Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8256–8275.
DOI:10.3934/math.2021478
Received: 30 March 2021
Accepted: 14 May 2021
Published: 26 May 2021

Research article

On properties of solutions of complex differential equations in the unit disc

Jianren Long*, **Pengcheng Wu and Sangui Zeng**

School of Mathematical Science, Guizhou Normal University, Guiyang, 550025, China

* **Correspondence:** Email: longjianren2004@163.com.

Abstract: The properties of solutions of the following differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z)$$

are studied, where $A_j(z)$ and $F(z)$ are analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, $j = 0, 1, \ldots, k-1$. First, the growth of solutions of the equation is estimated. Second, some coefficient's conditions such that the solution of the equation belong to Hardy type spaces are showed. Finally, some related question are studied in this paper.

Keywords: complex differential equations; order of growth; Hardy space; analytic function; the unit disc
Mathematics Subject Classification: 34M10, 30D35

1. Introduction

In 1982, Pommerenke studied the properties of solutions of second order differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an analytic function in $\mathbb{D} = \{z : |z| < 1\}$, some conditions of $A(z)$ such that every solution of (1.1) belong to the Hardy space H^2 have been obtained in [15]. The concepts concerning Hardy space and other related spaces will be given below. Later on, Heittokangas investigated the properties of solutions of higher order differential equation

$$f^{(k)} + A(z)f = 0, \tag{1.2}$$

where $A(z)$ is analytic function in $\mathbb{D}$, some conditions of the coefficient $A(z)$ such that all solutions of (1.2) belong to some analytic spaces, for example, weighted Hardy space, Bloch space and so on, see [7] for more details. At the same time, the following equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z) \tag{1.3}$$

was also studied, where $A_j(z)$ and $F(z)$ are analytic in $\mathbb{D}$, $j = 0, 1, \ldots, k-1$, $k \geq 2$. It was shown that all solutions of (1.3) are analytic in $\mathbb{D}$, for more details refers to [7, Theorem 7.1, p. 42]. From that time, it has been very interesting to investigate function space's properties of solutions of linear differential equations in $\mathbb{D}$, and more and more results have been obtained by many different researchers, for example, see [5, 8, 9, 12, 14, 16] and references therein. Here the properties of solutions of (1.3) are studied again, in which the growth of solutions and function space's properties of solutions are considered. According to the function space's properties, usually, we consider two kind questions, one is called direct problem, in which we ask the properties of solution by using condition of coefficients. Another aspect is called the inverse problem, in which we ask the properties of coefficients by using the conditions of solutions.

In order to state our results, some concepts will be recalled. Let $H(\mathbb{D})$ denotes a set of all holomorphic functions in $\mathbb{D}$. Let $0 < p < \infty$, the Hardy space H^p is defined as

$$H^p = \left\{ f(z) \in H(\mathbb{D}) : \|f\|_{H^p} = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi \right)^{\frac{1}{p}} < \infty \right\},$$

see [4] for more details. For the case of $p = \infty$, $f(z) \in H(\mathbb{D})$ is said to belong to H^∞ if and only if

$$\sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

Obviously, H^∞ denotes the set of bounded analytic functions in $\mathbb{D}$. For $0 \leq q < \infty$, the weighted Hardy space H_q^∞ is defined as

$$H_q^\infty = \left\{ f(z) \in H(\mathbb{D}) : \|f\|_{H_q^\infty} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^q < \infty \right\}.$$

Obviously, $H_0^\infty = H^\infty$. Moreover, $f(z)$ is said to belong to G_p if

$$p = \inf\{q > 0 : f(z) \in H_q^\infty\},$$

more details of G_p can be found in [2].

The Bloch space is defined as

$$\mathcal{B} = \{f(z) \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty\}.$$

For $\alpha > 0$, the α-Bloch space is also defined by

$$\mathcal{B}^\alpha = \{f(z) \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty\},$$

which can be found in [17].

Now, we introduce the $\mathcal{E}^{\beta}$ space and the order of growth of $f(z) \in H(\mathbb{D})$. Let $\beta \in (0, \infty)$ be a constant. Then $f(z) \in H(\mathbb{D})$ is said to belong to $\mathcal{E}^{\beta}$ space if and only if

$$|f(z)| \le \exp\left(\frac{\alpha}{(1-r)^{\beta}}\right)$$

for some constant $\alpha \in (0, \infty)$, which can be found in [7, p. 12]. Let $f(z) \in H(\mathbb{D})$. Then the order $\sigma_M(f)$ of $f(z)$ can be defined by

$$\sigma_M(f) = \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f)}{\log \frac{1}{1-r}},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$. For a real number $x \ge 0$, the positive logarithm is defined as follows,

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x, & x \ge 1, \\ 0, & 0 \le x < 1, \end{cases}$$

which can be found in [6, p. 3].

We say that A and B is comparable if there is a positive constant C such that $\frac{B}{C} \le A \le CB$. The paper is organized as follows, the growth of solutions of differential equations is estimated in Section 2. The direct problem and inverse problem of differential equations are studied in Section 3 and Section 4 respectively. Finally, the higher order derivative of solutions of differential equations is characterized in Section 5.

2. Growth of solutions

Auxiliary results. In order to prove Theorems 2.3 below. The following Lemma 2.1 plays an important role in dealing with the coefficients A_j of (1.3).

Lemma 2.1. *[17] Let $A(z)$ be an analytic function in $\mathbb{D}$, $1 < \alpha < \infty$ and $n \in \mathbb{N}$. Then the following quantities are comparable:*

(i) $\|A\|_{H^{\infty}_{\alpha-1}}$,

(ii) $\|A\|_{\mathcal{B}^{\alpha}} + |A(0)|$,

(iii) $\sup_{z \in \mathbb{D}} |A^{(n)}(z)|(1-|z|^2)^{n-1+\alpha} + \sum_{j=0}^{n-1} |A^{(j)}(0)|$.

In 2003, Chyzhykov-Gundersen-Heittokangas investigated the growth of solutions of (1.2) when the coefficient $A(z) \in G_p$, and obtained that every solution $f(z)$ of (1.3) satisfies $\sigma_M(f) \le \frac{p}{k} - 1$ for $p \ge k$, see [2, Theorem 2.2, p. 738] for more details. In 2010, Chyzhykov-Heittokangas-Rättyä studied the growth of solutions of (1.3) for $F(z) = 0$, and obtained more sharp results as Theorem 2.2 below.

Theorem 2.2. *[3, Theorem 1.4, p. 147-148] Let $A_j(z) \in G_{p_j}$, $j = 0, 1, \ldots, k-1$, $F(z) = 0$ and $1 \le \alpha < \infty$, $p_k = 0$. If $f(z)$ is a solution of* (1.3)*, then the following statements hold.*

(i) $\sigma_M(f) \le \left\{0, \max_{0 \le j \le k-1} \left\{\frac{p_j}{k-j} - 1\right\}\right\}$ *and*

$$\max\{\sigma_M(f), 1\} \ge \min_{1 \le j \le k} \left\{\frac{p_0 - p_j}{j} - 1\right\};$$

(ii) *Suppose that* $\min\limits_{1\le j\le k}\left\{\frac{p_0-p_j}{j}\right\}\ge 2$, *then* $\sigma_M(f)\le\alpha$ *if and only if* $\max\limits_{0\le j\le k-1}\left\{\frac{p_j}{k-j}-1\right\}\le\alpha$;

(iii) *Suppose that* $\min\limits_{1\le j\le k}\left\{\frac{p_0-p_j}{j}\right\}\ge 2$ *holds. If* $n\in\{0,1,\ldots,k-1\}$ *is the smallest index for which* $\frac{p_n}{k-n}=\max\limits_{0\le j\le k-1}\left\{\frac{p_j}{k-j}-1\right\}$, *then in every solution basis of* (1.3) *there are at least* $k-n$ *linearly independent solutions* $f(z)$ *such that* $\sigma_M(f)=\max\limits_{0\le j\le k-1}\left\{\frac{p_j}{k-j}-1\right\}$.

Main result. Here we estimate the growth of solutions of (1.3) for the case of $F(z)\neq 0$, and prove the following result.

Theorem 2.3. *Let* $A_j(z)$ *and* $F(z)$ *be analytic in* $\mathbb{D}$, $j=0,1,\ldots,k-1$. *Then the following statements hold.*

(i) *If* $A_j(z)\in H_q^\infty$ *and* $F(z)\in H_q^\infty$, *where* $q\ge 1$, *then every solution* $f(z)$ *of* (1.3) *satisfies* $\sigma_M(f)\le q-1$;

(ii) *If* $A_j(z)\in H_q^\infty$ *and* $F(z)\in\varepsilon^\beta$, *where* $q\ge 1$ *and* $\beta>0$, *then every solution* $f(z)$ *of* (1.3) *satisfies* $\sigma_M(f)\le\max\{q-1,\beta\}$.

Proof. (i) Since $A_j(z)\in H_q^\infty$ and $F(z)\in H_q^\infty$, then by Lemma 2.1, there exist constant $C_1>0$ and $C_2>0$ such that for any natural numbers n and $j=0,1,\ldots,k-1$,

$$|A_j^{(n)}(z)|\le\frac{C_1}{(1-|z|^2)^{q+n}},\quad |F(z)|\le\frac{C_2}{(1-|z|^2)^q}. \tag{2.1}$$

It follows from [10, Theorem 1 (a)] that there exist constant $C_3>0$ and $C_4>0$ such that for all $z=re^{i\theta}\in\mathbb{D}$,

$$\begin{aligned}|f(re^{i\theta})|\le&\left(C_3+\frac{1}{(k-1)!}\int_0^r|F(te^{i\theta})|(1-t)^{k-1}dt\right)\\&\cdot\exp\left(C_4\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r|A_j^{(n)}(te^{i\theta})|(1-t)^{k-j+n-1}dt\right).\end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), there exist constant $C_5>0$ and constant $C_6>0$ such that

$$|f(re^{i\theta})|\le\begin{cases}\dfrac{C_5}{(q-k)(1-r)^{q-k}}\cdot\exp\left(\dfrac{C_6}{(q-1)(1-r)^{q-1}}\right), & q>k,\\ C_5\log\dfrac{1}{1-r}\cdot\exp\left(\dfrac{C_6}{(q-1)(1-r)^{q-1}}\right), & q=k,\\ \dfrac{C_5}{k-q}\cdot\exp\left(\dfrac{C_6}{(q-1)(1-r)^{q-1}}\right), & 1<q<k,\\ \dfrac{C_5}{(k-q)(1-r)^{C_6}}, & q=1,\end{cases}$$

which implies that

$$\sigma_M(f)\le q-1.$$

(ii) Since $F(z) \in \varepsilon^{\beta}$, then there exists a constant $\alpha > 0$ such that for all $z = re^{i\theta} \in \mathbb{D}$,

$$|F(z)| \leq \exp\left(\frac{\alpha}{(1-|z|)^{\beta}}\right). \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), there exists a constant $C_7 > 0$ such that

$$|f(re^{i\theta})| \leq \begin{cases} \exp\left(\dfrac{C_7}{(1-r)^{\beta}} + \dfrac{C_6}{(q-1)(1-r)^{q-1}}\right), & q > 1, \\ \exp\left(\dfrac{C_7}{(1-r)^{\beta}}\right) \cdot \dfrac{1}{(1-r)^{C_6}}, & q = 1, \end{cases}$$

which implies that

$$\sigma_M(f) \leq \max\{q-1, \beta\}.$$

□

The following example shows that Theorem 2.3 is sharpness.

Examples. We consider the differential equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = F(z). \tag{2.4}$$

(1) Let $A_0(z) = 0$, $A_1(z) = \frac{6}{(1-z)^2}$, $A_2(z) = \frac{-2}{(1-z)^3} + \frac{-6}{1-z}$, $F(z) = \frac{12}{(1-z)^2} + 54$. Obviously, $f(z) = e^{\frac{1}{(1-z)^2}} + (z-1)^3$ is a solution of (2.4). It is easy to see that $A_0(z), A_1(z), A_2(z), F(z) \in H_3^{\infty}$ and $\sigma_M(f) = 2$.

(2) Let $A_0(z) = \frac{54}{(1-z)^3}$, $A_1(z) = \frac{6}{(1-z)^2}$, $A_2(z) = \frac{-2}{(1-z)^3} + \frac{-6}{1-z}$ and $F(z) = \frac{54}{(1-z)^3}e^{\frac{1}{(1-z)^2}} + \frac{12}{(1-z)^2}$. It is easy to see that $A_0(z), A_1(z), A_2(z) \in H_3^{\infty}$, $F(z) \in \varepsilon^2$ and $f(z) = e^{\frac{1}{(1-z)^2}} + (z-1)^3$ is a solution of (2.4) with $\sigma_M(f) = 2$.

3. Direct problem

Auxiliary results. Here we study the properties of function space of solution of (1.3) by limiting the condition of coefficients. In order to deal with the term $F(z)$ of (1.3), the Lemma 3.1 below is needed.

Lemma 3.1. *[13, Lemma 2.5] Let $A(z)$ be analytic in $\mathbb{D}$. If $A^{(k)}(z) \in H^p(\frac{1}{k} \leq p \leq \infty, k \geq 2)$, then $A(z) \in H^{\infty}$.*

The following Lemma 3.2 can be proved by using the similar reason as in the proof of [11, Lemma 5.10 (Gronwall), p. 86], here we omit the details.

Lemma 3.2. *Let u and v be nonnegative integrable functions in $[0,1)$, and let $c > 0$ be a constant. If*

$$u(t) \leq c + \int_0^t u(s)v(s)ds, \quad t \in [0,1),$$

then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right), \quad t \in [0,1).$$

Lemma 3.3. *Let $g(\xi)$ be annalytic in $\mathbb{D}$. For any z, $z_0 \in \mathbb{D}$ and $|\xi| \leq |z|$, set $h(z) = \int_{z_0}^{z}(z-\xi)^k g(\xi)d\xi$, where $k \in \mathbb{N}^+$. Then*

$$h^{(n)}(z) = \frac{k!}{(k-n)!}\int_{z_0}^{z}(z-\xi)^{k-n}g(\xi)d\xi$$

holds for any $n \in \mathbb{N}$ and $0 \leq n \leq k$.

Proof. If $k = 1$, then

$$\begin{aligned} h'(z) &= \left(\int_{z_0}^{z}(z-\xi)g(\xi)d\xi\right)' \\ &= \int_{z_0}^{z} g(\xi)d\xi + \sum_{n=0}^{1} C_1^n z^n(-z)^{1-n}g(z) \\ &= \int_{z_0}^{z} g(\xi)d\xi. \end{aligned}$$

Obviously, the conclusion holds.

If $k = 2$, then

$$\begin{aligned} h'(z) &= \left(\int_{z_0}^{z}(z-\xi)^2 g(\xi)d\xi\right)' \\ &= \left(\int_{z_0}^{z}(z^2 + 2z(-\xi) + \xi^2)g(\xi)d\xi\right)' \\ &= 2\int_{z_0}^{z}(z-\xi)g(\xi)d\xi + \sum_{n=0}^{2} C_1^n z^n(-z)^{2-n}g(z) \\ &= 2\int_{z_0}^{z}(z-\xi)g(\xi)d\xi. \end{aligned}$$

And by the case $k = 1$, we get

$$h''(z) = \left(2\int_{z_0}^{z}(z-\xi)g(\xi)d\xi\right)' = 2\int_{z_0}^{z} g(\xi)d\xi.$$

Obviously, the conclusion holds.

If $k > 2$, then

$$\begin{aligned} h(z) &= \int_{z_0}^{z}(z-\xi)^k g(\xi)d\xi \\ &= \int_{z_0}^{z}\sum_{n=0}^{k} C_k^n z^n(-\xi)^{k-n}g(\xi)d\xi \\ &= \sum_{n=0}^{k} C_k^n z^n \int_{z_0}^{z}(-\xi)^{k-n}g(\xi)d\xi. \end{aligned}$$

Therefore,

$$\begin{aligned} h'(z) &= k\sum_{n=1}^{k} C_k^n \frac{n}{k} z^{n-1}\int_{z_0}^{z}(-\xi)^{k-n}g(\xi)d\xi + \sum_{n=0}^{k} C_k^n z^n(-z)^{k-n}g(z)\\ &= k\sum_{n=1}^{k} C_{k-1}^{n-1} z^{n-1}\int_{z_0}^{z}(-\xi)^{k-n}g(\xi)d\xi + \sum_{n=0}^{k} C_k^n z^n(-z)^{k-n}g(z)\\ &= k\int_{z_0}^{z}(z-\xi)^{k-1}g(\xi)d\xi + \sum_{n=0}^{k} C_k^n z^n(-z)^{k-n}g(z)\\ &= k\int_{z_0}^{z}(z-\xi)^{k-1}g(\xi)d\xi. \end{aligned}$$

Then we summarize that

$$\begin{aligned} &h''(z) = k(k-1)\int_{z_0}^{z}(z-\xi)^{k-2}g(\xi)d\xi,\\ &\cdots\\ &h^{(n)}(z) = k(k-1)\cdots(k-n+1)\int_{z_0}^{z}(z-\xi)^{k-n}g(\xi)d\xi. \end{aligned}$$

By these equalities above, this proof is completed. □

In 2014, the higher order non-homogenous linear differential equation

$$f^{(k)} + A(z)f = F(z) \tag{3.1}$$

is studied by Li-Xiao in [13], where $A(z)$ and $F(z) \in H(\mathbb{D})$, which is improvement of previous results from [7, Theorem 4.3, p. 21].

Theorem 3.4. *[13, Theorem 1.10] Let $A(z)$ and $F(z)$ be analytic in $\mathbb{D}$ satisfying*

$$|A(z)| \le \frac{\alpha}{(1-|z|)^{\beta}}, \quad F(z) \in H^p,$$

where $\alpha > 0$ and $\beta \ge 0$ are finite constants, and $\frac{1}{k} \le p \le +\infty$. If $f(z)$ is a solution of (3.1), then the following statements hold.

(i) *If $0 \le \beta < k$, then $f(z) \in H^{\infty}$;*
(ii) *If $\beta = k$, then $f(z) \in H^{\infty}_{\frac{\alpha}{(k-1)}}$;*
(iii) *If $k < \beta < \infty$, then $f(z) \in \varepsilon^{\beta-k}$.*

Main results. Here we investigate the properties of solutions of (1.3) similarly to Theorem 3.4.

Theorem 3.5. *Let $A_j(z)$ and $F(z)$ be analytic in $\mathbb{D}$ satisfying*

$$|A_j(z)| \le \frac{\alpha}{(1-|z|)^{\beta-j}}, \quad F(z) \in H^p,$$

where $\alpha > 0$ and $\beta \ge 0$ are finite constants, and $\frac{1}{k} \le p \le +\infty$, $j = 0, 1, \ldots, k-1$. If $f(z)$ is a solution of (1.3), then the following statements hold.

(i) *If* $0 \le \beta < k$, *then* $f(z) \in H^{\infty}$;
(ii) *If* $\beta = k$, *then* $f(z) \in H_q^{\infty}$, *where* q *only depending on* k, α *and* β;
(iii) *If* $\beta > k$, *then* $f(z) \in \varepsilon^{\beta-k}$.

Proof. Set $g(z) = \frac{1}{(k-1)!}\int_0^z (z-\xi)^{k-1}F(\xi)d\xi,\ z \in \mathbb{D}$. By Lemma 3.3, we get

$$g^{(k)}(z) = F(z).$$

Combining Lemma 3.1 and $F(z) \in H^p$, we have

$$g(z) = \frac{1}{(k-1)!}\int_0^z (z-\xi)^{k-1}F(\xi)d\xi \in H^{\infty}.$$

Hence, there exists a constant $C_1 > 0$, such that

$$|g(z)| = \left|\frac{1}{(k-1)!}\int_0^z (z-\xi)^{k-1}F(\xi)d\xi\right| \le C_1. \tag{3.2}$$

It follows from [10, Theorem 9, p. 152-153] that

$$\begin{aligned} f(z) = &\sum_{n=0}^{k-1} c_n(z-z_0)^n + \frac{1}{(k-1)!}\int_{z_0}^z F(\xi)(z-\xi)^{k-1}d\xi \\ &+ \sum_{j=0}^{k-1}\sum_{n=0}^{j} d_{j,n}\int_{z_0}^z A_j^{(n)}(\xi)f(\xi)(z-\xi)^{k-j+n-1}d\xi, \end{aligned}$$

where the constants $c_n \in \mathbb{C}$ depend on the initial values of $f(z)$, $f'(z)$,..., $f^{(k-1)}(z)$, the constants $d_{j,n} \in \mathbf{Q}^+$, and the path of integration is a piecewise smooth curve in $\mathbb{D}$ joining z_0 and z. Let $z_0 = 0$, $z = re^{i\theta}$ and $\xi = te^{i\theta}$. Combining this and (3.2), we obtain

$$|f(z)| \le C_2 + C_3\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r \left|A_j^{(n)}(te^{i\theta})\right| \cdot \left|f(te^{i\theta})\right|(1-t)^{k-j+n-1}d\xi,$$

where C_2 and C_3 are positive constants. It follows from this and Lemma 3.2 that

$$|f(z)| \le C_2 \exp\left(C_3\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r \left|A_j^{(n)}(te^{i\theta})\right|(1-t)^{k-j+n-1}dt\right). \tag{3.3}$$

Next, we divide into three cases to estimate the derivative of the coefficients A_j.
Case 1: If $\beta = 0$, it is easy to see that

$$A_j(z) \in H^{\infty} \subset H_{q_0}^{\infty}, \quad j_1 = 0, 1, \ldots, k-1$$

holds for any $q_0 \in (0, 1)$. Then, there exists a constant $C_4' > 0$ such that for $n \in N$, and any $q_0 \in (0, 1)$,

$$|A_j^{(n)}(z)| \le \frac{C_4'}{(1-|z|^2)^{q_0+n}}, \quad j = 0, 1, \ldots, k-1. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$|f(z)| < +\infty \tag{3.5}$$

for $\beta = 0$. Therefore, $f(z) \in H^\infty$ for $0 = \beta < k$.

Case 2: If $\max\{j : A_j \neq 0\} \geq \beta > 0$, then there exist a constant $s \in \{j : A_j \neq 0\}$ satisfying $s-1 < \beta \leq s$. By using the similar reason as in the proof of case 1, we get

$$A_{j_1}(z) \in H^\infty_{\beta-j_1}, \quad j_1 = 0, 1, \ldots, s-1,\ j_1 < \beta$$

and for any $q_0 \in (0, 1)$

$$A_{j_2}(z) \in H^\infty_{q_0}, \quad j_2 = s, s+1, \ldots, k-1,\ j_2 \geq \beta.$$

By Lemma 2.1, there exists a constant $C_4'' > 0$ such that for every $n \in \mathbb{N}$,

$$|A^{(n)}_{j_1}(z)| \leq \frac{C_4''}{(1-|z|^2)^{\beta+n-j_1}} \tag{3.6}$$

and for any $q_0 \in (0, 1)$,

$$|A^{(n)}_{j_2}(z)| \leq \frac{C_4''}{(1-|z|^2)^{q_0+n}}. \tag{3.7}$$

Combining (3.3), (3.6) and (3.7), we get

$$|f(z)| < +\infty \tag{3.8}$$

for $\max\{j : A_j \neq 0\} \geq \beta > 0$. Therefore, $f(z) \in H^\infty$ for $0 < \beta \leq \max\{j : A_j \neq 0\} < k$.

Case 3: If $\max\{j : A_j \neq 0\} < \beta$, then it follows from $|A_j(z)| \leq \frac{\alpha}{(1-|z|)^{\beta-j}}$ that

$$A_j(z) \in H^\infty_{\beta-j}, \quad j = 0, 1, \ldots, k-1.$$

Then, by Lemma 2.1, there exists a constant $C_4''' > 0$ such that for every $n \in \mathbb{N}$ and $j = 0, 1, \ldots, k-1$,

$$|A^{(n)}_j(z)| \leq \frac{C_4'''}{(1-|z|^2)^{\beta+n-j}}. \tag{3.9}$$

It follows from (3.3) and (3.9) that there exists a constant $C_5 > 0$ such that for all $z = re^{i\theta} \in \mathbb{D}$,

$$|f(z)| \leq C_2 \exp\left(\int_0^r \frac{C_5}{(1-t)^{\beta-k+1}} dt\right),$$

and then

$$|f(z)| \leq \begin{cases} C_2 \exp\left(C_5 \log \dfrac{1}{1-r}\right), & \beta = k, \\ C_2 \exp\left(\dfrac{1}{\beta-k} \cdot \dfrac{C_5}{(1-r)^{\beta-k}}\right), & \beta > k, \\ C_2 \exp\left(\dfrac{C_5}{k-\beta}\right), & \max\{j : A_j \neq 0\} < \beta < k, \end{cases} \tag{3.10}$$

which implies that

$$|f(z)| < +\infty \tag{3.11}$$

for $\max\{j : A_j \neq 0\} < \beta < k$. Therefore, $f(z) \in H^\infty$ for $\max\{j : A_j \neq 0\} < \beta < k$.

Therefore, by (3.5), (3.8) and (3.11), we get the conclusions (i) holds when $0 \leq \beta < k$. It follows from (3.10) that the conclusions (ii) holds when $\beta = k$; the conclusions (iii) holds when $\beta > k$. This proof is completed. □

The following examples show Theorem 3.5 is sharpness.

Examples. (1) Let

$$A_0(z) = \frac{1}{(1-z)^{\frac{3}{2}}},\ A_1(z) = \frac{8}{(1-z)^{\frac{3}{2}}},\ A_2(z) = \frac{18}{(1-z)^{\frac{1}{2}}}$$

and

$$F(z) = \frac{1}{(1-z)^{\frac{3}{2}}} + 1.$$

It follows from [7, Lemma 1.1.2, p. 8] that $F(z) \in H^p$ for any $p \in [\frac{1}{2}, \frac{2}{3})$, thus $A_j(z)$ and $F(z)$ satisfy the conditions of Theorem 3.5(i) with $\beta = \frac{5}{2}$, where $j = 0, 1, 2$. Obviously, $f(z) = (1-z)^{\frac{3}{2}} + \frac{5}{8}$ is a solution of (2.4) and $f(z) \in H^\infty$.

(2) Let

$$A_0(z) = \frac{6}{(1-z)^3},\ A_1(z) = \frac{-6}{(1-z)^2},\ A_2(z) = \frac{-3}{1-z},\ F(z) = -84(z-1)^2.$$

Here $A_j(z)$ and $F(z)$ satisfy the conditions of Theorem 3.5(ii) for any $p \in [\frac{1}{2}, +\infty)$ and $\beta = 3$, $j = 0, 1, 2$. Obviously, $f(z) = \frac{1}{(1-z)^2} + \frac{1}{1-z} + (1-z)^5$ is a solution of (2.4) and $f(z) \in H_2^\infty$.

(3) Let

$$A_0(z) = \frac{-6}{(1-z)^4},\ A_1(z) = \frac{-4}{(1-z)^3},\ A_2(z) = \frac{-1}{(1-z)^2},\ F(z) = \frac{1}{1-z} + \frac{1}{4}.$$

It is easy to see that $A_0(z)$, $A_1(z)$, $A_2(z)$ and $F(z)$ satisfy the conditions of Theorem 3.5(iii) with $\beta = 4$ and $\frac{1}{2} \leq p < 1$ ([4, exercise 1, p. 13] shows that $\frac{1}{1-z} \in H^p$ for any $p < 1$). And $f(z) = e^{\frac{1}{1-z}} + \frac{1}{24}(z-1)^3$ solves the equation (2.4) with $f(z) \in \varepsilon^1$.

It is well known that $f(z) = \sum a_n z^n \in H^2$ if and only if $\sum |a_n|^2 < +\infty$ [4, p. 93]. Therefore, we obtained the following Corollary 3.6 by the Theorem 3.5.

Corollary 3.6. *Let $A_j(z)$ and $F(z) = \sum a_n z^n$ be analytic in $\mathbb{D}$ satisfying*

$$|A_j(z)| \leq \frac{\alpha}{(1-|z|)^{\beta-j}}, \quad \sum |a_n|^2 < \infty,$$

where $\alpha > 0$ and $\beta \geq 0$ are finite constants, and $j = 0, 1, \ldots, k-1$. If $f(z)$ is solution of (1.3), then the following statements hold.

(i) *If $0 \leq \beta < k$, then $f(z) \in H^\infty$;*
(ii) *If $\beta = k$, then $f(z) \in H_q^\infty$, where q only depending on k, α and β;*
(iii) *If $\beta > k$, then $f(z) \in \varepsilon^{\beta-k}$.*

Next, we get the following result by changing the condition of $F(z)$.

Theorem 3.7. *Let $A_j(z)$ and $F(z)$ be analytic in $\mathbb{D}$ satisfying*

$$|A_j(z)| \leq \frac{\alpha_1}{(1-|z|)^{\beta_1 - j}}, \quad |F(z)| \leq \frac{\alpha_2}{(1-|z|)^{\beta_2}},$$

where $\alpha_1 > 0$, $\alpha_1 > 0$, $\beta_2 \geq 0$, $\beta_1 \geq 0$ are finite constants, $j = 0, 1, \ \ldots, \ k-1$. If $f(z)$ is solution of (1.3)*, then the following statements hold.*

(i) *If $0 \leq \beta_1 < k$ and $0 \leq \beta_2 < k$, then $f(z) \in H^\infty$;*
(ii) *If $0 \leq \beta_1 < k$ and $\beta_2 = k$, then $f(z) \in H_q^\infty$ for any $q > 0$;*
(iii) *If $0 \leq \beta_1 < k$ and $\beta_2 > k$, then $f(z) \in H_{\beta_2 - k}^\infty$;*
(iv) *If $\beta_1 = k$, then for all finite number β_2, there exist a constant $q > 0$ such that $f(z) \in H_q^\infty$;*
(v) *If $\beta_1 > k$, then for any finite number β_2, $f(z) \in \mathcal{E}^{\beta_1 - k}$.*

Proof. By the conditions of Theorem 3.7 and using the similar way as in the proof of the case 3 of Theorem 3.5, then there exists a constant $C_6 > 0$ such that for all non-negative integers n and $j = 0, 1, \ldots, k-1$,

$$|A_j^{(n)}(z)| \leq \frac{C_6}{(1-|z|^2)^{\beta_1 + n - j}} \tag{3.12}$$

holds for the case $\max\{j : A_j \neq 0\} < \beta_1$.

Combining (3.12), [10, Theorem 1 (a)] and the conditions of Theorem 3.7, there exist constant $C_7 > 0$ and $C_8 > 0$ such that

$$|f(z)| \leq \left(\int_0^r \frac{C_7}{(1-t)^{\beta_2 - k + 1}} dt \right) \cdot \exp\left(\int_0^r \frac{C_8}{(1-t)^{\beta_1 - k + 1}} dt \right).$$

Let $h(z) = \exp\left(\int_0^r \frac{C_8}{(1-t)^{\beta_1 - k + 1}} dt \right)$. By using the similar reason as in the proof of Theorem 3.5, we get

$$|h(z)| \leq \begin{cases} \exp\left(\dfrac{C_8}{(\beta_1 - k)(1-r)^{\beta_1 - k}} \right), & \beta_1 > k, \\ \exp\left(C_8 \log \dfrac{1}{1-r} \right), & \beta_1 = k, \\ \exp\left(\dfrac{C_8}{k - \beta_1} \right), & \max\{j : A_j \neq 0\} < \beta_1 < k. \end{cases} \tag{3.13}$$

Let $g(z) = \int_0^r \frac{C_7}{(1-t)^{\beta_2 - k + 1}} dt$ and by the condition of $F(z)$, we get

$$|g(z)| \leq \begin{cases} \dfrac{C_7}{(\beta_2 - k)(1-r)^{\beta_2 - k}}, & \beta_2 > k, \\ C_7 \log \dfrac{1}{1-r}, & \beta_2 = k, \\ \dfrac{C_7}{k - \beta_2}, & 0 \leq \beta_2 < k. \end{cases} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we get

$$|f(re^{i\theta})| \leq O\left(\frac{1}{(1-r)^{C_8 + \max\{1, \beta_2 - k,\}}} \right)$$

for $\beta_1 = k$ and any finite β_2, and

$$|f(re^{i\theta})| \le O\left(\exp\left(\frac{1}{(1-r)^{\beta_1-k}}\right)\right)$$

for $\beta_1 > k$ and any finite β_2. Therefore, these conclusions (iv)-(v) hold.

By the inequality (3.13), we get

$$|h(z)| < \infty \tag{3.15}$$

for $\max\{j : A_j \neq 0\} < \beta_1 < k$. Now, we claim that if $|f(z)| \le \log \frac{1}{1-r}$ for all $z = re^{i\theta} \in \mathbb{D}$, then $f(z) \in H_q^\infty$ for any $q > 0$. In fact,

$$\begin{aligned}\lim_{r\to 1^-}(1-r)^q \log\frac{1}{1-r} &= \lim_{r\to 1^-}\frac{\log\frac{1}{1-r}}{\frac{1}{(1-r)^q}}\\ &= \lim_{r\to 1^-}\frac{\frac{1}{1-r}}{\frac{q}{(1-r)^{1-q}}}\\ &= \lim_{r\to 1^-}\frac{(1-r)^q}{q}\\ &= 0.\end{aligned}$$

Combining with (3.14) and (3.15), these conclusions (i)-(iii) can be deduced when $\max\{j : A_j \neq 0\} < \beta_1 < k$.

For the case $0 \le \beta_1 \le \max\{j : A_j \neq 0\}$, by (3.14), [10, Theorem 1] and using the similar way in the case 1 and case 2 of the proof of Theorem 3.5, there exists a constant $C^* > 0$ such that

$$|f(z)| \le \begin{cases} \dfrac{C^* C_7}{(\beta_2 - k)(1-r)^{\beta_2 - k}}, & \beta_2 > k,\\ C^* C_7 \log\dfrac{1}{1-r}, & \beta_2 = k,\\ \dfrac{C^* C_7}{k-\beta_2}, & 0 \le \beta_2 < k,\end{cases}$$

which implies that these conclusions (i)-(iii) hold for $0 \le \beta_1 \le \max\{j : A_j \neq 0\}$. Then, these conclusions (i)-(iii) can be deduced from these inequalities above. This proof is completed. □

Next, some examples for Theorem 3.7 are given.

Examples. We consider the following second order differential equation

$$f'' + A_1(z)f' + A_0(z)f = F(z). \tag{3.16}$$

(1) Let $A_0(z) = \frac{1}{4(1-z)^{\frac{3}{2}}}$, $A_1(z) = \frac{1}{4(1-z)^{\frac{1}{2}}}$ and $F(z) = \frac{1}{4(1-z)^{\frac{3}{2}}}$ satisfying the conditions of Theorem 3.7 (i) with $\beta_1 = \frac{3}{2}$ and $\beta_2 = \frac{3}{2}$. Then $f(z) = \alpha e^{(1-z)^{\frac{1}{2}}} + 1$ is a solution of (3.16) and $f(z) \in H^\infty$, where $\alpha \neq 0$ is a finite constant.

(2) Let $A_0(z) = \frac{1}{1-z}$, $A_1(z) = \log(1-z)$ and $F(z) = \frac{2}{(1-z)^2}$ satisfying the conditions of Theorem 3.7 (ii) with $\beta_1 \in (1,2)$ and $\beta_2 = 2$. Then $f(z) = \log\frac{1}{1-z}$ is a solution of (3.16) and $f(z) \in H_q^\infty$ for any $q > 0$.

(3) Let $A_0(z) = \frac{1}{z-1}$, $A_1(z) = \frac{1}{3}$ and $F(z) = \frac{12}{(1-z)^5}$ satisfying the conditions of Theorem 3.7 (iii) with $\beta_1 = 1$ and $\beta_2 = 5$. Then $f(z) = \frac{1}{(1-z)^3}$ is a solution of (3.16) and $f(z) \in H_3^\infty$.

(4) Let $A_0(z) = \frac{-5}{(1-z)^2}$, $A_1(z) = \frac{1}{2(z-1)}$ and $F(z) = \frac{-5}{(1-z)^2}$ satisfying the conditions of Theorem 3.7 (iv) with $\beta_1 = 2$ and $\beta_2 = 2$. Then $f(z) = \frac{1}{(1-z)^2} + 1$ is a solution of (3.16) and $f(z) \in H_2^\infty$.

(5) Let $A_0(z) = \frac{-6}{(1-z)^4}$, $A_1(z) = \frac{-2}{(1-z)^3}$ and $F(z) = \frac{-6}{(1-z)^4}$ satisfying the conditions of Theorem 3.7 (v) with $\beta_1 = 4$ and $\beta_2 = 4$. Then $f(z) = e^{\frac{1}{(1-z)^2}} + 1$ is a solution of (3.16) and $f(z) \in \varepsilon^2$.

The following Theorem 3.8 is the generalization of [8, Theorem 3.3, p. 96], which they considered only the case of $F(z) = 0$ and $\beta = 1$.

Theorem 3.8. *Let $0 < \delta < 1$. Suppose that $A_j(z)$ and $F(z)$ are analytic functions in $\mathbb{D}$ satisfying*

$$\sup_{|z|\geq\delta} |A_j(z)|(1-|z|^2)^{\beta(k-j)} \leq \alpha, \ j = 0, 1, \ldots, k-1,$$

and

$$|F(z)| \leq \frac{\alpha_1}{(1-|z|)^{\beta_1}},$$

where $\alpha_1 > 0$, $\alpha > 0$, $\beta_1 \geq 0$ and $\beta \geq 0$ are finite constants. If $f(z)$ is a solution of (1.3)*, then the following statements hold.*

(i) *If $0 < \beta < 1$, then $f(z) \in H_{\beta_1}^\infty$;*

(ii) *If $\beta = 1$, then $f(z) \in H_q^\infty$, where* $q = \begin{cases} \beta_1 + k^2\alpha^{\frac{1}{k}}, & 0 < \alpha < 1, \\ \beta_1 + k^2\alpha, & \alpha \geq 1, \end{cases}$;

(iii) *If $\beta > 1$, then for any finite β_1, $f(z) \in \varepsilon^{\beta-1}$.*

Proof. By [10, Theorem 2], there exists a constant $C_9 > 0$ such that for all $\theta \in [0, 2\pi)$ and $0 \leq s \leq r < 1$,

$$|f(re^{i\theta})| \leq C_9 \left(1 + \max_{0\leq t\leq r} |F(te^{i\theta})|\right) \cdot \exp\left(r + k\int_0^r \max_{0\leq j\leq k-1} |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds\right).$$

Combining the inequality above and the conditions of Theorem 3.8, we get

$$\begin{aligned} |f(re^{i\theta})| &\leq e \cdot C_9 \cdot \left(1 + \frac{\alpha_1}{(1-r)^{\beta_1}}\right) \\ &\quad \cdot \exp\left(k\int_0^\delta \max_{0\leq j\leq k-1} |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds + k\int_\delta^r \max_{0\leq j\leq k-1} |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds\right) \\ &\leq C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right) \cdot \exp\left(k\int_\delta^r \max_{0\leq j\leq k-1} |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds\right), \end{aligned}$$

where $C_{10} > 0$ is a constant.

If $0 < \alpha < 1$, then

$$|f(re^{i\theta})| \leq C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right) \cdot \exp\left(k^2\alpha^{\frac{1}{k}} \int_\delta^r \frac{1}{(1-s)^\beta} ds\right)$$

$$\leq \begin{cases} C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right)\cdot \exp\left(k^2\alpha^{\frac{1}{k}}\log\frac{1}{1-r}\right), & \beta = 1, \\ C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right)\cdot \exp\left(\frac{k^2\alpha^{\frac{1}{k}}}{(\beta-1)(1-r)^{\beta-1}}\right), & \beta > 1, \\ C_{10}\frac{k^2\alpha^{\frac{1}{k}}}{(1-\beta)}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right), & 0<\beta<1, \end{cases}$$

which means that $f(z) \in \begin{cases} H^{\infty}_{\beta_1+k^2\alpha^{\frac{1}{k}}}, & \beta = 1, \\ H^{\infty}_{\beta_1}, & 0<\beta<1, \\ \varepsilon^{\beta-1}, & \beta>1. \end{cases}$

If $\alpha \geq 1$, then

$$\begin{aligned} |f(re^{i\theta})| &\leq C_{10}\left(1+\frac{\alpha_1}{(1-r)^{\beta}}\right)\cdot \exp\left(k^2\alpha\int_{\delta}^{r}\frac{1}{(1-s)^{\beta}}ds\right) \\ &\leq \begin{cases} C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right)\cdot \exp\left(k^2\alpha\log\frac{1}{1-r}\right), & \beta = 1, \\ C_{10}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right)\cdot \exp\left(\frac{k^2\alpha}{(\beta-1)(1-r)^{\beta-1}}\right), & \beta > 1, \\ C_{10}\frac{k^2\alpha}{(\beta-1)}\left(\frac{\alpha_1}{(1-r)^{\beta_1}}\right), & 0<\beta<1, \end{cases} \end{aligned}$$

which means that $f(z) \in \begin{cases} H^{\infty}_{\beta_1+k^2\alpha}, & \beta = 1, \\ H^{\infty}_{\beta_1}, & 0<\beta<1, \\ \varepsilon^{\beta-1}, & \beta>1. \end{cases}$ The proof is completed. □

4. Inverse problem

Auxiliary result. In order to prove Theorem 4.3, we need the following Lemma 4.1, which can be deduced from [2, Theorem 3.1].

Lemma 4.1. *[2, Theorem 3.1] Let k be integer satisfying $k \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If $f(z)$ is meromorphic in $\mathbb{D}$ such that f does not vanish identically, then there exist a set $E \subset [0,2\pi)$, which has linear measure zero, such that if $\theta \in [0,2\pi)\backslash E$, then there is a constant $r_0 = r(\theta) \in (0,1)$ such that for all* $\arg z = \theta$ *and* $|z| \in (r_0, 1)$*,*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\cdot \max\left\{\log\frac{1}{1-r}, T\left(s(r), f\right)\right\}\right)^{k},$$

where $s(r) = 1 - d(1-r)$*.*

Main result. Now, we consider the following equation

$$f^{(k)} + A(z)f = F(z), \tag{4.1}$$

where $A(z)$ and $F(z)$ are analytic functions in $\mathbb{D}$, in which the inverse problem is studied.

Theorem 4.2. *Let $A(z)$ and $F(z)$ be analytic in $\mathbb{D}$, and $A(z) \in H_q^\infty$. If every solution $f(z)$ of* (4.1) *satisfies $f(z) \in \varepsilon^\beta$, then $F(z) \in \varepsilon^\beta$. Furthermore, if $q \leq k$, then every solution $f(z)$ of* (4.1) *satisfies $f(z) \in \varepsilon^\beta$ if and only if $F(z) \in \varepsilon^\beta$.*

Proof. Suppose that $A(z) \in H_q^\infty$ and $f(z)$ is any solution of (4.1) satisfies $f(z) \in \varepsilon^\beta$. First, we prove that $f^{(k)}(z) \in \varepsilon^\beta$ holds for any integers $k \in N^+$. If $f^{(k)}(z) \notin \varepsilon^\beta$, then there exist a constant $\beta' > \beta$ such that $|f^{(k)}(z)| \geq \exp\left(\frac{\alpha'}{(1-|z|)^{\beta'}}\right)$, which implies that $\sigma_M(f^{(k)}) \geq \beta'$. It follows from [1, Proposition 1.2] that $\sigma_M(f) = \sigma_M(f^{(k)}) \geq \beta'$. This contradicts with our hypothesis, and then $f^{(k)}(z) \in \varepsilon^\beta$ holds for any integers $k \in N^+$.

Next, we prove that $F(z) \in \varepsilon^\beta$. By (4.1) and the conditions of Theorem 4.2, we get

$$|F(z)| \leq |f^{(k)}(z)| + |A(z)| \cdot |f(z)| \leq \exp\left(\frac{\alpha_0}{(1-|z|)^\beta}\right),$$

where $\alpha_0 > 0$ is a constant depending only on q and α. This implies that $F(z) \in \varepsilon^\beta$.

Finally, we assume that $q \leq k$ and $F(z) \in \varepsilon^\beta$. By using the similar method as in the proof of Theorem 2.3, we get every solutions $f(z)$ of (4.1) satisfies $f(z) \in \varepsilon^\beta$. The proof is completed. □

Theorem 4.3. *Let $A(z)$ and $F(z)$ be analytic in $\mathbb{D}$. If every non-trivial solution $f(z)$ of* (4.1) *satisfies*

$$\limsup_{r\to 1^-} \frac{|f(re^{i\theta})|}{\exp\left(\frac{\alpha}{(1-r)^\beta}\right)} \leq 1 \quad \text{and} \quad \frac{F(z)}{f(z)} \in H_p^\infty,$$

where $\alpha > 0$ and $\beta > 0$ are finite constants. Then $A(z) \in H_q^\infty$, where q is a positive constant depending only on k, β and p.

Proof. Let $f(z)$ is a non-trivial solution of (4.1). By the conditions of Theorem 4.3, for any $\varepsilon > 0$, there exist a constant $r_1 \in (0,1)$ such that for all $r \in (r_1, 1)$,

$$T(s(r), f) \leq \log^+ M(s(r), f) \leq (1+\varepsilon)\frac{\alpha}{d^\beta(1-r)^\beta}. \tag{4.2}$$

By Lemma 4.1, for $d \in (0,1)$, there exist a constant $r_2 \in [0,1)$ and a set E of measure zero such that for all $r \in (r_2, 1)$ and $\theta \in [0, 2\pi)\backslash E$,

$$\begin{aligned}
\left|\frac{f^{(k)}(re^{i\theta})}{f(re^{i\theta})}\right| &\leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-r}, T\left(s(r), f\right)\right\}\right)^k \\
&\leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \left(\log\frac{1}{1-r} + T\left(s(r), f\right)\right)\right)^k \\
&\leq \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} \left(\log\frac{1}{1-r} + T\left(s(r), f\right)\right)^k.
\end{aligned}$$

Combining (4.2), for all $r \in (\max\{r_1, r_2\}, 1)$, we get

$$\left|\frac{f^{(k)}(re^{i\theta})}{f(re^{i\theta})}\right| \leq \frac{C}{(1-r)^{k\beta+k(2+\varepsilon)}}. \tag{4.3}$$

By Eq (4.1), we have

$$|A(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|\frac{F(z)}{f(z)}\right|.$$

It follows from the inequality above, (4.3) and the conditions of Theorem 4.3 that the conclusion holds. □

5. Higher derivative problem

For an analytic function $f(z)$, it is easy to see that $f'(z)$ and $f(z)$ do not necessarily belong to the same space. Let $f(z) = \frac{1}{1-z}$. Then $f(z) \in H_1^\infty$, and $f'(z) = \frac{1}{(1-z)^2}$. It is easy to get $f'(z) \in H_2^\infty$ and $f'(z) \notin H_1^\infty$. Therefore, a natural problem is: what conditions on coefficients guaranteeing solution of differential equations and its derivative belong to the same space? Here we investigate this question to the linear differential equation

$$f^{(k)} + A(z)f = F(z), \tag{5.1}$$

where $A(z)$ and $F(z)$ are analytic functions in $\mathbb{D}$. Firstly, we introduce some auxiliary results.
Auxiliary results. Here, an auxiliary results is given for the proof of our results.

Lemma 5.1. *Let $f(z)$ be a solution of* (5.1). *Then*

$$\begin{aligned}|f^{(m)}(z)| \leq &\left(C + \frac{1}{(k-m-1)!}\int_0^r |F(se^{i\theta})|(1-s)^{k-m-1}ds\right)\\ &\cdot \exp\left(\frac{1}{(k-m-1)!}\int_0^r |A(se^{i\theta})|(1-s)^{k-m-1}ds\right),\end{aligned}$$

where C is a positive constant, and $m \in \mathbb{N}$ is a natural number satisfying $0 \leq m \leq k-1$.

Proof. Let $f(z)$ be a solution of (5.1). By [10, Theorem 9], we get

$$\begin{aligned}f(z) = &\sum_{n=0}^{k-1}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\\ &+ \frac{1}{(k-1)!}\int_{z_0}^z F(\xi)(z-\xi)^{k-1}d\xi\\ &- \frac{1}{(k-1)!}\int_{z_0}^z f(\xi)A(\xi)(z-\xi)^{k-1}d\xi.\end{aligned} \tag{5.2}$$

Combining Lemma 3.3, we deduce that

$$\begin{aligned}f^{(m)}(z) = &\sum_{n=m}^{k-1}\frac{f^{(n)}(z_0)}{(n-m)!}(z-z_0)^{n-m}\\ &+ \frac{1}{(k-m-1)!}\int_{z_0}^z F(\xi)(z-\xi)^{k-m-1}d\xi\\ &- \frac{1}{(k-m-1)!}\int_{z_0}^z f(\xi)A(\xi)(z-\xi)^{k-m-1}d\xi,\end{aligned} \tag{5.3}$$

where $0 \leq m \leq k-1$. And then, we choose $z_0 = 0$ and the path of integration to be the line segment $[0, z]$, for all $z = re^{i\theta}$ and $z\xi = se^{i\theta}$ $(0 \leq s \leq r)$,

$$\begin{aligned} |f^{(m)}(z)| \leq C &+ \frac{1}{(k-m-1)!} \int_0^r |F(se^{i\theta})|(1-s)^{k-m-1} ds \\ &+ \frac{1}{(k-m-1)!} \int_0^r |f(se^{i\theta})| \cdot |A(se^{i\theta})|(1-s)^{k-m-1} ds, \end{aligned}$$

where C is a positive constant only depends on $f(0), f'(0), \ldots, f^{(k-1)}(0)$. The assertion follows from Lemma 3.2. □

Main results. In fact, a simple estimation shows that $f(z) \in H^\infty_{q-1}$ when $f'(z) \in H^\infty_q$ $(q > 1)$, but $f(z) \notin H^\infty_0 = H^\infty$ when $f'(z) \in H^\infty_1$, for example $f(z) = \log \frac{1}{1-r}$. So, it is very meaningful to study the properties of function space of derivative of solutions of differential equations.

Theorem 5.2. *Let $A(z)$ and $F(z)$ be analytic in $\mathbb{D}$ satisfying*

$$A(z) \leq \frac{\alpha}{(1-|z|)^\beta} \quad \text{and} \quad F(z) \in H^p,$$

where m is a positive integer satisfies $0 < m \leq k-2$ and $\frac{1}{k-m} \leq p \leq +\infty$, and α and β are finite constants. If $f(z)$ is a solution of (5.1), then the following statements hold.

(i) *If $0 \leq \beta < k-m$, then $f^{(m)}(z) \in H^\infty$;*
(ii) *If $\beta = k-m$, then $f^{(m)}(z) \in H^\infty_q$, where $q = \frac{\alpha}{(k-m-1)!}$;*
(iii) *If $\beta > k-m$, then $f^{(m)}(z) \in \varepsilon^{\beta-k+m}$.*

Proof. By (5.3) and using similar method as in the proof of Theorem 3.5, we get

$$|f^{(m)}(z)| \leq \begin{cases} C_1 \exp\left(C_2 \dfrac{1}{(1-r)^{\beta-k+m}}\right), & \beta > k-m, \\ C_1 \left(\dfrac{1}{1-r}\right)^{\frac{\alpha_1}{(k-m-1)!}}, & \beta = k-m, \\ C_1 \exp(-C_2), & 0 \leq \beta < k-m, \end{cases}$$

where $0 < C_1 < +\infty$ and $C_2 = \frac{\alpha}{(k-m-1)!(\beta-k+m)}$. It follows from the inequality above that these conclusion (i)-(iii) hold. □

The following Corollary 5.3 is also given by Theorem 5.2.

Corollary 5.3. *Let $A(z)$ and $F(z) = \sum a_n z^n$ be analytic in $\mathbb{D}$ satisfying*

$$A(z) \leq \frac{\alpha}{(1-|z|)^\beta} \quad \text{and} \quad \sum |a_n|^2 < \infty,$$

where α and β are finite constants. If $f(z)$ is a solution of (5.1), then for $0 \leq m \leq k-2$ the following statements hold.

(i) *If $0 \leq \beta < k-m$, then $f^{(m)}(z) \in H^\infty$;*

(ii) *If* $\beta = k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{\frac{\alpha}{(k-m-1)!}}$;
(iii) *If* $\beta > k - m$, *then* $f^{(m)}(z) \in \varepsilon^{\beta-k+m}$.

Theorem 5.4. *Let* $A(z)$ *and* $F(z)$ *be analytic in* $\mathbb{D}$ *satisfying*

$$A(z) \leq \frac{\alpha_1}{(1-|z|)^{\beta_1}} \quad \text{and} \quad |F(z)| \leq \frac{\alpha_2}{(1-|z|)^{\beta_2}},$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 \geq 0$ *and* $\beta_2 \geq 0$ *are finite constants. If* $f(z)$ *is a solution of* (5.1), m *is a non-negative integer satisfies* $m < k$, *then the following statements hold.*

(i) *If* $0 \leq \beta_1 < k - m$ *and* $0 \leq \beta_2 < k - m$, *then* $f^{(m)}(z) \in H^{\infty}$;
(ii) *If* $0 \leq \beta_1 < k - m$ *and* $\beta_2 = k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{q}$ *for any* $q > 0$;
(iii) *If* $0 \leq \beta_1 < k - m$ *and* $\beta_2 > k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{\beta_2-k+m}$;
(iv) *If* $\beta_1 = k - m$ *and* $0 \leq \beta_2 < k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{\frac{\alpha_1}{(k-m-1)!}}$;
(v) *If* $\beta_1 = k - m$ *and* $\beta_2 = k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{\frac{\alpha_1}{(k-m-1)!}+1}$;
(vi) *If* $\beta_1 = k - m$ *and* $\beta_2 > k - m$, *then* $f^{(m)}(z) \in H^{\infty}_{\frac{\alpha_1}{(k-m-1)!}+\beta_2-k+m}$;
(vii) *If* $\beta_1 > k - m$, *then for any finite* β_2, $f^{(m)}(z) \in \varepsilon^{\beta_1-k+m}$.

Proof. For all $z = re^{i\theta} \in \mathbb{D}$, set

$$g(z) = C_3 + \frac{1}{(k-m-1)!}\int_0^r |F(se^{i\theta})|(1-s)^{k-m-1}ds$$

and

$$h(z) = \exp\left(\frac{1}{(k-m-1)!}\int_0^r |A(se^{i\theta})|(1-s)^{k-m-1}ds\right).$$

By Lemma 5.1, there exists a constant $C_3 > 0$ such that

$$|f^{(m)}(z)| \leq |g(z)| \cdot |h(z)|.$$

From the conditions of Theorem 5.4, we get

$$|g(z)| \leq \begin{cases} C_3 + \dfrac{C_4}{(1-r)^{\beta_2-k+m}}, & \beta_2 > k-m, \\ C_3 + C_6 \log \dfrac{1}{1-r}, & \beta_2 = k-m, \\ C_3 - C_4, & \beta_2 < k-m, \end{cases}$$

and

$$|h(z)| \leq \begin{cases} \exp\left(\dfrac{C_5}{(1-r)^{\beta_1-k+m}}\right), & \beta_1 > k-m, \\ \left(\dfrac{1}{1-r}\right)^{C_7}, & \beta_1 = k-m, \\ \exp(-C_5), & 0 \leq \beta_1 < k-m, \end{cases}$$

where $C_3 > 0$, $C_4 = \frac{\alpha_2}{(k-m-1)!(\beta_2-k+m)}$, $C_5 = \frac{\alpha_1}{(k-m-1)!(\beta_1-k+m)}$, $C_6 = \frac{\alpha_2}{(k-m-1)!}$ and $C_7 = \frac{\alpha_1}{(k-m-1)!}$ are constants. It follows from these inequalities that these conclusions (i)-(vii) hold. □

Acknowledgments

The work is supported by the National Natural Science Foundation of China (Grant No. 11861023), the Foundation of Science and Technology of Guizhou Province of China (Grant No. [2015]2112) and the Foundation of Science and Technology project of Guizhou Province of China (Grant No. [2018]5769-05).

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. T. B. Cao, H. X Yi, The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc, *J. Math. Anal. Appl.,* **319** (2006), 278–294.
2. I. Chyzhykov, G. G. Gundersen, J. Heittokangas, Linear differential equations and logarithmic derivate of estimates, *Proc. Lond. Math. Soc.,* **86** (2003), 735–754.
3. I. Chyzhykov, J. Heittokangas, J. Rättyä, Sharp logarithmic derivative estimates with applications to ordinary differential equations in the unit disc, *J. Aust. Math. Soc.,* **88** (2010), 145–167.
4. P. L. Duren, *Theory of H^p Spaces*, New York-London: Academic Press, 1970.
5. J. Gröhn, J.-M. Huusko, J. Rättyä, Linear differential equations with slowly growing solutions, *Trans. Amer. Math. Soc.,* **370** (2018), 7201–7227.
6. W. K. Hayman, *Meromorphic Functions*, Oxford: Clarendon Press, 1964.
7. J. Heittokangas, On complex differential equations in the unit disc, *Ann. Acad. Sci. Fenn. Math. Diss.,* **122** (2000), 1–54.
8. J. Heittokangas, R. Korhonen, J. Rättyä, Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space, *Nagoya Math. J.,* **187** (2007), 91–113.
9. J. Heittokangas, R. Korhonen, J. Rättyä, Linear differential equations with coefficients in weighted Bergman and Hardy space, *Trans. Amer. Math. Soc.,* **360** (2008), 1035–1055.
10. J. Heittokangas, R. Korhonen, J. Rättyä, Growth estimates for solutions of nonhomogeneous linear complex differential equations, *Ann. Acad. Sci. Fenn. Math.,* **34** (2009), 145–156.
11. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Berlin-New York: Walter de Gruyter, 1993.
12. J. R. Long, Y. Sun, S. M. Zhang, G. M. Hu, Second-order linear differential equations with solutions in analytic function spaces, *J. Funct. Space.,* **2019** (2019), 1–9.
13. M. X. Li, L. P. Xiao, Solutions of linear differential equations in the unit disc, *J. Math. Reseach Appl.,* **34** (2014), 729–735.
14. H. Li, H. Wulan, Linear differential equations with solutions in the Q_k spaces, *J. Math. Anal. Appl.,* **375** (2011), 478–489.

15. Ch. Pommerenke, On the mean growth of the solutions of complex linear differential equations in the disk, *Complex Var.,* **1** (1982), 23–38.

16. Y. Sun, J. R. Long, G. M. Hu, Nonlinear complex differential equations with solutions in some function spaces, *Complex Var. Ellip. Equa.,* **64** (2019), 300–314.

17. K. Zhu, Bloch type spaces of analytic functions, *Rochy Mountain J. Math.,* **23** (1993), 1143–1177.

AIMS Press