Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8544–8566.
DOI:10.3934/math.2021496
Received: 18 March 2021
Accepted: 23 May 2021
Published: 07 June 2021

Research article

Values and bounds for depth and Stanley depth of some classes of edge ideals

Naeem Ud Din*, **Muhammad Ishaq and Zunaira Sajid**

Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad, Pakistan

* **Correspondence:** Email: naeemuddin03088@gmail.com; Tel:+923088944090.

Abstract: In this paper we study depth and Stanley depth of the quotient rings of the edge ideals associated with the corona product of some classes of graphs with arbitrary non-trivial connected graph G. These classes include caterpillar, firecracker and some newly defined unicyclic graphs. We compute formulae for the values of depth that depend on the depth of the quotient ring of the edge ideal $I(G)$. We also compute values of depth and Stanley depth of the quotient rings associated with some classes of edge ideals of caterpillar graphs and prove that both of these invariants are equal for these classes of graphs.

Keywords: depth; Stanley depth; Stanley decomposition; monomial ideal; edge ideal
Mathematics Subject Classification: Primary 13C15; Secondary 13P10, 13F20

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. The K subspace $aK[W]$ which is generated by all elements of the form aw where a is a homogeneous element in M, w is a monomial in $K[W]$ and $W \subseteq \{x_1, x_2, \dots, x_n\}$. $K[W]$, is called a Stanley space of dimension $|W|$ if it is a free $K[W]$-module. A decomposition $\mathcal{D}$ of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D} : M = \bigoplus_{j=1}^{r} a_jK[W_j]$, is called Stanley decomposition of M. Stanley depth of $\mathcal{D}$ is the minimum dimension of all the Stanley spaces. The quantity

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) \,|\, \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of M.

Depth of a finitely generated R-module M, where R is the local Noetherian ring with unique maximal ideal $m := (x_1, \dots, x_n)$, is the common length of all maximal M-sequences in m. For introduction to depth and Stanley depth we recommend the readers [5, 9, 15]. Stanley conjectured in [17] that for any $\mathbb{Z}^n$-graded S-module M, sdepth(M) $\geq$ depth(M). This conjecture has been studied in various special cases; see [6, 12, 14], this conjecture was later disproved by Duval et al. [4] in 2016, but it is still important to find classes of $\mathbb{Z}^n$-graded modules which satisfy the Stanley inequality. Let $I \subset J \subset S$ be monomial ideals. Herzog et al. [10] showed that the invariant Stanley depth of J/I is combinatorial in nature. The most important thing about Stanley depth is that it shares some properties and bounds with homological invariant depth; see [1, 6, 16].

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. A graph is called *simple* if it has no loops and multiple edges. Through out this paper all graphs are simple. A graph G is said to be *connected* if there is a path between any two vertices of G. If $V_G = \{v_1, v_2, \dots, v_n\}$ and $S = K[x_1, x_2, \dots, x_n]$, then edge ideal $I(G)$ of the graph G is the ideal of S generated by all monomials of the form x_ix_j such that $\{v_i, v_j\} \in E_G$. Let $n \geq 2$. A path on n vertices say $\{u_1, u_2, \dots, u_u\}$ is a graph denoted by P_n such that $E_{P_n} = \{\{u_i, u_{i+1}\} : 1 \leq i \leq n-1\}$. Let $n \geq 3$. A cycle on n vertices $\{u_1, u_2, \dots, u_n\}$ is a graph denoted by C_n such that $E_{C_n} = \{\{u_i, u_{i+1}\} : 1 \leq i \leq n-1\} \cup \{u_1, u_n\}$. A simple and connected graph $\mathcal{T}$ is said to be a *tree* if there exists a unique path between any two vertices of $\mathcal{T}$. If $u, v \in V_G$ then the *distance* between u and v is the length of the shortest path between u and v. The maximum distance between any two vertices of G is called *diameter* of G, denoted by $d(G)$. The *degree* of a vertex u in a graph G is the number of edges incident on u, degree of u is denoted by deg(u). A graph with only one vertex is called a *trivial graph*. We denote the trivial graph by T. Any vertex with degree 1 is said to be a *leaf* or *pendant* vertex of G. *Internal vertex* is a vertex that is not a leaf. A tree with one internal vertex and $k-1$ leaves incident on it is called *k-star*, we denoted $k-$star by S_k.

The aim of this paper is to study depth and Stanley depth of the quotient rings of the edge ideals associated with the corona product of firecracker graphs, some classes of caterpillar graphs and some newly defined unicyclic graphs with an arbitrary non-trivial connected graph G. We compute formulae for the values of depth that are the functions of depth of the quotient ring of the edge ideal $I(G)$ see Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4. As a consequence we also prove that if the Stanley's inequality holds for the quotient ring of the edge ideal $I(G)$, then it also holds for the quotient rings of the edge ideals associated to the corona product of the graphs we considered with G. We also compute values of depth and Stanley depth and verify Stanley's inequality for the quotient ring of the edge ideals associated with some special classes of caterpillar graphs, see Theorem 4.1 and Theorem 4.2.

2. Definitions and notations

In this section some definitions from Graph Theory are presented. For more details we refer the readers to [7,8,18]. We also present some known results from Commutative Algebra that are frequently used in this paper. Note that by abuse of notation, x_i will at times be used to denote both a vertex of a graph G and the corresponding variable of the polynomial ring S. For a given graph G, $K[V_G]$ will denote the polynomial ring whose variables are the vertices of the graph G.

Definition 2.1 ([7]). Let G_1 and G_2 be two graphs with order n and m respectively. The corona product of G_1 and G_2 denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and n copies of G_2; and then by joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2; see Figure 1.

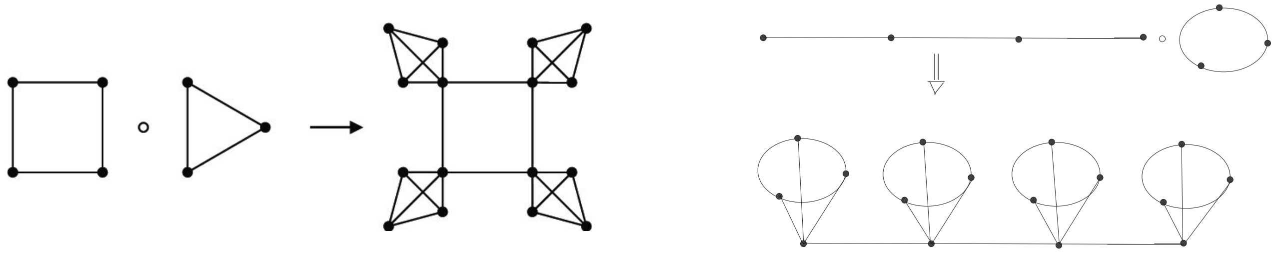

Figure 1. From left to right $C_4 \circ C_3$ and $P_4 \circ C_3$.

Definition 2.2. Let $z \geq 1$ and $k \geq 2$ be integers and P_z be a path on z vertices $u_1, u_2, \ldots, u_z$ that is, $E_{P_z} = \{u_i u_{i+1} : 1 \leq i \leq z-1\}$ (for $z = 1$, $E_{P_z} = \emptyset$). We define a graph on zk vertices by attaching $k-1$ pendant vertices at each u_i. We denote this graph by $P_{z,k}$; see Figure 2.

Definition 2.3. Let $z \geq 3$ and $k \geq 2$ be integers and C_z be a cycle on z vertices $u_1, u_2, \ldots, u_z$ that is, $E_{C_z} = \{u_i u_{i+1} : 1 \leq i \leq z-1\} \cup \{u_1 u_z\}$. We define a graph on zk vertices by attaching $k-1$ pendant vertices at each u_i. We denote this graph by $C_{z,k}$; see Figure 2.

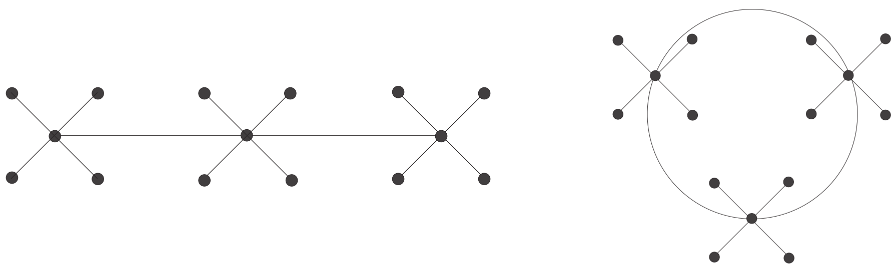

Figure 2. From left to right $P_{3,5}$ and $C_{3,5}$.

Definition 2.4 ([18])**.** Firecracker is a graph formed by the concatenation of α number of k-stars by linking exactly one leaf from each star. It is denoted by $F_{\alpha,k}$; see Figure 3.

Definition 2.5. The graph obtained by joining the end vertices of the path joining the leaves of the α stars in $F_{\alpha,k}$. We call this graph circular firecracker and is denoted by $CF_{\alpha,k}$; see Figure 3.

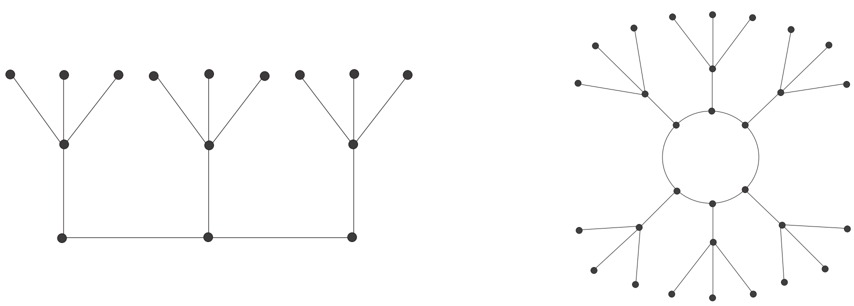

Figure 3. From left to right $F_{3,5}$ and $CF_{3,5}$.

Definition 2.6. Let $z \geq 3$ be an odd integer and $k_1, k_3, k_5, \ldots, k_z$ be integers greater than 1. Let P_z be a path on z vertices $u_1, u_2, \ldots, u_z$ that is, $E_{P_z} = \{u_i u_{i+1} : 1 \leq i \leq z-1\}$. Let $a \in \{1, 3, 5, \ldots, z\}$, we define a graph by attaching $k_a - 1$ pendant vertices at each vertex u_a of P_z. We denote this graph by $\mathcal{P}_z$; see Figure 4.

Definition 2.7. Let $z \geq 2$ and $k \geq 3$ be integers and P_z be a path on z vertices $\{u_1, u_2, \ldots, u_z\}$ that is, $E_{P_z} = \{u_i u_{i+1} : 1 \leq i \leq z-1\}$. We denote by $\mathcal{P}_{z,k}$ the graph obtained by attaching $k + i - 2$ pendant vertices at each u_i of P_z; see Figure 4.

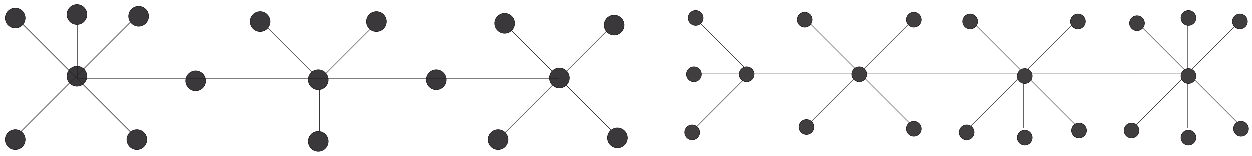

Figure 4. From left to right $\mathcal{P}_5$ and $\mathcal{P}_{5,4}$.

Here we recall some known results that will be used in this paper.

Lemma 2.8 ([2, Proposition 1.2.9]). *(Depth Lemma) If* $0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$ *is a short exact sequence of modules over a local ring* S*, or a Noetherian graded ring with* S_0 *local then*

(1) $depth(E_1) \geq \min\{depth(E_2), 1 + depth(E_3)\}$.

(2) $depth(E_2) \geq \min\{depth(E_1), depth(E_3)\}$.

(3) $depth(E_3) \geq \min\{depth(E_1) - 1, depth(E_2)\}$.

Lemma 2.9 ([14, Lemma 2.4]). *If* $0 \to E_1 \to E \to E_2 \to 0$ *is a short exact sequence of* $\mathbb{Z}^n$*−graded* S*-module, then*

$$sdepth(E) \geq \min\{sdepth(E_1), sdepth(E_2)\}.$$

Proposition 1 ([16, Corollary 1.3]). *If* $I \subset S$ *is a monomial ideal and* $u \in S$ *is a monomial such that* $u \notin I$*, then* $depth_S(S/(I : u)) \geq depth_S(S/I)$.

Proposition 2 ([3, Proposition 2.7]). *If* $I \subset S$ *is a monomial ideal and* $u \in S$ *is monomial such that* $u \notin I$*, then* $sdepth_S(S/(I : u)) \geq sdepth_S(S/I)$.

Lemma 2.10 ([13, Lemma 3.6]). *Let* $I \subset S$ *be a monomial ideal. If* $S' = S \otimes_K K[x_{n+1}] \cong S[x_{n+1}]$*, then* $depth(S'/I'S') = depth(S/I) + 1$ *and* $sdepth(S'/I'S') = sdepth(S/I) + 1$.

Lemma 2.11 ([3, Proposition 1.1]). *If* $I' \subset S' = K[x_1, \ldots, x_m]$ *and* $I'' \subset S'' = K[x_{m+1}, \ldots, x_n]$ *are monomial ideals, with* $1 \leq m < n$*, then*

$$depth_S(S/(I'S + I''S)) = depth_{S'}(S'/I') + depth_{S''}(S''/I'').$$

Lemma 2.12 ([3, Proposition 1.1]). *If* $I' \subset S' = K[x_1, \ldots, x_m]$ *and* $I'' \subset S'' = K[x_{m+1}, \ldots, x_n]$ *are monomial ideals, with* $1 \leq m < n$*, then*

$$depth(S'/I' \otimes_K S''/I'') = depth_S(S/(I'S + I''S)) = depth_{S'}(S'/I') + depth_{S''}(S''/I'').$$

Proof. Proof follows by [19, Proposition 2.2.20] and [19, Theorem 2.2.21]. □

Theorem 2.1 ([16, Theorem 3.1]). *If $I' \subset S' = K[x_1, \ldots, x_m]$ and $I'' \subset S'' = K[x_{m+1}, \ldots, x_n]$ are monomial ideals, with $1 \le m < n$, then*

$$sdepth_S(S/(I'S + I''S)) \ge sdepth_{S'}(S'/I') + sdepth_{S''}(S''/I'').$$

Lemma 2.13. *If $I' \subset S' = K[x_1, \ldots, x_m]$ and $I'' \subset S'' = K[x_{m+1}, \ldots, x_n]$ are monomial ideals, with $1 \le m < n$, then*

$$sdepth(S'/I' \otimes_K S''/I'')) \ge sdepth_{S'}(S'/I') + sdepth_{S''}(S''/I'').$$

Proof. By [19, Proposition 2.2.20], we have $S'/I' \otimes_K S''/I'' \cong S/(I'S + I''S)$, by Theorem 2.1 the required result follows. □

Let $m \ge 2$ be an integer, and consider $\{M_j : 1 \le j \le m\}$ and $\{N_i : 0 \le i \le m\}$ be sequence of $\mathbb{Z}^n$−graded $S - modules$ and consider the chain of short exact sequences of the form

$$0 \longrightarrow M_1 \longrightarrow N_0 \longrightarrow N_1 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow M_{m-1} \longrightarrow N_{m-2} \longrightarrow N_{m-1} \longrightarrow 0$$

$$0 \longrightarrow M_m \longrightarrow N_{m-1} \longrightarrow N_m \longrightarrow 0.$$

Then the following lemmas play key role in the proofs of our theorems.

Lemma 2.14 ([11, Lemma 3.1]). *If $depthM_m \le depthN_m$ and $depthM_{j-1} \le depthM_j$, for all $2 \le j \le m$, then $depthM_1 = depthN_0$.*

Lemma 2.15. $sdepthN_0 \ge \min\{sdepthM_j, sdepthN_m : 1 \le j \le m\}$.

Proof. Proof follows by applying Lemma 2.9 on the above chain of short exact sequences. □

Proposition 3 ([1]). *If I is an edge ideal of n-star, then $depth(S/I) = sdepth(S/I) = 1$, and $depth(S/I^t), sdepth(S/I^t) \ge 1$.*

Corollary 2.16 ([6, Theorem 3.2]). *Let G be a connected graph. If $I = I(G)$ and d is the diameter of G, then*

$$depth(S/I) \ge \lceil \frac{d+1}{3} \rceil.$$

Theorem 2.2 ([6, Theorem 4.18]). *Let G be a connected graph. If $I = I(G)$ and d is the diameter of G, then for $1 \le t \le 3$ we have*

$$sdepth(S/I^t) \ge \lceil \frac{d-4t+5}{3} \rceil.$$

Corollary 2.17. *Let G be connected graph. If $I = I(G)$ and d is the diameter of G, then we have*

$$sdepth(S/I) \ge \lceil \frac{d+1}{3} \rceil.$$

3. Caterpillar and firecrackers graphs and the corona product

In this section we prove our main results related to corona product of graphs. We start this section with some elementary results that are necessary for our main results. Let T be a trivial graph and G any non-trivial and connected graph. The first lemma of this section give depth and Stanley depth of the cyclic modules associated with $T \circ G$. For examples of $T \circ G$; see Figure 5.

Lemma 3.1. *Let T be a trivial graph and G be any connected non-trivial graph. If $I = I(T \circ G)$ and $S := K[V(T \circ G)]$, then depth$(S/I) = 1$ and sdepth$(S/I) = 1$.*

Proof. By definition of $T \circ G$ the only vertex x of T has an edge with every vertex of G. Consider the following short exact sequence

$$0 \longrightarrow S/(I : x) \longrightarrow S/I \longrightarrow S/(I, x) \longrightarrow 0.$$

Therefore $S/(I : x) \cong K[x]$, and depth$(S/(I : x)) = 1$. Now $S/(I, x) \cong S_x/I(G)$, where $S_x := S/(x)$. We have depth$(S/(I, x)) =$ depth$(S_x/I(G)) \geq 1$, by Corollary 2.16. Now by using Depth Lemma, we have depth$(S/I) = 1$. For the Stanley depth since $S/(I : x) \cong K[x]$ we have sdepth$(S/(I : x)) = 1$. Now $S/(I, x) \cong S_x/I(G)$. We have sdepth$(S/(I, x)) =$ sdepth$(S_x/I(G)) \geq 1$, by using Lemma 2.9 and Proposition 2, we have sdepth$(S/I) = 1$. □

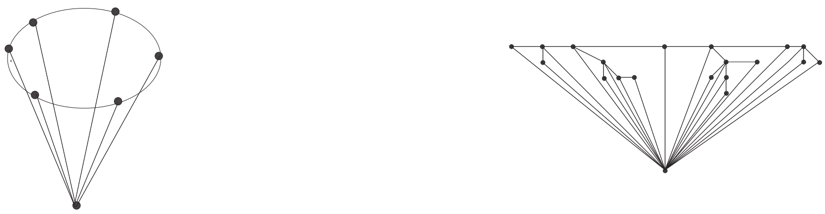

Figure 5. From left to right $T \circ C_6$ and $T \circ \mathcal{T}_{19}$ ($\mathcal{T}_{19}$ is a tree on 19 vertices).

Proposition 4. *For $n, k \geq 2$, let G be a non-trivial connected graph. If $S := K[V(S_k \circ G)]$, then*

$$depth(S/I(S_k \circ G)) = k - 1 + t,$$

where $t = depth(K[V(G)]/I(G))$. Also

$$sdepth(S/I(S_k \circ G)) \geq k - 1 + s,$$

where $s = sdepth(K[V(G)]/I(G))$.

Proof. First we prove the result for depth. Let $k = 2$. If e be a variable corresponding to a leaf in S_2. Consider the following short exact sequence

$$0 \longrightarrow S/(I : e) \longrightarrow S/I \longrightarrow S/(I, e) \longrightarrow 0$$

it is easy to see that $S/(I : e) \cong K[V(G)]/I(G) \otimes_K K[e]$ and

$$S/(I, e) \cong K[V(T \circ G)]/I(T \circ G) \otimes_K K[V(G)]/I(G)).$$

By Lemma 3.1, Lemma 2.10 and [19, Theorem 2.2.21], we have $\operatorname{depth}(S/(I : e)) = 1 + t$ and $\operatorname{depth}(S/(I, e)) = 1 + t = \operatorname{depth}(S/(I : e))$. Thus by Depth Lemma we have $\operatorname{depth}(S/I) = 1 + t$.

Let $k \geq 3$. We will prove the required result by induction on k. Let e be a variable corresponding to a leaf in S_k. Consider the following short exact sequence

$$0 \longrightarrow S/(I : e) \longrightarrow S/I \longrightarrow S/(I, e) \longrightarrow 0$$

we have

$$S/(I : e) \cong \overset{k-2}{\underset{j=1}{\otimes_K}} K[V(T \circ G)]/I(T \circ G) \otimes_K K[V(G)]/I(G) \otimes_K K[e].$$

By Lemmma [19, Theorem 2.2.21], we have

$$\operatorname{depth}(S/(I : e)) = \sum_{j=1}^{k-2} \operatorname{depth}(K[V(T \circ G)]/I(T \circ G)) + \operatorname{depth}(K[V(G)]/I(G)) + \operatorname{depth} K[e],$$

by Lemma 3.1, we get $\operatorname{depth}(S/(I : e)) = k - 2 + t + 1 = k - 1 + t$. It can easily be seen that

$$S/(I, e) \cong K[V(S_{k-1} \circ G)]/I(S_{k-1} \circ G) \otimes_K (K[V(G)]/I(G)).$$

Thus by [19, Theorem 2.2.21]

$$\operatorname{depth}(S/(I, e)) \cong \operatorname{depth}(K[V(S_{k-1} \circ G)]/I(S_{k-1} \circ G)) + \operatorname{depth}((K[V(G)]/I(G))),$$

applying induction on k we get

$$\operatorname{depth}(S/(I, e)) = (k - 2 + t) + t = k + 2t - 2 \geq k - 1 + t = \operatorname{depth}(S/(I : e)).$$

Hence by Depth Lemma we have $\operatorname{depth}(S/I) = k - 1 + t$. This completes the proof for depth.
For Stanley depth the result follows by Lemma 2.13 instead of [19, Theorem 2.2.21] and Lemma 2.9 instead of Depth Lemma. □

Corollary 3.2. *If Stanley's inequality holds for $K[V(G)]/I(G)$ then it also holds for $S/I(S_k \circ G)$.*

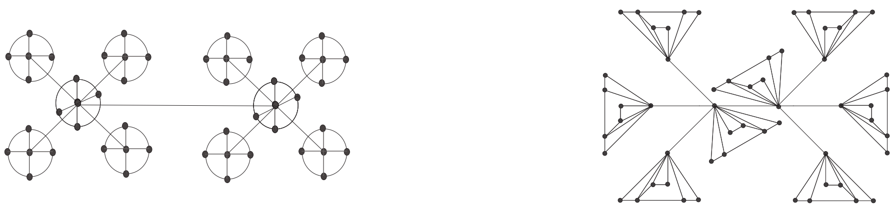

Figure 6. From left to right $P_{2,5} \circ C_3$ and $P_{2,5} \circ \mathcal{T}_6$.

Theorem 3.1. *Let $z \geq 1$ and $k \geq 2$ be integers. If G is a connected graph with $|V(G)| \geq 2$ and $S := K[V(P_{z,k} \circ G)]$, then*

$$\operatorname{depth}(S/I(P_{z,k} \circ G)) = z(k - 1 + t),$$

where $t = \operatorname{depth}((K[V(G)])/I(G))$ and

$$\operatorname{sdepth}(S/I(P_{z,k} \circ G)) \geq z(k - 1 + s),$$

where $s = \operatorname{depth}((K[V(G)])/I(G))$; see Figure 6.

Proof. First we prove the result for depth. We consider the following cases.

1. If $z = 1$ and $k \geq 2$ then the result follows from Proposition 4.
2. Let $z = 2$. We consider the following subcases:
   (a) If $k = 2$ and e is a variable corresponding to a leaf in $P_{2,2}$. Consider the following short exact sequence
   $$0 \longrightarrow S/(I:e) \longrightarrow S/I \longrightarrow S/(I,e) \longrightarrow 0$$
   then $S/(I:e) \cong K[V(S_2 \circ G)]/I(S_2 \circ G) \otimes_K K[V(G)]/I(G) \otimes_K K[e]$, $S/(I,e) \cong K[V(S_3 \circ G)]/I(S_3 \circ G) \otimes_K K[V(G)]/I(G))$. By [19, Theorem 2.2.21],
   $$\begin{aligned}\operatorname{depth}(S/(I:e)) &= \operatorname{depth}(K[V(S_2 \circ G)]/I(S_2 \circ G)) \\ &\quad + \operatorname{depth}(K[V(G)]/I(G)) + \operatorname{depth}(K[e]),\\ \operatorname{depth}(S/(I,e)) &= \operatorname{depth}(K[V(S_3 \circ G)]/I(S_3 \circ G)) + \operatorname{depth}(K[V(G)]/I(G))).\end{aligned}$$
   By Proposition 4 we have $\operatorname{depth}(S/(I:e)) = t+1+t+1 = 2(1+t)$ and $\operatorname{depth}(S/(I,e)) = 2+t+2 = 2(1+t) = \operatorname{depth}(S/(I:e))$. Hence by Depth Lemma we have $\operatorname{depth}(S/I) = 2(1+t)$ and we are done in this special case.

   (b) Let $k \geq 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_2 in $P_{2,k}$ and $I = I(P_{2,k} \circ G)$. For $0 \leq i \leq k-2$, $I_i := (I_i, e_{i+1})$, where $I_0 = I$. Consider the chain of short exact sequences of the form
   $$\begin{array}{ccccccccc} 0 & \longrightarrow & S/(I_0:e_1) & \longrightarrow & S/I_0 & \longrightarrow & S/(I_0,e_1) & \longrightarrow & 0\\ 0 & \longrightarrow & S/(I_1:e_2) & \longrightarrow & S/I_1 & \longrightarrow & S/(I_1,e_2) & \longrightarrow & 0\\ & & & & \vdots & & & & \\ 0 & \longrightarrow & S/(I_{k-2}:e_{k-1}) & \longrightarrow & S/I_{k-2} & \longrightarrow & S/(I_{k-2},e_{k-1}) & \longrightarrow & 0\\ 0 & \longrightarrow & S/(I_{k-1}:u_2) & \longrightarrow & S/I_{k-1} & \longrightarrow & S/(I_{k-1},u_2) & \longrightarrow & 0\end{array}$$
   $$\begin{aligned}S/(I_i:e_{i+1}) \;&\cong\; K[V(S_k \circ G)]/I(S_k \circ G) \bigotimes_{K,\, j=1}^{k-2-i} K[V(T \circ G)]/I(T \circ G)\\ &\quad \bigotimes_{K,\, j=1}^{i+1} K[V(G)]/I(G) \otimes_K K[e_{i+1}].\end{aligned}$$
   By [19, Theorem 2.2.21]
   $$\begin{aligned}\operatorname{depth}(S/(I_i:e_{i+1})) &= \operatorname{depth}(K[V(S_k \circ G)]/I(S_k \circ G)) + \sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/I(G))\\ &\quad + \sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T \circ G)]/I(T \circ G)) + \operatorname{depth}(k[e_{i+1}]) \qquad (3.1)\end{aligned}$$

hence by Lemma 3.1 and Proposition 4, we get

$$\begin{aligned}\operatorname{depth} S/(I_i : e_{i+1}) &= k-1+t+\sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 \\ &= k+t+k-2-i+(i+1)t = 2(k-1+t)+i(t-1). \quad (3.2)\end{aligned}$$

Also we have

$$S/(I_{k-1} : u_2) \cong \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/I(T\circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/I(G) \otimes_K K[u_1],$$

$$S/(I_{k-1}, u_2) \cong K[V(S_k\circ G)]/I(S_k\circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/I(G).$$

By [19, Theorem 2.2.21] we have

$$\operatorname{depth}(S/(I_{k-1} : u_2)) = (\sum_{j=1}^{k-1} \operatorname{depth}(K[V(T\circ G)]/I(T\circ G))+ \\ (\sum_{j=1}^{k} \operatorname{depth}(K[V(G)]/I(G))) + \operatorname{depth}(K[u_1])$$

and similarly

$$\operatorname{depth} S/(I_{k-1}, u_2) = \operatorname{depth} K[V(S_k\circ G)]/I(S_k\circ G) + \sum_{j=1}^{k} \operatorname{depth} K[V(G)]/I(G)$$

by Proposition 4, we get

$$\operatorname{depth}(S/(I_{k-1} : u_2)) = k + kt = 2(k-1+t) + (k-2)(t-1), \quad (3.3)$$

$$\operatorname{depth}(S/(I_{k-1}, u_2)) = k-1+t+kt = 2(k-1+t) + (k-1)(t-1). \quad (3.4)$$

Hence by Lemma 2.14, we have

$$\operatorname{depth}(S/I(P_{2,k}\circ G)) = 2(k-1+t).$$

This completes the proof for $z = 2$.

3. Let $z \geq 3$. We consider the following subcases:

   (a) If $k = 2$, We will prove the result by induction on z. Let u_z be the vertex in the definition of $P_{z,2}$. Consider the following short exact sequence

$$0 \longrightarrow S/(I : u_z) \longrightarrow S/I \longrightarrow S/(I, u_z) \longrightarrow 0$$

   we have $S/(I : u_z) \cong K[V(P_{z-2,2}\circ G)]/I(P_{z-2,2}\circ G) \bigotimes_{j=1}^{2}{}_K K[V(G)]/I(G)$ $\otimes_K K[V(T\circ G)]/I(T\circ G) \otimes_K K[e]$,

$$S/(I, u_z) \cong K[V(P_{z-1,2}\circ G)]/I(P_{z-1,2}\circ G) \otimes_K K[V(G)]/I(G)) \otimes_K K[V(T\circ G)]/I(T\circ G).$$

By induction on z, [19, Theorem 2.2.21], and Lemma 3.1, we have

$$\operatorname{depth}(S/(I : u_z)) = (z-2)(t+1) + 2t + 2 = z(t+1)$$

and similarly

$$\operatorname{depth}(S/(I, u_z)) = z(1+t) = \operatorname{depth}(S/(I : u_z)).$$

Thus by Depth Lemma we have $\operatorname{depth}(S/I) = z(1+t)$ and the result is proved for the case $k = 2$.

(b) Now consider $k \geq 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_z and $I = I(P_{z,k} \circ G)$. For $0 \leq i \leq k-2$, $I_i := (I_i, e_{i+1})$ where $I_0 = I$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S/(I_0 : e_1) & \longrightarrow & S/I_0 & \longrightarrow & S/(I_0, e_1) & \longrightarrow & 0\\
0 & \longrightarrow & S/(I_1 : e_2) & \longrightarrow & S/I_1 & \longrightarrow & S/(I_1, e_2) & \longrightarrow & 0\\
 & & & & \vdots & & & & \\
0 & \longrightarrow & S/(I_{k-2} : e_{k-1}) & \longrightarrow & S/I_{k-2} & \longrightarrow & S/(I_{k-2}, e_{k-1}) & \longrightarrow & 0\\
0 & \longrightarrow & S/(I_{k-1} : u_z) & \longrightarrow & S/I_{k-1} & \longrightarrow & S/(I_{k-1}, u_z) & \longrightarrow & 0
\end{array}$$

we have,

$$S/(I_i : e_{i+1}) \cong K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G) \bigotimes_{K, j=1}^{k-2-i} K[V(T \circ G)]/I(T \circ G)$$
$$\bigotimes_{K, j=1}^{i+1} K[V(G)]/I(G) \otimes_K K[e_{i+1}].$$

By [19, Theorem 2.2.21] we have

$$\operatorname{depth}(S/(I_i : e_{i+1})) = \operatorname{depth}(K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G)) +$$
$$\sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/I(G)) + \sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T \circ G)]/I(T \circ G)) + 1.$$

Thus by Lemma 3.1, Proposition 4 and induction on z we get,

$$\begin{aligned}
\operatorname{depth} S/(I_i : e_{i+1}) &= (z-1)(k-1+t) + \sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 \\
&= (z-1)(k-1+t) + k - 2 - i + (i+1)t + 1 \\
&= z(k-1+t) + i(t-1). \qquad (3.5)
\end{aligned}$$

Also we have

$$S/(I_{k-1} : u_z) \cong K[V(P_{z-2,k} \circ G)]/I(P_{z-2,k} \circ G) \bigotimes_{K, j=1}^{k-1} K[V(T \circ G)]/I(T \circ G)$$
$$\bigotimes_{K, j=1}^{k} K[V(G)]/I(G) \otimes_K K[u_z]$$

and similarly

$$S/(I_{k-1}, u_z) \cong K[V(P_{z-1,k} \circ G)]/I(P_z - 1, k \circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/I(G).$$

By [19, Theorem 2.2.21] and Proposition 4, we get

$$\text{depth}(S/(I_{k-1} : u_z)) = z(k - 1 + t) + (k - 2)(t - 1), \tag{3.6}$$

$$\text{depth}S/(I_{k-1}, u_z) = \text{depth}K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G) + \sum_{j=1}^{k} \text{depth}K[V(G)]/I(G)$$

$$\text{depth}(S/(I_{k-1}, u_z)) = (z - 1)(k - 1 + t) + kt = z(k - 1 + t) + (k - 1)(t - 1). \tag{3.7}$$

Hence by Lemma 2.14, we get

$$\text{depth}(S/I(P_{z,k} \circ G)) = z(k - 1 + t).$$

This completes the proof.

For Stanley depth the result follows by Lemma 2.13 instead of [19, Theorem 2.2.21] and Lemma 2.15 instead of Lemma 2.14. □

Corollary 3.3. *If Stanley's inequality holds for $K[V(G)]/I(G)$ then it also holds for $S/I(P_{z,k} \circ G)$.*

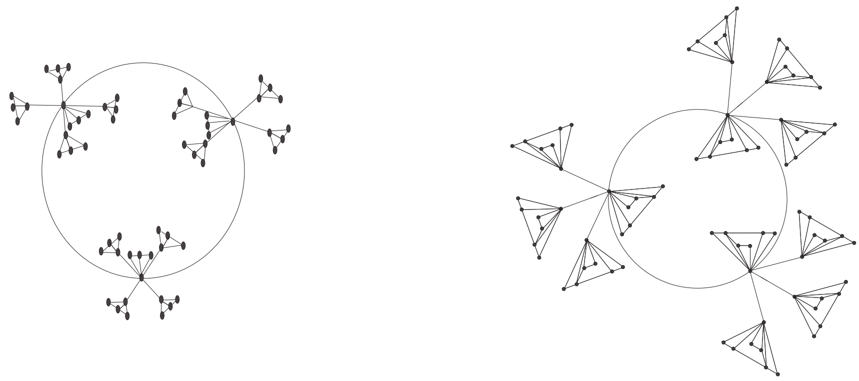

Figure 7. From left to right $C_{3,5} \circ P_3$ and $C_{3,5} \circ \mathcal{T}_6$.

Theorem 3.2. *Let $z \geq 3$ and $k \geq 2$ be integers and G be a connected graph with $|V(G)| \geq 2$. Consider $S := K[V(C_{z,k} \circ G)]$. We have*

$$depth(S/I(C_{z,k} \circ G)) = z(k - 1 + t),$$

where $t = depth((K[V(G)])/I(G))$ and

$$sdepth(S/I(C_{z,k} \circ G)) \geq z(k - 1 + s),$$

where $s = sdepth((K[V(G)])/I(G))$; see Figure 7.

Proof. First we prove the result for depth.

1. Let $z = 3$. We consider the following subcases:

   (a) Let $k = 2$. Let u be a variable corresponding to a vertex of C_3 in $C_{3,2}$. Consider the following short exact sequence

$$0 \longrightarrow S/(I:u) \longrightarrow S/I \longrightarrow S/(I,u) \longrightarrow 0$$

   we have

$$S/(I:u) \cong \bigotimes_{j=1}^{2}{}_K K[V(T \circ G)]/I(T \circ G) \bigotimes_{j=1}^{3}{}_K K[V(G)]/I(G) \otimes_K K[e],$$

$$S/(I,u) \cong K[V(P_{2,2} \circ G)]/I(P_{2,2} \circ G) \otimes_K K[V(G)]/I(G) \otimes_K K[V(T \circ G)]/I(T \circ G).$$

   Hence by using Lemma 3.1, [19, Theorem 2.2.21] and Theorem 3.1, we have

$$\operatorname{depth}(S/(I:u)) = 2 + 3t + 1 = 3(t+1),$$

$$\operatorname{depth}(S/(I,u)) = 3(1+t) = \operatorname{depth}(S/(I:e)).$$

   Thus by Depth Lemma we have $\operatorname{depth}(S/I) = 3(1+t)$.

   (b) Let $k \geq 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_3 in $C_{3,k}$ and $I = I(C_{3,k} \circ G)$. For $0 \leq i \leq k-2$, $I_i := (I_i, e_{i+1})$ where $I_0 = I$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccccc}
0 \longrightarrow & S/(I_0 : e_1) & \longrightarrow & S/I_0 & \longrightarrow & S/(I_0, e_1) & \longrightarrow 0 \\
0 \longrightarrow & S/(I_1 : e_2) & \longrightarrow & S/I_1 & \longrightarrow & S/(I_1, e_2) & \longrightarrow 0 \\
 & & & \vdots & & & \\
0 \longrightarrow & S/(I_{k-2} : e_{k-1}) & \longrightarrow & S/I_{k-2} & \longrightarrow & S/(I_{k-2}, e_{k-1}) & \longrightarrow 0 \\
0 \longrightarrow & S/(I_{k-1} : u_3) & \longrightarrow & S/I_{k-1} & \longrightarrow & S/(I_{k-1}, u_3) & \longrightarrow 0
\end{array}$$

   we have,

$$\begin{aligned}
S/(I_i : e_{i+1}) \;\cong\; & K[V(P_{2,k} \circ G)]/I(P_{2,k} \circ G) \bigotimes_{j=1}^{k-2-i}{}_K K[V(T \circ G)]/I(T \circ G) \\
& \bigotimes_{j=1}^{i+1}{}_K K[V(G)]/I(G) \underset{K}{\otimes_K} K[e_{i+1}].
\end{aligned} \tag{3.8}$$

   By using [19, Theorem 2.2.21]

$$\begin{aligned}
\operatorname{depth}(S/(I_i : e_{i+1})) = {} & \operatorname{depth}(K[V(P_{k,2} \circ G)]/I(P_{k,2} \circ G)) + \\
& \sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/I(G)) + \sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T \circ G)]/I(T \circ G)) + \operatorname{depth} K[e_{i+1}]
\end{aligned}$$

hence by Lemma 3.1, Proposition 4 and Theorem 3.1, we get

$$\begin{aligned}\operatorname{depth} S/(I_i : e_{i+1}) &= 2(k-1+t) + \sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} +1 \\ &= 2(k-1+t) + k - 2 - i + it + t + 1 \\ &= 3(k-1+t) + i(t-1). \end{aligned} \quad (3.9)$$

$$\begin{aligned} S/(I_{k-1} : u_3) \;\cong\; & \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/I(T\circ G) \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/I(T\circ G) \\ & \bigotimes_{j=1}^{k+1}{}_K K[V(G)]/I(G) \otimes_K K[u_3], \end{aligned}$$

$$S/(I_{k-1}, u_3) \cong K[V(P_{2,k}\circ G)]/I(P_{2,k}\circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/I(G).$$

By [19, Theorem 2.2.21]

$$\operatorname{depth} S/(I_{k-1}, u_3) = \operatorname{depth} K[V(P_{2,k}\circ G)]/I(P_{2,k}\circ G) + \sum_{j=1}^{k} \operatorname{depth} K[V(G)]/I(G)$$

by Lemma 3.1 and Theorem 3.1, we get

$$\operatorname{depth}(S/(I_{k-1} : u_1)) = 3(k-1+t) + (k-2)(t-1). \quad (3.10)$$

$$\operatorname{depth}(S/(I_{k-1}, u_3)) = 2(k-1+t) + kt = 3(k-1+t) + (k-1)(t-1). \quad (3.11)$$

Hence by Lemma 2.14, we get

$$\operatorname{depth}(S/I(C_{3,k}\circ G)) = 3(k-1+t).$$

2. Let $z \geq 4$. We consider the following subcases:

   (a) Let $k = 2$. Let u be a variable corresponding to the vertex of C_z in $C_{z,2}$. Consider the following short exact sequence

   $$0 \longrightarrow S/(I : u) \longrightarrow S/I \longrightarrow S/(I, u) \longrightarrow 0$$

   we have $S/(I : u) \cong K[V(P_{z-3,2}\circ G)]/I(P_{z-3,2}\circ G) \bigotimes_{j=1}^{3}{}_K K[V(G)]/I(G)$
   $\bigotimes_{j=1}^{2}{}_K K[V(T\circ G)]/I(T\circ G) \otimes_K K[e]$,

   $$S/(I, u) \cong K[V(P_{z-1,2}\circ G)]/I(P_{z-1,2}\circ G) \otimes_K K[V(G)]/I(G)) \otimes_K K[V(T\circ G)]/I(T\circ G).$$

   Hence by using Lemma 3.1, [19, Theorem 2.2.21] and Theorem 3.1, we have

   $$\operatorname{depth}(S/(I : u)) = (z-3)(t+1) + 3t + 2 + 1 = z(t+1)$$

   $$\operatorname{depth}(S/(I, u)) = z(1+t) = \operatorname{depth}(S/(I : u)).$$

   Thus by Depth Lemma we have $\operatorname{depth}(S/I) = z(1+t)$.

(b) Let $k \geq 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_z in $C_{z,k}$ and $I = I(C_{z,k} \circ G)$. For $0 \leq i \leq k-2$, $I_i := (I_i, e_{i+1})$ where $I_0 = I$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccc}
0 \longrightarrow & S/(I_0 : e_1) \longrightarrow & S/I_0 & \longrightarrow & S/(I_0, e_1) \longrightarrow & 0 \\
0 \longrightarrow & S/(I_1 : e_2) \longrightarrow & S/I_1 & \longrightarrow & S/(I_1, e_2) \longrightarrow & 0 \\
 & & \vdots & & & \\
0 \longrightarrow & S/(I_{k-2} : e_{k-1}) \longrightarrow & S/I_{k-2} & \longrightarrow & S/(I_{k-2}, e_{k-1}) \longrightarrow & 0 \\
0 \longrightarrow & S/(I_{k-1} : u_z) \longrightarrow & S/I_{k-1} & \longrightarrow & S/(I_{k-1}, u_z) \longrightarrow & 0
\end{array}$$

$$\begin{aligned} S/(I_i : e_{i+1}) \cong\ & K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G) \\ & \bigotimes_{j=1}^{k-2-i}{}_K K[V(T \circ G)]/I(T \circ G) \bigotimes_{j=1}^{i+1}{}_K K[V(G)]/I(G) \otimes_K K[e_{i+1}]. \end{aligned}$$

By using [19, Theorem 2.2.21]

$$\begin{aligned} &\mathrm{depth}(S/(I_i : e_{i+1})) = \mathrm{depth}(K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G)) + \\ &\sum_{j=1}^{i+1} \mathrm{depth}(K[V(G)]/I(G)) + \sum_{j=1}^{k-2-i} \mathrm{depth}(K[V(T \circ G)]/I(T \circ G)) + \mathrm{depth} K[e_{i+1}] \end{aligned} \tag{3.12}$$

by Lemma 3.1, Proposition 4 and Theorem 3.1, we get

$$\begin{aligned} \mathrm{depth} S/(I_i : e_{i+1}) &= (z-1)(k-1+t) + \sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 \\ &= (z-1)(k-1+t) + k - 2 - i + it + t + 1 \\ &= z(k-1+t) + i(t-1). \end{aligned} \tag{3.13}$$

$$\begin{aligned} S/(I_{k-1} : u_z) \cong\ & K[V(P_{z-3,k} \circ G)]/I(P_{z-3,k} \circ G) \bigotimes_{j=1}^{k-1}{}_K K[V(T \circ G)]/I(T \circ G) \\ & \bigotimes_{j=1}^{k-1}{}_K K[V(T) \circ G]/I(T \circ G) \quad \bigotimes_{j=1}^{k+1}{}_K K[V(G)]/I(G) \otimes_K K[u_z], \end{aligned}$$

$$S/(I_{k-1}, u_z) \cong K[V(P_{z-1,k} \circ G)]/I(P_{z-1,k} \circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/I(G)$$

by Lemma 3.1, [19, Theorem 2.2.21] and Theorem 3.1, we have

$$\mathrm{depth}(S/(I_{k-1} : u_z)) = z(k-1+t) + (k-2)(t-1). \tag{3.14}$$

$$\mathrm{depth}(S/(I_{k-1}, u_z)) = (z-1)(k-1+t) + kt = z(k-1+t) + (k-1)(t-1). \tag{3.15}$$

Hence by Lemma 2.14, we will have the required result

$$\mathrm{depth}(S/I(C_{z,k} \circ G)) = z(k-1+t).$$

For Stanley depth the result follows by Lemma 2.13 instead of [19, Theorem 2.2.21] and Lemma 2.15 instead of Lemma 2.14. □

Corollary 3.4. *Stanley's inequality holds for $S/I(C_{z,k} \circ G)$ if it holds for $K[V(G)]/I(G)$.*

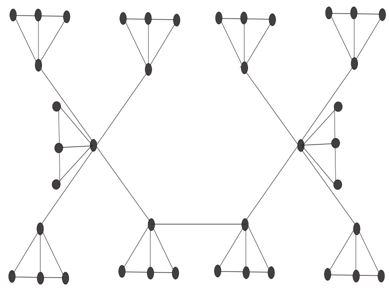

Figure 8. $F_{3,5} \circ P_3$.

Theorem 3.3. *Let $\alpha \geq 2$ and $k \geq 3$ be integers and G be a connected graph with $|V(G)| \geq 2$ and $S := K[V(F_{\alpha,k} \circ G)]$. Then*

$$depth S/I(F_{\alpha,k} \circ G) = \alpha(k-1+t) + \lceil \frac{\alpha-1}{2} \rceil (t-1),$$

where $t = depth((K[V(G)])/I(G))$ and

$$sdepth S/I(F_{\alpha,k} \circ G) \geq \alpha(k-1+s) + \lceil \frac{\alpha-1}{2} \rceil (s-1),$$

where $s = sdepth((K[V(G)])/I(G))$ and $\lceil \alpha \rceil = \{n \in \mathbb{Z} : n \geq \alpha\}$; see Figure 8.

Proof. We consider the following cases:

1. Let $\alpha = 2$. Let $e_1, e_2, \dots, e_{k-1}$ be leaves attached to u_2 in $F(2,k)$ and $I = I(F_{2,k} \circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow S/(I : e_1) \longrightarrow S/I \longrightarrow S/(I, e_1) \longrightarrow 0$$

where e_1 is leave of second star that is attached to the previous star.

$$S/(I : e_1) \cong K[V(S_{k-1} \circ G)]/I(S_{k-1} \circ G) \bigotimes_{j=1}^{k-2}{}_K K[V(T \circ G)]/I(T \circ G)$$
$$\bigotimes_{j=1}^{2}{}_K K[V(G)]/I(G) \otimes_K K[e_1],$$

$$S/(I, e_1) \cong K[V(S_k \circ G)]/I(S_k \circ G) \otimes_K K[V(S_{k-1} \circ G)]/I(S_{k-1}) \otimes_K K[V(G)]/I(G).$$

By [19, Theorem 2.2.21]

$$\text{depth}(S/(I : e_1)) = \text{depth}(K[V(S_{k-1} \circ G)]/I(S_{k-1} \circ G)) + 2\text{depth}(K[V(G)]/I(G))$$
$$+ \sum_{j=1}^{k-2} \text{depth}(K[V(T \circ G)]/I(T \circ G)) + K[e_1]$$

hence by Lemma 3.1 and Proposition 4, we get

$$\begin{aligned}\text{depth}S/(I:e_1) &= k-2+t+\sum_{j=1}^{k-2}1+2t+1\\ &= k-2+t+k-2+2t+1=2(k-1+t)+(t-1)\end{aligned}$$

and similarly

$$\text{depth}S/(I,e_1)=(k-1+t)+(k-2+t)+t=2(k-1+t)+(t-1)$$

So by using Depth Lemma, we have

$$\text{depth}S/I(F_{2,k}\circ G)=2(k-1+t)+(t-1).$$

2. Let $\alpha\geq 3$. Let $e_1.e_2,...,e_{k-1}$ be leaves attached to u_α in $F(z,k)$ and $I=I(F_{\alpha,k}\circ G)$. Consider the short exact sequence of the form

$$0\longrightarrow S/(I:e_1)\longrightarrow S/I\longrightarrow S/(I,e_1)\longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star in $F_{\alpha,k}$. We have

$$\begin{aligned}S/(I:e_1)\cong{}& K[V(F_{\alpha-2,k}\circ G)]/I(F_{\alpha-2,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)\\ &\bigotimes_{K,\,j=1}^{k-2}K[V(T\circ G)]/I(T\circ G)\bigotimes_{K,\,j=1}^{2}K[V(G)]/I(G)\otimes_K K[e_1],\end{aligned}$$

$$\begin{aligned}S/(I,e_1)\cong{}& K[V(F_{\alpha-1,k}\circ G)]/I(F_{\alpha-1,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)\\ &\otimes_K K[V(G)]/I(G).\end{aligned}$$

By [19, Theorem 2.2.21]

$$\begin{aligned}\text{depth}(S/(I:e_1))={}&\text{depth}K[V(F_{\alpha-2,k}\circ G)]/I(F_{\alpha-2,k}\circ G)+\\ &\text{depth}(K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G))\\ &+\sum_{j=1}^{k-2}\text{depth}(K[V(T\circ G)]/I(T\circ G))+2\text{depth}(K[V(G)]/I(G))+\text{depth}K[e_1]\end{aligned}$$

hence by Lemma 3.1, Proposition 4 and induction on α, we get

$$\begin{aligned}\text{depth}S/(I:e_1)&=(\alpha-2)(k-1+t)+\lceil\frac{\alpha-3}{2}\rceil(t-1)+(k-2+t)+\sum_{j=1}^{k-2}1+2t+1\\ &=\alpha(k-1+t)+\lceil\frac{\alpha-1}{2}\rceil(t-1)\end{aligned}$$

and similarly

$$\text{depth}S/(I,e_1)=(\alpha-1)(k-1+t)+\lceil\frac{\alpha-2}{2}\rceil(t-1)+(k-2+t)+t$$

$$\text{depth}S/(I:e_1) \;=\; \alpha(k-1+t)+\lceil\frac{\alpha}{2}\rceil(t-1). \tag{3.16}$$

So by using Depth Lemma, we have

$$\text{depth}S/(F_{\alpha,k}\circ G)=\alpha(k-1+t)+\lceil\frac{\alpha-1}{2}\rceil(t-1).$$

For Stanley depth the result follows by Lemma 2.13 instead of [19, Theorem 2.2.21] and Lemma 2.9 instead of Depth Lemma. □

Corollary 3.5. *Stanley's inequality holds for $S/I(F_{\alpha,k}\circ G)$ if it holds for $K[V(G)]/I(G)$.*

Theorem 3.4. *Let $\alpha\geq 3$ and $k\geq 3$ be integers and G be a connected graph with $|V(G)|\geq 2$. Consider $S := K[V(CF_{\alpha,k}\circ G)]$. Then*

$$depthS/I(CF_{\alpha,k}\circ G)=\alpha(k-1+t)+\lceil\frac{\alpha}{2}\rceil(t-1),$$

where $t = depth((K[V(G)])/I(G))$ and

$$sdepthS/I(C_{\alpha,k}\circ G)\geq\alpha(k-1+s)+\lceil\frac{\alpha}{2}\rceil(s-1),$$

where $s = sdepth((K[V(G)])/I(G))$; see Figure 9.

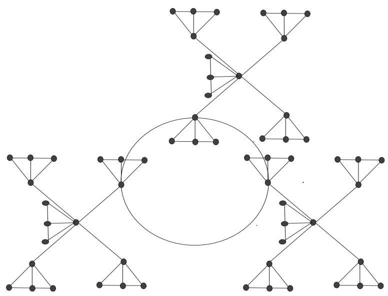

Figure 9. $CF_{3,5}\circ P_3$.

Proof. We consider the following cases:

1. Consider $\alpha=3$. Let $e_1,e_2,...,e_{k-1}$ be leaves attached to u_3 in $CF(3,k)$ and $I=I(CF_{3,k}\circ G)$. Consider the short exact sequence of the form

$$0\longrightarrow\; S/(I:e_1)\longrightarrow\; S/I\longrightarrow\; S/(I,e_1)\longrightarrow\; 0$$

where e_1 is leave of third star that is attached to the previous star and first star in $CF_{3,k}$. We have

$$S/(I:e_1)\cong\bigotimes_{j=1}^{2}{}_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[(T\circ G)]/I(T\circ G)\bigotimes_{j=1}^{3}{}_K K[V(G)]/I(G)\otimes_K K[e_1],$$

$$S/(I,e_1) \cong K[V(F_{2,k}\circ G)]/I(F_{2,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)\otimes_K K[V(G)]/I(G).$$

By [19, Theorem 2.2.21]

$$\operatorname{depth}(S/(I:e_1)) = 2\operatorname{depth}(K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)) + 3\operatorname{depth}(K[V(G)]/I(G))$$
$$\sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/I(T\circ G)) + \operatorname{depth}K[e_1]$$

hence by Lemma 3.1 and Proposition 4, we get

$$\operatorname{depth}S/(I:e_1) = 2(k-2+t) + \sum_{j=1}^{k-2} 1 + 3t + 1 = 3(k-1+t) + 2(t-1) \tag{3.17}$$

and similarly

$$\begin{aligned}\operatorname{depth}S/(I,e_1) &= 2(k-1+t) + (t-1) + (k-2+t) + t\\ \operatorname{depth}S/(I,e_1) &= 3(k-1+t) + 2(t-1).\end{aligned} \tag{3.18}$$

So by using Depth Lemma 2.8, we have

$$\operatorname{depth}S/(CF_{3,k}\circ G) = 3(k-1+t) + 2(t-1).$$

2. Let $\alpha \geq 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_α in $CF(\alpha, k)$ and $I = I(CF_{\alpha,k}\circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow S/(I:e_1) \longrightarrow S/I \longrightarrow S/(I,e_1) \longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star and first star in $CF_{\alpha,k}$. We have

$$S/(I:e_1) \cong K[V(F_{\alpha-3,k}\circ G)]/I(F_{\alpha-3,k}\circ G)\bigotimes_{j=1}^{2}{}_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[V(T\circ G)]/I(T\circ G)\bigotimes_{j=1}^{3}{}_K K[V(G)]/I(G)\otimes_K K[e_1],$$

$$S/(I,e_1) \cong K[V(F_{\alpha-1,k}\circ G)]/I(F_{\alpha-1,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)\otimes_K K[V(G)]/I(G).$$

By using [19, Theorem 2.2.21]

$$\operatorname{depth}(S/(I:e_1)) = \operatorname{depth}K[V(F_{\alpha-3,k}\circ G)]/I(F_{\alpha-3,k}\circ G) + \sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/I(T\circ G))$$
$$+ 2\operatorname{depth}(K[V(S_{k-1}\circ G)]/I(S_{k-1}\circ G)) + 3\operatorname{depth}(K[V(G)]/I(G)) + 1$$

hence by Lemma 3.1, Proposition 4 and Theorem 3.3, we get

$$\operatorname{depth}S/(I:e_1) = (\alpha-3)(k-1+t) + \lceil\frac{\alpha-4}{2}\rceil(t-1) + 2(k-2+t) + \sum_{j=1}^{k-2} 1 + 3t + 1$$

$$= \quad \alpha(k-1+t)+\lceil\frac{\alpha}{2}\rceil(t-1) \tag{3.19}$$

and similarly

$$\begin{aligned} \operatorname{depth} S/(I,e_1) &= (\alpha-1)(k-1+t)+\lceil\frac{\alpha-2}{2}\rceil(t-1)+(k-2+t)+t \\ \operatorname{depth} S/(I,e_1) &= \alpha(k-1+t)+\lceil\frac{\alpha}{2}\rceil(t-1). \end{aligned} \tag{3.20}$$

So by using Depth Lemma 2.8, we have

$$\operatorname{depth} S/(CF_{\alpha,k}\circ G)=\alpha(k-1+t)+\lceil\frac{\alpha}{2}\rceil(t-1).$$

For Stanley depth the result follows by Lemma 2.13 instead of [19, Theorem 2.2.21] and Lemma 2.9 instead of Depth Lemma. □

Corollary 3.6. *Stanley's inequality holds for $S/I(CF_{\alpha,k}\circ G)$ if it holds for $K[V(G)]/I(G)$.*

4. Some special classes of caterpillar trees

In this section we calculate values of depth and Stanley depth of the quotient rings associated with edge ideals of some class of caterpillar graphs. We also prove that the values of both depth and Stanley depth for these classes of graphs are exactly the same. As a consequence the Stanley's inequality holds for the quotient ring of edge ideals of these classes of graphs.

Theorem 4.1. *Let $z\geq 3$ and $S=K[V(\mathcal{P}_z)]$. For $a\in\{1,3,5,\dots,z\}$, if $k_a>1$ and $I=I(\mathcal{P}_z)$, then*

$$\mathit{depth}(S/I)=\mathit{sdepth}(S/I)=\frac{z+1}{2}.$$

Proof. The proof is done by induction on z. Let $z=3$. Consider the following short exact sequence

$$0\longrightarrow S/(I:u_3)\longrightarrow S/I\longrightarrow S/(I,u_3)\longrightarrow 0$$

We have $(I:u_3)=(x:x\in N(u_3))+I(S_{k_1})$ and $S/(I:u_3)\cong K[V(S_{k_1})\cup\{u_3\}]/I(S_{k_1})$, thus by Lemma 2.10 and Proposition 3, $\operatorname{depth} S/(I:u_3)=\operatorname{sdepth} S/(I:u_3)=1+1=2$. Clearly $(I,u_3)=(I(S_{k_1+1}),u_3)$ and $S/(I,u_3)\cong K[V(S_{k_1+1})\cup\{u_3\}\cup\{e_1,e_2,\dots,e_{k_3}\}]/I(S_{k_1+1})$ by Lemma 2.10 and Proposition 3, $\operatorname{depth} S/(I,u_3)=\operatorname{sdepth} S/(I,u_3)=1+k_3-1=k_3$, by using Depth Lemma, Lemma 2.9 and Proposition 2 we have

$$\operatorname{depth} S/I(\mathcal{P}_3)=\operatorname{sdepth} S/I(\mathcal{P}_3)=2.$$

Now assume that $z\geq 5$, consider a short exact sequence of the form

$$0\longrightarrow S/(I:u_z)\longrightarrow S/I\longrightarrow S/(I,u_z)\longrightarrow 0$$

it is easy to see that $(I:u_z)=(x:x\in N(u_z))+I(\mathcal{P}_{z-2})$ and $S/(I:u_z)\cong K[V(\mathcal{P}_{z-2})\cup\{u_z\}]/I(\mathcal{P}_{z-2})$ so by Lemma 2.10 and induction on z, we get

$$\operatorname{depth} S/(I:u_z)=\operatorname{sdepth} S/(I:u_z)=\frac{z-2+1}{2}+1=\frac{z+1}{2}.$$

Since $(I, u_z) = (I(\mathcal{P}_{z-2}), u_z)$ and

$$S/(I, u_z) \cong K[V(\mathcal{P}_{z-2}) \cup \{u_z\} \cup \{e_1, e_2, \dots, e_{k_z}\}]/(I(\mathcal{P}_{z-2}), u_z),$$

therefore by using Lemma 2.10 and induction on z, we get

$$\text{depth} S/(I, u_z) = \text{sdepth} S/(I, u_z) = \frac{z-2+1}{2} + k_z - 1 = \frac{z+1}{2} + k_z - 2.$$

Hence by Depth Lemma we have depth$S/I = \frac{z+1}{2}$ and by Lemma 2.9 sdepth$S/I \geq \frac{z+1}{2}$. Now for the upper bound by Proposition 2 we have sdepth$S/(I) \leq$
sdepth$S/(I : u_z) = \frac{z+1}{2}$ and hence sdepth$(S/I) = \frac{z+1}{2}$. □

Theorem 4.2. *Let $z \geq 2$, $k \geq 3$ and $S := K[V(\mathcal{P}_{z,k})]$. If $I = I(\mathcal{P}_{z,k})$, then*

$$\text{depth}(S/I) = \text{sdepth}(S/I) = \begin{cases} k, & \text{if } z = 2; \\ \lfloor \frac{z}{2} \rfloor (k-2) + z + \sum_{m=1}^{\lceil \frac{z}{2} \rceil - 1} (z - 2m), & \text{if } z \geq 3. \end{cases}$$

Where $\lfloor \alpha \rfloor = \{n \in \mathbb{Z} : n \leq \alpha\}$.

Proof. The proof is done by induction on z. Let $z = 2$. Consider the following short exact sequence

$$0 \longrightarrow S/(I : u_2) \xrightarrow{\cdot u_2} S/I \longrightarrow S/(I, u_2) \longrightarrow 0$$

we have $(I : u_2) = (x : x \in N(u_2))$ and $S/(I : u_2) \cong K[\mathcal{L}(u_1) \cup \{u_2\}]$, where $N(u_2)$ are the neighbours of u_2 and $\mathcal{L}(u_1)$ represent the number of leaves at u_1. Thus by Lemma 2.10, depth$(S/(I : u_2)) = 1 + k - 1 = k$. Also $(I, u_2) = (I(S_k), u_2)$ and $S/(I, u_2) \cong K[V(S_k) \cup \mathcal{L}(u_2)]/I(S_k)$, therefore by Proposition 2.10, depth$(S/(I, u_2)) = 1 + (k + 1 - 1) = k + 1$ thus by Depth Lemma depth$(S/I) = k$. Now by Lemma 2.9 sdepth$(S/I) \geq k$ and by using Proposition 2 and Lemma 3 we have sdepth$(S/I) \leq k$. Thus sdepth$(S/I) = k$. Let $z = 3$. Consider the following short exact sequence

$$0 \longrightarrow S/(I : u_3) \xrightarrow{\cdot u_3} S/I \longrightarrow S/(I, u_3) \longrightarrow 0$$

we have $(I : u_3) = I(S_k) + (x : x \in N(u_3))$ and $S/(I : u_3) \cong [V(\mathcal{S}_k) \cup \mathcal{L}(u_2) \cup \{u_3\}]/I(\mathcal{S}_k)$. Thus by Lemma 2.10 and Proposition,3, depth$(S/(I : u_3)) =$ sdepth$(S/(I : u_3)) = 1 + (k + 1 - 1) + 1 = k + 2$. Further $(I, u_3) = (I(\mathcal{P}_{2,k}), u_3)$ and $S/(I, u_3) \cong K[V(\mathcal{P}_{2,k}) \cup \mathcal{L}(u_3)]/I(\mathcal{P}_{2,k})$. Therefore by Lemma 2.10, and the above case we have depth$(S/(I, u_3)) =$ sdepth$(S/(I, u_3)) = k + (k + 2 - 1) = 2k + 1$. Applying Depth Lemma we get depth$(S/I) = k + 2$. Now by Lemma 2.9 and Proposition 2 we get sdepth$(S/I) = k + 2$. Let $z \geq 4$. Consider the following short exact sequence

$$0 \longrightarrow S/(I : u_z) \xrightarrow{\cdot u_z} S/I \longrightarrow S/(I, u_z) \longrightarrow 0$$

it is easy to see that $(I : u_z) = (x : x \in N(u_z)) + I(\mathcal{P}_{z-2,k})$ and $S/(I : u_z) \cong K[V(\mathcal{P}_{z-2,k}) \cup \mathcal{L}(u_{z-1}) \cup \{u_z\}]/I(\mathcal{P}_{z-2,k})$ also $(I, u_z) = (I(\mathcal{P}_{z-1,k}), u_z)$ and $S/(I, u_z) \cong K[V(\mathcal{P}_{z-1,k}) \cup \mathcal{L}(u_z)]/I(\mathcal{P}_{z-1,k})$.

Thus by using induction on z and Lemma 2.10

$$\begin{aligned}
\operatorname{depth}(S/(I:u_z)) &= \operatorname{depth}K[V(\mathcal{P}_{z-2,k})]/I(\mathcal{P}_{z-2,k}) + |\mathcal{L}(u_{z-1})|+1\\
&= \lfloor\frac{z-2}{2}\rfloor(k-2)+(z-2)+\sum_{m=1}^{\lceil\frac{z-2}{2}\rceil-1}(z-2-2m)+(k+z-3)+1\\
&= \lfloor\frac{z-2}{2}\rfloor(k-2)+z-2+\sum_{m=1}^{\lceil\frac{z-2}{2}\rceil-1}(z-2-2m)+k+z-2\\
&= \lfloor\frac{z}{2}\rfloor(k-2)-(k-2)+\sum_{m=0}^{\lceil\frac{z-2}{2}\rceil-1}(z-2-2m)+k+z-2\\
&= \lfloor\frac{z}{2}\rfloor(k-2)+z+\sum_{m=0}^{\lceil\frac{z-2}{2}\rceil-1}(z-2-2m)
\end{aligned}$$

introducing the transformation $j := m+1$ we get $\operatorname{depth}(S/(I:u_z)) = \lfloor\frac{z}{2}\rfloor(k-2)+z+\sum_{j=1}^{\lceil\frac{z}{2}\rceil-1}(z-2j)$, where j is dummy variable so by replacing j with m we get

$$\operatorname{depth}(S/(I:u_z)) = \lfloor\frac{z}{2}\rfloor(k-2)+z+\sum_{m=1}^{\lceil\frac{z}{2}\rceil-1}(z-2m).$$

Now by considering the inequality $\lceil x+y\rceil \geq \lceil x\rceil + \lceil y\rceil - 1$, we get

$$\begin{aligned}
\operatorname{depth}(S/(I,u_z)) &= \operatorname{depth}K[V(\mathcal{P}_{z-1,k})/I(\mathcal{P}_{z-1,k})) + |\mathcal{L}(u_z)|\\
&= \lfloor\frac{z-1}{2}\rfloor(k-2)+z-1+\sum_{m=1}^{\lceil\frac{z-1}{2}\rceil-1}(z-1-2m)+k+z-2\\
&\geq \lfloor\frac{z-2}{2}\rfloor(k-2)+\sum_{m=0}^{\lceil\frac{z-1+2-2}{2}\rceil-1}(z-1-2m)+k+z-2\\
&\geq \lfloor\frac{z}{2}\rfloor(k-2)-(k-2)+\sum_{m=0}^{\lceil\frac{z}{2}\rceil-2}(z-1-2m)+k+z-2\\
&= \lfloor\frac{z}{2}\rfloor(k-2)+z+\sum_{m=0}^{\lceil\frac{z}{2}\rceil-2}(z-1-2m)\\
&\geq \lfloor\frac{z}{2}\rfloor(k-2)+z+\sum_{m=1}^{\lceil\frac{z}{2}\rceil-1}(z-2m).
\end{aligned}$$

Thus by Depth Lemma

$$\operatorname{depth}(S/I) = \lfloor\frac{z}{2}\rfloor k+\sum_{m=1}^{\lceil\frac{z}{2}\rceil-1}(z-2m)+z-2\lfloor\frac{z}{2}\rfloor.$$

For Stanley depth the result follows by Lemma 2.9 and 2 instead of Depth Lemma. Clearly, one can see that Stanley's inequality holds for these classes of graphs. □

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. Alipour, A. Tehranian, Depth and Stanley Depth of Edge Ideals of Star Graphs, *International Journal of Applied Mathematics and Statistics*, **56** (2017), 63–69.
2. W. Bruns, H. J. Herzog, *Cohen-macaulay rings*, No. 39, Cambridge university press, 1998.
3. M. Cimpoeas, Several inequalities regarding Stanley depth, *Romanian Journal of Math and Computer Science*, **2** (2012), 28–40.
4. A. M. Duval, B. Goeckner, C. J. Klivans, J. L. Martine, A non-partitionable CohenMacaulay simplicial complex, *Adv. Math.*, **299** (2016), 381–395.
5. S. A. S. Fakhari, On the Stanley Depth of Powers of Monomial Ideals, *Mathematics*, **7** (2019), 607.
6. L. Fouli, S. Morey, A lower bound for depths of powers of edge ideals, *J. Algebr. Comb.*, **42** (2015), 829–848.
7. R. Frucht, F. Harary, On the corona of two graphs, *Aeq. Math.*, **4** (1970), 322–325.
8. R. Hammack, W. Imrich, S. Klavar, *Handbook of Product Graphs*, Second Edition, CRC Press, Boca Raton, FL, 2011.
9. J. Herzog, A survey on Stanley depth, In *Monomial ideals, computations and applications*, (2013), 3–45. Springer, Heidelberg.
10. J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, *J. Algebra*, **322** (2009), 3151–3169.
11. Z. Iqbal, M. Ishaq, Depth and Stanley depth of the edge ideals of the powers of paths and cycles, *An. Sti. U. Ovid. Co-Mat*, **27** (2019), 113–135.
12. Z. Iqbal, M. Ishaq, M. A. Binyamin, Depth and Stanley depth of the edge ideals of the strong product of some graphs, *Hacet. J. Math. Stat.*, **50** (2021), 92–109.
13. S. Morey, Depths of powers of the edge ideal of a tree, *Commun. Algebra*, **38** (2010), 4042–4055.
14. R. Okazaki, A lower bound of Stanley depth of monomial ideals, *J. Commut. Algebr.*, **3** (2011), 83–88.
15. M. R. Pournaki, S. A. S. Fakhari, M. Tousi, S. Yassemi, What is Stanley depth? *Not. Am. Math. Soc.*, **56** (2009), 1106–1108.
16. A. Rauf, Depth and Stanley depth of multigraded modules, *Commun. Algebra*, **38** (2010), 773–784.
17. R. P. Stanley, Linear Diophantine equations and local cohomolog, *Invent. Math.*, **68** (1982), 175–193.
18. V. Swaminathan, P. Jeyanthi, Super edge-magic strength of fire crackers, banana trees and unicyclic graphs, *Discrete math.*, **306** (2006), 1624–1636.
19. R. H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, Vol. 238, 2011.

20. D. B. West, *Introduction to graph theory*, Upper Saddle River, NJ: Prentice hall, Vol. 2, 1996.