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Research article

Measure pseudo almost automorphic solution to second order fractional impulsive neutral differential equation

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Abstract: We discuss the concept of pseudo almost automorphic solution to fractional neutral differential equation with impulses using measure theory. Our principal results are obtained via semigroup theory and the fixed point theorem due to Krasnoselskii and their combination with the properties of measure theory. An example is provided to outline the thought developed on this work.

Keywords: measure theory; measure-pseudo almost automorphic functions; fractional differential equation; impulsive conditions
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1. Introduction

The idea about almost periodic functions prompted to varied fundamental generalization within the year 1924-25 by H.Bohr [7]. The notion of almost automorphic (in short $\mathcal{AA}$) function is one of its crucial generalization by S. Bochner [6]. The concept of weighted pseudo almost automorphic (in short $\mathcal{WPAA}$) functions is one of the further important generalization of $\mathcal{AA}$ introduced by Blot et.al. [4]. These functions are a lot of typical and complex than weighted pseudo almost periodic functions. In 2012, Blot, Cieutat and Ezzinbi [5] applied the abstract measure theory to define an ergodic function and established fundamental properties of measure pseudo almost automorphic functions (in short $\mu_1-\mathcal{PAA}$), and thus the classical theories of pseudo almost automorphic functions and weighted pseudo almost automorphic functions become particular cases of this approach. After that, the $\mu_1-\mathcal{PAA}$ function has been developed in different ways, see for instance [10, 21, 26] and references therein.

The Dirac delta functions and "leaps" are two main directions in mathematical theory of impulsive differential equations. For describing the impulsive effects, the Dirac delta functions are a fundamental mathematical tool. In 1960, second direction of research "leaps" processes with some results for the solutions of stability was given by V. D. Milman and A. D. Myshkis [24]. In reality, many processes and phenomena are affected by short-term external factors. While comparing to total duration of phenomena and processes, this duration is negligible and therefore they form the impulses. Ecology, population dynamics, epidemiology, pharmacokinetics, economics, mechanics, control theory and other fields of science are all concerned in the dynamical states developed by such "leaps and bounds", see the monographs [3, 18] and the articles [12, 13, 17].

Fractional calculus deals with integro-differential equations can be considered as a branch of mathematical physics which has been effectively developed and plays a very important role in distinct fields such as biophysics, mechanics, electro chemistry, notable control theory and visco elasticity and so on. The fractional calculus is a generalization of the traditional calculus, but with a much wider applicability. The fractional methodology is suitable for a lot of applications in image processing, complex system dynamics and nonlinear dynamics. Thus it leads to the sustained interest in studying the theory of fractional differential equations [2, 8, 14–16, 20].

The authors Wang and Agarwal [25] investigated the existence of piecewise weighted pseudo almost automorphic mild solutions to impulsive ∇-dynamic equations. Chang and Feng [11] study the existence and uniqueness of measure pseudo almost automorphic solutions of the fractional differential equations. Our main results can be described as generalization of work in [11, 25]. Motivated by the works [5, 11, 21, 25, 26] the main purpose of this article is to establish the piecewise $\mu_1 - \mathcal{PAA}$ properties for the following impulsive fractional neutral differential equation

$$\begin{cases} D_t^{\alpha}[z(t) - g_1(t, z(t))] = A[z(t) - g_1(t, z(t))] + D_t^{\alpha-1}G_1(t, z(t)), t \in \mathcal{R}, t \neq t_j,\ j \in \mathbb{Z} \\ \triangle z(t_j)|_{t=t_j} = I_j(z(t_j)),\ t = t_j \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$ and $A : D(A) \subset Y \to Y$ is a linear densely defined operator of sectorial type on a complex Banach Space $(Y, \|\cdot\|)$. The functions $I_j : Y \to Y, G_1 : \mathcal{R} \times Y \to Y, g_1 : \mathcal{R} \times Y \to Y$ is a $\mu_1 - \mathcal{PAA}$ function in t for each $z \in Y$ satisfying suitable conditions. $\Delta z(t)|_{t=t_j} = z(t_j^+) - z(t_j^-), (j = 1, 2, ...), 0 = t_0 < t_1 < ... < t_n < ....$ Here $z(t_j^+)$ and $z(t_j^-)$ represent right and left limits of $z(t)$ at $t = t_j$ respectively. The fractional derivative D_t^{α} is considered as Caputo's sense.

The rest of this work is organized as follows. In Section 2, we define some definitions, terminologies, previous results, basic properties of $\mu_1 - \mathcal{PAA}$ functions and assumptions. In Section 3, we investigate the important results which are needed to prove the main results. In Section 4, we establish the existence of $\mu_1 - \mathcal{PAA}$ mild solutions to the model (1.1). In Section 5, we provide an example to illustrate our results.

2. Preliminaries

In this section, we review few notations, definitions and Lemmas which will be utilized throughout this paper.

2.1. Terminology and definitions

In this segment, we define basic definitions.

Let $(Y, \|\cdot\|_Y)$ be a Banach space, T be a subset of Y. The symbol $C(\mathcal{R}, Y)$(resp $C(\mathcal{R}\times T, Y)$) stands for the set of all continuous function from $\mathcal{R}$ to Y(resp from $\mathcal{R} \times T$ to Y) and $PC(\mathcal{R}, Y)$(resp $PC(\mathcal{R} \times T, Y)$) stands for set of all piecewise continuous function from $\mathcal{R}$ to Y(resp from $\mathcal{R} \times T$ to $\mathcal{R}$).

The fractional integral of order $q > 0$ in the Riemann–Liouville sense is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds$$

also, the fractional derivative of function f of order $q > 0$ in the Caputo sense is defined as

$$D_t^q = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} \frac{d^n f(s)}{ds^n} ds,$$

where $\Gamma(q)$ is a gamma function.

Moreover, the Riemann-Liouville definition entails physically unacceptable initial conditions (fractional order initial conditions); conversely of the Liouville-Caputo representation where the initial conditions are expressed in terms of integer-order derivatives having direct physical significance. The Caputo definition of fractional derivatives not only provides initial conditions with clear physical interpretation but it is also bounded, meaning that the derivative of a constant is equal to 0. Further, Caputo fractional derivative has lots of applications in real world problems, such as Groundwater flowing within an unconfined aquifer, measles epidemiological autonomous dynamical system etc.,

Definition 2.1. *[1] A sequence $t_1 : \mathbb{Z}^+ \to Y$ is said to be $\mathcal{AA}$ sequence, if t_1 is bounded and for every sequence of integer numbers $\{j'_n\}$, there exist a sub-sequence $\{j_n\} \subseteq \{j'_n\}$ such that*

$$\lim_{n\to\infty} t_1(j + j_n) = f(j), \quad \textit{for all} \quad n \in \mathbb{Z}$$

is well defined and

$$\lim_{n\to\infty} f(j - j_n) = t_1(j)$$

for each $j \in \mathbb{Z}^+$. Denote this collection of sequences by $AA_S^o(\mathbb{Z}, Y)$.

Definition 2.2. *[1] A piecewise continuous bounded function $G_1 \in PC(\mathcal{R}, Y)$ is said to be $\mathcal{AA}$ if*

- *sequence of impulsive moments $\{t_j\}$ is an $\mathcal{AA}$ sequence*
- *for each sequence of real numbers $\{u_n\}$, there exist a sub-sequence $\{u_{n_k}\} \subseteq \{u_n\}$ such that*

$$F_1(t) = \lim_{n\to\infty} G_1(t + u_{n_k}), \quad \textit{for all} \quad t \in \mathcal{R}$$

is well defined and

$$\lim_{n\to\infty} F_1(t - u_{n_k}) = G_1(t), \quad \textit{for all} \quad t \in \mathcal{R}.$$

Denote this collection of functions by $AA_\Omega^o(\mathcal{R}, Y)$.

Definition 2.3. *[1] A piecewise continuous bounded function $G_1 \in PC(\mathcal{R} \times T, Y)$ is said to be $\mathcal{AA}$ in compact subsets of Y in t uniformly for t_1 if*

- *sequence of impulsive moments $\{t_j\}$ is an $\mathcal{AA}$ sequence*

- *for each compact set $Q \subseteq Y$ and every sequence of real numbers $\{u_n\}$, there exist a sub-sequence $\{u_{n_k}\} \subseteq \{u_n\}$ such that*

$$F_1(t,t_1) = \lim_{n\to\infty} G_1(t+u_{n_k},t_1), \quad \text{for all} \quad t\in\mathcal{R},\ t_1\in Q$$

is well defined and

$$\lim_{n\to\infty} F_1(t-u_{n_k},t_1) = G_1(t,t_1), \quad \text{for all} \quad t\in\mathcal{R},\ t_1\in Q.$$

Denote this collection of functions by $AA^o_\Omega(\mathcal{R}\times T, Y)$.

We denote $\mathcal{M}_1$ by the set of all positive measures μ_1 on $\mathcal{B}_1$, where $\mathcal{B}_1$ is the Lebesgue σ-field of $\mathcal{R}$ satisfying, $\mu_1([a,b]) < \infty$, and $\mu_1(\mathcal{R}) = +\infty$ for all $b,a\in\mathcal{R}$ $(b\geq a)$.

Definition 2.4. *$\phi:\mathcal{R}\to Y$, a bounded continuous function, is said to be μ_1-ergodic if*

$$\lim_{l_1\to\infty}\frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]}\|\phi(t)\|d\mu_1(t) = 0,$$

where $\mu_1\in\mathcal{M}_1$. Denote this collection of functions by $\kappa(\mathcal{R},Y,\mu_1)$.

Definition 2.5. *Let $\mu_1\in\mathcal{M}_1$. A piecewise continuous bounded function $G_1:\mathcal{R}\to Y$ is said to be μ_1-$\mathcal{PAA}$ if G_1 is written in the form, $G_1 = H_1 + H_2$, where $H_1\in\mathcal{AA}^o_\Omega(\mathcal{R},Y)$ and $H_2\in\kappa(\mathcal{R},Y,\mu_1)$. Denote collection of such functions as $\mathcal{PAA}^o_\Omega(\mathcal{R},Y,\mu_1)$.*

Remark 2.1. *Define the positive measure μ_1 by*

$$\mu_1(C) = \int_C \rho(t)dt \quad for \quad C\in\mathcal{B}_1, \tag{2.1}$$

where ρ is a nonnegative $\mathcal{B}_1$-measurable function and dt denotes the Lebesgue measure on $\mathcal{R}$. With respect to the Lebesgue measure on $\mathcal{R}$, the function ρ in (2.1) is called the Radon-Nikodym derivative of μ_1. In this case, ρ is locally Lebesgue-integrable on $\mathcal{R}$ and $\int_{-\infty}^{+\infty}\rho(t)dt = +\infty$ if and only if its positive measure $\mu_1\in\mathcal{M}_1$.

A bounded sequence $h_1:\mathbb{Z}\to Y$ is said to be in $\kappa_S(\mathbb{Z},Y,\mu_1)$ for $\mu_1\in\mathcal{M}_1$ if $-l_1 = t_1 < t_2 < ... < t_n = l_1$ be a sequence of real numbers then

$$\lim_{l_1\to\infty}\frac{1}{\mu_1([-l_1,l_1])}\sum_{t_j\in[-l_1,l_1]}\|h_1(t_j)\| = 0.$$

Definition 2.6. *A bounded sequence $x:\mathbb{Z}\to Y$ is said to be μ_1-$\mathcal{PAA}$ sequence if it can be decomposed as $x = x_1 + x_2$ where $x_1\in AA^o_S(\mathbb{Z},Y)$ and $x_2\in\kappa_S(\mathbb{Z},Y,\mu_1)$. Denote collection of such functions as $\mathcal{PAA}^o_S(\mathbb{Z},Y,\mu_1)$.*

2.2. Previous results

In this section, we present some preliminary results which are needed in the sequel.

Proposition 2.1. *[5] Let $\mu_1 \in \mathcal{M}_1$. Then $(\kappa(\mathcal{R}, Y, \mu_1), \|.\|_\infty)$ is a Banach space.*

Proposition 2.2. *[5] For $\varsigma \in \mathcal{R}$ and $\mu_1 \in \mathcal{M}_1$, we denote $\mu_{1\varsigma}$ be the positive measure on $(\mathcal{R}, \mathcal{B}_1)$ defined by*

$$\mu_{1\varsigma}(C) = \mu_1(\{a + \varsigma : a \in C\}), \quad for \quad C \in \mathcal{B}_1.$$

We give the following assumption from $\mu_1 \in \mathcal{M}_1$:

(H) For all $\varsigma \in \mathcal{R}$ and a bounded interval I_1, there exist $\beta > 0$ such that

$$\beta\mu_1(C) \geq \mu_{1\varsigma}(C), when \quad C \in \mathcal{B}_1 \quad satisfies \quad C \cap I_1 = \emptyset.$$

Theorem 2.1. *[5] Let I_1 be a bounded interval (eventually $I_1 = \emptyset$) and $\mu_1 \in \mathcal{M}_1$. Assume that $G_1 \in PC(\mathcal{R}, Y)$. Then the following statements are equivalent.*

- $G_1 \in \kappa(\mathcal{R}, Y, \mu_1)$.
- $\lim_{l_1 \to +\infty} \dfrac{1}{\mu_1([-l_1, l_1] \backslash I_1)} \displaystyle\int_{[-l_1, l_1] \backslash I_1} \|G_1(t)\| d\mu_1(t) = 0$
- *For any* $\varepsilon > 0, \lim_{l_1 \to +\infty} \dfrac{\mu_1(\{t \in [-l_1, l_1] \backslash I_1 : \|G_1(t)\| > \varepsilon\})}{\mu_1([-l_1, l_1] \backslash I_1)} = 0.$

Remark 2.2. *The fact that $\mu_1([-l_1, l_1]) = \mu_1([-l_1, l_1] \backslash I_1) + \mu_1(I_1)$ for l_1 sufficiently large and from $\mu_1 \in \mathcal{M}_1$, we deduce that* $\lim_{l_1 \to +\infty} \mu_1([-l_1, l_1] \backslash I_1) = +\infty$.

Definition 2.7. *[5] Let μ_2 and $\mu_3 \in \mathcal{M}_1$. μ_2 is said to be equivalent to $\mu_3(\mu_2 \sim \mu_3)$ if there exists constants $\beta, \alpha > 0$ and a bounded interval I_1 (eventually $I_1 = \emptyset$) such that $\beta\mu_2(C) \geq \mu_3(C) \geq \alpha\mu_2(C)$, for $C \in \mathcal{B}_1$ satisfying $C \cap I_1 = \emptyset$.*

Theorem 2.2. *Let μ_2, $\mu_3 \in \mathcal{M}_1$. If μ_2 and μ_3 are equivalent, then $\kappa(\mathcal{R}, Y, \mu_2) = \kappa(\mathcal{R}, Y, \mu_3)$ and $\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_2) = \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_3)$.*

Lemma 2.1. *[5] Let $\mu_1 \in \mathcal{M}_1$. The measures $\mu_2 \sim \mu_3$ are equivalent for all $\varsigma \in \mathcal{R}$ if and only if μ_1 satisfies (H).*

Theorem 2.3. *[5] Assume $\mu_1 \in \mathcal{M}_1$ and (H) holds. If $\kappa(\mathcal{R}, Y, \mu_1)$ is translation invariant, then $\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ is also translation invariant.*

Theorem 2.4. *[5] Let $\mu_1 \in \mathcal{M}_1$ and $G_1 = H_1 + H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$, where $H_1 \in AA^o_\Omega(\mathcal{R}, Y)$ and $H_2 \in \kappa(\mathcal{R}, Y, \mu_1)$. If $\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ is translation invariant, then*

$$\overline{\{G_1(t) : t \in \mathcal{R}\}} \supset \{H_1 : t \in \mathcal{R}\} \tag{2.2}$$

Theorem 2.5. *[5] Assume that $\mu_1 \in \mathcal{M}_1$ and $\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ is translation invariant. Then $(\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1), \|.\|_\infty)$ is a Banach space.*

Theorem 2.6. *[9] Let $G_1 = H_1 + H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$, where $\mu_1 \in \mathcal{M}_1$. Assume that $G_1(t, z)$ and $H_1(t, z)$ are uniformly continuous on any bounded subset $K_* \subset T$ uniformly in $t \in \mathcal{R}$. If $\Phi_* \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ then $G_1(\cdot, \Phi_*(\cdot)) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$.*

Theorem 2.7. *(Krasnoselskii, [22]) Let $\mathbb{X}$ be a convex closed nonempty subset of a Banach space $(Y, \|\cdot\|)$. Suppose that A_2 and B_2 map $\mathbb{X}$ into Y such that*

- $A_2x + B_2y \in \mathbb{X} \quad (\forall\ x, y \in \mathbb{X})$
- *A_2 is continuous and $A_2\mathbb{X}$ is contained in a compact set*
- *B_2 is a contraction mapping.*

Then there is a $y \in \mathbb{X}$ with $A_2y + B_2y = y$.

2.3. Assumptions

To prove the main results, we consider the following assumptions:

(H1) $G_1 : \mathcal{R} \times Y \to Y$. Let $L_{G_1} > 0$ be such that $\|G_1(t, z_1) - G_1(t, z_2)\| \leq L_{G_1}\|z_1 - z_2\|$, $t \in \mathcal{R}$, $z_1, z_2 \in Y$.
(H2) The sequence I_j is $\mu_1 - \mathcal{PAA}$ and there exists $L_1 > 0$ such that $\|I_j(z) - I_j(t_1)\| \leq L_1\|z - t_1\|$, $j \in \mathbb{Z}$ and $z, t_1 \in Y$.
(H3) $g_1 : \mathcal{R} \times Y \to Y$. Let $L_g > 0$ be such that $\|g_1(t, z) - g_1(t, t_1)\| \leq L_g\|z - t_1\|$, $t \in \mathcal{R}$ and $z, t_1 \in Y$.

3. Useful results

In this section, we present the important results which are needed to prove the main results.

Lemma 3.2. *If a bounded sequence $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \kappa(\mathcal{R}, Y, \mu_1)$, then there exists a uniformly continuous function $g \in \kappa(\mathcal{R}, Y, \mu_1)$ such that $g(t_n) = \{\varphi(n)\}_{n\in\mathbb{Z}}, \quad t_n \in \mathcal{R}$.*

Proof. We define a function $g(t) = \varphi(n) + \int_n^t H_2(t, \varphi(t))dy$ for $t \in [n, n + 1)$. If $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \kappa(\mathcal{R}, Y, \mu_1)$, then it follows from the boundedness of H_2 that $g(t)$ is bounded on $\mathcal{R}$. From Theorem 2.6, we have $H_2(., \varphi(.)) \in \kappa(\mathcal{R}, Y, \mu_1)$. Let the set $B_j = \{t \in [-l_1, l_1] : u_1(t) \in O_j\}$ is open in $[-l_1, l_1]$ and $[-l_1, l_1] = U_{j=1}^m B_j$. Let $E_1 = B_1, E_j = B_j \setminus \cup_{j=1}^{k-1} B_j \quad (2 \leq k \leq m)$. Then $E_i \cap E_j = \phi$ when $i \neq j, 1 \leq i, j \leq m$. Now we have,

$$
\begin{aligned}
&\frac{1}{\mu_1([-l_1, l_1])} \int_{[-l_1,l_1]} \|g(t)\| d\mu_1(t) \\
&\quad \leq \frac{1}{\mu_1([-l_1, l_1])} \left[\sum_{j=1}^{m} \int_{E_j} \|\varphi(j)\|\, d\mu_1 + \int_{[-l_1,l_1]} \left\| \int_n^t H_2(t, \varphi(t))dy \right\| d\mu_1 \right] \\
&\quad \leq \frac{1}{\mu_1([-l_1, l_1])} \left[\sum_{j=1}^{m} \int_{E_j} \|\varphi(j)\|\, d\mu_1 + \int_{[-l_1,l_1]} \|H_2(t, \varphi(t))\|\, (n + 1 - n)dy \right] \\
&\quad \leq \frac{1}{\mu_1([-l_1, l_1])} \left[\sum_{j=1}^{m} \int_{E_j} \|\varphi(j)\|\, d\mu_1 + \sum_{j=1}^{m} \int_{E_j} \|H_2(t, \varphi(t))\|\, dy \right]
\end{aligned}
$$

as $l_1 \to \infty$, we deduce that $\lim_{l_1\to\infty} \frac{1}{\mu_1[-l_1, l_1]} \int_{[-l_1,l_1]} \|g(t)\| d\mu_1 = 0$. That is $\quad g \in \kappa(\mathcal{R}, Y, \mu_1)$. □

Theorem 3.1. *Let $I_j : Y \to Y$ be a $\mu_1 - \mathcal{PAA}$ sequence and satisfying (H2). If $\phi \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \mu_1)$ then $I_j(\phi(t_j))$ is a μ_1-$\mathcal{PAA}$ sequence.*

Proof. Since $\phi = p_1 + p_2$, where $p_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$, $p_2 \in \kappa(\mathcal{R}, Y, \mu_1)$, it follows that $I_j(\phi(t_j)) = I_j(p_1(t_j)) + I_j(p_2(t_j))$. By [23, Lemma 3.2], $I_j(p_1(t_j))$ is an $\mathcal{AA}$ sequence. Now it remains to show that $I_j(p_2(t_j)) \in \kappa(\mathcal{R}, Y, \mu_1)$. Since $p_2 \in \kappa(\mathcal{R}, Y, \mu_1)$, we have $\lim_{l_1\to\infty} \frac{1}{\mu_1([-l_1, l_1])} \int_{[-l_1,l_1]} \|p_2(t)\| d\mu_1(t) = 0$.

Let $-l_1 = t_1 < t_2 < .... < t_n = l_1$ be a sequence of real numbers, we have

$$\int_{[-l_1,l_1]} \|p_2(t)\| d\mu_1 = \sum_{t_j\in[-l_1,l_1]} \|p_2(t_j)\|.$$

Thus we obtain

$$\frac{1}{\mu_1([-l_1, l_1])} \int_{[-l_1,l_1]} \|p_2(t)\| d\mu_1 = \frac{1}{\mu_1([-l_1, l_1])} \sum_{t_j\in[-l_1,l_1]} \|p_2(t_j)\|. \tag{3.1}$$

Taking the limit when $l_1 \to \infty$, from Eq 3.1, we obtain $p_2(t_j) \in \kappa(\mathbb{Z}, Y, \mu_1)$.
Now from (H2),

$$\|I_j p_2(t_j)\| \le \|I_j(p_2(t_j)) - I_j(0)\| + \|I_j(0)\| \le L_1\|p_2(t_j)\| + \|I_j(0)\|.$$

We see that the sequence $I_j(\phi(t_j))$ is a μ_1-$\mathcal{PAA}$ sequence. □

Remark 3.3. *In order to prove the main results, we need to present an important estimate from [19] as follows:*

$$\|E_\alpha(t)\|_{\mathcal{L}(Y)} \le \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha}, \quad t \ge 0. \tag{3.2}$$

Lemma 3.3. *Let $E_\alpha(t)$ be strongly continuous family of bounded linear operators satisfying (3.2). If $z \in AA^0_\Omega(\mathcal{R}, Y)$ and $u_0 : \mathcal{R} \to Y$ is defined by*

$$u_0(t) = \sum_{t>t_j} E_\alpha(t - t_j) I_j(z(t_j))$$

then $u_0(\cdot) \in AA^0_\Omega(\mathcal{R}, Y)$.

Proof. Since z is $\mathcal{AA}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $h_1(t) = \lim_{n\to\infty} z(t + u_{n_k})$ is well defined for every $t \in \mathcal{R}$. We consider

$$u_0(t + u_{n_k}) = \sum_{t+u_{n_k}>t_j} E_\alpha(t + u_{n_k} - t_j) I_j(z(t_j)) = \sum_{t>t_j} E_\alpha(t - t_j) I_j(z(t_j + u_{n_k})).$$

And

$$\begin{aligned} \|u_0(t + u_{n_k})\| &= \Big\| \sum_{t>t_j} E_\alpha(t - t_j) I_j(z(t_j + u_{n_k})) \Big\| \le \sum_{t>t_j} \|E_\alpha(t - t_j)\| \|I_j(z(t_j + u_{n_k}))\| \\ &\le CMI \sum_{t>t_j} \frac{1}{1 + |\omega|(t - t_j)^\alpha}. \end{aligned}$$

Since $z \in AA^0_\Omega(\mathcal{R}, Y)$, $\lim_{n\to\infty} z(t + u_{n_k}) = h_1(t_j)$ for every $j \in \mathbb{Z}$. Therefore, for any $t > t_j$, $j \in \mathbb{Z}$, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} u_0(t + u_{n_k}) = \sum_{t>t_j} E_\alpha(t - t_j)I_j(h_1(t_j)).$$

Therefore, $u_0(\cdot) \in AA^0_\Omega(\mathcal{R}, Y)$. □

Lemma 3.4. *Let $f = g_2 + h \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ with $g_2 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$ and $h \in \kappa(\mathcal{R}, Y, \mu_1)$. Then*

$$u_0(.) = \int_{-\infty}^{t} E_\alpha(t - s)g_2(z(s))ds \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1).$$

Proof. Now, let $u_0(t) = \int_{-\infty}^{t} E_\alpha(t - s)\gamma_1(s)ds + \int_{-\infty}^{t} E_\alpha(t - s)\gamma_2(s)ds = u_1(t) + v_1(t)$,
where $u_1(t) = \int_{-\infty}^{t} E_\alpha(t - s)\gamma_1(s)ds$ and $v_1(t) = \int_{-\infty}^{t} E_\alpha(t - s)\gamma_2(s)ds$

Let $(\tau_n^{'})$ be an arbitrary sequence on $\mathcal{R}$. Since $\gamma_1 \in AA^o_\Omega(\mathcal{R}, Y)$, there exists a subsequence (τ_n) of $(\tau_n^{'})$ such that

$$\bar{\gamma}_1(t) = \lim_{n\to\infty} \gamma_1(t + \tau_n) \text{ is well defined}$$

and

$$\lim_{n\to\infty} \bar{\gamma}_1(t - \tau_n) = \gamma_1(t), \text{for each } t \in \mathcal{R}.$$

Define $\bar{u_1}(t) = \int_{-\infty}^{t} E_\alpha(t - s)\bar{\gamma}_1(s)ds$.
Consider,

$$\begin{aligned} u_1(t + \tau_n) &= \int_{-\infty}^{t+\tau_n} E_\alpha(t + \tau_n - s)\gamma_1(s)ds \\ &= \int_{-\infty}^{t} E_\alpha(t - u)\gamma_{1_n}(u)du \end{aligned}$$

where $\gamma_{1_n}(u) = \gamma_1(u + \tau_n), n = 1, 2, ...$

$$u_1(t + \tau_n) = \int_0^\infty E_\alpha(u)\gamma_{1_n}(t - u)du$$

Now, we have

$$\begin{aligned} \|u_1(t + \tau_n)\| &\le \int_0^\infty \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha} \left\|\gamma_{1_n}(t - u)\right\| du \\ &\le C(\theta, \alpha)M \frac{|\omega|^{-1/\alpha}\pi}{\alpha sin(\pi/\alpha)} \|\gamma_1\|_\infty \end{aligned} \tag{3.3}$$

and since $E_\alpha(\cdot)z$ is continuous, we get $E_\alpha(t-u)\gamma_{1_n}(u) \to E_\alpha(t-u)\overline{\gamma_1}(u)$ as $n \to \infty$ for any $t \ge u$ and for all fixed $u \in \mathcal{R}$. Then by using the Lebesgue's dominated convergence theorem,

$$u_1(t + \tau_n) \to \overline{u_1}(t) \quad \text{as} \quad n \to \infty \quad \forall\, t \in R.$$

By this way we can show that,

$$\overline{u_1}(t-\tau_n) \to u(t) \quad \text{as} \quad n \to \infty \quad \forall\, t \in R.$$

Therefore $u_1(t) \in AA^0_\Omega(\mathcal{R}, Y)$.
Next we show $v_1 \in \kappa(\mathcal{R}, Y, \mu_1)$.

$$\begin{aligned}
\frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \|v_1(t)\| d\mu_1(t) &= \frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \left\| \int_{-\infty}^{t} E_\alpha(t-s)\gamma_2(s)ds \right\| d\mu_1(t).\\
&= \frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \int_{-\infty}^{t} \frac{C(\theta,\alpha)M}{1+|\omega|(t-s)^\alpha} \|\gamma_2(s)\|\, ds d\mu_1(t).\\
&= \frac{CM}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \int_{0}^{\infty} \frac{\|\gamma_2(t-s)\|}{1+|\omega|s^\alpha} ds d\mu_1(t)\\
&= \int_0^\infty \frac{CM}{1+|\omega|s^\alpha}\left[\frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \|\gamma_2(t-s)\|\, d\mu_1(t)\right] ds.
\end{aligned}$$

Since $\gamma_2 \in \kappa(\mathcal{R}, Y, \mu_1)$, we find $\frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \|\gamma_2(t-s)\|\, d\mu_1(t) = 0$ for all $t \in \mathcal{R}$. Therefore $v_1(t) \in \kappa(\mathcal{R}, Y, \mu_1)$, by Lebesgue's dominated convergence theorem. □

Theorem 3.2. *If* $G_1 = H_1 + H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ *with* $H_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y), H_2 \in \kappa(\mathcal{R}, Y, \mu_1)$. *Then* $Q_1(t) = \int_{-\infty}^{t} E_\alpha(t-s)G_1(t)ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))$ *is a* μ_1-$\mathcal{PAA}$ *function.*

Proof. From Lemma 3.4, it follows that $\int_{-\infty}^{t} E_\alpha(t-s)G_1(t)ds \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$.

Next we show that $\sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$. By Theorem 3.1, $I_j(z(t_j)) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$. Let $I_j(z(t_j)) = \beta_j + \gamma_j$, where $\beta_j \in \mathcal{AA}^o_S(\mathbb{Z}, Y)$ and $\gamma_j \in \kappa(\mathbb{Z}, Y, \mu_1)$, then

$$\begin{aligned}
\sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)) &= \sum_{t>t_j} E_\alpha(t-t_j)\beta_j + \sum_{t>t_j} E_\alpha(t-t_j)\gamma_j\\
&= R_2(t) + V_2(t).
\end{aligned}$$

By Lemma 3.3, $R_2(t) \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$. Next to show that $V_2(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$.
Since $\gamma_j \in \kappa(\mathbb{Z}, Y, \mu_1)$, by Lemma 3.2, there exists $g(t) = \gamma_j, \quad t \in [j, j+1)$ such that $g \in \kappa(\mathcal{R}, Y, \mu_1)$ and $g(j) = \gamma_j, j \in \mathbb{Z}$. Then,

$$\begin{aligned}
\frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \|V_2(t)\| d\mu_1 &= \frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \|\sum_{t>t_j} E_\alpha(t-t_j)\gamma_j\| d\mu_1\\
&\le \frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \sum_{t>t_j} \|E_\alpha(t-t_j)\|\|\gamma_j\| d\mu_1\\
&= \frac{1}{\mu_1([-l_1,l_1])}\int_{[-l_1,l_1]} \frac{CM}{1+|\omega|(t-t_j)^\alpha} \|g(t)\| d\mu_1
\end{aligned}$$

$$= CM\Big[\frac{1}{1+|\omega|m_1^{\alpha}} + \sum_{n=2}^{\infty}\frac{1}{1+|\omega|n^{\alpha}}\Big]\frac{1}{\mu_1[-l_1,l_1]}\int_{\mu_1([-l_1,l_1])}\|g(t)\|d\mu_1$$

where $m_1 = \{\min(t-t_j) : 0 < t-t_j \le 1\}$. Since $g(t) \in \kappa(\mathcal{R}, Y, \mu_1)$, we have $V_2(t) \in \kappa(\mathcal{R}, Y, \mu_1)$. Thus $\sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$. □

4. Main results

Here we give the mild solution of our model (1.1).

Definition 4.8. *A function $z : \mathcal{R} \to Y$ is said to be a mild solution of (1.1) if*

$$z(t) = g_1(t,z(t)) + \int_{-\infty}^{t} E_\alpha(t-s)G_1\left(s,z(s)\right)ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)), \quad \textit{for each } t\in\mathcal{R}. \tag{4.1}$$

Theorem 4.1. *Suppose* $(H1)-(H3)$ *are satisfied then the model* (1.1) *has a* $\mu_1 - \mathcal{PAA}$ *solution* z *on* $\mathcal{R}$ *provided* $\left(L_g + \frac{CM\pi L_{G_1}|\omega|^{1/\alpha}}{\alpha sin(\pi/\alpha)}\right) < 1.$

Proof. Let B_{q_0} be the closed convex and bounded subset of $\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$, where $B_{q_0} = \{z \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1) : \|z\| \le q_0\}$.
Now introduce the operator $\Gamma_1 : B_{q_0} \to \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$ as follows:

$$\Gamma_1 z(t) = g_1(t,z(t)) + \int_{-\infty}^{t} E_\alpha(t-s)G_1\left(s,z(s)\right)ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)).$$

We decompose $\Gamma_1 = \Gamma_1^* + \Gamma_2^*$ as

$$\Gamma_1^* z(t) = g_1(t,z(t)) + \int_{-\infty}^{t} E_\alpha(t-s)G_1\left(s,z(s)\right)ds,$$
$$\Gamma_2^* z(t) = \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)).$$

Step 1: For $z \in B_{q_0}$ implies $\Gamma_1^* z, \Gamma_2^* z \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$.
By Lemma 3.4 and Theorem 3.2 we have $\Gamma_1^* z, \Gamma_2^* z \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \mu_1)$.
Step 2: For $z_1, z_2 \in B_{q_0}$ implies $\Gamma_1^* z_1 + \Gamma_2^* z_2 \in B_{q_0}$.

$$\begin{aligned}
&\|\Gamma_1^* z_1(t) + \Gamma_2^* z_2(t)\| \\
&= \|g_1(t,z_1(t)) + \int_{-\infty}^{t} E_\alpha(t-s)G_1\left(s,z_1(s)\right)ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z_2(t_j))\| \\
&\le \|g_1(t,z_1(t))\| + \int_{-\infty}^{t} \|E_\alpha(t-s)\|\|G_1(s,z_1(s))\|ds + \sum_{t>t_j} \|E_\alpha(t-t_j)\|\|I_j(z_2(t_j))\| \\
&\le \|g_1(t,z_1(t)) - g_1(0,0)\| + \|g_1(0,0)\| + \int_{-\infty}^{t} \frac{CM}{1+|\omega|(t-s)^{\alpha}}\Big[\|G_1(s,z_1(s)) - G_1(s,0)\| + \|G_1(s,0)\|\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t>t_j}\|E_\alpha(t-t_j)\|\Big[\|I_j(z_2(t_j)-I_j(0)\|+\|I_j(0)\|\Big]\\
&\le L_g\|z_1\|+\|g_1(0,0)\|+\int_0^\infty \frac{CM}{1+|\omega|s^\alpha}\Big(L_{G_1}\|z_1\|+\|G_1(s,0)\|\Big)ds\\
&+\sum_{t>t_j}\frac{CM}{1+|\omega|(t-t_j)^\alpha}\Big(L_1\|z_2\|+\|I_j(0)\|\Big)\\
&\le L_g q_0+\frac{\pi CM|\omega|^{-\frac{1}{\alpha}}\Big(q_0L_{G_1}+\|G_1(s,0)\|\Big)}{\alpha sin(\frac{\pi}{\alpha})}\\
&+\sum_{t>t_j}\frac{CM}{1+|\omega|(t-t_j)^\alpha}\Big(q_0L_1+\|I_j(0)\|\Big)+\|g_1(0,0)\|\le q_0.
\end{aligned}
$$

Step 3: Γ_1^* is contraction on B_{q_0}.
For each $t\in\mathcal{R}$, let $z_1,z_2\in B_{q_0}$ then by (H1), (H3) and (3.2), we have

$$
\begin{aligned}
\|\Gamma_1^*z_1(t)-\Gamma_1^*z_2(t)\| &\le \|g_1(t,z_1(t))-g_1(t,z_2(t))\|+\int_{-\infty}^t\|E_\alpha(t-s)\|\|G_1\left(s,z_1(s)\right)-G_1\left(s,z_2(s)\right)\|ds\\
&\le L_g\|z_1(t)-z_2(t)\|+\int_0^\infty\frac{CM}{1+|\omega|s^\alpha}\|z_1(s)-z_2(s)\|ds\\
&\le\left[L_g+\frac{\pi CML_{G_1}|\omega|^{-\frac{1}{\alpha}}}{\alpha sin(\frac{\pi}{\alpha})}\right]\|z_1-z_2\|.
\end{aligned}
$$

Step 4: Γ_2^* is continuous on B_{q_0}.
Let $\{z^n(t)\}_0^\infty\subseteq B_{q_0}$ with $z^n\to z$ in B_{q_0} then by (H2) and (3.2), we have

$$
\begin{aligned}
\|\Gamma_2^*z^n(t)-\Gamma_2^*z(t)\| &\le\sum_{t>t_j}\|E_\alpha(t-t_j)\|\|I_j(z^n(t_j))-I_j(z(t_j))\|\\
&\le\left(\sum_{t>t_j}\frac{L_1CM}{1+|\omega|(t-t_j)^\alpha}\right)\|z^n-z\|.
\end{aligned}
$$

As $n\to\infty$, $\Gamma_2^*z^n\to\Gamma_2^*z$.

Step 5: Γ_2^* maps bounded sets into bounded sets.
It is enough to prove that for $t>0$, there exist positive constant γ such that, for each $z\in B_{q_0}=\{z\in\mathcal{PAA}_\Omega^o(\mathcal{R},Y,\mu_1):\|z\|\le q_0\}$ and we have $\|\Gamma_2^*z\|\le\gamma$. Now for $t\in\mathcal{R}$,

$$
\begin{aligned}
\|\Gamma_2^*z(t)\| &\le\sum_{t>t_j}\|E_\alpha(t-t_j)\|\|I_j(z(t_j))\|\\
&\le\sum_{t>t_j}\|E_\alpha(t-t_j)\|\|I_j(z(t_j))-I_j(0)\|+\|I_j(0)\|
\end{aligned}
$$

$$\le \sum_{t>t_j} \frac{CM}{1+|\omega|(t-t_j)^{\alpha}}\Big(q_0 L_1 + \|I_j(0)\|\Big)\| = \gamma.$$

Step 6: $\Gamma_2^* z$ maps bounded sets into equi-continuous sets.
Let $z \in B_{q_0}$ and for $t_j < \tau_1 < \tau_2 \le t_{j+1}$, we receive

$$\begin{aligned}
\|\Gamma_1 z(\tau_2) - \Gamma_1 z(\tau_1)\| &= \|\sum_{\tau_2>t_j} E_\alpha(\tau_2 - t_j) I_j(z(t_j)) - \sum_{\tau_1>t_j} E_\alpha(\tau_1 - t_j) I_j(z(t_j))\| \\
&\le \|\sum_{-\infty<t_j<\tau_1} E_\alpha(\tau_2 - t_j) I_j(z(t_j)) + \sum_{\tau_1\le t_j<\tau_2} E_\alpha(\tau_2 - t_j) I_j(z(t_j)) \\
&\quad - \sum_{-\infty<t_j<\tau_1} E_\alpha(\tau_1 - t_j) I_j(z(t_j))\| \\
&\le \sum_{-\infty<t_j<\tau_1} \|E_\alpha(\tau_2 - t_j) - E_\alpha(\tau_1 - t_j)\|\|I_j(z(t_j))\| \\
&\quad + \sum_{\tau_1\le t_j<\tau_2} \|E_\alpha(\tau_2 - t_j)\|\|I_j(z(t_j))\|.
\end{aligned}$$

The right hand side does not depend on z and $\to 0$ as $\tau_2 \to \tau_1$. Hence by utilizing the general form of Arzela-Ascoli theorem for equi-continuous function (Diethelm, [20, Theorem D.10]), we find that Γ_1 is relatively compact. Therefore the operator Γ_1 is compact. Now, by Theorem 2.7, the model (1.1) admits at least one mild solution. □

5. Example

Consider the following model:

$$\begin{cases}
\partial_t^\alpha\Big[r(t,z) - \varphi\Big(\sin\frac{1}{2-\sin t - \sin \pi t} r(t,z) + e^{-t}\sin(r(t,z))\Big)\Big] = (\partial_t^2 - \omega)\Big[r(t,z) \\
\quad -\varphi\Big(\sin\frac{1}{2-\sin t-\sin\pi t} r(t,z) + e^{-t}\sin(r(t,z))\Big)\Big] + \\
\qquad \partial_t^{\alpha-1}\Big(\beta\Big(\cos\frac{1}{\sin t + \sin\sqrt{2t}} r(t,z) + \frac{\sin(r(t,z))}{1+t^2}\Big)\Big), t>0, t\ne t_j, \\
\triangle r(t_j,z) = I_j(r(t_j,z)) = \varrho\Big(\sin\frac{1}{2+\sin j} r(t_j,z) + \frac{\cos(r(t_j,z))}{1+j^2}\Big), \quad j = 1,2,\ldots,
\end{cases} \tag{5.1}$$

where β, φ and ϱ are positive constant. Let the Radon-Nikodym derivative ρ of the measure μ_1 be defined by $\rho(t) = e^{\sin t}$. Clearly μ_1 satisfy (H). Take $Y = L^2([0,\pi])$ and define the operators A by $A\psi = \frac{\partial^2\psi}{\partial t^2} - \omega\psi, \psi \in D(A)$, where $D(A) = \{\psi \in Y : \psi'' \in Y, \psi(0) = \psi(\pi)\} \subset Y$.
Let

$$\begin{aligned}
G_1(t, z(\varsigma)) &= \beta\Big(\cos\frac{1}{\sin t + \sin\sqrt{2t}} z(\varsigma) + \frac{\sin(z(\varsigma))}{1+t^2}\Big), \\
g_1(t, z(\varsigma)) &= \varphi\Big(\sin\frac{1}{2-\sin t - \sin\pi t} z(\varsigma) + e^{-t}\sin(z(\varsigma))\Big), \\
I_j(z(\varsigma)) &= \varrho\Big(\sin\frac{1}{2+\cos j} z(\varsigma) + \frac{\cos(z(\varsigma))}{1+j^2}\Big)
\end{aligned}$$

It is not difficult to see that the function G_1, g_1 and I_j are continuous function and $\mu_1 - \mathcal{PAA}$ in t.

$$\begin{aligned}\|G_1(t,z_1) - G_1(t,z_2)\|_2^2 &\le \int_0^{\pi} |\beta|^2 \Big|\cos\frac{1}{\sin t + \sin\sqrt{2}t}\Big|^2 \Big|z_1(v) - z_2(v)\Big|^2 \\ &+ |\beta|^2 \Big|\frac{1}{1+t^2}\Big|^2 |\sin(z_1(v)) - \sin(z_2)(v)|^2 dv \\ &\le |\beta|^2 \Big[\Big|\cos\frac{1}{\sin t + \sin\sqrt{2}t}\Big|^2 + \Big|\frac{1}{1+t^2}\Big|^2\Big]\Big[\|z_1 - z_2\|_2^2\Big]\end{aligned}$$

Hence

$$\|G_1(t,z_1) - G_1(t,z_2)\|_2 \le 2|\beta|\Big[\|z_1 - z_2\|_2\Big].$$

Also

$$\begin{aligned}\|g_1(t,z_1) - g_1(t,z_2)\|_2^2 &\le \int_0^{\pi} |\varphi|^2 |\sin\frac{1}{2 - \sin t - \sin\pi t}|^2 |z_1(\varsigma) - z_2(\varsigma)|^2 \\ &+ |\varphi|^2 |e^{-t}|^2 |\sin z_1(\varsigma) - \sin z_2(\varsigma)|^2 d\varsigma \\ \|g_1(t,z_1) - g_1(t,z_2)\|_2 &\le 2|\varphi|\|z_1 - z_2\|_2.\end{aligned}$$

Furthermore

$$\begin{aligned}\|I_j(z_1) - I_j(z_2)\|_2^2 &\le \int_0^{\pi} |\varrho|^2 \Big|\sin\frac{1}{2+\cos j}\Big|^2 \Big|z_1(v) - z_2(v)\Big|^2 + \Big|\frac{\cos r(t_j, z)}{1+j^2}\Big|^2 \\ &|\cos(z_1(v)) - \cos(z_2)(v)|^2 dv \\ &\le |\varrho|^2 \Big[\Big|\sin\frac{1}{2+cos j}\Big|^2 + \Big|\frac{1}{1+j}\Big|^2\Big]\|z_1 - z_2\|_2^2.\end{aligned}$$

Hence

$$\|I_j(z_1) - I_j(z_2)\|_2 \le 2|\varrho|\|z_1 - z_2\|_2.$$

Thus G_1, g_1, I_j satisfies Lipschitz conditions with $L_{G_1} = 2|v| = 1/10, L_g = 2|\varphi| = 1/10, L_l = 2|\varrho| = 1/10$. Let $\omega = -1$ and $\alpha = 5/4$, then $\left(L_g + \dfrac{CM\pi L_{G_1}|\omega|^{1/\alpha}}{\alpha\sin(\pi/\alpha)}\right) = 0.5276 < 1$. Therefore by Theorem 4.1, the model (5.1) has $\mu_1 - \mathcal{PAA}$ mild solution.

6. Conclusions

In this paper, we investigate many important results on the new theory of measure pseudo almost automorphic functions with impulses. Those results have an important impact on the theory of systems. Numerous researchers are particularly involved in discussing the existence, stability and controllability results for various systems under different hypotheses. We assure that, these existence results can be further extended to stepanov type measure pseudo almost periodic and automorphic functions for integer and non-integer systems.

Conflict of interest

The authors declare no conflict of interest.

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