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Research article

Certain properties of multivalent analytic functions defined by q-difference operator involving the Janowski function

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Abstract: A new subclass of multivalent analytic functions is defined by means of q-difference operator and Janowski function. Some properties of functions in this new subclass such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity , partial sums and closure theorems are studied.

Keywords: q-difference operator; Janowski function; multivalent analytic function; distortion theorem; radii of starlikeness and convexity; partial sum; closure theorem
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1. Introduction

Quantum calculus is ordinary classical calculus without the concept of limits. In recent years, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [9, 10]. He was the first to develop q-derivative and q-integral in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. Aral [5] and Anastassiou and Gal [2, 3] generalized some complex operators which are known as q-Picard and q-Gauss-Weierstrass singular integral operators. Moreover, Srivastava et al. published a set of articles [13, 15–19] in which they concentrated upon the classes of q-starlike functions related with the Janowski functions [11] from several different aspects. Additionally, a recently-published survey-cum-expository review article by Srivastava [20] is potentially useful for researchers and scholars working on these topics. In this survey-cum-expository review article [20], the mathematical explanation and applications of the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory was systematically investigated. Refer to further q-theory can be found in [1, 6–8, 12, 14].

Let A_p denote the class of multivalent analytic functions $f(z)$ given by Taylor-Maclaurin's series

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, 3, \cdots\})$$

in the open unit disk $D = \{z : |z| < 1\}$. For $p = 1$, we write $A := A_1$.

A function $f(z) \in A_p$ is said to be multivalent starlike function of order σ and is written as $f(z) \in S_p^*(\sigma)$, if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma \quad (0 \leq \sigma < p)$$

for all $z \in D$ (see [4]).

A function $f(z) \in A_p$ is known as multivalent convex function of order σ and is denoted by $f(z) \in C_p(\sigma)$, if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \sigma \quad (0 \leq \sigma < p)$$

for all $z \in D$.

Given two functions $f(z)$ and $g(z)$, which are analytic in D, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g(z) \prec f(z)$ $(z \in D)$, if there exists a Schwarz function $w(z)$ such that $g(z) = f(w(z))$ $(z \in D)$. In particular, if $f(z)$ is univalent in D, then we have the following equivalence:

$$g(z) \prec f(z) \quad (z \in D) \Longleftrightarrow g(0) = f(0) \quad \text{and} \quad g(D) \subset f(D).$$

Definition 1. A function $h(z)$ is said to be in the class $P[A, B]$, if it is analytic in D with $h(0) = 1$ and

$$h(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1),$$

equivalently we can write

$$\left|\frac{h(z) - 1}{A - Bh(z)}\right| < 1.$$

Definition 2. Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in C), \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in N). \end{cases}$$

Particularly, when $\lambda = 0$, we have $[0]_q = 0$.

Definition 3 *[9, 10]***.** Let $q \in (0, 1)$. The q-difference operator ∂_q of a function $f(z)$ is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

provided that $f'(0)$ exists.

From Definition 3, we can see that

$$\lim_{q \to 1^-} \partial_q f(z) = \lim_{q \to 1^-} \frac{f(qz) - f(z)}{(q - 1)z} = f'(z)$$

for a differentiable function $f(z)$ in a given subset of C. Also, for $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, one can observe that (see [21])

$$\partial_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty} [p+n]_q a_{p+n} z^{p+n-1} \quad (z \neq 0),$$

where $[p]_q = \frac{1-q^p}{1-q} = 1 + q + q^2 + \cdots + q^{p-1}$.

In q-calculus sense, we now define the following subclass of A_p associated with the q-difference operator ∂_q.

Definition 4. A function $f(z) \in A_p$ $(p \geq 2)$ is said to belong to the class $T_{p,q}(\alpha, A, B)$, if it satisfies

$$\frac{1}{1-\alpha}\left(\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}}\right) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \quad 0 \leq \alpha < 1, \quad q \in (0,1),$$

or equivalently

$$\left| \frac{\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}}\right)} \right| < 1. \tag{1.1}$$

Remark. For $\alpha = 0$, $A = 1$, $B = -1$ and $q \to 1^-$, we can see from Definition 4 that $T_{p,q}(\alpha, A, B)$ reduces to the subclass of p-valently close-to-convex functions.

In this paper we shall study some geometric properties of functions in $T_{p,q}(\alpha, A, B)$ such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, partial sums and closure theorems.

2. Main results

Theorem 1. Let $p \geq 2$ and $\frac{[p-1]_q}{[p]_q} \leq \alpha < 1$. Also let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in A_p$. Then $f(z) \in T_{p,q}(\alpha, A, B)$ if and only if

$$\sum_{n=1}^{\infty} \Big((1+B)[p+n]_q \big(\alpha[p+n-1]_q - [p-1]_q\big)\Big) |a_{p+n}| \leq (1-\alpha)(A-B)[p]_q [p-1]_q. \tag{2.1}$$

Proof. Assuming that the inequality (2.1) holds true, then we only need to show the inequality (1.1). Now we have

$$\begin{aligned}
&\left| \frac{\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q [p-1]_q z^{p-2}}\right)} \right| \\
&= \left| \frac{[p-1]_q \partial_q f(z) - \alpha z \partial_q^2 f(z) - (1-\alpha)[p]_q [p-1]_q z^{p-1}}{(1-\alpha)A[p]_q [p-1]_q z^{p-1} - B([p-1]_q \partial_q f(z) - \alpha z \partial_q^2 f(z))} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty} [p+n]_q (\alpha[p+n-1]_q - [p-1]_q) |a_{p+n}| z^{p+n-1}}{(1-\alpha)(A-B)[p]_q [p-1]_q z^{p-1} - \sum_{n=1}^{\infty} B[p+n]_q (\alpha[p+n-1]_q - [p-1]_q) |a_{p+n}| z^{p+n-1}} \right|
\end{aligned}$$

$$= \left| \frac{\sum_{n=1}^{\infty}[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)|a_{p+n}|z^n}{(1-\alpha)(A-B)[p]_q[p-1]_q - \sum_{n=1}^{\infty} B[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)|a_{p+n}|z^n} \right| < 1,$$

which shows that $f(z) \in T_{p,q}(\alpha, A, B)$.

Conversely, let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$. Then from (1.1), we have

$$\begin{aligned} &\left| \frac{\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q[p-1]_q z^{p-2}} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{\partial_q f(z)}{[p]_q z^{p-1}} - \alpha \frac{\partial_q^2 f(z)}{[p]_q[p-1]_q z^{p-2}}\right)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty}[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)|a_{p+n}|z^n}{(1-\alpha)(A-B)[p]_q[p-1]_q - \sum_{n=1}^{\infty} B[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)|a_{p+n}|z^n} \right| \\ &< 1. \end{aligned} \tag{2.2}$$

The inequality (2.2) is true for all $z \in D$. Now we choose $z = \text{Re}z \to 1^-$ and obtain the inequality (2.1). Thus the proof of Theorem 1 is completed

Corollary 1. Let $\frac{[p-1]_q}{[p]_q} < \alpha < 1$ and $-1 < B < A \leq 1$. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$, then

$$|a_{p+n}| \leq \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)} \quad (n = 1, 2, \cdots).$$

The results are sharp for the function $f(z)$ defined by

$$f(z) = z^p - \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)} z^{p+n} \quad (n = 1, 2, \cdots).$$

Theorem 2. Let $\frac{[p-1]_q}{[p]_q} < \alpha < 1$ and $-1 < B < A \leq 1$. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$, then for $|z| = r$, we have

$$r^p - \tau_1 r^{p+1} \leq |f(z)| \leq r^p + \tau_1 r^{p+1}.$$

where

$$\tau_1 = \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+1]_q(\alpha[p]_q - [p-1]_q)}.$$

The bounds are sharp for the function

$$f(z) = z^p - \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+1]_q(\alpha[p]_q - [p-1]_q)} z^{p+1}.$$

Proof. Let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n}$. By applying the triangle inequality, we have

$$|f(z)| = \left| z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \right| \leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}||z|^{p+n}.$$

Since $|z| = r < 1$, we can see that $r^{p+n} \leq r^{p+1}$. Thus we have

$$|f(z)| \leq r^p + r^{p+1} \sum_{n=1}^{\infty} |a_{p+n}|. \tag{2.3}$$

and

$$|f(z)| \geq r^p - r^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|. \tag{2.4}$$

Considering $f(z) \in T_{p,q}(\alpha, A, B)$, we know from Theorem 1 that

$$\sum_{n=1}^{\infty}\Big((1+B)[p+n]_q\Big(\alpha[p+n-1]_q-[p-1]_q\Big)\Big)|a_{p+n}| \leq (1-\alpha)(A-B)[p]_q[p-1]_q.$$

As we know that $\{(1+B)[p+n]_q\Big(\alpha[p+n-1]_q-[p-1]_q\Big)\}$ is an increasing sequence with respect to n $(n \geq 1)$, so

$$\Big((1+B)[p+n]_q\Big(\alpha[p]_q-[p-1]_q\Big)\Big)\sum_{n=1}^{\infty}|a_{p+n}| \leq \sum_{n=1}^{\infty}\Big((1+B)[p+n]_q\Big(\alpha[p+n-1]_q-[p-1]_q\Big)\Big)|a_{p+n}|.$$

Hence by transitivity we obtain

$$\Big((1+B)[p+n]_q\Big(\alpha[p]_q-[p-1]_q\Big)\Big)\sum_{n=1}^{\infty}|a_{p+n}| \leq (1-\alpha)(A-B)[p]_q[p-1]_q,$$

which implies that

$$\sum_{n=1}^{\infty}|a_{p+n}| \leq \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+1]_q(\alpha[p]_q-[p-1]_q)}. \tag{2.5}$$

Substituting inequality (2.5) into inequalities (2.3) and (2.4), we get the required results. The proof of Theorem 2 is completed.

Theorem 3. Let $\frac{[p-1]_q}{[p]_q} < \alpha < 1$ and $-1 < B < A \leq 1$. If $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$, then for $|z| = r$, we have

$$[p]_q r^{p-1} - \tau_2 r^p \leq |\partial_q f(z)| \leq [p]_q r^{p-1} + \tau_2 r^p,$$

where

$$\tau_2 = \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)(\alpha[p]_q-[p-1]_q)}.$$

The results are sharp for the function

$$f(z) = z^p - \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)[p+1]_q(\alpha[p]_q-[p-1]_q)}z^{p+1}.$$

Proof. Let $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$. From Definition 3, we can write

$$\partial_q f(z) = [p]_q z^{p-1} - \sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|z^{p+n-1}.$$

By applying the triangle inequality, we get

$$|\partial_q f(z)| = \left|[p]_q z^{p-1} - \sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|z^{p+n-1}\right| \leq [p]_q|z|^{p-1} + \sum_{n=1}^{\infty}[p+n]_q|a_{p+n}||z|^{p+n-1}.$$

Further, we have that

$$|\partial_q f(z)| \leq [p]_q r^{p-1} + r^p \sum_{n=1}^{\infty} [p+n]_q |a_{p+n}|. \tag{2.6}$$

and

$$|\partial_q f(z)| \geq [p]_q r^{p-1} - r^p \sum_{n=1}^{\infty} [p+n]_q |a_{p+n}|. \tag{2.7}$$

Since $f(z) \in T_{p,q}(\alpha, A, B)$, we know from Theorem 1 that

$$\sum_{n=1}^{\infty} \left((1+B)\left(\alpha[p+n-1]_q - [p-1]_q\right)\right)[p+n]_q |a_{p+n}| \leq (1-\alpha)(A-B)[p]_q[p-1]_q.$$

As we know that $\{(1+B)\left(\alpha[p+n-1]_q - [p-1]_q\right)\}$ is an increasing sequence in regard to n $(n \geq 1)$, so

$$\left((1+B)\left(\alpha[p]_q - [p-1]_q\right)\right)\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}| \leq \sum_{n=1}^{\infty}\left((1+B)\left(\alpha[p+n-1]_q - [p-1]_q\right)\right)[p+n]_q|a_{p+n}|.$$

Thus by transitivity we have

$$\left((1+B)\left(\alpha[p]_q - [p-1]_q\right)\right)\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}| \leq (1-\alpha)(A-B)[p]_q[p-1]_q,$$

which implies that

$$\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}| \leq \frac{(1-\alpha)(A-B)[p]_q[p-1]_q}{(1+B)(\alpha[p]_q - [p-1]_q)}. \tag{2.8}$$

By putting (2.8) in (2.6) and (2.7), we obtain the required results. Now Theorem 3 is proved.

Theorem 4. Let $\frac{[p-1]_q}{[p]_q} < \alpha < 1$, $-1 < B < A \leq 1$ and $0 \leq \sigma < p$. If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in T_{p,q}(\alpha, A, B),$$

then for $0 < |z| < r_1$, $f(z)$ is p-valently starlike function of order σ, where

$$|z| < r_1 = \min\left\{\inf_{n\geq 1}\left(\frac{(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)}{(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}}, 1\right\}.$$

Proof. Let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in T_{p,q}(\alpha, A, B)$. In order to prove $f(z) \in S_p^*(\sigma)$, we only need to show

$$\frac{\frac{zf'(z)}{f(z)} - \sigma}{p-\sigma} \prec \frac{1+z}{1-z}, \quad 0 \leq \sigma < p.$$

The subordination above is equivalent to $\left|\frac{zf'(z) - pf(z)}{zf'(z) + (p-2\sigma)f(z)}\right| < 1$. After some calculations and simplifications, we obtain

$$\sum_{n=1}^{\infty} \frac{n+p-\sigma}{p-\sigma} |a_{p+n}| |z|^n < 1. \tag{2.9}$$

From the inequality (2.1), we can obviously know that

$$\sum_{n=1}^{\infty} \frac{(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}|a_{p+n}| < 1.$$

Inequality (2.9) wants to be true if it satisfies the following inequality

$$\sum_{n=1}^{\infty} \frac{n+p-\sigma}{p-\sigma}|a_{p+n}||z|^n < \sum_{n=1}^{\infty} \frac{(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}|a_{p+n}|,$$

which implies that

$$|z|^n < \frac{(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}$$

or

$$|z| < \left(\frac{(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}}.$$

Let $r_1 = \min\left\{\inf_{n\geq 1}\left(\frac{(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}}, 1\right\}$, then we get the required result. The proof of Theorem 4 is completed.

Theorem 5. Let $\frac{[p-1]_q}{[p]_q} < \alpha < 1$, $-1 < B < A \leq 1$ and $0 \leq \sigma < p$. If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B),$$

then for $0 < |z| < r_2$, $f(z)$ is p-valently convex function of order σ, where

$$|z| < r_2 = \min\left\{\inf_{n\geq 1}\left(\frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}}, 1\right\}.$$

Proof. Let $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \in T_{p,q}(\alpha, A, B)$. To prove $f(z) \in C_p(\sigma)$, we must show that

$$\frac{1+\frac{zf''(z)}{f'(z)}-\sigma}{p-\sigma} \prec \frac{1+z}{1-z}, \quad 0 \leq \sigma < p.$$

This is equivalent to the inequality $\left|\frac{zf''(z)-(p-1)f'(z)}{zf''(z)+(1-2\sigma+p)f'(z)}\right| < 1$. After some calculations and simplifications, we have

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p-\sigma)}{p(p-\sigma)}|a_{p+n}||z|^n < 1. \tag{2.10}$$

From the inequality (2.1), we can easily obtain that

$$\sum_{n=1}^{\infty} \frac{(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}|a_{p+n}| < 1.$$

For inequality (2.10) to be true, it will be enough if

$$\sum_{n=1}^{\infty}\frac{(p+n)(n+p-\sigma)}{p(p-\sigma)}|a_{p+n}||z|^n<\sum_{n=1}^{\infty}\frac{(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}|a_{p+n}|,$$

which implies that

$$|z|^n<\frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}$$

or

$$|z|<\left(\frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}}.$$

Let $r_2=\min\left\{\inf_{n\geq1}\left(\frac{p(p-\sigma)(1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)}{(p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q}\right)^{\frac{1}{n}},1\right\}$, then we obtain the required result. Now Theorem 5 is proved.

Next, we will study the ratio of a function $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ to its sequence of partial sums $f_k(z)=z^p-\sum_{n=1}^{k}|a_{p+n}|z^{p+n}(k=1,2,3,\cdots)$ for all $z\in D$.

Theorem 6. Let $\frac{(1+B)[p+1]_q[p-1]_q+(A-B)[p]_q[p-1]_q}{(1+B)[p+1]_q[p]_q+(A-B)[p]_q[p-1]_q}<\alpha<1$, $-1<B<A\leq1$. If $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}\in T_{p,q}(\alpha,A,B)$, then we have

$$\operatorname{Re}\left(\frac{f(z)}{f_k(z)}\right)\geq1-\frac{1}{\varphi_{k+1}}\quad(z\in D)\tag{2.11}$$

and

$$\operatorname{Re}\left(\frac{f_k(z)}{f(z)}\right)\geq\frac{\varphi_{k+1}}{1+\varphi_{k+1}}\quad(z\in D),\tag{2.12}$$

where

$$\varphi_{k+1}=\frac{(1+B)[p+k+1]_q(\alpha[p+k]_q-[p-1]_q)}{(1-\alpha)(A-B)[p]_q[p-1]_q}.\tag{2.13}$$

Proof. In order to prove (2.11), we set

$$\varphi_{k+1}\left[\frac{f(z)}{f_k(z)}-\left(1-\frac{1}{\varphi_{k+1}}\right)\right]=\frac{1-\sum_{n=1}^{k}|a_{p+n}|z^n-\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}{1-\sum_{n=1}^{k}|a_{p+n}|z^n}=\frac{1+w(z)}{1-w(z)}.$$

After some simplifications, we have

$$w(z)=\frac{-\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}{2-2\sum_{n=1}^{k}|a_{p+n}|z^n-\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}$$

and

$$|w(z)|\leq\frac{\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|}{2-2\sum_{n=1}^{k}|a_{p+n}|-\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|}.$$

Now we can see that $|w(z)|<1$, if and only if

$$\sum_{n=1}^{k}|a_{p+n}|+\varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|\leq1.\tag{2.14}$$

From (2.1) we have $\sum_{n=1}^{\infty}\varphi_n|a_{p+n}| \le 1$. It is not difficult to see that φ_n is an increasing sequence with respect to n and that $\varphi_n \ge 1$ $(n = 1, 2, \cdots)$. Therefore, we get

$$\sum_{n=1}^{k}|a_{p+n}| + \varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}| \le \sum_{n=1}^{k}\varphi_n|a_{p+n}| + \sum_{n=k+1}^{\infty}\varphi_n|a_{p+n}| = \sum_{n=1}^{\infty}\varphi_n|a_{p+n}| \le 1.$$

Thus, the inequality (2.14) is true. This proves (2.11).

Next, in order to prove the inequality (2.12), we consider

$$(1+\varphi_{k+1})\left[\frac{f_k(z)}{f(z)} - \frac{\varphi_{k+1}}{1+\varphi_{k+1}}\right] = \frac{1-\sum_{n=1}^{k}|a_{p+n}|z^n + \varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}{1-\sum_{n=1}^{\infty}|a_{p+n}|z^n} = \frac{1+w(z)}{1-w(z)}.$$

After some simplifications , we can find that

$$w(z) = \frac{(1+\varphi_{k+1})\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}{2-2\sum_{n=1}^{k}|a_{p+n}|z^n + (\varphi_{k+1}-1)\sum_{n=k+1}^{\infty}|a_{p+n}|z^n}$$

and

$$|w(z)| \le \frac{(1+\varphi_{k+1})\sum_{n=k+1}^{\infty}|a_{p+n}|}{2-2\sum_{n=1}^{k}|a_{p+n}| - (\varphi_{k+1}-1)\sum_{n=k+1}^{\infty}|a_{p+n}|}.$$

Now we can see that $|w(z)| < 1$ if it satisfies

$$\sum_{n=1}^{k}|a_{p+n}| + \varphi_{k+1}\sum_{n=k+1}^{\infty}|a_{p+n}| \le 1.$$

The remaining part of the proof is much akin to that of (2.11) and hence we omit it. The proof of the Theorem is completed.

Theorem 7. Let $\frac{[p-1]_q}{[p]_q} \le \alpha < 1$. If $f_j(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B)$ $(j = 1, 2)$, then for $0 \le \lambda \le 1$,the function $H(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \in T_{p,q}(\alpha, A, B)$.

Proof. For $0 \le \lambda \le 1$, the function $H(z)$ can be written as

$$H(z) = \lambda f_1(z) + (1-\lambda)f_2(z) = z^p - \sum_{n=1}^{\infty}(\lambda|a_{p+n,1}| + (1-\lambda)|a_{p+n,2}|)z^{p+n}.$$

For functions $f_1(z), f_2(z) \in T_{p,q}(\alpha, A, B)$, by Theorem 1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q))|\lambda|a_{p+n,1}| + (1-\lambda)|a_{p+n,2}||\\
&= \lambda\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q))|a_{p+n,1}|\\
&+ (1-\lambda)\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q))|a_{p+n,2}|\\
&\le \lambda(1-\alpha)(A-B)[p]_q[p-1]_q + (1-\lambda)(1-\alpha)(A-B)[p]_q[p-1]_q\\
&= (1-\alpha)(A-B)[p]_q[p-1]_q,
\end{aligned}$$

which shows that $H(z) \in T_{p,q}(\alpha, A, B)$.

Corollary 2. Let $\frac{[p-1]_q}{[p]_q} \leq \alpha < 1$. If $f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B)$ $(j = 1, 2, \cdots, t)$, then the function $F(z) = \sum_{j=1}^{t} \lambda_j f_j(z) \in T_{p,q}(\alpha, A, B)$, where $\lambda_j \geq 0$ and $\sum_{j=1}^{t} \lambda_j = 1$.

Theorem 8. Let $\frac{[p-1]_q}{[p]_q} \leq \alpha < 1$. If $f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}|z^{p+n} \in T_{p,q}(\alpha, A, B)$ $(j = 1, 2)$, then for $-1 \leq m \leq 1$, we have

$$Q_m(z) = \frac{(1-m)f_1(z) + (1+m)f_2(z)}{2} \in T_{p,q}(\alpha, A, B).$$

Proof. For $-1 \leq m \leq 1$, the function $Q_m(z)$ can be written as

$$Q_m(z) = \frac{(1-m)f_1(z) + (1+m)f_2(z)}{2} = z^p - \sum_{n=1}^{\infty} \left(\frac{1-m}{2}|a_{p+n,1}| + \frac{1+m}{2}|a_{p+n,2}| \right) z^{p+n}.$$

In view of $f_1(z), f_2(z) \in T_{p,q}(\alpha, A, B)$, by applying Theorem 1, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q)) \left| \frac{1-m}{2}|a_{p+n,1}| + \frac{1+m}{2}|a_{p+n,2}| \right| \\
&= \frac{1-m}{2} \sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q))|a_{p+n,1}| \\
&+ \frac{1+m}{2} \sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q))|a_{p+n,2}| \\
&\leq \frac{1-m}{2}(1-\alpha)(A-B)[p]_q[p-1]_q + \frac{1+m}{2}(1-\alpha)(A-B)[p]_q[p-1]_q \\
&= (1-\alpha)(A-B)[p]_q[p-1]_q,
\end{aligned}$$

which shows that $Q_m(z) \in T_{p,q}(\alpha, A, B)$.

3. Conclusions

Our objective is to generalize some classical interesting results in geometric function theory from the ordinary analysis to q-analysis. By using the q-difference operator, a new subclass $T_{p,q}(\alpha, A, B)$ of multivalent analytic functions is introduced. Some geometric properties of functions in $T_{p,q}(\alpha, A, B)$ such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, partial sums and closure theorems are given. In particular, if we let $\alpha = 0$, $A = 1$, $B = -1$ and $q \to 1^-$, then $T_{p,q}(\alpha, A, B)$ reduces to the subclass of p-valently close-to-convex functions.

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Conflict of interest

The authors declare no conflict of interest.

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