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Research article

Commutative ideals of BCK-algebras and BCI-algebras based on soju structures

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Abstract: The concept of a commutative soju ideal in a BCK-algebra and a BCI-algebra is introduced, and their properties are investigated. The relationship between a soju ideal and a commutative soju ideal are discussed, and examples to show that any soju ideal may not be a commutative soju ideal are provided. Conditions for a soju ideal to be a commutative soju ideal are considered, and characterizations of a commutative soju ideal are studied. A new commutative soju ideal using the given commutative soju ideal is maded, and the extension property for a commutative soju ideal is established. A commutative soju ideal is established by using a commutative ideal of a BCI-algebra. The notion of a closed soju ideal in a BCI-algebra is also introduced, and it is used in studying the characterization of a commutative soju ideal.

Keywords: soju ideal; commutative soju ideal; closed soju ideal
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1. Introduction

As a representative tool for dealing with uncertainty, we can think of intuitionistic fuzzy set and soft set. Intuitionistic fuzzy set, which is one of several generalizations of fuzzy set theory for various objectives, is introduced by Atanassov [4,5]. Jun et al. [13] conducted a study that applied Atanassov's intuitionistic fuzzy set to BCK-algebras. soft set theory is also a generalization of fuzzy set theory, that was proposed by Molodtsov [23] in 1999 to deal with uncertainty in a parametric manner. Intuitionistic fuzzy set and soft set theory have been applied in many ways (see [1–3, 6–8, 10–12, 14, 15, 17, 24–26]). As intuitionistic fuzzy set and soft set have emerged as tools to deal with uncertainty very effectively, we have considered the need to make uncertainty more convenient and effective by developing a new hybrid structure that uses both of these tools. Based on this idea, Jun et al. [16] introduced a new structure called soju structure that can handle intuitionistic fuzzy set and soft set simultaneously and applied it to BCK/BCI-algebras. They introduced the notion of soju subalgebra and soju ideal in BCK/BCI-algebras, and considered the relation between soju subalgebra and soju ideal. They provided conditions for a soju structure to be a soju ideal in a BCK-algebra, and discussed characterizations of soju subalgebra and soju ideal.

In this article, we introduce the concept of a commutative soju ideal in a BCK-algebra and a BCI-algebra, and investigate their properties. We discuss the relationship between a soju ideal and a commutative soju ideal. We provide examples to show that any soju ideal may not be a commutative soju ideal. We consider conditions for a soju ideal to be a commutative soju ideal, and consider characterizations of a commutative soju ideal. We make a new commutative soju ideal using the given commutative soju ideal. We establish the extension property for a commutative soju ideal. Using a commutative ideal of a BCI-algebra, we establish a commutative soju ideal. We introduce the notion of a closed soju ideal in a BCI-algebra and use it to study the characterization of a commutative soju ideal.

2. Preliminaries

2.1. Basic concepts about BCK-algebras

A *BCI-algebra* is defined to be an algebra $(X; *, 0)$ that satisfies the following conditions:

(I_1) $((\tilde{x} * \tilde{y}) * (\tilde{x} * \tilde{z})) * (\tilde{z} * \tilde{y}) = 0$,
(I_2) $(\tilde{x} * (\tilde{x} * \tilde{y})) * \tilde{y} = 0$,
(I_3) $\tilde{x} * \tilde{x} = 0$,
(I_4) $\tilde{x} * \tilde{y} = 0,\ \tilde{y} * \tilde{x} = 0 \Rightarrow \tilde{x} = \tilde{y}$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

If a BCI-algebra X satisfies the following identity:

(K) $(\forall \tilde{x} \in X)\ (0 * \tilde{x} = 0)$,

then X is called a *BCK-algebra.* We define an order relation "≤" on a BCK/BCI-algebra X as follows:

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} * \tilde{y} = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies:

$$(\forall \tilde{x} \in X)(\tilde{x} * 0 = \tilde{x}), \tag{2.2}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\tilde{x} \leq \tilde{y} \Rightarrow \tilde{x} * \tilde{z} \leq \tilde{y} * \tilde{z},\ \tilde{z} * \tilde{y} \leq \tilde{z} * \tilde{x}), \tag{2.3}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{y}) * \tilde{z} = (\tilde{x} * \tilde{z}) * \tilde{y}). \tag{2.4}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{z}) * (\tilde{y} * \tilde{z}) \leq \tilde{x} * y). \tag{2.5}$$

Every BCI-algebra X satisfies:

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} * (\tilde{x} * (\tilde{x} * \tilde{y})) = \tilde{x} * \tilde{y}), \tag{2.6}$$
$$(\forall \tilde{x}, \tilde{y} \in X)(0 * (\tilde{x} * \tilde{y}) = (0 * \tilde{x}) * (0 * \tilde{y})). \tag{2.7}$$

A BCK-algebra X is said to be *commutative* (see [21]) if $\tilde{x} * (\tilde{x} * \tilde{y}) = \tilde{y} * (\tilde{y} * \tilde{x})$ for all $\tilde{x}, \tilde{y} \in X$. We will abbreviate commutative BCK-algebra to cBCK-algebra.

A BCI-algebra X is said to be *commutative* (see [22]) if it satisfies:

$$(\forall \tilde{x}, \tilde{y} \in X)\,(\tilde{x} \leq \tilde{y} \;\Rightarrow\; \tilde{x} = \tilde{y} * (\tilde{y} * \tilde{x}))\,. \tag{2.8}$$

We will abbreviate commutative BCI-algebra to cBCI-algebra.

A subset L of a BCK/BCI-algebra X is called a *subalgebra* of X if $\tilde{x} * \tilde{y} \in L$ for all $\tilde{x}, \tilde{y} \in L$. A subset L of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in L, \tag{2.9}$$

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} * \tilde{y} \in L,\ \tilde{y} \in L \;\Rightarrow\; \tilde{x} \in L). \tag{2.10}$$

A subset L of a BCK-algebra X is called a *commutative ideal* of X (see [18]) if it satisfies (2.9) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{y}) * \tilde{z} \in L,\ \tilde{z} \in L \;\Rightarrow\; \tilde{x} * (\tilde{y} * (\tilde{y} * \tilde{x})) \in L). \tag{2.11}$$

A subset L of a BCI-algebra X is called a *commutative ideal* of X (see [19]) if it satisfies (2.9) and

$$(\tilde{x} * \tilde{y}) * \tilde{z} \in L,\ \tilde{z} \in L \;\Rightarrow\; \tilde{x} * ((\tilde{y} * (\tilde{y} * \tilde{x})) * (0 * (0 * (\tilde{x} * \tilde{y})))) \in L \tag{2.12}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

For more information on BCI-algebra and BCK-algebra, please refer to the books [9] and [21].

2.2. *Basic concepts about soju structures*

In what follows, let W be an initial universe set unless otherwise specified.

Definition 2.1 ([16]). Let X be a set of parameters. For any subset B of X, let $\lambda := (\zeta_\lambda, \xi_\lambda)$ be an intuitionistic fuzzy set in B and $(\tilde{G}, B)$ be a soft set over W. Then the pair $(B, \langle \lambda; \tilde{G} \rangle)$ is called a *soju structure* over $([0,1], W)$.

Given a soju structure $(X, \langle \lambda; \tilde{G} \rangle)$ over $([0,1], W)$, $(t, s) \in [0,1] \times [0,1]$ with $t + s \leq 1$ and $\alpha \in 2^W$, consider the following sets:

$$X(\zeta_\lambda \uparrow t) := \{x \in X \mid \zeta_\lambda(x) \geq t\},$$

$$X(\xi_\lambda \downarrow s) := \{x \in X \mid \xi_\lambda(x) \leq s\},$$

$$X(\tilde{G}; \alpha) := \{x \in X \mid \tilde{G}(x) \supseteq \alpha\}.$$

Definition 2.2 ([16]). Let B be a subset of a BCK/BCI-algebra X. A soju structure $(B, \langle \lambda; \tilde{G} \rangle)$ over $([0,1], W)$ is called a *soju subalgebra* based on B (briefly, *soju B-subalgebra*) of X if the following condition is valid.

$$(\forall x, y \in B)\left(x * y \in B \;\Rightarrow \left\{ \begin{array}{l} \zeta_\lambda(x * y) \geq \min\{\zeta_\lambda(x), \zeta_\lambda(y)\} \\ \xi_\lambda(x * y) \leq \max\{\xi_\lambda(x), \xi_\lambda(y)\} \\ \tilde{G}(x * y) \supseteq \tilde{G}(x) \cap \tilde{G}(y) \end{array} \right. \right). \tag{2.13}$$

Definition 2.3 ([16]). Let B be a subalgebra of a BCK/BCI-algebra X. A soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0,1], W)$ is called a *soju ideal* based on B (briefly, *soju B-ideal*) of X if the following conditions are valid.

$$(\forall x \in B)(\zeta_\lambda(0) \geq \zeta_\lambda(x),\ \xi_\lambda(0) \leq \xi_\lambda(x),\ \tilde{G}(0) \supseteq \tilde{G}(x)), \tag{2.14}$$

$$(\forall x, y \in B)\left(\begin{array}{l} \zeta_\lambda(x) \geq \min\{\zeta_\lambda(x * y), \zeta_\lambda(y)\} \\ \xi_\lambda(x) \leq \max\{\xi_\lambda(x * y), \xi_\lambda(y)\} \\ \tilde{G}(x) \supseteq \tilde{G}(x * y) \cap \tilde{G}(y) \end{array}\right). \tag{2.15}$$

If $B = X$, soju B-ideal would simply be called soju ideal.

3. Commutative soju ideals of BCK-algebras

In this section, let X represent BCK-algebra unless it is otherwise specified.

Definition 3.1. Let B be a subalgebra of X. A soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0,1], W)$ is called a *commutative soju ideal* based on B (briefly, *commutative soju B-ideal*) of X if it satisfies the condition (2.14) and

$$(\forall x, y, z \in B)\left(\begin{array}{l} \zeta_\lambda(x * (y * (y * x))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\} \\ \xi_\lambda(x * (y * (y * x))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\} \\ \tilde{G}(x * (y * (y * x))) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z) \end{array}\right). \tag{3.1}$$

If $B = X$, commutative soju B-ideal would simply be called commutative soju ideal.

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a set with the Cayley table which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then X is a BCK-algebra (see [21]). For $B = X$, consider a soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0,1], W)$ which is defined by Table 2 where $\alpha_1 \supsetneq \alpha_2 \supsetneq \alpha_3 \neq \emptyset$ in 2^W.

Table 2. Tabular representation of $(B, \langle\lambda; \tilde{G}\rangle)$.

$B(= X)$	$\zeta_\lambda(x)$	$\xi_\lambda(x)$	$\tilde{G}(x)$
0	0.78	0.19	α_1
1	0.56	0.43	α_2
2	0.56	0.43	α_2
3	0.41	0.37	α_3

It is routine to verify that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X.

We discuss the relationship between soju ideal and commutative soju ideal.

Theorem 3.1. *Every commutative soju B-ideal is a soju B-ideal for every subalgebra B of X.*

Proof. For any subalgebra B of X, let $(B, \langle\lambda;\tilde{G}\rangle)$ be a commutative soju B-ideal of X. If we take $y = 0$ in (3.1) and use (K) and (2.2), then

$$\begin{aligned}\zeta_\lambda(x) &= \zeta_\lambda(x * (0 * (0 * x)))\\ &\geq \min\{\zeta_\lambda((x * 0) * z), \zeta_\lambda(z)\}\\ &= \min\{\zeta_\lambda(x * z), \zeta_\lambda(z)\},\end{aligned}$$

$$\begin{aligned}\xi_\lambda(x) &= \xi_\lambda(x * (0 * (0 * x)))\\ &\leq \max\{\xi_\lambda((x * 0) * z), \xi_\lambda(z)\}\\ &= \max\{\xi_\lambda(x * z), \xi_\lambda(z)\}\end{aligned}$$

and $\tilde{G}(x) = \tilde{G}(x * (0 * (0 * x))) \supseteq \tilde{G}((x * 0) * z) \cap \tilde{G}(z) = \tilde{G}(x * z) \cap \tilde{G}(z)$ for all $x, y, z \in B$. Hence $(B, \langle\lambda;\tilde{G}\rangle)$ is a soju ideal of X. □

The example below faces the existence of soju ideal rather than commutative soju ideal.

Example 3.2. Let $W = \mathbb{Z}$ be the initial universe set and let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 3.

Table 3. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X, *, 0)$ is a BCK-algebra (see [21]). For $B = X$, let $(B, \langle\lambda;\tilde{G}\rangle)$ be a soju structure over $([0, 1], W)$ which is defined by Table 4.

Table 4. Tabular representation of $(B, \langle\lambda;\tilde{G}\rangle)$.

$B(= X)$	$\zeta_\lambda(x)$	$\xi_\lambda(x)$	$\tilde{G}(x)$
0	0.65	0.29	$\mathbb{N}$
1	0.46	0.48	$2\mathbb{N}$
2	0.53	0.36	$4\mathbb{N}$
3	0.33	0.48	$8\mathbb{N}$
4	0.33	0.48	$8\mathbb{N}$

It is routine to check that $(B, \langle\lambda;\tilde{G}\rangle)$ is a soju B-ideal of X. But it is not a commutative soju B-ideal of X since

$$\zeta_\lambda(2 * (3 * (3 * 2))) = \zeta_\lambda(2) = 0.53 \ngeq 0.65 = \min\{\zeta_\lambda((2 * 3) * 0), \zeta_\lambda(0)\},$$

$$\xi_\lambda(2 * (3 * (3 * 2))) = \xi_\lambda(2) = 0.36 \nleq 0.29 = \max\{\xi_\lambda((2 * 3) * 0), \xi_\lambda(0)\},$$

and/or $\tilde{G}(2 * (3 * (3 * 2))) = \tilde{G}(2) = 4\mathbb{N} \not\supseteq \mathbb{N} = \tilde{G}((2 * 3) * 0) \cap \tilde{G}(0)$.

We consider conditions for soju ideal to be commutative soju ideal.

Lemma 3.1 ([16]). *Given a subalgebra B of a BCK/BCI-algebra X, every soju B-ideal $(B, \langle\lambda; \tilde{G}\rangle)$ of X satisfies the following assertion.*

$$(\forall x, y, z \in B)\left(x * y \leq z \;\Rightarrow\; \left\{\begin{array}{l} \zeta_\lambda(x) \geq \min\{\zeta_\lambda(y), \zeta_\lambda(z)\} \\ \xi_\lambda(x) \leq \max\{\xi_\lambda(y), \xi_\lambda(z)\} \\ \tilde{G}(x) \supseteq \tilde{G}(y) \cap \tilde{G}(z) \end{array}\right.\right). \tag{3.2}$$

Theorem 3.2. *In a cBCK-algebra, every soju ideal is a commutative soju ideal.*

Proof. Let B be a subalgebra of a cBCK-algebra X and let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju B-ideal of X. Note that

$$(x * (y * (y * x))) * ((x * y) * z) \leq z$$

for all $x, y, z \in B$. It follows from (3.2) that

$$\zeta_\lambda(x * (y * (y * x))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\},$$

$$\xi_\lambda(x * (y * (y * x))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\},$$

and $\tilde{G}(x * (y * (y * x))) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z)$. Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

Corollary 3.1. *Let B be a subalgebra of X which satisfies:*

$$(\forall x, y \in B)(x * (x * y) \leq y * (y * x)).$$

Then every soju B-ideal is a commutative soju B-ideal.

Proof. Straightforward. □

Corollary 3.2. *If a BCK-algebra X is a lower semilattice with respect to the order relation "≤", then every soju B-ideal is a commutative soju B-ideal for every subalgebra B of X.*

Proof. Assume that a BCK-algebra X is a lower semilattice with respect to the order relation "≤" and let $x, y \in B$ for every subalgebra B of X. Then $y * (y * x)$ is the greatest lower bound of x and y. Since $x * (x * y)$ is a common lower bound of x and y, it follows that $x * (x * y) \leq y * (y * x)$. Hence every soju B-ideal is a commutative soju B-ideal by Corollary 3.1. □

Theorem 3.3. *For any subalgebra B of X, if a soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0, 1], W)$ is a soju B-ideal of X that satisfies the condition*

$$(\forall x, y, z \in B)\left(\begin{array}{l} \zeta_\lambda((x * z) * (y * (y * x))) \geq \zeta_\lambda((x * y) * z) \\ \xi_\lambda((x * z) * (y * (y * x))) \leq \xi_\lambda((x * y) * z) \\ \tilde{G}(((x * z) * (y * (y * x))) \supseteq \tilde{G}((x * y) * z) \end{array}\right), \tag{3.3}$$

then $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X.

Proof. Assume that $(B, \langle\lambda;\tilde{G}\rangle)$ is a soju B-ideal of X satisfying the condition (3.3). Using (2.4), (2.15) and (3.3), we have

$$\begin{aligned}\zeta_\lambda(x*(y*(y*x))) &\geq \min\{\zeta_\lambda((x*(y*(y*x)))*z), \zeta_\lambda(z)\}\\ &= \min\{\zeta_\lambda((x*z)*(y*(y*x))), \zeta_\lambda(z)\}\\ &\geq \min\{\zeta_\lambda((x*y)*z), \zeta_\lambda(z)\},\end{aligned}$$

$$\begin{aligned}\xi_\lambda(x*(y*(y*x))) &\leq \max\{\xi_\lambda((x*(y*(y*x)))*z), \xi_\lambda(z)\}\\ &= \max\{\xi_\lambda((x*z)*(y*(y*x))), \xi_\lambda(z)\}\\ &\leq \max\{\xi_\lambda((x*y)*z), \xi_\lambda(z)\},\end{aligned}$$

and

$$\begin{aligned}\tilde{G}(x*(y*(y*x))) &\supseteq \tilde{G}((x*(y*(y*x)))*z)\cap\tilde{G}(z)\\ &= \tilde{G}((x*z)*(y*(y*x)))\cap\tilde{G}(z)\\ &\supseteq \tilde{G}((x*y)*z)\cap\tilde{G}(z)\end{aligned}$$

for all $x, y, z \in B$. Therefore $(B, \langle\lambda;\tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

We consider characterizations of a commutative soju ideal.

Theorem 3.4. *For any subalgebra B of X, a soju structure $(B, \langle\lambda;\tilde{G}\rangle)$ over $([0,1], W)$ is a commutative soju B-ideal of X if and only if it is a soju B-ideal of X that satisfies the following assertion.*

$$(\forall x, y \in B)\begin{pmatrix}\zeta_\lambda(x*y) \leq \zeta_\lambda(x*(y*(y*x)))\\ \xi_\lambda(x*y) \geq \xi_\lambda(x*(y*(y*x)))\\ \tilde{G}(x*y) \subseteq \tilde{G}(x*(y*(y*x)))\end{pmatrix}. \tag{3.4}$$

Proof. Assume that $(B, \langle\lambda;\tilde{G}\rangle)$ is a commutative soju B-ideal of X. Then $(B, \langle\lambda;\tilde{G}\rangle)$ is a soju B-ideal of X by Theorem 3.1. The condition (3.4) is induced by taking $z = 0$ in (3.1) and using (2.2) and (2.14).

Conversely, let $(B, \langle\lambda;\tilde{G}\rangle)$ be a soju B-ideal of X that satisfies the condition (3.4). Then

$$\zeta_\lambda(x*(y*(y*x))) \geq \zeta_\lambda(x*y) \geq \min\{\zeta_\lambda((x*y)*z), \zeta_\lambda(z)\},$$

$$\xi_\lambda(x*(y*(y*x))) \leq \xi_\lambda(x*y) \leq \max\{\xi_\lambda((x*y)*z), \xi_\lambda(z)\},$$

and $\tilde{G}(x*(y*(y*x))) \supseteq \tilde{G}(x*y) \supseteq \tilde{G}((x*y)*z)\cap\tilde{G}(z)$ for all $x, y, z \in B$. Therefore $(B, \langle\lambda;\tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

Theorem 3.5. *Let $(B, \langle\lambda;\tilde{G}\rangle)$ be a soju structure over $([0,1], W)$ for $B = X$. If $(B, \langle\lambda;\tilde{G}\rangle)$ is a commutative soju B-ideal of X, then the sets $X(\zeta_\lambda \uparrow \zeta_\lambda(w))$, $X(\xi_\lambda \downarrow \xi_\lambda(w))$ and $X(\tilde{G}, \tilde{G}(w))$ are commutative ideals of X for any $w \in X$.*

Proof. It is clear that $0 \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$ for all $w \in X$. Let $x, y, z \in X$ be such that $z \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$ and

$$(x * y) * z \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$$

for all $w \in X$. Then $\zeta_\lambda(z) \geq \zeta_\lambda(w)$, $\zeta_\lambda((x * y) * z) \geq \zeta_\lambda(w)$, $\xi_\lambda(z) \leq \xi_\lambda(w)$, $\xi_\lambda((x * y) * z) \leq \xi_\lambda(w)$, $\tilde{G}(z) \supseteq \tilde{G}(w)$, and $\tilde{G}((x*y)*z) \supseteq \tilde{G}(w)$. Since $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X, it follows from (3.1) that

$$\begin{aligned}
&\zeta_\lambda(x * (y * (y * x))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\} \geq \zeta_\lambda(w),\\
&\xi_\lambda(x * (y * (y * x))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\} \leq \xi_\lambda(w),\\
&\tilde{G}(x * (y * (y * x))) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z) \supseteq \tilde{G}(w).
\end{aligned}$$

Hence $x * (y * (y * x)) \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$, and therefore $X(\zeta_\lambda \uparrow \zeta_\lambda(w))$, $X(\xi_\lambda \downarrow \xi_\lambda(w))$ and $X(\tilde{G}, \tilde{G}(w))$ are commutative ideals of X for all $w \in X$. □

Corollary 3.3. *Let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0, 1], W)$ for $B = X$. If $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X, then the set*

$$X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w)) \tag{3.5}$$

is a commutative ideal of X for all $w \in X$.

Proof. Straightforward. □

The converse of Corollary 3.3 is not true in general as seen in the following example, that is, there exists a soju structure $(X, \langle\lambda; \tilde{G}\rangle)$ over $([0, 1], W)$ such that the set in (3.5) is a commutative ideal of X for any $w \in X$, and $(X, \langle\lambda; \tilde{G}\rangle)$ is not a commutative soju B-ideal of X.

Example 3.3. Let $W = \mathbb{Z}$ be the initial universe set and let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 5.

Table 5. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [21]). For $B = X$, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0, 1], W)$ which is defined by Table 6.
Then

$$X(\zeta_\lambda \uparrow \zeta_\lambda(w)) = \begin{cases} \{0\} & \text{if } w = 0,\\ \{0, 1, 4\} & \text{if } w = 1,\\ \{0, 1, 2, 4\} & \text{if } w = 2,\\ X & \text{if } w = 3,\\ \{0, 4\} & \text{if } w = 4, \end{cases}$$

Table 6. Tabular representation of $(B, \langle\lambda; \tilde{G}\rangle)$.

$B(=X)$	$\zeta_\lambda(x)$	$\xi_\lambda(x)$	$\tilde{G}(x)$
0	0.71	0.24	$\mathbb{Z}$
1	0.58	0.24	$2\mathbb{Z}$
2	0.37	0.36	$8\mathbb{Z}$
3	0.26	0.52	$8\mathbb{Z}$
4	0.63	0.29	$4\mathbb{Z}$

$$X(\xi_\lambda \downarrow \xi_\lambda(w)) = \begin{cases} \{0\} & \text{if } w \in \{0,1\}, \\ \{0,1,2,4\} & \text{if } w = 2, \\ X & \text{if } w = 3, \\ \{0,1,4\} & \text{if } w = 4, \end{cases}$$

and

$$X(\tilde{G}, \tilde{G}(w)) = \begin{cases} \{0\} & \text{if } w = 0, \\ \{0,1\} & \text{if } w = 1, \\ X & \text{if } w \in \{2,3\}, \\ \{0,1,4\} & \text{if } w = 4. \end{cases}$$

Hence

$$X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w)) = \begin{cases} \{0\} & \text{if } w \in \{0,1\}, \\ \{0,1,2,4\} & \text{if } w = 2, \\ X & \text{if } w = 3, \\ \{0,4\} & \text{if } w = 4, \end{cases}$$

and so $X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$ is a commutative ideal of X. But $(X, \langle\lambda; \tilde{G}\rangle)$ is not a commutative soju B-ideal of X since $\xi_\lambda(2*(3*(3*2))) = \xi_\lambda(2) = 0.36 \nleq 0.24 = \max\{\xi_\lambda((2*3)*1), \xi_\lambda(1)\}$ and/or $\tilde{G}(2*(3*(3*2))) = \tilde{G}(2) = 8\mathbb{Z} \not\supseteq \mathbb{Z} = \tilde{G}((2*3)*1) \cap \tilde{G}(1)$.

Lemma 3.2 ([20]). *An ideal L of X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)(x * y \in L \;\Rightarrow\; x * (y * (y * x)) \in L). \tag{3.6}$$

We provide conditions for a soju structure to be a commutative soju ideal.

Theorem 3.6. *Let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0,1], W)$ for $B = X$. Given $\alpha \in 2^W$ and $(t, s) \in [0,1] \times [0,1]$ with $t + s \leq 1$, if the nonempty sets $X(\zeta_\lambda \uparrow t)$, $X(\xi_\lambda \downarrow s)$ and $X(\tilde{G}, \alpha)$ are commutative ideals of X, then $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X.*

Proof. Assume that the nonempty sets $X(\zeta_\lambda \uparrow t)$, $X(\xi_\lambda \downarrow s)$ and $X(\tilde{G}, \alpha)$ are commutative ideals of X for all $\alpha \in 2^W$ and $(t, s) \in [0,1] \times [0,1]$ with $t + s \leq 1$. Then $X(\zeta_\lambda \uparrow t)$, $X(\xi_\lambda \downarrow s)$ and $X(\tilde{G}, \alpha)$ are ideals of X, and hence they are subalgebras of X. For any $x, y \in B = X$, let $\zeta_\lambda(x) = t_x$, $\zeta_\lambda(y) = t_y$, $\xi_\lambda(x) = s_x$,

$\xi_\lambda(y) = s_y$, $\tilde{G}(x) = \alpha_x$ and $\tilde{G}(y) = \alpha_y$. If we take $t := \min\{t_x, t_y\}$, $s := \max\{s_x, s_y\}$ and $\alpha := \alpha_x \cap \alpha_y$, then $x, y \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$, and so $x * y \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$. Hence

$$\begin{aligned} &\zeta_\lambda(x * y) \geq t = \min\{t_x, t_y\} = \min\{\zeta_\lambda(x), \zeta_\lambda(y)\}, \\ &\xi_\lambda(x * y) \leq s = \max\{s_x, s_y\} = \max\{\xi_\lambda(x), \xi_\lambda(y)\}, \\ &\tilde{G}(x * y) \supseteq \alpha = \alpha_x \cap \alpha_y = \tilde{G}(x) \cap \tilde{G}(y). \end{aligned} \tag{3.7}$$

Taking $x = y$ in (3.7) and using (I_3) will induce $\zeta_\lambda(0) \geq \zeta_\lambda(x)$, $\xi_\lambda(0) \leq \xi_\lambda(x)$ and $\tilde{G}(0) \supseteq \tilde{G}(x)$ for all $x \in B = X$. Let $x, y \in B = X$ be such that $\zeta_\lambda(x * y) = t_x$, $\zeta_\lambda(y) = t_y$, $\xi_\lambda(x * y) = s_x$, $\xi_\lambda(y) = s_y$, $\tilde{G}(x * y) = \alpha_x$ and $\tilde{G}(y) = \alpha_y$. If we take $t := \min\{t_x, t_y\}$, $s := \max\{s_x, s_y\}$ and $\alpha := \alpha_x \cap \alpha_y$, then $x * y$, $y \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$, and so $x \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$. It follows that $\zeta_\lambda(x) \geq t = \min\{t_x, t_y\} = \min\{\zeta_\lambda(x * y), \zeta_\lambda(y)\}$, $\xi_\lambda(x) \leq s = \max\{s_x, s_y\} = \max\{\xi_\lambda(x * y), \xi_\lambda(y)\}$, and $\tilde{G}(x) \supseteq \alpha = \alpha_x \cap \alpha_y = \tilde{G}(x * y) \cap \tilde{G}(y)$. Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a soju B-ideal of X. Let $x, y \in B = X$ be such that $\zeta_\lambda(x * y) = t$, $\xi_\lambda(x * y) = s$ and $\tilde{G}(x * y) = \alpha$. Then $x * y \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$, and so $x * (y * (y * x)) \in X(\zeta_\lambda \uparrow t) \cap X(\xi_\lambda \downarrow s) \cap X(\tilde{G}, \alpha)$ by Lemma 3.2. Thus $\zeta_\lambda(x * (y * (y * x))) \geq t = \zeta_\lambda(x * y)$, $\xi_\lambda(x * (y * (y * x))) \leq s = \xi_\lambda(x * y)$, and $\tilde{G}(x * (y * (y * x))) \supseteq \alpha = \tilde{G}(x * y)$. It follows from Theorem 3.4 that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

We make a new commutative soju ideal using the given commutative soju ideal.

Theorem 3.7. *Given a soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0, 1], W)$ for $B = X$, let $(B, \langle\lambda^*; \tilde{G}^*\rangle)$, where $\lambda^* := (\zeta^*_\lambda, \xi^*_\lambda)$, be a soju structure over $([0, 1], W)$ defined as follows:*

$$\lambda^* := (\zeta^*_\lambda, \xi^*_\lambda) : B \to [0, 1] \times [0, 1],\ x \mapsto \begin{cases} (\zeta_\lambda(x), \xi_\lambda(x)) & \text{if } x \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)), \\ (m, n) & \text{otherwise.} \end{cases}$$

$$\tilde{G}^* : B \to 2^W,\ x \mapsto \begin{cases} \tilde{G}(x) & \text{if } x \in X(\tilde{G}, \tilde{G}(w)), \\ \beta & \text{otherwise.} \end{cases}$$

where $w \in B = X$, $\beta \in 2^W$ and $m, n \in [0, 1]$ with $\beta \subsetneq \tilde{G}(x)$, $m + n \leq 1$, $m < \zeta_\lambda(x)$ and $n > \xi_\lambda(x)$. If $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X, then so is $(B, \langle\lambda^*; \tilde{G}^*\rangle)$.

Proof. Suppose that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. Then the sets $X(\zeta_\lambda \uparrow \zeta_\lambda(w))$, $X(\xi_\lambda \downarrow \xi_\lambda(w))$ and $X(\tilde{G}, \tilde{G}(w))$ are commutative ideals of X for any $w \in B = X$ by Theorem 3.5. Hence $0 \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w)) \cap X(\tilde{G}, \tilde{G}(w))$, and so $\zeta^*_\lambda(0) = \zeta_\lambda(0) \geq \zeta_\lambda(x) \geq \zeta^*_\lambda(x)$, $\xi^*_\lambda(0) = \xi_\lambda(0) \leq \xi_\lambda(x) \leq \xi^*_\lambda(x)$, and $\tilde{G}^*(0) = \tilde{G}(0) \supseteq \tilde{G}(x) \supseteq \tilde{G}^*(x)$ for all $x \in B = X$. Let $x, y, z \in B = X$. If $(x * y) * z \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w))$ and $z \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w))$, then $x * (y * (y * x)) \in X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w))$. Thus

$$\zeta^*_\lambda(x * (y * (y * x))) = \zeta_\lambda(x * (y * (y * x))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\} = \min\{\zeta^*_\lambda((x * y) * z), \zeta^*_\lambda(z)\}$$

and

$$\xi^*_\lambda(x * (y * (y * x))) = \xi_\lambda(x * (y * (y * x))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\} = \max\{\xi^*_\lambda((x * y) * z), \xi^*_\lambda(z)\}.$$

If $(x * y) * z \notin X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w))$ or $z \notin X(\zeta_\lambda \uparrow \zeta_\lambda(w)) \cap X(\xi_\lambda \downarrow \xi_\lambda(w))$, then

$$(\zeta^*_\lambda, \xi^*_\lambda)((x * y) * z) = (m, n) \text{ or } (\zeta^*_\lambda, \xi^*_\lambda)(z) = (m, n).$$

It follows that

$$\zeta^*_\lambda(x*(y*(y*x))) \geq m = \min\{\zeta^*_\lambda((x*y)*z), \zeta^*_\lambda(z)\}$$

and

$$\xi^*_\lambda(x*(y*(y*x))) \leq n = \max\{\xi^*_\lambda((x*y)*z), \xi^*_\lambda(z)\}.$$

Also, if $(x*y)*z \in X(\tilde{G}, \tilde{G}(w))$ and $z \in X(\tilde{G}, \tilde{G}(w))$, then $x*(y*(y*x)) \in X(\tilde{G}, \tilde{G}(w))$. Thus

$$\tilde{G}^*(x*(y*(y*x))) = \tilde{G}(x*(y*(y*x))) \supseteq \tilde{G}((x*y)*z) \cap \tilde{G}(z) = \tilde{G}^*((x*y)*z) \cap \tilde{G}^*(z).$$

Assume that $(x*y)*z \notin X(\tilde{G}, \tilde{G}(w))$ or $z \notin X(\tilde{G}, \tilde{G}(w))$. Then $\tilde{G}^*((x*y)*z) = \beta$ or $\tilde{G}^*(z) = \beta$ which imply that $\tilde{G}^*(x*(y*(y*x))) \supseteq \beta = \tilde{G}^*((x*y)*z) \cap \tilde{G}^*(z)$. Therefore $(B, \langle\lambda^*; \tilde{G}^*\rangle)$ is a commutative soju B-ideal of X. □

Lemma 3.3 ([20]). *An ideal L of a BCK-algebra X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)(x*y \in L \;\Rightarrow\; x*(y*(y*x)) \in L). \tag{3.8}$$

Note that a soju ideal might not be a commutative soju ideal (see Example 3.2). But we have the following extension property for a commutative soju ideal.

Theorem 3.8. *Given a BCK-algebra X and B = X, let $(B, \langle\lambda; \tilde{G}\rangle)$ and $(B, \langle\sigma; \tilde{H}\rangle)$ be soju ideals of X such that*

(i) $\zeta_\lambda(0) = \zeta_\sigma(0)$, $\xi_\lambda(0) = \xi_\sigma(0)$, $\tilde{G}(0) = \tilde{H}(0)$.
(ii) $(\forall x \in B)\ (\zeta_\lambda(x) \leq \zeta_\sigma(x), \xi_\lambda(x) \geq \xi_\sigma(x), \tilde{G}(x) \subseteq \tilde{H}(x))$.

If $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju ideal of X, then so is $(B, \langle\sigma; \tilde{H}\rangle)$.

Proof. Assume that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju ideal of X. Then $X(\zeta_\lambda \uparrow t)$, $X(\xi_\lambda \downarrow s)$ and $X(\tilde{G}, \alpha)$ are commutative ideals of X for all $(t, s, \alpha) \in [0,1] \times [0,1] \times 2^W$ with $t + s \leq 1$ and $X(\zeta_\lambda \uparrow t)$, $X(\xi_\lambda \downarrow s)$ and $X(\tilde{G}, \alpha)$ are nonempty. Since $(B, \langle\sigma; \tilde{H}\rangle)$ is a soju ideal of X, we know that $X(\zeta_\sigma \uparrow t)$, $X(\xi_\sigma \downarrow s)$ and $X(\tilde{H}, \alpha)$ are ideals of X for all $(t, s, \alpha) \in [0,1] \times [0,1] \times 2^W$ with $t + s \leq 1$ and $X(\zeta_\sigma \uparrow t)$, $X(\xi_\sigma \downarrow s)$ and $X(\tilde{H}, \alpha)$ are nonempty. Let $x, y, a, b, u, v \in X$ be such that $x*y \in X(\zeta_\sigma \uparrow t)$, $a*b \in X(\xi_\sigma \downarrow s)$ and $u*v \in X(\tilde{H}, \alpha)$. Using (I_3) and (2.4), we have

$$(x*(x*y))*y = (x*y)*(x*y) = 0 \in X(\zeta_\lambda \uparrow t),$$

$$(a*(a*b))*b = (a*b)*(a*b) = 0 \in X(\xi_\lambda \uparrow t),$$

and $(u*(u*v))*v = (u*v)*(u*v) = 0 \in X(\tilde{H}, \alpha)$. It follows from (2.4), Lemma 3.3 and (ii) that

$$(x*(y*(y*(x*(x*y)))))*(x*y) = (x*(x*y))*(y*(y*(x*(x*y)))) \in X(\zeta_\lambda \uparrow t) \subseteq X(\zeta_\sigma \uparrow t),$$

$$(a*(b*(b*(a*(a*b)))))*(a*b) = (a*(a*b))*(b*(b*(a*(a*b)))) \in X(\xi_\lambda \downarrow s) \subseteq X(\xi_\sigma \downarrow s),$$

and

$$(u*(v*(v*(u*(u*v)))))*(u*v) = (u*(u*v))*(v*(v*(u*(u*v)))) \in X(\tilde{G}, \alpha) \subseteq X(\tilde{H}, \alpha).$$

Hence $x*(y*(y*(x*(x*y)))) \in X(\zeta_\sigma \uparrow t)$, $a*(b*(b*(a*(a*b)))) \in X(\xi_\sigma \downarrow s)$ and $u*(v*(v*(u*(u*v)))) \in X(\tilde{H}, \alpha)$. Since $\tilde{x}*(\tilde{x}*\tilde{y}) \leq \tilde{x}$ for all $\tilde{x}, \tilde{y} \in X$, we get $\tilde{x}*(\tilde{y}*(\tilde{y}*\tilde{x})) \leq \tilde{x}*(\tilde{y}*(\tilde{y}*(\tilde{x}*(\tilde{x}*\tilde{y}))))$ for all $\tilde{x}, \tilde{y} \in X$ by (2.3). It follows that $x*(y*(y*x)) \in X(\zeta_\sigma \uparrow t)$, $a*(b*(b*a)) \in X(\xi_\sigma \downarrow s)$ and $u*(v*(v*u)) \in X(\tilde{H}, \alpha)$. Hence $X(\zeta_\sigma \uparrow t)$, $X(\xi_\sigma \downarrow s)$ and $X(\tilde{H}, \alpha)$ are commutative ideals of X by Lemma 3.3, and therefore $(B, \langle\sigma; \tilde{H}\rangle)$ is a commutative soju ideal of X by Theorem 3.6. □

4. Commutative soju ideals in BCI-algebras

Definition 4.1. Let B be a subalgebra of a BCI-algebra X. A soju structure $(B, \langle\lambda; \tilde{G}\rangle)$ over $([0,1], W)$ is called a *commutative soju ideal* based on B (briefly, *commutative soju B-ideal*) of X if it satisfies the condition (2.14) and

$$(\forall x, y, z \in B)\left(\begin{array}{l} \zeta_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\} \\ \xi_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\} \\ \tilde{G}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z) \end{array}\right). \tag{4.1}$$

Example 4.1. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 7.

Table 7. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Then X is a BCI-algebra (see [9]). For $B = X$ and $W = \mathbb{Z}$, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0,1], W)$ which is defined by Table 8. It is routine to verify that $(B, \langle\lambda; \tilde{G}\rangle)$ satisfies the conditions

Table 8. Tabular representation of $(B, \langle\lambda; \tilde{G}\rangle)$.

$B(= X)$	$\zeta_\lambda(x)$	$\xi_\lambda(x)$	$\tilde{G}(x)$
0	0.63	0.17	$2\mathbb{Z}$
1	0.48	0.24	$4\mathbb{Z}$
2	0.27	0.56	$8\mathbb{Z}$
3	0.27	0.56	$8\mathbb{Z}$
4	0.27	0.56	$8\mathbb{Z}$

(2.14) and (4.1). Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X.

Proposition 4.1. *In a BCI-algebra X, every commutative soju B-ideal* $(B, \langle\lambda; \tilde{G}\rangle)$*, where* $B = X$*, of X satisfies:*

$$(\forall x, y \in B)\left(\begin{array}{l} \zeta_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \zeta_\lambda(x * y) \\ \xi_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq \xi_\lambda(x * y) \\ \tilde{G}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq \tilde{G}(x * y) \end{array}\right). \tag{4.2}$$

Proof. If we take $z = 0$ in (4.1) and use (2.2) and (2.14), then we get (4.2). □

Using a commutative ideal of a BCI-algebra, we establish a commutative soju ideal.

Theorem 4.1. *Given a commutative ideal L of a BCI-algebra X and B = X, let* $(B, \langle\lambda; \tilde{G}\rangle)$ *be a soju structure over* $([0,1], W)$ *which is defined as follows:*

$$\lambda := (\zeta_\lambda, \xi_\lambda) : B \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (t,s) & \text{if } x \in L, \\ (0,1) & \text{otherwise.} \end{cases}$$

$$\tilde{G} : B \to 2^W,\ x \mapsto \begin{cases} \alpha & \text{if } x \in L, \\ \beta & \text{otherwise,} \end{cases}$$

where $(t,s) \in (0,1] \times [0,1)$ *and* $\alpha, \beta \in 2^W$ *with* $t + s \leq 1$ *and* $\alpha \supsetneq \beta$*. Then* $(B, \langle\lambda; \tilde{G}\rangle)$ *is a commutative soju B-ideal of X.*

Proof. It is clear that $\zeta_\lambda(0) \geq \zeta_\lambda(x)$, $\xi_\lambda(0) \leq \xi_\lambda(x)$ and $\tilde{G}(0) \supseteq \tilde{G}(x)$ for all $x \in B = X$. Let $x, y, z \in B = X$. If $(x * y) * z \in L$ and $z \in L$, then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in L$ since L is a commutative ideal of X. Thus $\zeta_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) = t = \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\}$, $\xi_\lambda(x*((y*(y*x))*(0*(0*(x*y))))) = s = \max\{\xi_\lambda((x*y)*z), \xi_\lambda(z)\}$, and $\tilde{G}(x*((y*(y*x))*(0*(0*(x*y))))) = \alpha = \tilde{G}((x*y)*z) \cap \tilde{G}(z)$. Suppose that $(x*y)*z \notin L$ or $z \notin L$. Then $\lambda((x*y)*z) = (0,1)$ or $\lambda(z) = (0,1)$, and $\tilde{G}((x * y) * z) = \beta$ or $\tilde{G}(z) = \beta$. It follows that

$$\zeta_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\},$$

$$\xi_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\},$$

and $\tilde{G}(x*((y*(y*x))*(0*(0*(x*y))))) \supseteq \tilde{G}((x*y)*z) \cap \tilde{G}(z)$. Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

Theorem 4.2. *In a BCI-algebra X, every commutative soju B-ideal is a soju B-ideal for B = X.*

Proof. Let $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. For every $x, y \in B = X$, we have

$$\begin{aligned} \zeta_\lambda(x) &= \zeta_\lambda(x * 0) = \zeta_\lambda(x * ((0 * (0 * x)) * (0 * (0 * (x * 0))))) \\ &\geq \min\{\zeta_\lambda((x * 0) * y), \zeta_\lambda(y)\} = \min\{\zeta_\lambda(x * y), \zeta_\lambda(y)\}, \end{aligned}$$

$$\begin{aligned} \xi_\lambda(x) &= \xi_\lambda(x * 0) = \xi_\lambda(x * ((0 * (0 * x)) * (0 * (0 * (x * 0))))) \\ &\leq \max\{\xi_\lambda((x * 0) * y), \xi_\lambda(y)\} = \max\{\xi_\lambda(x * y), \xi_\lambda(y)\}, \end{aligned}$$

and $\tilde{G}(x) = \tilde{G}(x*0) = \tilde{G}(x*((0*(0*x))*(0*(0*(x*0))))) \supseteq \tilde{G}((x*0)*y) \cap \tilde{G}(y)\} = \tilde{G}(x*y) \cap \tilde{G}(y)\}$ by using (I_3), (2.2) and (4.1). Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a soju B-ideal of X. □

The following example shows that the converse of Theorem 4.2 is not established.

Example 4.2. Consider the BCK-algebra, and hence a BCI-algebra, $X = \{0,1,2,3,4\}$ which is given in Example 3.2. For $B = X$ and $W = \mathbb{N}$, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0,1], W)$ which is defined by Table 9.
It is routine to check that $(B, \langle\lambda; \tilde{G}\rangle)$ is a soju B-ideal of X. We know that

$$\zeta_\lambda(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = 0.37 < 0.67 = \zeta_\lambda(2 * 3),$$

$$\xi_\lambda(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = 0.53 > 0.21 = \xi_\lambda(2 * 3),$$

and/or $\tilde{G}(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = 8\mathbb{N} \subsetneq 2\mathbb{N} = \tilde{G}(2 * 3)$. Hence $(B, \langle\lambda; \tilde{G}\rangle)$ is not a commutative soju B-ideal of X by Proposition 4.1.

Table 9. Tabular representation of $(B, \langle\lambda; \tilde{G}\rangle)$.

$B(= X)$	$\zeta_\lambda(x)$	$\xi_\lambda(x)$	$\tilde{G}(x)$
0	0.67	0.21	$2\mathbb{N}$
1	0.53	0.29	$4\mathbb{N}$
2	0.37	0.53	$8\mathbb{N}$
3	0.37	0.53	$8\mathbb{N}$
4	0.37	0.53	$8\mathbb{N}$

We find and present the conditions under which a commutative soju ideal can be made from a soju ideal.

Theorem 4.3. *Let B be a subalgebra of a BCI-algebra X. Given a soju B-ideal $(B, \langle\lambda; \tilde{G}\rangle)$ of X, the following are equivalent.*

(i) $(B, \langle\lambda; \tilde{G}\rangle)$ *is a commutative soju B-ideal of X.*
(ii) $(B, \langle\lambda; \tilde{G}\rangle)$ *satisfies the condition* (4.2).

Proof. Assume that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. Then the condition (4.2) is valid by Proposition 4.1.

Conversely, suppose that the soju B-ideal $(B, \langle\lambda; \tilde{G}\rangle)$ of X satisfies the condition (4.2). Then $\zeta_\lambda(x * y) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\}$, $\xi_\lambda(x * y) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\}$, and $\tilde{G}(x * y) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z)$ by (2.15). It follows from (4.2) that

$$\zeta_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\zeta_\lambda((x * y) * z), \zeta_\lambda(z)\},$$

$$\xi_\lambda(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq \max\{\xi_\lambda((x * y) * z), \xi_\lambda(z)\}$$

and $\tilde{G}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq \tilde{G}((x * y) * z) \cap \tilde{G}(z)$. Consequently, $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. □

Definition 4.2. Let B be a subalgebra of a BCI-algebra X. A soju B-ideal $(B, \langle\lambda; \tilde{G}\rangle)$ of X is said to be *closed* if

$$(\forall x \in B)\Big(\zeta_\lambda(0 * x) \geq \zeta_\lambda(x),\ \xi_\lambda(0 * x) \leq \xi_\lambda(x),\ \tilde{G}(0 * x) \supseteq \tilde{G}(x)\Big). \tag{4.3}$$

Example 4.3. Let B be a subalgebra of a BCI-algebra X and let $(B, \langle\lambda; \tilde{G}\rangle)$ be a soju structure over $([0, 1], W)$ which is defined as follows:

$$\lambda := (\zeta_\lambda, \xi_\lambda) : B \to [0, 1] \times [0, 1],\ x \mapsto \begin{cases} (t, s) & \text{if } x \in X_+, \\ (0, 1) & \text{otherwise.} \end{cases}$$

$$\tilde{G} : B \to 2^W,\ x \mapsto \begin{cases} \alpha & \text{if } x \in X_+, \\ \beta & \text{otherwise,} \end{cases}$$

where $X_+ := \{x \in X \mid 0 \leq x\}$, $(t, s) \in (0, 1] \times [0, 1)$ and $\alpha, \beta \in 2^W$ with $t + s \leq 1$ and $\alpha \supsetneq \beta$. It is routine to show that $(B, \langle\lambda; \tilde{G}\rangle)$ is a closed soju B-ideal of X.

Theorem 4.4. *Given a subalgebra B of a BCI-algebra X, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a closed soju B-ideal of X. Then $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X if and only if it satisfies:*

$$(\forall x, y \in B)\left(\begin{array}{l} \zeta_\lambda(x*(y*(y*x))) \geq \zeta_\lambda(x*y) \\ \xi_\lambda(x*(y*(y*x))) \leq \xi_\lambda(x*y) \\ \tilde{G}(x*(y*(y*x))) \supseteq \tilde{G}(x*y) \end{array}\right). \tag{4.4}$$

Proof. Assume that $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X. Using (I_1), (2.4), (2.6) induces

$$\begin{aligned} &(x*(y*(y*x)))*(x*((y*(y*x))*(0*(0*(x*y))))) \\ &\leq ((y*(y*x))*(0*(0*(x*y))))*(y*(y*x)) \\ &= ((y*(y*x))*(y*(y*x)))*(0*(0*(x*y))) \\ &= 0*(0*(0*(x*y))) = 0*(x*y) \end{aligned}$$

for all $x, y \in X$. It follows from Lemma 3.1, Theorem 4.3 and (4.3) that

$$\begin{aligned} \zeta_\lambda(x*(y*(y*x))) &\geq \min\{\zeta_\lambda(x*((y*(y*x))*(0*(0*(x*y))))), \zeta_\lambda(0*(x*y))\} \\ &\geq \min\{\zeta_\lambda(x*y), \zeta_\lambda(0*(x*y))\} = \zeta_\lambda(x*y), \end{aligned}$$

$$\begin{aligned} \xi_\lambda(x*(y*(y*x))) &\leq \max\{\xi_\lambda(x*((y*(y*x))*(0*(0*(x*y))))), \xi_\lambda(0*(x*y))\} \\ &\leq \max\{\xi_\lambda(x*y), \xi_\lambda(0*(x*y))\} = \xi_\lambda(x*y), \end{aligned}$$

and

$$\begin{aligned} \tilde{G}(x*(y*(y*x))) &\supseteq \tilde{G}(x*((y*(y*x))*(0*(0*(x*y))))) \cap \tilde{G}(0*(x*y)) \\ &\supseteq \tilde{G}(x*y) \cap \tilde{G}(0*(x*y)) = \tilde{G}(x*y) \end{aligned}$$

for all $x, y \in B$. Hence (4.4) is valid.

Conversely, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a closed soju B-ideal of X that satisfies the condition (4.4). For any $x, y \in X$, we have

$$\begin{aligned} &(x*((y*(y*x))*(0*(0*(x*y)))))*(x*(y*(y*x))) \\ &\leq (y*(y*x))*((y*(y*x))*(0*(0*(x*y)))) \\ &\leq 0*(0*(x*y)) \end{aligned}$$

by (I_1) and (I_2). It follows from Lemma 3.1, (4.3) and (4.4) that

$$\begin{aligned} &\zeta_\lambda(x*((y*(y*x))*(0*(0*(x*y))))) \\ &\geq \min\{\zeta_\lambda(x*(y*(y*x))), \zeta_\lambda(0*(0*(x*y)))\} \\ &\geq \min\{\zeta_\lambda(x*y), \zeta_\lambda(0*(0*(x*y)))\} = \zeta_\lambda(x*y), \end{aligned}$$

$$\begin{aligned} &\xi_\lambda(x*((y*(y*x))*(0*(0*(x*y))))) \\ &\leq \max\{\xi_\lambda(x*(y*(y*x))), \xi_\lambda(0*(0*(x*y)))\} \end{aligned}$$

$$\leq \max\{\xi_\lambda(x * y), \xi_\lambda(0 * (0 * (x * y)))\} = \xi_\lambda(x * y),$$

and

$$\begin{aligned}
&\tilde{G}(x * ((y * (y * x)) * (0 * (0 * (x * y)))))\\
&\supseteq \tilde{G}(x * (y * (y * x))) \cap \tilde{G}(0 * (0 * (x * y)))\\
&\supseteq \tilde{G}(x * y) \cap \tilde{G}(0 * (0 * (x * y))) = \tilde{G}(x * y)
\end{aligned}$$

for all $x, y \in B$. Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X by Theorem 4.3. □

Lemma 4.1 ([22]). *A BCI-algebra X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)(x * (x * y) = y * (y * (x * (x * y)))). \tag{4.5}$$

Theorem 4.5. *If B is a subalgebra of a cBCI-algebra X, then every closed soju B-ideal is a commutative soju B-ideal.*

Proof. Given a subalgebra B of a cBCI-algebra X, let $(B, \langle\lambda; \tilde{G}\rangle)$ be a closed soju B-ideal of X. Using (I_1), (I_3), (2.4), (2.6) and Lemma 4.1, we get

$$\begin{aligned}
&(x * (y * (y * x))) * (x * y) = (x * (x * y)) * (y * (y * x))\\
&= (y * (y * (x * (x * y)))) * (y * (y * x))\\
&= (y * (y * (y * x))) * (y * (x * (x * y)))\\
&= (y * x) * (y * (x * (x * y)))\\
&\leq (x * (x * y)) * x = (x * x) * (x * y) = 0 * (x * y)
\end{aligned}$$

for all $x, y \in X$. It follows from Lemma 3.1 and (4.3) that

$$\zeta_\lambda(x * (y * (y * x))) \geq \min\{\zeta_\lambda(x * y), \zeta_\lambda(0 * (x * y))\} = \zeta_\lambda(x * y),$$

$$\xi_\lambda(x * (y * (y * x))) \leq \max\{\xi_\lambda(x * y), \xi_\lambda(0 * (x * y))\} = \xi_\lambda(x * y),$$

and $\tilde{G}(x * (y * (y * x))) \supseteq \tilde{G}(x * y) \cap \tilde{G}(0 * (x * y)) = \tilde{G}(x * y)$ for all $x, y \in B$. Therefore $(B, \langle\lambda; \tilde{G}\rangle)$ is a commutative soju B-ideal of X by Theorem 4.4. □

5. Conclusions

Jun et al. [16] have introduced a new structure called soju structure that can handle intuitionistic fuzzy set and soft set simultaneously and applied it to BCK/BCI-algebras. In this manuscript, we have introduced the concept of a commutative soju ideal in a BCK-algebra and a BCI-algebra, and we have investigated their properties. We have discussed the relationship between a soju ideal and a commutative soju ideal, and we have provided examples to show that any soju ideal may not be a commutative soju ideal. We have considered conditions for a soju ideal to be a commutative soju ideal, and we have considered characterizations of a commutative soju ideal. We have made a new commutative soju ideal using the given commutative soju ideal, and we have established the extension property for a commutative soju ideal. Using a commutative ideal of a BCI-algebra, we have constructed a commutative soju ideal. We have introduce the notion of a closed soju ideal in a BCI-algebra and have used it to study the characterization of a commutative soju ideal. In our future works, using the ideas and results in this paper, we first study several kinds of substructures such as a-ideal, p-ideal, q-ideal, subimplicative ideal, filter, quasi-associative ideal and fantastic ideal in BCK/BCI-algebras. Secondly, we will use the ideas and results in this paper to study substructures such as ideals, filters, and duductive systems, etc. in MV-algebra, BL-algebra, equality algebra, EQ-algebra, etc. which have deep relevance to BCK/BCI-algebra.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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