$. An application Lemma 2.2 of [21], we can estimate the first term on the left side of the Eq (3.4) as follow:

$$C\|y_h(u) - y_h\|_0^2 \le a(y_h(u) - y_h, I_h w_h). \tag{3.6}$$

Note that $u \geq 0$, there holds that

$$C\|y_h(u) - y_h\|_0^2 \leq (u(y_h(u) - y_h), I_h w_h). \tag{3.7}$$

By using δ-Cauchy inequality, we also can estimate the first term on the right side of the Eq (3.4) as follow:

$$\begin{aligned}(y_h(u_h - u), I_h w_h) \leq& C\|u_h - u\| \cdot \|y_h(u) - y_h\| \\ \leq& C\|u_h - u\|^2 + \delta\|y_h(u) - y_h\|^2.\end{aligned} \tag{3.8}$$

where $\delta \in (0, c)$, c is a sufficiently small positive constant. Substitute (3.6)–(3.8) into (3.4), we can obtain

$$\|y_h(u) - y_h\|_0 \leq C\|u - u_h\|, \tag{3.9}$$

where $\| y_h\| \leq C$, and C is a positive constant. Note that

$$\begin{aligned}(y_h(u) - y_h, I_h q_h) \leq& C\|y_h(u) - y_h\|_0 \cdot \|p_h(u) - p_h\|_0 \\ \leq& C\|u - u_h\| \cdot \|p_h(u) - p_h\|_0,\end{aligned} \tag{3.10}$$

and

$$\|p_h(u) - p_h\| \leq \|p_h(u) - p_h\|_0. \tag{3.11}$$

Similarly, collecting (3.5), (3.10), and (3.11), we can obtain

$$\|p_h(u) - p_h\|_0 \leq C\|u - u_h\|. \tag{3.12}$$

Then, combining (3.9) and (3.12), we have

$$\|p_h(u) - p_h\|_1 + \|y_h(u) - y_h\|_0 \leq C\|u - u_h\|. \tag{3.13}$$

This completes the proof. □

Similar to the proof of Corollary 3.6 in [21] with $\alpha = 1$, we can obtain the following result.

Lemma 3.2. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. Assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$. Then it holds that*

$$\|p_h(u) - p\| + \|y_h(u) - y\| \leq Ch^2. \tag{3.14}$$

Proof. More details can be found in [21]. □

Define the directional derivative of $F(\cdot)$ at the point $u \in U_{ad}$ in the direction $v \in U_{ad}$ as following:

$$\lim_{t \to 0^+} \frac{F(u + tv) - F(u)}{t} = F'(u)(v).$$

If $F'(u)(\cdot)$ is a continuous linear functional on U_{ad}, then we obtain that F is D-differentiable at u.

Let $(p(u), y(u))$ and $(p_h(u), y_h(u))$ be the solutions of (2.13), (2.14) and (2.18), (2.19), respectively. Let $J(\cdot) : U_{ad} \to \mathbb{R}$ be a D-differential convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2}\|y(u) - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}.$$

Then we have a sequence of convex functional $J_h : U_{ad} \to \mathbb{R}$:

$$J_h(u) = \frac{1}{2}\|y_h(u) - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)},$$
$$J_h(u_h) = \frac{1}{2}\|y_h(u_h) - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_h\|^2_{L^2(\Omega)}.$$

According to [29], such that

$$(J'(u), v) = (u - y(u)p(u), v),$$
$$(J'_h(u), v) = (u - y_h(u)p_h(u), v),$$
$$(J'_h(u_h), v) = (u_h - y_h(u_h)p_h(u_h), v).$$

In the following we estimate $\|u - u_h\|$. We assume that the function J is strictly convex near the solution u, i.e., for the solution u there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c > 0$ satisfies:

$$(J'(u) - J'(v), u - v) \geq c\|u - v\|^2, \tag{3.15}$$

for all v in the neighborhood of u. The convexity of $J(\cdot)$ and $J_h(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the similar problems. For example, in [4, 5, 8], the authors are concerned with some optimal control problems governed by semilinear elliptic equations and give the first-order and second-order optimality conditions of the optimal solutions. Casas, Tröltzsch, and Unger discuss necessary and sufficient optimality conditions for general nonlinear optimal control problems, and derive the first order optimality conditions by using the Lagrangian formulation and the second order optimality conditions by using the Lagrangian and Hamiltonian functions in [7]. Then, the following second order sufficiently optimality condition (see [19, 30]) satisfies: there exists a $c > 0$ such that $J''(u)v^2 \geq c\|v\|_0^2$.

It follows from the assumption (3.15), which is proved in [1], there exists a constant $c > 0$ satisfies

$$(J'_h(v) - J'_h(u), v - u) \geq c\|v - u\|^2, \quad \forall v \in U_{ad}. \tag{3.16}$$

We next estimate the error of the approximate control in L^2-norm.

Theorem 3.1. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. We assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$. Then we obtain the following error estimate:*

$$\|u - u_h\| \leq Ch^2. \tag{3.17}$$

Proof. Let $v = u_h$ in (2.7) and $v = u$ in (2.15), such that

$$(u - yp, u_h - u) \geq 0, \tag{3.18}$$
$$(u_h - y_h p_h, u - u_h) \geq 0. \tag{3.19}$$

By using (3.16), (3.18), and (3.19), we obtain

$$\begin{aligned}
c\|u - u_h\|^2 &\leq (J_h'(u), u - u_h) - (J_h'(u_h), u - u_h) \\
&= (u - y_h(u)p_h(u), u - u_h) - (u_h - y_h p_h, u - u_h) \\
&= (u, u - u_h) - (u_h, u - u_h) + (y_h p_h - y_h(u)p_h(u), u - u_h) \\
&\leq (yp, u - u_h) - (y_h p_h, u - u_h) + (y_h p_h - y_h(u)p_h(u), u - u_h) \\
&= (yp - y_h(u)p_h(u), u - u_h) \\
&= (yp - yp_h(u), u - u_h) + (yp_h(u) - y_h(u)p_h(u), u - u_h) \\
&\equiv E_1 + E_2.
\end{aligned} \tag{3.20}$$

We now estimate all terms at the right side of (3.20). From Lemma 3.2 and δ-Cauchy inequality, there holds that

$$\begin{aligned}
E_1 &= (yp - yp_h(u), u - u_h) \\
&\leq C\|p - p_h(u)\| \cdot \|u - u_h\| \\
&\leq Ch^2\|u - u_h\| \\
&\leq Ch^4 + \delta\|u - u_h\|^2,
\end{aligned} \tag{3.21}$$

where $\delta \in (0, c)$, c is a sufficiently small positive constant. Similarly, for E_2, we obtain that

$$\begin{aligned}
E_2 &= (yp_h(u) - y_h(u)p_h(u), u - u_h) \\
&\leq C\|y - y_h(u)\| \cdot \|u - u_h\| \\
&\leq Ch^2\|u - u_h\| \\
&\leq Ch^4 + \delta\|u - u_h\|^2.
\end{aligned} \tag{3.22}$$

Finally, we can derive the result (3.17) from (3.20)–(3.22). □

Theorem 3.2. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. Assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$. Then there exists a $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$\|u - u_h\| + \|y - y_h\| + \|p - p_h\| \leq Ch^2. \tag{3.23}$$

Proof. Using the triangle inequality leads to

$$\begin{aligned}
\|y - y_h\| &\leq \|y - y_h(u)\| + \|y_h(u) - y_h\|, \\
\|p - p_h\| &\leq \|p - p_h(u)\| + \|p_h(u) - p_h\|.
\end{aligned}$$

An application Lemma 3.1 yields

$$\|y - y_h\| \leq \|y - y_h(u)\| + C\|u - u_h\|, \tag{3.24}$$
$$\|p - p_h\| \leq \|p - p_h(u)\| + C\|u - u_h\|. \tag{3.25}$$

Note that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$, by using Lemma 3.2, it holds that

$$\|y - y_h(u)\| \leq Ch^2 \text{ and } \|p - p_h(u)\| \leq Ch^2. \tag{3.26}$$

It follows from (3.17) and (3.24), such that

$$\|y - y_h\| \leq Ch^2. \tag{3.27}$$

By using (3.25), (3.17), and $\|p - p_h(u)\| \leq Ch^2$, we derive

$$\|p - p_h\| \leq Ch^2. \tag{3.28}$$

From (3.27), (3.28) and (3.17), we can obtain (3.23). □

4. L^∞ error estimates

In the section, we will discuss L^∞ error estimates of the bilinear elliptic optimal control problem. Firstly, we introduce the error estimates of the numerical solutions of the state and costate in H^1-norm.

Theorem 4.1. *Assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ are the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. Then there exists a $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$\|y - y_h\|_1 + \|p - p_h\|_1 \leq Ch. \tag{4.1}$$

Proof. Using the triangle inequality yields

$$\|y - y_h\|_1 \leq \|y - y_h(u)\|_1 + \|y_h(u) - y_h\|_1,$$
$$\|p - p_h\|_1 \leq \|p - p_h(u)\|_1 + \|p_h(u) - p_h\|_1.$$

An application of Lemma 3.1 leads to

$$\|y - y_h\|_1 \leq \|y - y_h(u)\|_1 + C\|u - u_h\|, \tag{4.2}$$
$$\|p - p_h\|_1 \leq \|p - p_h(u)\|_1 + C\|u - u_h\|. \tag{4.3}$$

By using Theorem 3.3 of [21] yields

$$\|y - y_h(u)\|_1 \leq Ch, \quad \|p - p_h(u)\|_1 \leq Ch. \tag{4.4}$$

From Theorem 3.2 and (4.2)–(4.4), we can easily obtain (4.1). □

Then, we estimate the error of the numerical solutions of control, state and costate in $L^\infty(\Omega)$-norm.

Theorem 4.2. *Assume that $A \in W^{2,\infty}(\Omega)$ and $f, y_d \in H^1(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)–(2.7) and (2.13)–(2.15), respectively. Then there exists a $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$\|u - u_h\|_\infty + \|y - y_h\|_\infty + \|p - p_h\|_\infty \leq Ch^2 \sqrt{|\ln h|}. \tag{4.5}$$

Proof. According to Lemma 3.1, Lemma 10.5.1 of [36], and Theorem 3.1, there holds that

$$\begin{aligned}
\|y - y_h\|_\infty &\leq \|y - y_h(u)\|_\infty + \|y_h(u) - y_h\|_\infty \\
&\leq \|y - y_h(u)\|_\infty + C(|\ln h|)^{1/2}\|y_h(u) - y_h\|_1 \\
&\leq \|y - y_h(u)\|_\infty + C(|\ln h|)^{1/2}\|u - u_h\| \\
&\leq Ch^2(|\ln h|)^{1/2}.
\end{aligned}$$

Similarly, by using (2.16), (2.17), we can get

$$\begin{aligned}
\|u - u_h\|_\infty &\leq C\|yp - y_h p_h\|_\infty \\
&\leq C(\|yp - yp_h(u)\|_\infty + \|yp_h(u) - yp_h\|_\infty + \|yp_h - y_h p_h\|_\infty) \\
&\leq C\|p - p_h(u)\|_\infty + C(|\ln h|)^{1/2}\|p_h(u) - p_h\|_1 + C\|y - y_h\|_\infty \\
&\leq C\|p - p_h(u)\|_\infty + C(|\ln h|)^{1/2}\|y - y_h\| + C\|y - y_h\|_\infty \\
&\leq Ch^2(|\ln h|)^{1/2}.
\end{aligned}$$

Using again Lemma 3.1, Lemma 10.5.1 of [36], and Theorem 3.1, such that

$$\begin{aligned}
\|p - p_h\|_\infty &\leq \|p - p_h(u)\|_\infty + \|p_h(u) - p_h\|_\infty \\
&\leq \|p - p_h(u)\|_\infty + C(|\ln h|)^{1/2}\|p_h(u) - p_h\|_1 \\
&\leq \|p - p_h(u)\|_\infty + C(|\ln h|)^{1/2}\|y - y_h\| \\
&\leq Ch^2(|\ln h|)^{1/2}.
\end{aligned}$$

Then we complete the proof of (4.5). $\square$

5. Numerical example

In order to test the theory of the previous section, we present one numerical example to illustrate them. The optimal problem was solved numerically by a precondition projection algorithm, with codes developed based on AFEPack.

In this example, we consider the bilinear elliptic optimal control problem:

$$\min_{u \in U_{ad}} \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|_{L^2(\Omega)},$$

subject to the state equation

$$\begin{aligned}
-\Delta y + y^3 = u + f, \quad -\Delta p + 3y^2 p &= y - y_d, \qquad &&\text{in } \Omega, \\
y &= 0, &&\text{on } \Gamma,
\end{aligned}$$

where $\Omega = [0, 1] \times [0, 1]$ and $U_{ad} = \{u : u \geq 0\}$.

Then we assume that

$$
\begin{aligned}
&y = \sin(\pi x_1)\sin(\pi x_2),\\
&p = 2\sin(\pi x_1)\sin(\pi x_2),\\
&y_d = y + \Delta p - 3y^2 p,\\
&u = \max(-p, 0),\\
&f = -\Delta y + y^3 - u.
\end{aligned}
$$

We present the error for the numerical solution of the triple (u_h, y_h, p_h) in Tables 1 and 2. In the numerical implementation, the errors $||u - u_h||_{L^2}$, $||y - y_h||_{L^2}$ and $||p - p_h||_{L^2}$ obtained on a sequence of uniformly refined meshes are presented in Table 1. While the errors $||u-u_h||_{L^\infty}$, $||y-y_h||_{L^\infty}$ and $||p-p_h||_{L^\infty}$ are presented in Table 2. In Table 1, the ith line is four times of the $(i + 1)$th line, $i = 3, 4, 5$, it is clear that $||u - u_h||_{L^2}$, $||y - y_h||_{L^2}$ and $||p - p_h||_{L^2}$ have the convergence order of $O(h^2)$. In Table 2, the ith line is almost four times of the $(i+1)$th line, $i = 3, 4, 5$, which means that the convergent rates are $O(h^2\sqrt{|lnh|})$. So, $||u - u_h||_{L^\infty}$, $||y - y_h||_{L^\infty}$ and $||p - p_h||_{L^\infty}$ have the convergence order of $O(h^2\sqrt{|lnh|})$. The numerical results show the a priori error estimates is reliable, which is consistent with our theoretical results.

Table 1. The L^2-errors for state and control variables.

dof	Errors		
	$\|\|u-u_h\|\|_{L^2}$	$\|\|y-y_h\|\|_{L^2}$	$\|\|p-p_h\|\|_{L^2}$
49	3.48573E-02	2.40798E-02	4.78575E-02
225	8.71856E-03	6.08546E-03	1.25367E-02
961	2.21423E-03	1.50809E-03	3.05864E-03
3969	5.37573E-04	3.85875E-04	7.53643E-04

Table 2. The L^∞-errors for state and control variables.

dof	Errors		
	$\|\|u-u_h\|\|_{L^\infty}$	$\|\|y-y_h\|\|_{L^\infty}$	$\|\|p-p_h\|\|_{L^\infty}$
49	9.01533E-02	3.85715E-02	7.69858E-02
225	2.36452E-02	9.83623E-03	1.93561E-02
961	5.68646E-03	2.41389E-03	4.92130E-03
3969	1.51334E-03	6.15235E-04	1.23641E-03

The corresponding convergent rates of these approximations are presented in Figure 3. In Figure 3, the slope of the solid line is -1, which means the convergent rate is $O(h^2)$ or $O(h^2\sqrt{|lnh|})$.

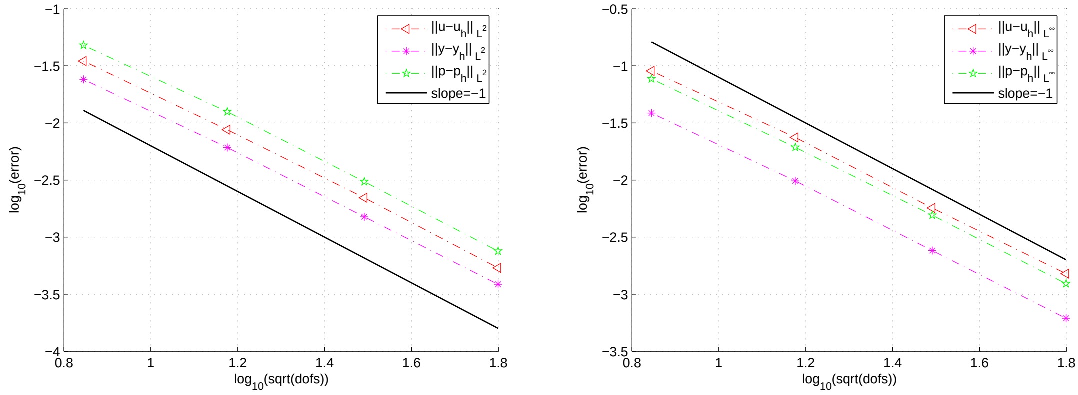

Figure 3. The convergent rates in the L^2-norm on the left hand side and in the L^∞-norm on the right hand side for the finite volume element approximation.

Seen from the numerical results listed in Table 1, Table 2 and Figure 3, the convergent orders match the theories derived in the previous sections.

6. Conclusions and future works

In this paper, we considered a priori error estimates for the finite volume element approximation of the bilinear elliptic optimal control problem. Then we used the finite volume method to discretize the state and adjoint equation of the system. Under some reasonable assumptions, we obtained some optimal order error estimates. The approximate orders for the state, costate and control variables were $O(h^2)$ and $O(h^2\sqrt{|lnh|})$ in the sense of L^2-norm and L^∞-norm. To our best knowledge in the context of optimal control problems, there has no literature that considers the priori error estimates of the finite volume method governed by the bilinear elliptic optimal control problem.

In the future, we shall consider the finite volume element method for bilinear parabolic optimal control problem. Furthermore, we shall consider a posteriori error estimates and superconvergence of the finite volume element solutions for bilinear parabolic optimal control problem.

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Conflict of interest

The authors declared that they have no conflict of interest.

References

1. N. Arada, E. Casas, F. Tröltzsch, Error estimates for the numerical approximation of a semilinear elliptic control problem, *Comput. Optim. Appl.*, **23** (2002), 201–229.
2. R. E. Bank, D. J. Rose, Some error estimates for the box method, *SIAM J. Numer. Anal.*, **24** (1987), 777–787.
3. F. Boyer, F. Hubert, Finite volume method for 2D linear and nonlinear elliptic problems with discontinuities, *SIAM J. Numer. Anal.*, **46** (2008), 3032–3070.
4. E. Casas, F. Tröltzsch, Second-order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations, *Appl. Math. Optim.*, **39** (1999), 211–227.
5. Y. Danping, Y. Chang, W. Liu , A priori error estimate and superconvergence analysis for an optimal control problem of bilinear type, *J. Comput. Math.*, **26** (2008), 3–19.
6. U. Langer, O. Steinbach, F. Trltzsch, Unstructured space-time finite element methods for optimal control of parabolic equations, *SIAM J. Sci. Comput.*, **43** (2021), 744–771.
7. E. Casas, F. Tröltzsch, A. Unger, Second order sufficient optimality condition for a nonlinear elliptic boundary control problem, *Z. Anal. Anwend.*, **15** (1998), 687–707.
8. H. Guan, D. Shi, An efficient NFEM for optimal control problems governed by a bilinear state equation, *Comput. Math. Appl.*, **77** (2019), 1821–1827.
9. Z. Cai, On the finite volume element method, *Numer. Math.*, **58** (1991), 713–735.
10. J. Liu, Z. Zhou, Finite element approximation of time fractional optimal control problem with integral state constraint, *AIMS Math.*, **6** (2021), 979–997.
11. P. Chatzipantelidis, A finite volume method based on the Crouzeix-Raviart element for elliptic PDEs in two dimensions, *Numer. Math.*, **82** (1999), 409–432.
12. Y. Chen, Z. Lu, *High efficient and accuracy numerical methods for optimal control problems*, Science Press, Beijing, 2015.
13. Y. Chen, Z. Lu, Error estimates of fully discrete mixed finite element methods for semilinear quadratic parabolic optimal control problems, *Comput. Methods Appl. Mech. Eng.*, **199** (2010), 1415–1423.
14. Y. Chen, Z. Lu, Error estimates for parabolic optimal control problem by fully discrete mixed finite element methods, *Finite Elem. Anal. Des.*, **46** (2010), 957–965.
15. Y. Chen, Z. Lu, Y. Huang, Superconvergence of triangular Raviart-Thomas mixed finite element methods for bilinear constrained optimal control problem, *Comput. Math. Appl.*, **66** (2013), 1498–1513.
16. Y. Chen, N. Yi, W. Liu, A Legendre-Galerkin spectral method for optimal control problems governed by elliptic equations, *SIAM J. Numer. Anal.*, **46** (2008), 2254–2275.
17. Z. Chen, R. Li, A. Zhou, A note on the optimal L^2 estimate of the finite volume element method, *Adv. Comput. Math.*, **16** (2002), 291–303.
18. S. Chou, Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: A unified approach, *Math. Comput.*, **69** (2000), 103–120.
19. S. Chou, X. Ye, Unified analysis of finite volume methods for second order elliptic problems, *SIAM J. Numer. Anal.*, **45** (2007), 1639–1653.

20. D. Estep, M. Pernice, P. Du, A posteriori error analysis of a cell-centered finite volume method for semilinear elliptic problems, *J. Comput. Appl. Math.*, **233** (2009), 459–472.
21. R. E. Ewing, T. Lin, Y. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, *SIAM J. Numer. Anal.*, **39** (2002), 1865–1888.
22. F. S. Falk, Approximation of a class of optimal control problems with order of convergence estimates, *J. Math. Anal. Appl.*, **44** (1973), 28–47.
23. T. Geveci, On the approximation of the solution of an optimal control problem governed by an elliptic equation, *RAIRO: Anal. Numer.*, **13** (1979), 313–328.
24. M. Hinze, A variational discretization concept in control constrained optimization: The linear-quadratic case, *Comput. Optim. Appl.*, **30** (2005), 45–61.
25. J. L. Lions, *Optimal control of systems governed by partial differential equtions*, Springer, Berlin, 1971.
26. R. Li, W. Liu, H. Ma, T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems, *SIAM J. Control Optim.*, **41** (2002), 1321–1349.
27. W. Liu, N. Yan, A posteriori error estimates for convex boundary control problems, *SIAM J. Numer. Anal.*, **39** (2001), 73–99.
28. W. Liu, N. Yan, A posteriori error estimates for control problems governed by nonlinear elliptic equations, *Appl. Numer. Math.*, **47** (2003), 173–187.
29. W. Liu, N. Yan, *Adaptive finite element methods for optimal control governed by PDEs*, Science Press, Beijing, 2008.
30. W. Liu, D. Tiba, Error estimates for the finite element approximation of a class of nonlinear optimal control problems, *J. Numer. Func. Optim.*, **22** (2001), 935–972.
31. Z. Lu, Y. Chen, W. Zheng, A posteriori error estimates of lowest order Raviart-Thomas mixed finite element methods for bilinear optimal control problems, *East Asia J. Appl. Math.*, **2** (2012), 108–125.
32. Z. Lu, S. Zhang, L^∞-error estimates of rectangular mixed finite element methods for bilinear optimal control problem, *Appl. Math. Comput.*, **300** (2017), 79–94.
33. X. Luo, Y. Chen, Y. Huang, Some error estimates of finite volume element approximation for elliptic optimal control problems, *Int. J. Numer. Anal. Mod.*, **10** (2013), 697–711.
34. S. Phongthanapanich, R. Eymard, A comparative study of characteristic finite element and characteristic finite volume methods for convection-diffusion-reaction problems on triangular grids, *Appl. Sci. Eng. Prog.*, **12** (2019), 235–242.
35. J. M. Sargado, I. Berre, J. M. Nordbotten, A combined finite element-finite volume framework for phase-field fracture, *Comput. Methods. Appl. Mech. Eng.*, **373** (2021), 113474.
36. Z. Shi, M. Wang, *Finite element methods*, Science Press, Beijing, 2010.

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