Research article

Certain properties of multivalent analytic functions defined by $q$-difference operator involving the Janowski function

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Abstract: A new subclass of multivalent analytic functions is defined by means of $q$-difference operator and Janowski function. Some properties of functions in this new subclass such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, partial sums and closure theorems are studied.

Keywords: $q$-difference operator; Janowski function; multivalent analytic function; distortion theorem; radii of starlikeness and convexity; partial sum; closure theorem

Mathematics Subject Classification: 05A30, 11B65, 30C45, 47B38

1. Introduction

Quantum calculus is ordinary classical calculus without the concept of limits. In recent years, the area of $q$-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of $q$-calculus was initiated by Jackson [9, 10]. He was the first to develop $q$-derivative and $q$-integral in a systematic way. Later, geometrical interpretation of $q$-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and $q$-analysis. Aral [5] and Anastassiou and Gal [2, 3] generalized some complex operators which are known as $q$-Picard and $q$-Gauss-Weierstrass singular integral operators. Moreover, Srivastava et al. published a set of articles [13, 15–19] in which they concentrated upon the classes of $q$-starlike functions related with the Janowski functions [11] from several different aspects. Additionally, a recently-published survey-cum-expository review article by Srivastava [20] is potentially useful for researchers and scholars working on these topics. In this survey-cum-expository review article [20], the mathematical explanation and applications of the fractional $q$-calculus and the fractional $q$-derivative operators in Geometric Function Theory was systematically investigated. Refer to further $q$-theory can be found in [1, 6–8, 12, 14].
Let $A_p$ denote the class of multivalent analytic functions $f(z)$ given by Taylor-Maclaurin’s series
\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \]
in the open unit disk $D = \{z : |z| < 1\}$. For $p = 1$, we write $A := A_1$.

A function $f(z) \in A_p$ is said to be multivalent starlike function of order $\sigma$ and is written as $f(z) \in S_p^\sigma(\sigma)$, if it satisfies
\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \sigma \quad (0 \leq \sigma < p) \]
for all $z \in D$ (see [4]).

A function $f(z) \in A_p$ is known as multivalent convex function of order $\sigma$ and is denoted by $f(z) \in C_p(\sigma)$, if it satisfies
\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \sigma \quad (0 \leq \sigma < p) \]
for all $z \in D$.

Given two functions $f(z)$ and $g(z)$, which are analytic in $D$, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g(z) \prec f(z)$ ($z \in D$), if there exists a Schwarz function $w(z)$ such that $g(z) = f(w(z))$ ($z \in D$). In particular, if $f(z)$ is univalent in $D$, then we have the following equivalence:
\[ g(z) \prec f(z) \quad (z \in D) \iff g(0) = f(0) \quad \text{and} \quad g(D) \subset f(D). \]

**Definition 1.** A function $h(z)$ is said to be in the class $P[A, B]$, if it is analytic in $D$ with $h(0) = 1$ and
\[ h(z) < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1), \]
equivalently we can write
\[ \left| \frac{h(z) - 1}{A - Bh(z)} \right| < 1. \]

**Definition 2.** Let $q \in (0, 1)$ and define the $q$-number $[\lambda]_q$ by
\[ [\lambda]_q = \begin{cases} 1-q^\lambda, & (\lambda \in C), \\ \frac{1-q}{1-q^\lambda}, & \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} \quad (\lambda = n \in \mathbb{N}). \end{cases} \]

Particularly, when $\lambda = 0$, we have $[0]_q = 0$.

**Definition 3** [9, 10]. Let $q \in (0, 1)$. The $q$-difference operator $\partial_q$ of a function $f(z)$ is defined by
\[ \partial_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & (z \neq 0) \\ f'(0), & (z = 0), \end{cases} \]
provided that $f'(0)$ exists.

From Definition 3, we can see that
\[ \lim_{q \to 1} \partial_q f(z) = \lim_{q \to 1} \frac{f(qz) - f(z)}{(q-1)z} = f'(z) \]
for a differentiable function $f(z)$ in a given subset of $C$. Also, for $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}$, one can observe that (see [21])

$$\partial_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty} [p+n]_q a_{p+n} z^{p+n-1} \quad (z \neq 0),$$

where $[p]_q = \frac{1-q^p}{1-q} = 1 + q + q^2 + \cdots + q^{p-1}$.

In $q$-calculus sense, we now define the following subclass of $A_p$ associated with the $q$-difference operator $\partial_q$.

**Definition 4.** A function $f(z) \in A_p$ ($p \geq 2$) is said to belong to the class $T_{p,q}(\alpha, A, B)$, if it satisfies

$$\frac{1}{1-\alpha} \left( \frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} \right) < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad 0 \leq \alpha < 1, \quad q \in (0, 1),$$

or equivalently

$$\left| \frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} - (1 - \alpha) \right| < A - B \left( \frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} \right). \quad (1.1)$$

**Remark.** For $\alpha = 0$, $A = 1$, $B = -1$ and $q \to 1^{-}$, we can see from Definition 4 that $T_{p,q}(\alpha, A, B)$ reduces to the subclass of $p$-valently close-to-convex functions.

In this paper we shall study some geometric properties of functions in $T_{p,q}(\alpha, A, B)$ such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, partial sums and closure theorems.

2. Main results

**Theorem 1.** Let $p \geq 2$ and $\frac{[p-1]_q}{[p]_q} \leq \alpha < 1$. Also let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in A_p$. Then $f(z) \in T_{p,q}(\alpha, A, B)$ if and only if

$$\sum_{n=1}^{\infty} \left( (1 + B)[p+n]_q (\alpha [p+n-1]_q - [p-1]_q) |a_{p+n}| \right) \leq (1 - \alpha) (A - B)[p]_q [p-1]_q. \quad (2.1)$$

**Proof.** Assuming that the inequality (2.1) holds true, then we only need to show the inequality (1.1). Now we have

$$\left| \frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} - (1 - \alpha) \right| \leq \frac{\sum_{n=1}^{\infty} [p+n]_q (\alpha [p+n-1]_q - [p-1]_q) |a_{p+n}| z^{p+n-1}}{(1 - \alpha) (A - B)[p]_q [p-1]_q z^{p-1} - \sum_{n=1}^{\infty} B[p+n]_q (\alpha [p+n-1]_q - [p-1]_q) |a_{p+n}| z^{p+n-1}}.$$
Let Corollary 1.

Thus the proof of Theorem 1 is completed.

The results are sharp for the function

Proof.

Let

Since

Theorem 2.

for

which shows that $f(z) \in T_{p,q}(\alpha, A, B)$.

Conversely, let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$. Then from (1.1), we have

The inequality (2.2) is true for all $z \in D$. Now we choose $z = \text{Re}z \to 1^-$ and obtain the inequality (2.1). Thus the proof of Theorem 1 is completed.

Corollary 1. Let $\frac{p-1}{p} < \alpha < 1$ and $-1 < B < A \leq 1$. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$, then

The results are sharp for the function $f(z)$ defined by

Theorem 2. Let $\frac{p-1}{p} < \alpha < 1$ and $-1 < B < A \leq 1$. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$, then for $|z| = r$, we have

where

The bounds are sharp for the function

Proof. Let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n}$. By applying the triangle inequality, we have

Since $|z| = r < 1$, we can see that $r^{p+n} \leq r^{p+1}$. Thus we have

\begin{align*}
|f(z)| &\leq r^p + r^{p+1} \sum_{n=1}^{\infty} |a_{p+n}|.
\end{align*}
and
\[ |f(z)| \geq r^p - r^{p+1} \sum_{n=1}^{\infty} |a_{p+n}|. \] (2.4)

Considering \( f(z) \in T_{p,q}(\alpha, A, B) \), we know from Theorem 1 that
\[ \sum_{n=1}^{\infty} \left( (1 + B)[p + n]_q \left( \alpha[p + n - 1]_q - [p - 1]_q \right) \right) |a_{p+n}| \leq (1 - \alpha)(A - B)[p]_q[p - 1]_q. \]

As we know that \( \{(1 + B)[p + n]_q \left( \alpha[p + n - 1]_q - [p - 1]_q \right) \} \) is an increasing sequence with respect to \( n \ (n \geq 1) \), so
\[ (1 + B)[p + n]_q \left( \alpha[p]_q - [p - 1]_q \right) \sum_{n=1}^{\infty} |a_{p+n}| \leq (1 + B)[p + n]_q \left( \alpha[p + n - 1]_q - [p - 1]_q \right) |a_{p+n}|. \]

Hence by transitivity we obtain
\[ (1 + B)[p + n]_q \left( \alpha[p]_q - [p - 1]_q \right) \sum_{n=1}^{\infty} |a_{p+n}| \leq (1 - \alpha)(A - B)[p]_q[p - 1]_q, \]

which implies that
\[ \sum_{n=1}^{\infty} |a_{p+n}| \leq \frac{(1 - \alpha)(A - B)[p]_q[p - 1]_q}{(1 + B)(\alpha[p]_q - [p - 1]_q)}. \] (2.5)

Substituting inequality (2.5) into inequalities (2.3) and (2.4), we get the required results. The proof of Theorem 2 is completed.

**Theorem 3.** Let \( \frac{[p]_q}{[p]_q} < \alpha < 1 \) and \(-1 < B < A \leq 1\). If \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B) \), then for \( |z| = r \), we have
\[ [p]_q r^{p-1} - \tau_2 r^p \leq |\partial_q f(z)| \leq [p]_q r^{p-1} + \tau_2 r^p, \]
where
\[ \tau_2 = \frac{(1 - \alpha)(A - B)[p]_q[p - 1]_q}{(1 + B)(\alpha[p]_q - [p - 1]_q)}. \]

The results are sharp for the function
\[ f(z) = z^p - \frac{(1 - \alpha)(A - B)[p]_q[p - 1]_q}{(1 + B)(\alpha[p]_q - [p - 1]_q)} z^{p+1}. \]

**Proof.** Let \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \). From Definition 3, we can write
\[ \partial_q f(z) = [p]_q z^{p-1} - \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| z^{p+n-1}. \]

By applying the triangle inequality, we get
\[ |\partial_q f(z)| = \left| [p]_q z^{p-1} - \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| z^{p+n-1} \right| \leq [p]_q |z|^{p-1} + \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| |z|^{p+n-1}. \]
Further, we have that
\[
|\partial_q f(z)| \leq [p]_q r^{p-1} + r^p \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}|.
\]  
and
\[
|\partial_q f(z)| \geq [p]_q r^{p-1} - r^p \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}|.
\]  
Since \( f(z) \in T_{p,q}(\alpha, A, B) \), we know from Theorem 1 that
\[
\sum_{n=1}^{\infty} \left((1 + B)(\alpha[p + n - 1] - [p - 1]_q)\right) [p + n]_q |a_{p+n}| \leq (1 - \alpha)(A - B)[p]_q[p - 1]_q.
\]
As we know that \((1 + B)(\alpha[p + n - 1] - [p - 1]_q)\) is an increasing sequence in regard to \( n (n \geq 1) \), so
\[
\sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| \leq \sum_{n=1}^{\infty} \left((1 + B)(\alpha[p + n - 1] - [p - 1]_q)\right) [p + n]_q |a_{p+n}|.
\]
Thus by transitivity we have
\[
\sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| \leq \frac{(1 - \alpha)(A - B)[p]_q[p - 1]_q}{(1 + B)(\alpha[p]_q - [p - 1]_q)}.
\]  
By putting (2.8) in (2.6) and (2.7), we obtain the required results. Now Theorem 3 is proved.

**Theorem 4.** Let \( \frac{[p-1]_q}{[p]_q} < \alpha < 1, -1 < B < A \leq 1 \) and \( 0 \leq \sigma < p \). If
\[
f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B),
\]
then for \( 0 < |z| < r_1 \), \( f(z) \) is \( p \)-valently starlike function of order \( \sigma \), where
\[
|z| < r_1 = \min \left\{ \inf_{n \geq 1} \left( \frac{(p - \sigma)(1 + B)(\alpha+p+n-1)q(\alpha+p+n-1)q-[p-1]_q)}{(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q} \right)^{1/n}, 1 \right\}.
\]

**Proof.** Let \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B) \). In order to prove \( f(z) \in S^*_p(\sigma) \), we only need to show
\[
\frac{zf'(z)}{f(z)} - \sigma < \frac{1 + z}{1 - z}, \quad 0 \leq \sigma < p.
\]

The subordination above is equivalent to
\[
\frac{zf'(z) - pf(z)}{zf'(z) - pf(z) - 2zf'(z)} < 1. \quad \text{After some calculations and simplifications, we obtain}
\]
\[
\sum_{n=1}^{\infty} \frac{n + p - \sigma}{p - \sigma} |a_{p+n}|z^n < 1.
\]  

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Volume 6, Issue 8, 8497–8508.
From the inequality (2.1), we can obviously know that
\[
\sum_{n=1}^{\infty} \frac{(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(1 - \alpha)(A - B)[p]_q[p - 1]_q}|a_{p+n}| < 1.
\]

Inequality (2.9) wants to be true if it satisfies the following inequality
\[
\sum_{n=1}^{\infty} \frac{n + p - \sigma}{p - \sigma}|a_{p+n}||z|^n < \sum_{n=1}^{\infty} \frac{(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(1 - \alpha)(A - B)[p]_q[p - 1]_q}|a_{p+n}|,
\]
which implies that
\[
|z|^p < \frac{(p - \sigma)(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(n + p - \sigma)(1 - \alpha)(A - B)[p]_q[p - 1]_q}
\]
or
\[
|z| < \left( \frac{(p - \sigma)(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(n + p - \sigma)(1 - \alpha)(A - B)[p]_q[p - 1]_q} \right)^{\frac{1}{p}}.
\]

Let \( r_1 = \min \left\{ \inf_{n \geq 1} \left( \frac{(p - \sigma)(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(n + p - \sigma)(1 - \alpha)(A - B)[p]_q[p - 1]_q} \right)^{\frac{1}{p}}, 1 \right\} \), then we get the required result. The proof of Theorem 4 is completed.

**Theorem 5.** Let \( \frac{p-1}{\alpha} < \alpha < 1 \), \(-1 < B < A \leq 1 \) and \( 0 \leq \sigma < p \). If
\[
f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B),
\]
then for \( 0 < |z| < r_2 \), \( f(z) \) is \( p \)-valently convex function of order \( \sigma \), where
\[
|z| < r_2 = \min \left\{ \inf_{n \geq 1} \left( \frac{p(p - \sigma)(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(n + p - \sigma)(1 - \alpha)(A - B)[p]_q[p - 1]_q} \right)^{\frac{1}{p}}, 1 \right\}.
\]

**Proof.** Let \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B) \). To prove \( f(z) \in C_p(\sigma) \), we must show that
\[
\frac{1 + \frac{zf''(z)}{f'(z)}}{p - \sigma} < \frac{1 + z}{1 - z}, \quad 0 \leq \sigma < p.
\]
This is equivalent to the inequality
\[
\left| \frac{zf''(z)(p-1)+f'(z)}{zf''(z)+p-f'(z)} \right| < 1.
\]
After some calculations and simplifications, we have
\[
\sum_{n=1}^{\infty} \frac{(p + n)(n + p - \sigma)}{p(p - \sigma)}|a_{p+n}||z|^n < 1. \tag{2.10}
\]
From the inequality (2.1), we can easily obtain that
\[
\sum_{n=1}^{\infty} \frac{(1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)}{(1 - \alpha)(A - B)[p]_q[p - 1]_q}|a_{p+n}| < 1.
\]
For inequality (2.10) to be true, it will be enough if
\[
\sum_{n=1}^{\infty} \frac{(p+n)(n+p-\sigma)}{p(p-\sigma)}|a_{p+n}|z^n < \sum_{n=1}^{\infty} \frac{(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}|a_{p+n}|,
\]
which implies that
\[
|z|^p < \frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}
\]
or
\[
|z| < \left( \frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q} \right)^{\frac{1}{p}}.
\]
Let \( r_2 = \min \left\{ \inf_{n \geq 1} \left( \frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q} \right)^{\frac{1}{p}}, 1 \right\} \), then we obtain the required result. Now Theorem 5 is proved.

Next, we will study the ratio of a function \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \) to its sequence of partial sums \( f_k(z) = z^p - \sum_{n=1}^{k} |a_{p+n}|z^{p+n} \) for all \( z \in D \).

**Theorem 6.** Let \( \frac{(1+B)[p+k+1]_q[A-B][p]_q[p-1]_q}{(1+B)[p+1]_q[A-B][p]_q[p-1]_q} < \alpha < 1, -1 < B < A \leq 1 \). If \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B) \), then we have
\[
\text{Re} \left( \frac{f(z)}{f_k(z)} \right) \geq 1 - \frac{1}{\varphi_{k+1}} \quad (z \in D)
\]
and
\[
\text{Re} \left( \frac{f_k(z)}{f(z)} \right) \geq \frac{\varphi_{k+1}}{1 + \varphi_{k+1}} \quad (z \in D),
\]
where
\[
\varphi_{k+1} = \frac{(1+B)[p+k+1]_q[A-B][p]_q[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}.
\]

**Proof.** In order to prove (2.11), we set
\[
\varphi_{k+1} \left[ \frac{f(z)}{f_k(z)} - \left( 1 - \frac{1}{\varphi_{k+1}} \right) \right] = \frac{1 - \sum_{n=1}^{k} |a_{p+n}|z^n - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|z^n}{1 - \sum_{n=1}^{k} |a_{p+n}|z^n} = 1 + w(z).
\]
After some simplifications, we have
\[
w(z) = \frac{-\varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|z^n}{2 - 2 \sum_{n=1}^{k} |a_{p+n}|z^n - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|z^n}
\]
and
\[
|w(z)| \leq \frac{\varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|}{2 - 2 \sum_{n=1}^{k} |a_{p+n}| - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|}.
\]
Now we can see that \( |w(z)| < 1 \), if and only if
\[
\sum_{n=1}^{k} |a_{p+n}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| \leq 1.
\]
From (2.1) we have \( \sum_{n=1}^{\infty} \varphi_{n} |a_{p+n}| \leq 1 \). It is not difficult to see that \( \varphi_{n} \) is an increasing sequence with respect to \( n \) and that \( \varphi_{n} \geq 1 \) (\( n = 1, 2, \cdots \)). Therefore, we get

\[
\sum_{n=1}^{k} |a_{p+n}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| \leq \sum_{n=1}^{k} \varphi_{n} |a_{p+n}| + \sum_{n=k+1}^{\infty} \varphi_{n} |a_{p+n}| = \sum_{n=1}^{\infty} \varphi_{n} |a_{p+n}| \leq 1.
\]

Thus, the inequality (2.14) is true. This proves (2.11).

Next, in order to prove the inequality (2.12), we consider

\[
(1 + \varphi_{k+1}) \left[ \frac{f_{k}(z)}{f(z)} - \frac{\varphi_{k+1}}{1 + \varphi_{k+1}} \right] = \frac{1 - \sum_{n=1}^{k} |a_{p+n}|z^{n} + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|z^{n}}{1 - \sum_{n=1}^{\infty} |a_{p+n}|z^{n}} = \frac{1 + w(z)}{1 - w(z)}.
\]

After some simplifications, we can find that

\[
w(z) = \frac{(1 + \varphi_{k+1}) \sum_{n=k+1}^{\infty} |a_{p+n}|z^{n}}{2 - 2 \sum_{n=1}^{k} |a_{p+n}|z^{n} + (\varphi_{k+1} - 1) \sum_{n=k+1}^{\infty} |a_{p+n}|z^{n}}
\]

and

\[
|w(z)| \leq \frac{(1 + \varphi_{k+1}) \sum_{n=k+1}^{\infty} |a_{p+n}|}{2 - 2 \sum_{n=1}^{k} |a_{p+n}| - (\varphi_{k+1} - 1) \sum_{n=k+1}^{\infty} |a_{p+n}|}.
\]

Now we can see that \( |w(z)| < 1 \) if it satisfies

\[
\sum_{n=1}^{k} |a_{p+n}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| \leq 1.
\]

The remaining part of the proof is much akin to that of (2.11) and hence we omit it. The proof of the Theorem is completed.

**Theorem 7.** Let \( \frac{p-1}{[p]_{q}} \leq \alpha < 1 \). If \( f_{j}(z) = z^{p} - \sum_{n=1}^{\infty} |a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B) \) (\( j = 1, 2 \)), then for \( 0 \leq \lambda \leq 1 \), the function \( H(z) = \lambda f_{1}(z) + (1 - \lambda) f_{2}(z) \in T_{p,q}(\alpha, A, B) \).

**Proof.** For \( 0 \leq \lambda \leq 1 \), the function \( H(z) \) can be written as

\[
H(z) = \lambda f_{1}(z) + (1 - \lambda) f_{2}(z) = z^{p} - \sum_{n=1}^{\infty} (\lambda |a_{p+n,1}| + (1 - \lambda) |a_{p+n,2}|)z^{p+n}.
\]

For functions \( f_{1}(z), f_{2}(z) \in T_{p,q}(\alpha, A, B) \), by Theorem 1, we have

\[
\sum_{n=1}^{\infty} ((1 + B)[p + n]_{q}(\alpha[p + n - 1]_{q} - [p - 1]_{q})|\lambda|a_{p+n,1}| + (1 - \lambda)|a_{p+n,2}|)
\]

\[
= \lambda \sum_{n=1}^{\infty} ((1 + B)[p + n]_{q}(\alpha[p + n - 1]_{q} - [p - 1]_{q}))|a_{p+n,1}|
\]

\[
+ (1 - \lambda) \sum_{n=1}^{\infty} ((1 + B)[p + n]_{q}(\alpha[p + n - 1]_{q} - [p - 1]_{q}))|a_{p+n,2}|
\]

\[
\leq \lambda(1 - \alpha)(A - B)|p|_{q}|p - 1|_{q} + (1 - \lambda)(1 - \alpha)(A - B)|p|_{q}|p - 1|_{q}
\]

\[
= (1 - \alpha)(A - B)|p|_{q}|p - 1|_{q}.
\]
which shows that \( H(z) \in T_{p,q}(\alpha, A, B) \).

**Corollary 2.** Let \( \frac{p-1}{|p|} \leq \alpha < 1 \). If \( f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B) \) \((j = 1, 2, \cdots, t)\), then the function \( F(z) = \sum_{j=1}^{t} \lambda_j f_j(z) \in T_{p,q}(\alpha, A, B) \), where \( \lambda_j \geq 0 \) and \( \sum_{j=1}^{t} \lambda_j = 1 \).

**Theorem 8.** Let \( \frac{p-1}{|p|} \leq \alpha < 1 \). If \( f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B) \) \((j = 1, 2)\), then for \(-1 \leq m \leq 1\), we have

\[
Q_m(z) = \frac{(1 - m)f_1(z) + (1 + m)f_2(z)}{2} \in T_{p,q}(\alpha, A, B).
\]

**Proof.** For \(-1 \leq m \leq 1\), the function \( Q_m(z) \) can be written as

\[
Q_m(z) = \frac{(1 - m)f_1(z) + (1 + m)f_2(z)}{2} = z^p - \sum_{n=1}^{\infty} \left( \frac{1 - m}{2}|a_{p+n,1}| + \frac{1 + m}{2}|a_{p+n,2}| \right) z^{p+n}.
\]

In view of \( f_1(z), f_2(z) \in T_{p,q}(\alpha, A, B) \), by applying Theorem 1, we get

\[
\begin{align*}
&\sum_{n=1}^{\infty} ((1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q)) \left| \frac{1 - m}{2}|a_{p+n,1}| + \frac{1 + m}{2}|a_{p+n,2}| \right|
\leq \frac{1 - m}{2} \sum_{n=1}^{\infty} ((1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q))|a_{p+n,1}|
\quad + \frac{1 + m}{2} \sum_{n=1}^{\infty} ((1 + B)[p + n]_q(\alpha[p + n - 1]_q - [p - 1]_q))|a_{p+n,2}|
\leq \frac{1 - m}{2}(1 - \alpha)(A - B)[p]_q[p - 1]_q + \frac{1 + m}{2}(1 - \alpha)(A - B)[p]_q[p - 1]_q
= (1 - \alpha)(A - B)[p]_q[p - 1]_q,
\end{align*}
\]

which shows that \( Q_m(z) \in T_{p,q}(\alpha, A, B) \).

3. Conclusions

Our objective is to generalize some classical interesting results in geometric function theory from the ordinary analysis to \( q \)-analysis. By using the \( q \)-difference operator, a new subclass \( T_{p,q}(\alpha, A, B) \) of multivalent analytic functions is introduced. Some geometric properties of functions in \( T_{p,q}(\alpha, A, B) \) such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radius of starlikeness and convexity, partial sums and closure theorems are given. In particular, if we let \( \alpha = 0 \), \( A = 1 \), \( B = -1 \) and \( q \to 1^- \), then \( T_{p,q}(\alpha, A, B) \) reduces to the subclass of \( p \)-valently close-to-convex functions.

**Acknowledgments**

The authors would like to express sincere thanks to the referees for careful reading and suggestions which helped us to improve the paper. This work was supported by National Natural Science Foundation of China (Grant No.11571299).
Conflict of interest

The authors declare no conflict of interest.

References


