*Mathematics*

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8331–8341.
DOI:10.3934/math.2021482
Received: 30 March 2021
Accepted: 17 May 2021
Published: 28 May 2021

***Research article***

# Global bifurcation result and nodal solutions for Kirchhoff-type equation

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**Abstract:** We investigate the global structure of nodal solutions for the Kirchhoff-type problem

$$\begin{cases} -(a+b\int_0^1 |u'|^2dx)u'' = \lambda f(u), \ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $a > 0, b > 0$ are real constants, $\lambda$ is a real parameter. $f \in C(\mathbb{R},\mathbb{R})$ and there exist four constants $s_1 \le s_2 < 0 < s_3 \le s_4$ such that $f(0) = f(s_i) = 0, i = 1,2,3,4,$ $f(s) > 0$ for $s \in (s_1, s_2) \cup (0, s_3) \cup (s_4, +\infty)$, $f(s) < 0$ for $s \in (-\infty, s_1) \cup (s_2, 0) \cup (s_3, s_4)$. Under some suitable assumptions on nonlinear terms, we prove the existence of unbounded continua of nodal solutions of this problem which bifurcate from the line of trivial solutions or from infinity, respectively.

**Keywords:** nodal solutions; bifurcation; nonlocal problem; eigenvalues
**Mathematics Subject Classification:** 34C10, 34C23, 47J10

## 1. Introduction

This paper is devoted to the Kirchhoff-type problem

$$\begin{cases} -(a+b\int_0^1 |u'|^2dx)u'' = \lambda f(u), \ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $a > 0, b > 0$ are real constants, $\lambda$ is a real parameter. In recent years, a lot of classical results have been concerned on a bounded domain for Kirchhoff equation. For example, the existence of solutions can be founded in [1–9] and the references therein.

When $a = 1, b = 0$ in problem (1.1), it reduces to the classic second-order semilinear problem. The conclusions of global bifurcation of such problems are well known, see [10–14] for details. In particular, Ma [10], Ma and Han [12] discussed the existence of nodal solutions when the nonlinear term of the problem (1.1) has two non-zero zeros.

In this article, we are interested in studying nodal solutions of problem (1.1) with the nonlinear term $f$ has some zeros in $\mathbb{R} \setminus \{0\}$. This work is motivated by the recent results of Cao and Dai [1] who concerned with determining values of $\lambda$ for which there exist nodal solutions of the Kirchhoff-type problem

$$\begin{cases} -(a + b\int_0^1 |u'|^2 dx)u'' = \lambda f(x, u), \ x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

(1.2) is often used to describe the stationary problem of a model introduced by Kirchhoff to describe the transversal oscillations of a stretched string. Where $f$ satisfies the following assumptions:
(A1) $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ with $sf(x, s) > 0$ for all $x \in (0, 1)$ and any $s \neq 0$.
(A2) There exist $f_0, f_\infty \in (0, \infty)$ such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{as}, \ f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{bs^3}$$

uniformly with respect to $x \in (0, 1)$.

It is well known that the problem

$$\begin{cases} -u'' = \lambda u, \ x \in (0, 1), \\ u(0) = u(1) = 0 \end{cases}$$

possesses infinitely many eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty$, all of which are simple. The eigenvalue $\phi_k$ corresponding to $\lambda_k$ has exactly $k - 1$ simple zeros in $(0, 1)$. According to the Theorem 1.2 of [1], we know that the eigenvalue problem

$$\begin{cases} -(\int_0^1 |u'|^2 dx)u'' = \mu u^3, \ x \in (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{1.3}$$

possesses infinitely many eigenvalues $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots \to +\infty$. Every $\mu_k$ is simple and the corresponding one-dimensional space of solutions of the problem (1.3) with $\mu = \mu_k$ is spanned by a function having precisely $k$ bumps in $(0, 1)$. Each $k-$bump solution is constructed by the reflection and compression of the eigenfunction $\psi_1$ associated with $\mu_1$.

Using the bifurcation results of [1], the authors further established the following result:
**Theorem A.** ( [1]. Theorem 1.3) Let (A1)-(A2) hold. Then for

$$\lambda \in (\frac{\lambda_k}{f_0}, \frac{\mu_k}{f_\infty}) \cup (\frac{\mu_k}{f_\infty}, \frac{\lambda_k}{f_0}),$$

problem (1.2) possesses at least two solutions $u_k^+$ and $u_k^-$ such that $u_k^+$ has exactly $k - 1$ simple zeros in $(0, 1)$ and is positive near 0, and $u_k^-$ has exactly $k - 1$ simple zeros in $(0, 1)$ and is negative near 0.

Based on the above works, of course the natural question is what would happen if $f$ is allowed to have some zeros in $\mathbb{R} \setminus \{0\}$? In this paper, we will establish the global bifurcation results about the components nodal solutions for the Kirchhoff-type problem (1.1). In order to obtain our main results, let us make the assumptions as follows:

(H1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $s_1 \leq s_2 < 0$ such that $f(0) = f(s_1) = f(s_2) = 0$, and $f(s) > 0$ for $s \in (s_1, s_2)$, $f(s) < 0$ for $s \in (-\infty, s_1) \cup (s_2, 0)$.

(H2) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $0 < s_3 \leq s_4$ such that $f(0) = f(s_3) = f(s_4) = 0$, and $f(s) > 0$ for $s \in (0, s_3) \cup (s_4, +\infty)$, $f(s) < 0$ for $s \in (s_3, s_4)$.

(H3) There exists $f_0 \in (0, \infty)$ such that $f_0 = \lim\limits_{|s| \to 0} \frac{f(s)}{s}$ uniformly with respect to all $x \in (0, 1)$.

(H4) There exists $f_\infty \in (0, \infty)$ such that $f_\infty = \lim\limits_{|s| \to +\infty} \frac{f(s)}{s^3}$ uniformly with respect to all $x \in (0, 1)$.

(H5) There exist $f_\infty = +\infty$ such that $f_\infty = \lim\limits_{|s| \to +\infty} \frac{f(s)}{s^3}$ uniformly with respect to all $x \in (0, 1)$.

The paper is organized as follows. In Section 2, we state some notations and preliminary results. Sections 3 and 4 are devoted to study the bifurcation from the trivial solution and infinity of problem (1.1), and we show the optimal intervals of $\lambda$ for which the nodal solutions exist.

## 2. Preliminary

In this section, we introduce some lemmas and well-known results which will be used in the subsequent section.

**Definition 2.1.** Let $X$ be a Banach space, $\{C_n | n = 1, 2, 3, \cdots\}$ be a family of subsets of $X$. Then the superior $D$ of $C_n$ is defined by

$$D := \limsup_{n \to \infty} C_n = \{x \in X | \ \exists n_i \subset N \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

**Definition 2.2.** The component of $M$ is the largest connected subset in $M$.

**Lemma 2.3.** *( [15]) Let X be a Banach space, $C_n$ is a component of X, assume that*

*(i) There exists $z_n \in C_n (n = 1, 2, \cdots)$ and $z^* \in X$, such that $z_n \to z^*$;*

*(ii) $\lim\limits_{n \to \infty} r_n = \infty$, where $r_n = \sup\{\|x\| : x \in C_n\}$;*

*(iii) For every $R > 0$, $(\bigcup\limits_{n=1}^{\infty} C_n) \bigcap \Omega_R$ is a relative compact set of X, where $\Omega_R = \{x \in X : \|x\| \leq R\}$.*

*Then $D := \limsup\limits_{n \to \infty} C_n$ contains an unbounded component C such that $z^* \in C$.*

Denote $Y = C[0, 1], E := \{u \in C_0^1[0, 1] : u(0) = u(1) = 0\}$ with the norm $\|u\|_\infty = \max\limits_{t \in [0,1]} |u(t)|$ and $\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty\}$, respectively.

Let $S_k^+$ denote the set of functions in $E$ which have exactly $k - 1$ interior nodal (i.e. non-degenerate) zeros in $(0, 1)$ and are positive near $t = 0$, and set $S_k^- = -S_k^+$, $S_k = S_k^+ \cup S_k^-$. Obviously, $S_k^+$ and $S_k^-$ are disjoint and open in $E$. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$.

When considering Kirchhoff-type problem, Dancer-type unilateral global bifurcation theorem is established in [1], which can be applied to similar problems.

**Lemma 2.4.** *( [1], Theorem 1.1) The pair $(a\lambda_k, 0)$ is a bifurcation point of problem*

$$\begin{cases} -(a + b\int_0^1 |u'|^2 dx)u'' = \lambda u + h(x, u, \lambda), \ x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

*where* $h : (0,1)\times\mathbb{R}^2 \to \mathbb{R}$ *is a continuous function satisfying* $\lim_{s\to 0}\frac{h(x,s,\lambda)}{s} = 0$ *uniformly for all* $x \in (0,1)$ *and* $\lambda$ *on bounded sets. Moreover, there are two distinct unbounded continua in* $\mathbb{R}\times H_0^1(0,1)$*,* $\mathfrak{C}_k^+$ *and* $\mathfrak{C}_k^-$*, consisting of the bifurcation branch* $\mathfrak{C}_k$ *emanating from* $(a\lambda_k, 0)$*, such that* $\mathfrak{C}_k^\nu \subset (\{(a\lambda_k,0)\}\cup\Phi_k^\nu)$*,* $\nu = +$ *or* $-$*.*

Let $\xi, \eta \in C(\mathbb{R},\mathbb{R})$ be such that

$$f(s) = f_0 s + \xi(s), \qquad f(s) = f_\infty s^3 + \eta(s).$$

Obviously,

$$\lim_{|s|\to 0}\frac{\xi(s)}{s} = 0 \quad\text{and}\quad \lim_{|s|\to\infty}\frac{\xi(s)}{s^3} = f_\infty \quad\text{uniformly on } [0,1],$$

$$\lim_{|s|\to 0}\frac{\eta(s)}{s} = f_0 \quad\text{and}\quad \lim_{|s|\to\infty}\frac{\eta(s)}{s^3} = 0 \quad\text{uniformly on } [0,1].$$

Let us consider

$$\begin{cases} -(a + b\int_0^1 |u'|^2dx)u'' = \lambda f_0 u + \lambda\xi(u), \ \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.1}$$

as a bifurcation problem from the trivial solution $u \equiv 0$, and

$$\begin{cases} -(a + b\int_0^1 |u'|^2dx)u'' = \lambda f_\infty u^3 + \lambda\eta(u), \ \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.2}$$

as a bifurcation problem from infinity. (2.1) and (2.2) are equivalent to the problem (1.1).

Let us discuss (2.1). According to Lemma 2.4, we can see that for each integer $k \geq 1$ and $\nu \in \{+,-\}$, there exists a continuum $C_k^\nu$ of solutions of (2.1) joining $(\frac{a\lambda_k}{f_0}, 0)$ to infinity. In addition, $C_k^\nu \setminus \{(\frac{a\lambda_k}{f_0},0)\} \subset \mathbb{R}\times S_k^\nu$.

Let us discuss (2.2). According to the proof of Theorem 1.3 of [1], we can see that for each integer $k \geq 1$ and $\nu \in \{+,-\}$, there exists a continuum $\mathcal{D}_k^\nu$ of solutions of (2.2) meeting $(\frac{b\mu_k}{f_\infty}, \infty)$. In addition, $\mathcal{D}_k^\nu \setminus \{(\frac{b\mu_k}{f_\infty},\infty)\} \subset \mathbb{R}\times S_k^\nu$.

**Remark 2.5.** *We note that when* $\lambda = 0$*, (1.1) has only trivial solution. Therefore,* $C_k^+$ *and* $C_k^-$ *are separated by the hyperplane* $\lambda = 0$*. Furthermore, we know that* $C_k^+$ *and* $C_k^-$ *are both unbounded.*

## 3. Global bifurcation results for $f_\infty \in (0,\infty)$

In this section, we will provide more details about the connected components of nodal solutions under the assumptions that $f$ has some zeros.

**Theorem 3.1.** *Let (H1)-(H4) hold. Then we have the following results:*

*(i) If* $(\lambda,u) \in C_k^+ \cup C_k^-$*, then*

$$s_2 < u(x) < s_3, \quad x \in [0,1];$$

*(ii) If* $(\lambda,u) \in \mathcal{D}_k^+ \cup \mathcal{D}_k^-$*, then either*

$$\max_{x\in[0,1]} u(x) > s_4 \quad \textit{or} \quad \min_{x\in[0,1]} u(x) < s_1.$$

*Proof.* (i) For $(\lambda, u) \in C^+_{k,+} \cup C^+_{k,-}$, we just need to prove that $\max\{u(x)|x \in [0,1]\} \neq s_3$ and $\min\{u(x)|x \in [0,1]\} \neq s_2$. Otherwise, there is $(\lambda, u) \in C^+_k \cup C^-_k$ such that

$$\max\{u(x)|x \in [0,1]\} = s_3 \tag{3.1}$$

or

$$\min\{u(x)|x \in [0,1]\} = s_2. \tag{3.2}$$

Denote

$$0 = \tau_1 < \tau_2 < \cdots < \tau_l = 1$$

as the zeros of $u$ in $[0,1]$.

If (3.1) holds, then there exists $j \in \{0, \cdots, l-1\}$ such that

$$\max\{u(x)|x \in [\tau_j, \tau_{j+1}]\} = s_3 \tag{3.3}$$

and

$$0 \le u(x) \le s_3, \quad x \in [\tau_j, \tau_{j+1}].$$

We consider the boundary value problem

$$\begin{cases} -(a + b\int_0^1 |u'|^2 dx)u'' = \lambda f(u(x)), \ x \in (\tau_j, \tau_{j+1}), \\ u(\tau_j) = u(\tau_{j+1}) = 0. \end{cases}$$

We claim that there exists a constant $m > 0$ such that

$$f(u) \le m(s_3 - u) \quad \text{and} \quad 0 \le u \le s_3 \ \text{ for all } x \in [\tau_j, \tau_{j+1}]. \tag{3.4}$$

It is seen form (H2) that the claim is true for the case $u = 0$ or $u = s_3$. Suppose on the contrary that there exists $s_3' \in (0, s_3)$ such that $f(s_3') > m(s_3 - s_3')$ for any $m > 0$. This gives that $m < \frac{f(s_3')}{s_3 - s_3'}$, which contradicts the arbitrariness of $m$.

Noting (3.4), we obtain that

$$-(a + b\int_0^1 |(s_3 - u)'|^2 dx)(s_3 - u)'' + \lambda m(s_3 - u) \ge \lambda m(s_3 - u) - \lambda f(u) \ge 0, \ x \in (\tau_j, \tau_{j+1}).$$

It is straightforward to see from $s_3 > 0$ that

$$s_3 - u(\tau_j) > 0, \quad s_3 - u(\tau_{j+1}) > 0.$$

By virtue of the strong maximum principle [16], we can show that $s_3 > u(x)$, $x \in [\tau_j, \tau_{j+1}]$. This contradicts (3.3).

If (3.2) holds, then there exists $j \in \{0, \cdots, l-1\}$ such that

$$\min\{u(x)|x \in [\tau_j, \tau_{j+1}]\} = s_2 \tag{3.5}$$

and

$$s_2 \le u(x) \le 0, \quad x \in [\tau_j, \tau_{j+1}].$$

Similarly, we claim that there exists a constant $m > 0$ such that

$$f(u) \geq m(s_2 - u) \quad \text{and} \quad s_2 \leq u \leq 0 \ \text{ for all } x \in [\tau_j, \tau_{j+1}]. \tag{3.6}$$

Noting (3.6), we obtain that

$$-(a + b \int_0^1 |(s_2 - u)'|^2 dx)(s_2 - u)'' + \lambda m(s_2 - u) \leq \lambda m(s_2 - u) - \lambda f(u) \leq 0, \ x \in (\tau_j, \tau_{j+1}).$$

It is straightforward to see from $s_2 < 0$ that

$$s_2 - u(\tau_j) < 0, \quad s_2 - u(\tau_{j+1}) < 0.$$

By virtue of the strong maximum principle [16], we can show that $s_2 < u(x)$, $x \in [\tau_j, \tau_{j+1}]$. This contradicts (3.5).

The argument of (ii) is similar to that of (i). □

**Remark 3.2.** *From Theorem 3.1, it is easy to see that*

$$||u||_\infty < \max\{|s_2|, s_3\} = s^*.$$

*Further,*

$$||u||_E < \max\{s^*, \lambda \max_{|s| \leq s^*} |f(s)|\}.$$

Combining Theorem 3.1 and Remark 3.2, by virtue of the similar argument of [10, Corollaries 2.1–2.2] with obvious changes, we conclude the following results:

**Theorem 3.3.** *Let (H1)-(H4) hold. Assume that* $\frac{a\lambda_k}{f_0} < \frac{b\mu_k}{f_\infty}$*, then*

*(i) if* $\lambda \in [\frac{a\lambda_k}{f_0}, \frac{b\mu_k}{f_\infty})$*, then problem (1.1) has at least two solutions* $u^+_{k,0}$ *and* $u^-_{k,0}$ *such that* $u^+_{k,0}$ *has exactly* $k - 1$ *zeros in (0,1) and is positive near 0,* $u^-_{k,0}$ *has exactly* $k - 1$ *zeros in (0,1) and is negative near 0;*

*(ii) if* $\lambda \in (\frac{b\mu_k}{f_\infty}, +\infty)$*, then problem (1.1) has at least four solutions* $u^+_{k,\infty}$*,* $u^-_{k,\infty}$*,* $u^+_{k,0}$ *and* $u^-_{k,0}$ *such that* $u^+_{k,\infty}, u^+_{k,0}$ *have exactly* $k - 1$ *zeros in (0,1) and are positive near 0;* $u^-_{k,\infty}, u^-_{k,0}$ *have exactly* $k - 1$ *zeros in (0,1) and are negative near 0.*

**Theorem 3.4.** *Let (H1)-(H4) hold. Assume that* $\frac{a\lambda_k}{f_0} > \frac{b\mu_k}{f_\infty}$*, then*

*(i) if* $\lambda \in (\frac{b\mu_k}{f_\infty}, \frac{a\lambda_k}{f_0}]$*, then problem (1.1) has at least two solutions* $u^+_{k,\infty}$ *and* $u^-_{k,\infty}$ *such that* $u^+_{k,\infty}$ *has exactly* $k - 1$ *zeros in (0,1) and is positive near 0,* $u^-_{k,\infty}$ *has exactly* $k - 1$ *zeros in (0,1) and is negative near 0;*

*(ii) if* $\lambda \in (\frac{a\lambda_k}{f_0}, +\infty)$*, then problem (1.1) has at least four solutions* $u^+_{k,\infty}$*,* $u^-_{k,\infty}$*,* $u^+_{k,0}$ *and* $u^-_{k,0}$ *such that* $u^+_{k,\infty}, u^+_{k,0}$ *have exactly* $k - 1$ *zeros in (0,1) and are positive near 0;* $u^-_{k,\infty}, u^-_{k,0}$ *have exactly* $k - 1$ *zeros in (0,1) and are negative near 0.*

## 4. Global bifurcation results for $f_\infty = +\infty$

**Theorem 4.1.** *Let (H1), (H3) and (H5) hold. Then,*

*(i) if $\lambda \in (0, \frac{a\lambda_k}{f_0})$, then problem (1.1) has at least two solutions $u^-_{k,\infty}$ and $u^+_k$ such that $u^+_k$ has exactly $k-1$ zeros in (0,1) and is positive near 0, $u^-_{k,\infty}$ has exactly $k-1$ zeros in (0,1) and is negative near 0;*

*(ii) if $\lambda = \frac{a\lambda_k}{f_0}$, then problem (1.1) has at least one solution $u^-_{k,\infty}$;*

*(iii) if $\lambda \in (\frac{a\lambda_k}{f_0}, +\infty)$, then problem (1.1) has at least two solutions $u^-_{k,\infty}, u^-_{k,0}$.*

*Proof.* For any $n \in \mathbb{N}^+$ and $n > -s_1$. Define the function $f^{[n]} : \mathbb{R} \to \mathbb{R}$ as follows

$$f^{[n]}(s) = \begin{cases} f(s), & |s| \le n, \\ \frac{1}{n^3} f(n) s^3, & |s| > n. \end{cases} \tag{4.1}$$

Thus $f^{[n]} \in C(\mathbb{R}, \mathbb{R})$. Further, $f^{[n]}(0) = f^{[n]}(s_1) = f^{[n]}(s_2) = 0$, and

$$(f^{[n]})_\infty = \frac{f(n)}{n^3}.$$

We can see from (H5) that $\lim_{n\to\infty} (f^{[n]})_\infty = +\infty$.

Consider the following auxiliary problem

$$\begin{cases} -(a + b\int_0^1 |u'|^2 dx)u'' = \lambda f^{[n]}(u), \ x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

Let $\eta^{[n]} \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f^{[n]}(u) = (f^{[n]})_\infty u^3 + \eta^{[n]}(u).$$

Then $\lim_{|u|\to\infty} \frac{\eta^{[n]}(u)}{u^3} = 0$ uniformly on $[0, 1]$.

We consider

$$\begin{cases} -(a + b\int_0^1 |u'|^2 dx)u'' = \lambda (f^{[n]})_\infty u^3 + \lambda \eta^{[n]}(u), \quad x \in (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{4.3}$$

as a bifurcation problem from infinity.

It is easy to see from [17, Theorem 1.6 and Corollary 1.8] that for each integer $k \ge 1$ and $n \in \mathbb{N}^+$ with $n > -s_1$, there exists a continuum $\mathscr{D}^{[n],-}_{k,\infty}$ of solutions of (4.2) meeting $(\frac{b\mu_k}{(f^{[n]})_\infty}, \infty)$ and $\mathscr{D}^{[n],-}_{k,\infty} \setminus \{(\frac{b\mu_k}{(f^{[n]})_\infty}, \infty)\} \subset (\mathbb{R} \times S^-_k)$.

Similar to the proof of Theorem 3.1, for any $(\lambda, u) \in \mathscr{D}^{[n],-}_{k,\infty}$, we obtain that $u(x_0) < s_1$ for some $x_0 \in (0, 1)$. Further, it is direct to check that

$$\sup\{\lambda \big| (\lambda, u) \in \mathscr{D}^{[n],-}_{k,\infty}\} = \infty. \tag{4.4}$$

It remains to be shown that for each $n \in \mathbb{N}^+$ with $n > -s_1$, there exists a positive constant $M$ such that

$$\sup\{\|u\|_\infty \big| (\lambda, u) \in \mathscr{D}^{[n],-}_{k,\infty} \text{ and } \lambda \in I\} \le M, \tag{4.5}$$

where $I \subset (\frac{b\mu_k}{(f^{[n]})_\infty}, \infty)$ is a closed and bounded interval.

Suppose on the contrary that there is a sequence $\{(\kappa_l, u_l)\} \subset \mathcal{D}^{[n],-}_{k,\infty} \cap (I \times E)$ satisfying

$$\|u_l\|_E \to \infty \quad \text{as} \quad l \to \infty. \tag{4.6}$$

We *claim* that

$$\|u_l\|_\infty \to \infty \quad \text{as} \quad l \to \infty. \tag{4.7}$$

In fact, it is straightforward to see that $(\kappa_l, u_l)$ satisfies

$$\begin{cases} -(a + b\int_0^1 |u_l'|^2 dx)u_l'' = \kappa_l f^{[n]}(u_l), \ x \in (0,1), \\ u_l(0) = u_l(1) = 0. \end{cases}$$

Thus, we know that there exists $x_l \in (0,1)$ such that $u_l'(x_l) = 0$ and

$$u_l'(x) = -\int_{x_l}^x \kappa_l \frac{1}{a + b\int_0^1 |u_l(\tau)'|^2 d\tau} f^{[n]}(u_l(s))ds.$$

There is a positive constant $N$ such that $\|u_l\|_\infty \leq N$ for each $l$. Further, combining the definition of $f^{[n]}$ and (4.7), gives

$$\|u_l'\|_\infty \leq N' \quad \text{for some} \quad N' > 0 \ \text{ and all } \ l.$$

This is a contradiction. Therefore, we complete the proof of (4.7).

Let $0 = \tau(0,l) < \tau(1,l) < \cdots < \tau(k,l) = 1$ denote the zeros of $u_l$. Taking a subsequence and relabeling if necessary, we assume that for each $i \in \{0, 1, \cdots, k\}$,

$$\lim_{l\to\infty} \tau(i,l) = \tau(i,\infty).$$

Moreover, it is interesting to see that there exists $\alpha \in \mathbb{R}$ such that

$$\min\{(-1)^i u_l(x) : x \in I(i,l)\} \geq \alpha \max\{|u_l(x)| : x \in [\tau(i,l), \tau(i+1,l)]\}, \tag{4.8}$$

where $I(i,l) = [\tau(i,l) + \frac{\tau(i+1,l)-\tau(i,l)}{4}, \tau(i+1,l) - \frac{\tau(i+1,l)-\tau(i,l)}{4}]$. By virtue of (4.7) and (4.8), we get that there is $i' \in \{0, 1, \cdots, k-1\}$ and a closed interval $I_1 \subset (\tau(i',\infty), \tau(i'+1,\infty))$ with positive length such that

$$(-1)^{i'} u_l(x) \to \infty \quad \text{as} \quad l \to \infty \ \text{ uniformly for } \ x \in I_1. \tag{4.9}$$

Since $\{\kappa_l\} \subset I$, then there must exist a $\kappa^*$ with $\kappa^* > \frac{b\mu_k}{(f^{[n]})_\infty}$ such that $\lim_{l\to\infty} \kappa_l = \kappa^*$. In view of the above arguments, we obtain that

$$\lim_{l\to\infty} \kappa_l \frac{f^{[n]}(u_l)}{u_l^3} = \kappa^*(f^{[n]})_\infty \ \text{ uniformly for } \ x \in I_1. \tag{4.10}$$

Since $\kappa^*(f^{[n]})_\infty > b\mu_k$ and $-(a + b\int_0^1 |u_l'|^2 dx)u_l'' = \kappa_l \frac{f(u_l)}{u_l^3} u_l^3$ for $x \in I_1$. We conclude that $u_l$ must changes its sign on $I_1$ with $l$ large enough. This is a contradiction. Therefore, we complete the proof of (4.5).

Next we prove that $\mathcal{D}_{k,\infty}^{[n],-}$ satisfies all the conditions of Lemma 2.3. Since

$$\lim_{n\to\infty}\frac{b\mu_k}{(f^{[n]})_\infty}=\lim_{n\to\infty}\frac{b\mu_k}{\frac{f(n)}{n^3}},$$

this together with (4.5) gives that there is a closed interval $J\subset(0,\infty)$ and a positive constant $\gamma$. Denote $\Sigma=\{u\in E|-s_1<\|u\|_\infty<\gamma\}$, thus there must exist $u_{n_j}\in\mathcal{D}_{k,\infty}^{[n],-}\cap(J\times\Sigma)$ such that $u_{n_j}\to u^*$. Therefore, condition (i) in Lemma 2.3 is satisfied. It is clear that

$$r_n=\sup\{\lambda+\|u\|_E:(\lambda,u)\in\mathcal{D}_{k,\infty}^{[n],-}\}\to\infty.$$

Thus, (ii) in Lemma 2.3 holds.

According to the Arzéla-Ascoli Theorem and the definition of $f^{[n]}$, (iii) is obviously valid. Therefore, with the help of Lemma 2.3, we get that $\lim\sup_{n\to+\infty}\mathcal{D}_{k,\infty}^{[n],-}$contains an unbounded connected components $\widetilde{\mathcal{D}}_{k,\infty}^{-}$ with

$$\sup\{\lambda\,|(\lambda,u)\in\widetilde{\mathcal{D}}_{k,\infty}^{-}\}=\infty.$$

In view of the similar arguments of the proof of Theorem 3.1, for $(\lambda,u)\in\widetilde{\mathcal{D}}_{k,\infty}^{-}$, one has that $u(x_0)<s_1$ for some $x_0\in(0,1)$.

Next we prove that $\lim_{(\lambda,u)\in\widetilde{\mathcal{D}}_{k,\infty}^{-},\|u\|_E\to\infty}\lambda=0$. Suppose on the contrary that there exists $\{(\lambda_n,u_n)\}\subset\widetilde{\mathcal{D}}_{k,\infty}^{-}$ such that $\|u_n\|_E\to\infty,\lambda_n\geq\delta$ for some constant $\delta>0$. Thus, (4.7)-(4.9) hold. We can see from (H5) and (4.9) that

$$\lim_{n\to\infty}\frac{f(u_n)}{u_n^3}=\infty\quad\text{uniformly on }\ x\in I_1.$$

This ensures that for all $n$ sufficiently large, the solution $u_n$ of

$$-(a+b\int_0^1|u_n'|^2dx)u_n''=\lambda_n\frac{f(u_n)}{u_n^3}u_n^3$$

must change its sign on $I_1$. This contradicts (4.9). Thus, $\lim_{(\lambda,u)\in\widetilde{\mathcal{D}}_{k,\infty}^{-},\|u\|_E\to\infty}\lambda=0$. According to the above arguments, we conclude that

$$\text{Proj}_{\mathbb{R}}\mathcal{D}_{k,\infty}^{-}=(0,\infty).\tag{4.11}$$

Moreover, it is straightforward from Theorem 3.1 to see that for any $(\lambda,u)\in C_{k,0}^{-}$,

$$s_2<u(x)<0.$$

Remark 3.2 yields that the set $\{(\rho,z)\in C_{k,0}^{-}\,|\,\rho\in[0,h]\}$ is bounded for any fixed $h\in(0,\infty)$. Combining the above with the fact that $C_{k,0}^{-}$ joins $(\frac{a\lambda_k}{f_0},0)$ to infinity gives

$$\text{Proj}_{\mathbb{R}}C_{k,0}^{-}\supset(\frac{a\lambda_k}{f_0},+\infty).\tag{4.12}$$

On the other hand, according to the Lemma 2.4, for each integer $k\geq1$, we conclude that there exists unbounded continuum $C_k^+$ joining $(\frac{a\lambda_k}{f_0},0)$ to infinity such that $C_k^+\setminus\{(\frac{a\lambda_k}{f_0},0)\}\subset(\mathbb{R}\times S_k^+)$. Next, we prove that $C_k^+$ joins $(\frac{a\lambda_k}{f_0},0)$ to $(0,\infty)$.

Let $\{(\rho_l, u_l)\} \subset C_k^+$ be such that $|\rho_l| + \|u_l\|_E \to \infty$ as $l \to \infty$. Suppose that $\{\|u_l\|_E\}$ is bounded, then we can assume that $\lim_{l\to\infty} \rho_l = \infty$. Since $\Sigma_{i=0}^{k-1}[\tau(i+1,\infty) - \tau(i,\infty)] = 1$, this implies that there is $i_0 \in \{0, \cdots, k-1\}$ such that $\tau(i_0,\infty) < \tau(i_0+1,\infty)$. Then there must exist $i_0 \in \mathbb{N}$ and a closed interval $I_0 \subset (\tau(i_0,\infty), \tau(i_0+1,\infty))$ with positive length such that $I_0 \subset (\tau(i_0,m), \tau(i_0+1,m))$ for all $i \geq i_0$. Further,

$$(-1)^{i_0+1} u_l > 0 \quad \text{for all} \quad i \geq i_0,\ x \in I_0. \tag{4.13}$$

In view of the relations $\lim_{l\to\infty} \rho_l = \infty$ and $-(a + b\int_0^1 |u_l'|^2 dx)u_l'' = \rho_l f(u_l)$ for $x \in I_0$, we get that $u_l$ must change its sign on $I_0$ if $l$ is large enough. This contradicts (4.13). Therefore, $\{\|u_l\|_E\}$ is unbounded. Similarly, we can show that $\lim_{l\to\infty} \rho_l = 0$ and

$$\mathrm{Proj}_{\mathbb{R}} C_k^+ \supset (0, \frac{a\lambda_k}{f_0}). \tag{4.14}$$

By virtue of the above arguments, it is easy to get the desired results.

□

As an immediate consequence of Theorem 4.1, we have the second main result in this section read as follows:

**Theorem 4.2.** *Let (H2), (H3) and (H5) hold. Then,*

*(i) if $\lambda \in (0, \frac{a\lambda_k}{f_0})$, then problem (1.1) has at least two solutions $u_{k,\infty}^+$ and $u_k^-$ such that $u_{k,\infty}^+$ has exactly $k-1$ zeros in (0,1) and is positive near 0, $u_k^-$ has exactly $k-1$ zeros in (0,1) and is negative near 0;*

*(ii) if $\lambda = \frac{a\lambda_k}{f_0}$, then problem (1.1) has at least one solution $u_{k,\infty}^+$;*

*(iii) if $\lambda \in (\frac{a\lambda_k}{f_0}, +\infty)$, then problem (1.1) has at least two solutions $u_{k,\infty}^+, u_{k,0}^+$.*

## Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable suggestions. This work is supported by the Natural Science Foundation of Gansu Province (20JR10RA086).

## Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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