Existence and uniqueness results for sequential $\psi$-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions

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Abstract: In this paper, we discuss the existence and uniqueness of boundary value problems for sequential $\psi$-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions. The existence results are obtained via the well known Krasnoselskii’s fixed point theorem while the uniqueness is demonstrated by using the Banach’s contraction mapping principle. Some examples are also given to demonstrate the application of the main results.

Keywords: existence; uniqueness; $\psi$-Hilfer Fractional derivative; pantograph equations; fixed point theory

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1. Introduction

Fractional calculus is a powerful tool to investigate several complex problems in numerous scientific and engineering disciplines such as physics, chemistry, biology, economics, and control theory. Differential equations of fractional order describe many real world processes more accurately compared to the classical order differential equations. For more details about the theory of fractional differential equations and applications, see [1–6].

In the literature, the most used derivatives of fractional order are the Caputo and the Riemann-Liouville derivatives. A generalization of these derivatives was introduced by R. Hilfer in [7], and this derivative is called the Hilfer fractional derivative. For more details we give the following references [8, 9].

In [10], the authors began the study of nonlocal boundary value problems involving the Hilfer fractional derivatives, by studying the following problem
\[
\begin{cases}
\frac{H^\alpha x(t)}{t^\beta} = f(t, x(t)), \quad t \in [a, b], \quad 1 < \alpha < 2, \quad 0 \leq \beta \leq 1, \\
x(a) = 0, x(b) = \sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i), \quad \varphi_i > 0, \quad \delta_i \in \mathbb{R}, \quad \xi_i \in [a, b],
\end{cases}
\tag{1.1}
\]

where \(\frac{H^\alpha}{t^\beta}\) is the Hilfer fractional derivative of order \(\alpha\), and parameter \(\beta\), \(I^{\varphi_i}\) is the Riemann-Liouville fractional integral of order \(\varphi_i > 0\), several fixed point theorems were used to prove the existence and uniqueness results.

In [11], the authors considered the existence and uniqueness for a class of system of Hilfer-Hadamard fractional differential equations with two point boundary conditions

\[
\begin{cases}
\left(\frac{H^\alpha_t + k_1 \frac{H^\alpha_1}{t^\beta}}{t^\beta} \right) u(t) = f(t, u(t), v(t)), \quad t \in [1, \epsilon], \\
\left(\frac{H^\alpha_1 + k_2 \frac{H^\alpha_2}{t^\beta}}{t^\beta} \right) v(t) = g(t, u(t), v(t)), \quad t \in [1, \epsilon], \\
u(1) = 0, u(\epsilon) = A_1, \\
v(1) = 0, v(\epsilon) = A_2,
\end{cases}
\tag{1.2}
\]

where \(\frac{H^\alpha}{t^\beta}\) is the Hilfer-Hadamard fractional derivative of order \(1 < \alpha_i \leq 2\), and type \(0 \leq \beta_i \leq 1\) for \(i \in \{1, 2\}, k_1, k_2, A_1, A_2 \in \mathbb{R}^+\), and \(f, g : [1, \epsilon] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are given continuous functions.

Another fractional derivative, which is a derivative with respect to another function, is the \(\psi\)-Hilfer fractional derivative, it was introduced in [12]. A lot of papers studied the existence and uniqueness of fractional differential equations using the \(\psi\)-Hilfer fractional derivatives, please see [13–20] and references therein.

On the other hand, another important class of fractional differential equations are the pantograph equations. The pantograph equations are an important class of delay equations and they are used in deterministic situations. Initial value problems for pantograph equations with the Hilfer fractional derivative were studied in [21–24].

Recently in [25], the authors studied the existence and uniqueness of solutions for a new class of boundary value problems of sequential \(\psi\)-Hilfer fractional differential equations with multi-point boundary conditions of the form

\[
\begin{cases}
\left(\frac{H^\alpha_0 + k \frac{H^\alpha_1}{t^\beta}}{t^\beta} \right) x(t) = f(t, x(t)), \quad t \in (a, b] \\
x(a) = 0, x(b) = \sum_{i=1}^{m} \lambda_i x(\theta_i).
\end{cases}
\tag{1.3}
\]

In this paper, we consider a new class of sequential \(\psi\)-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions as follows

\[
\begin{cases}
\left(\frac{H^\alpha_0 + p \frac{H^\alpha_1}{t^\beta}}{t^\beta} \right) x(t) = f(t, x(t), x(\sigma t)), \quad t \in (0, T], \quad 0 < \sigma < 1 \\
x(0) = 0, \quad \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{j=1}^{n} \omega_j^i \frac{H^\alpha_0}{t^\beta} x(\theta_j) + \sum_{k=1}^{r} \lambda_k \frac{H^\alpha_0}{t^\beta} x(\xi_k) = A,
\end{cases}
\tag{1.4}
\]
where $^H D_{0^+}^{\alpha,\beta}$ are the $\psi$-Hilfer derivatives of order $u = [\alpha, \mu_k]$, $1 < \mu_k < \alpha \leq 2$, $0 \leq \beta \leq 1$, $I_{0^+}^{\beta}$ are the $\psi$-Riemann Liouville fractional integrals of order $\beta_j$, with $\beta_j > 0$, for $j = 1, 2, \ldots, n$, $p, A, \delta, \omega_j, \lambda_k \in \mathbb{R}$ are given constants, the points $\eta, \theta_j, \xi_k$ are in $J$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r$ and the function $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, $J = [0, T]$, $T > 0$.

It is important for us to note that the problem considered in this paper provide more insight in the study of sequential $\psi$-Hilfer-type fractional differential equations, this paper can be viewed as a generalization of some existing papers in the literature. Our nonlocal boundary conditions are more useful and more general. We note that the mixed nonlocal boundary conditions include multi-point, fractional derivative of multi-order and fractional integral of multi-order boundary conditions.

This research paper is organized as follows, in section 2, we provide some definitions and lemmas that will be used throughout the paper, in section 3, we establish the existence and uniqueness results by means of the fixed point theorems, and last but not least, in section 4, we give some examples to illustrate the applicability of the results.

2. Preliminaries and auxiliary results

In this section, we introduce some definitions, lemmas and useful notations that will be used throughout the paper.

Let $C(J, \mathbb{R})$ denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm defined by $\|f\| = \sup_{t \in J} |f(t)|$.

We also define the $n$-times absolutely continuous functions given by

$$\mathcal{AC}^n(J, \mathbb{R}) = \{ f : J \longrightarrow \mathbb{R}; f^{(n)} \in \mathcal{AC}(J, \mathbb{R}) \}.$$  

**Definition 2.1.** (see [1]) Let $(a, b), (-\infty < a < b \leq \infty)$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a, b)$, having a continuous derivative $\psi'(x)$ on $(a, b)$. The $\psi$-Riemann-Liouville fractional integral of a function $f$ with respect to other function $\psi$ is defined by

$$I_{a^+}^{\alpha, \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \quad t > a > 0,$$

where $\Gamma(.)$ is the Gamma function.

**Definition 2.2.** (see [1]) Let $\psi(t) \neq 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function $f$ with respect to another function $\psi$ of order $\alpha$, is defined by

$$D_{a^+}^{\alpha, \psi} f(t) = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{\alpha-n, \psi} f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number $\alpha$.

**Definition 2.3.** (see [12]) Let $n - 1 < \alpha < n$ with $n \in \mathbb{N}$, $[a, b]$ is the interval such that $-\infty < a < b \leq \infty$ and $f, \psi \in C^n([a, b], \mathbb{R})$ two functions such that $\psi$ is increasing and $\psi'(t) \neq 0$, for all
Lemma 2.6. (see [12]) Let \( m \in \mathbb{N} \), \( n = [\alpha] + 1 \), \([\alpha]\) represents the integer part of the real number \( \alpha \) with \( \gamma = \alpha + \beta(n - \alpha) \).

Lemma 2.7. (Krasnoselskii’s fixed point theorem, see [27]) Let \( M \) be a closed, bounded, convex, and nonempty subset of a Banach space \( X \). Let \( A \) be the operators such that

\[
A \text{ is compact and continuous},
\]

(c) \( B \) is a contraction mapping.

Then, there exists \( z \in M \) such that \( z = Az + Bz \).

In order to convert the problem (1.4) into a fixed point problem, we must transform it into an equivalent integral equation. We provide the following Lemma, which is a linear variant of the boundary value problem (1.4).
Lemma 2.10. Let $1 < \mu_k < \alpha \leq 2$, $\gamma = \alpha + \beta(2 - \alpha), k = 1, 2, ..., r$, and $\Lambda \neq 0$. Suppose that $h \in C$. Then $x \in C^2$ is a solution of the problem

$$
\begin{cases}
(HD_0^{\alpha, \beta; \phi} + pHD_0^{\alpha-1, \beta; \phi}) x(t) = h(t), t \in (0, T], \\
x(0) = 0, \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{j=1}^{n} \omega_j I_0^{\beta; \phi} x(\theta_j) + \sum_{k=1}^{r} \lambda_k H D_0^{\mu_k, \beta} x(\xi_k) = A,
\end{cases}
$$

(2.1)

if and only if $x$ satisfies the integral equation

$$
x(t) = I^{\alpha; \phi} h(t) - p I_0^{1, \phi} x(t) + \frac{\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} \left[ A + p \sum_{i=1}^{m} \delta_i I_0^{1, \phi} x(\eta_i) \\
+ \sum_{k=1}^{r} \lambda_k I_0^{1-\mu_k, \phi} x(\xi_k) + \sum_{j=1}^{n} \omega_j I_0^{1+\beta; \phi} x(\theta_j) - \left( \sum_{i=1}^{m} \delta_i I_0^{\alpha; \phi} h(\eta_i) \\
+ \sum_{k=1}^{r} \lambda_k I_0^{\alpha-\mu_k, \phi} h(\xi_k) + \sum_{j=1}^{n} \omega_j I_0^{\alpha+\beta; \phi} h(\theta_j) \right) \right],
$$

where

$$
\Lambda = \sum_{i=1}^{m} \frac{\delta_i (\psi(\eta_i) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + \sum_{k=1}^{r} \frac{\lambda_k (\psi(\xi_k) - \psi(0))^{\gamma-\mu_k-1}}{\Gamma(\gamma - \mu_k)} + \sum_{j=1}^{n} \frac{\omega_j (\psi(\theta_j) - \psi(0))^{\gamma+\beta_j-1}}{\Gamma(\gamma + \beta_j)}.
$$

Proof. Let $x$ be a solution of the problem (2.1). By using Lemma 2.7, and operating $I_0^{\alpha; \phi}$ on both sides of Eq (2.1) we obtain

$$
x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + c_2 \frac{(\psi(t) - \psi(0))^{\gamma-2}}{\Gamma(\gamma - 1)} - p I_0^{1, \phi} x(t) + I_0^{\alpha; \phi} h(t),
$$

where $c_1, c_2$ are real constants.

For $t = 0$, we get $c_2 = 0$, and thus

$$
x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + I_0^{\alpha; \phi} h(t) - p I_0^{1, \phi} x(t).
$$

(2.2)

Applying the operators $HD_0^{\mu_k, \phi}$ and $I_0^{\beta_i; \phi}$ to (2.2), we obtain

$$
HD_0^{\mu_k, \phi} x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma-\mu_k-1}}{\Gamma(\gamma - \mu_k)} - p I_0^{1-\mu_k, \phi} x(t) + I_0^{\alpha-\mu_k, \phi} h(t),
$$

$$
I_0^{\beta_i; \phi} x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma+\beta_i-1}}{\Gamma(\gamma + \beta_i)} - p I_0^{1+\beta_i; \phi} x(t) + I_0^{\alpha+\beta_i; \phi} h(t).
$$

By using the second boundary condition in (2.1), we obtain
The proof is now completed.

3. Existence and uniqueness results

In view of Lemma 2 given in (2.2), we obtain the solution. Conversely, it is easy to show that the solution $x$ given in Lemma 2 satisfies the problem (2.1). The proof is now completed.

3. Existence and uniqueness results

In this section, we present the existence and uniqueness results to the problem (1.4).

For convenience, we are going to use the following expressions:

$$Q(\chi, \epsilon) = \frac{(\psi(\chi) - \psi(0))^r}{\Gamma(\epsilon + 1)},$$

$$\Omega_1 = \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) + \sum_{k=1}^{r} |\lambda_k| \frac{(\psi(\xi_k) - \psi(0))^{1-\mu_k}}{\Gamma(2 - \mu_k)} + \sum_{j=1}^{n} |\omega_j| \frac{(\psi(\theta_j) - \psi(0))^{1+\beta_j}}{\Gamma(2 + \beta_j)},$$

$$\Omega_2 = \sum_{i=1}^{m} |\delta_i| \frac{(\psi(\eta_i) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} + \sum_{k=1}^{r} |\lambda_k| \frac{(\psi(\xi_k) - \psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha - \mu_k + 1)} + \sum_{j=1}^{n} |\omega_j| \frac{(\psi(\theta_j) - \psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha + \beta_j + 1)}.$$

In view of Lemma 2.10, we define the operator $T : C \rightarrow C$ by

$$(Tx)(t) = I^{\alpha\phi}_0 F_x(s)(t) - p I^{1-\phi}_0 x(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \left[ A + p \sum_{i=1}^{m} \delta_i I^{1-\phi}_0 x(\eta_i) + \sum_{k=1}^{r} \lambda_k I^{1-\mu_k\phi}_0 x(\xi_k) + \sum_{j=1}^{n} \omega_j I^{1+\beta_j\phi}_0 x(\theta_j) \right]
+ \sum_{k=1}^{r} \lambda_k I^{1-\mu_k\phi}_0 F_x(s)(\xi_k) + \sum_{j=1}^{n} \omega_j I^{1+\beta_j\phi}_0 F_x(s)(\theta_j)).$$
where \( F_x(t) = f(t, x(t), x(\sigma t)), 0 < \sigma < 1 \).

It should be mentioned here that the problem (1.4) has solutions if and only if the operator \( T \) has fixed points.

### 3.1. Uniqueness results via the Banach’s contraction principle

By applying the Banach’s contraction principle, we establish the existence and uniqueness of solutions for the problem (1.4).

**Theorem 3.1.** We consider the following hypotheses:

\((H_1)\) The functions \( f : J \times \mathbb{R}^2 \to \mathbb{R} \) is continuous and there exists a constant \( L > 0 \) such that

\[
|f(t, x, y) - f(t, z, w)| \leq L(|x - z| + |y - w|), \text{ for all } t \in J, \text{ and } x, y, z, w \in \mathbb{R}.
\]

If we have

\[
2L(Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|}) + \left( |p| \Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p| (\psi(T) - \psi(0)) \right) < 1,
\]

where \( \Omega_1 \) and \( \Omega_2 \) are given by (3.2) and (3.3) respectively, then the problem (1.4) has a unique solution on \( J \).

**Proof.** First of all, we transform the problem (1.4) into a fixed point problem, \( x = Tx \), where the operator \( T \) is defined in the previous section. By applying the Banach’s contraction principle, we show that the operator \( T \) has a unique fixed point, which is the unique solution of problem (1.4).

Let \( \sup_{t \in J} |f(t, 0, 0)| = M < \infty \), and we set \( B_r := \{ x \in C : ||x|| \leq r \} \) with

\[
r \geq \frac{\left[ Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|} \right] M + |\Lambda| Q(T, \gamma - 1)}{1 - \left[ |p| \Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p| (\psi(T) - \psi(0)) \right] - 2L\left[ Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|} \right]},
\]

where \( Q(\chi, \epsilon), \Omega_1, \Omega_2 \) are given by (3.1),(3.2),(3.3) respectively. It is clear that \( B_r \) is a bounded, closed and convex subset of \( C \).

**Step I.** We first show that \( TB_r \subset B_r \).

We have from the hypothesis \((H_1)\) that

\[
|F_x(t)| \leq |f(t, x(t), x(\sigma t)) - f(t, 0, 0)| + |f(t, 0, 0)|
\leq L(|x(t)| + |x(\sigma t)|) + M
\leq 2L||x|| + M
\]

Then we have
\[ |(Tx)(t)| = \sum_{j=1}^{n} |\psi(t) - \psi(0)|^{\alpha} \left( L \|x\| + M \right) + |p| |(\psi(T) - \psi(0))^{\gamma-1}| |A| + |p| \left( \sum_{j=1}^{m} |\delta_j| |x(\eta_j)| + \sum_{k=1}^{r} |\lambda_k| |F_\gamma(s(\theta_j))| \right) \]
\[ \leq \frac{\Gamma(\alpha + 1)}{|A| |\Gamma(\gamma)|} \left( \sum_{j=1}^{n} |\delta_j| |x(\eta_j)| + \sum_{k=1}^{r} |\lambda_k| |F_\gamma(s(\theta_j))| \right) \]

which implies that \(TB_c \subset B_r\).

**Step II.** We show that the operator \(T: C \rightarrow C\) is a contraction.

For any \(x, y \in C\) and for each \(t \in J\), we have
\[(Tx(t)) - (Ty(t)) \leq \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} \left| F_x(s) - F_y(s) \right| (T) + |p| \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} |x(t) - y(t)| \]

\[+ \sum_{k=1}^r |\lambda_k| \frac{T^{\mu_0+\phi}}{\Gamma(\alpha + 1)} |x(\xi_k) - y(\xi_k)| + \sum_{j=1}^n |\omega_j| \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} |x(\theta_j) - y(\theta_j)| + \left( \sum_{i=1}^m |\delta_i| \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} \right) |x(\eta_i) - y(\eta_i)|\]

\[+ \sum_{k=1}^r |\lambda_k| \frac{T^{\mu_0+\phi}}{\Gamma(\alpha + 1)} |F_x(s) - F_y(s)(\xi_k)| + \sum_{j=1}^n |\omega_j| \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} |F_x(s) - F_y(s)(\theta_j)|\]

\[+ \sum_{j=1}^n |\omega_j| \frac{T^{\mu_0,\phi}}{\Gamma(\alpha + 1)} \left| F_x(s)(\theta_j) - F_y(s)(\theta_j) \right| \leq \frac{2L}{\Gamma(\alpha + 1)} \left| \psi(T) - \psi(0) \right|^\alpha + |p| \left( \left| \psi(T) - \psi(0) \right|^\alpha \left| x - y \right| \right)\]

which implies that

\[|\langle T \rangle x(t) - \langle T \rangle y(t) \rangle \leq \frac{2L}{\Gamma(\alpha + 1)} \left( \left| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| + \left| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| + \left| |p| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| \right) \left| \psi(T) - \psi(0) \right|^\alpha \]

\[+ \left( \left| \frac{2L}{\Gamma(\alpha + 1)} \left| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| + \left| \frac{2L}{\Gamma(\alpha + 1)} \left| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| \right| \right) \left| \psi(T) - \psi(0) \right| \]

And as

\[2L \left( \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right) + \left( \left| \frac{2L}{\Gamma(\alpha + 1)} \left| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| \right) + \left| |p| \frac{Q(T, \gamma - 1)}{\left| \Lambda \right|} \right| + \left| |p| \frac{\left( \psi(T) - \psi(0) \right)}{\left| \Lambda \right|} \right| < 1,\]

\[\text{AIMS Mathematics} \quad \text{Volume 6, Issue 8, 8239–8255.}\]
we get that the operator $T$ is a contraction.

Therefore, by the Banach’s contraction mapping principle, the operator $T$ has a unique fixed point, and hence the problem (1.4) has a unique solution on $J$. The proof is now completed. \hfill \Box

### 3.2. Existence results via the Krasnoselskii’s fixed point theorem

Now, we present an existence result based on the Krasnoselskii’s fixed point theorem.

**Theorem 3.2.** Let us assume that $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying:

\[(H_2) \ |f(t,u,v)| \leq \phi(t), \forall (t,u,v) \in J \times \mathbb{R}^2, \text{ and } \phi(t) \in C(J, \mathbb{R}^+) .\]

In addition if:

\[
\left( |p| \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p| (\psi(T) - \psi(0)) \right) < 1, \tag{3.4}
\]

where $Q(T, \gamma - 1)$ and $\Omega_1$ are defined by (3.1) and (3.2) respectively, then the problem (1.4) has at least one solution on $J$.

**Proof.** Let $\sup_{t \in J} |\phi(t)| = \| \phi \|$ and $B_r := \{ x \in C : \| x \| \leq r \}$, where

\[
r \geq \frac{\left[ Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|} \right] \| \phi \| + |A| \frac{Q(T, \gamma - 1)}{|\Lambda|} - 1}{\left[ |p| \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p| (\psi(T) - \psi(0)) \right]}. \]

We define the operators $T_1$ and $T_2$ on $B_r$ by

\[
(T_1 x)(t) = \frac{\Gamma_{\mu_1} \phi_{1, \phi} F_{x}(s)(t)}{\Lambda \Gamma(\gamma)} \left( \sum_{i=1}^{m} \delta_i \Gamma_{\mu_i}^{\alpha_i} F_{x}(s)(\eta_i) + \sum_{k=1}^{r} \lambda_k \Gamma_{\mu_k}^{\alpha_k} F_{x}(s)(\xi_k) + \sum_{j=1}^{n} \omega_j \Gamma_{\mu_j}^{1+\beta_j} F_{x}(s)(\theta_j) \right),
\]

\[
(T_2 x)(t) = -p \Gamma_{\mu_1} \phi_{1, \phi} x(t) + \frac{(\psi(t) - \psi(0))^{\gamma - 1}}{\Lambda \Gamma(\gamma)} \left( A + p \left( \sum_{i=1}^{m} \delta_i \Gamma_{\mu_i}^{\alpha_i} x(\eta_i) + \sum_{k=1}^{r} \lambda_k \Gamma_{\mu_k}^{1-\mu_k} x(\xi_k) + \sum_{j=1}^{n} \omega_j \Gamma_{\mu_j}^{1+\beta_j} x(\theta_j) \right) \right).
\]

We note that $T = T_1 + T_2$.

For any $x, y \in B_r$, we have:
\[
| (T_1 x)(t) + (T_2 y)(t) | \leq \frac{I_0^{\alpha, \psi} | F_s(s) | (T) + |p| I_0^{1, \psi} y(t)}{|A| \Gamma(\gamma)} + \frac{|\lambda| \Gamma(\gamma)}{m} \left( \sum_{i=1}^{m} |\delta_i| I_0^{\alpha, \psi} | y(\eta_i) | + \sum_{k=1}^{r} |\lambda_k| I_0^{1, \psi} y(\xi_k) \right) \\
+ \sum_{j=1}^{n} |\omega_j| I_0^{1, \psi} \left| y(\theta_j) \right| \\
+ \left( \sum_{i=1}^{m} |\delta_i| I_0^{\alpha, \psi} | F_s(s)(\eta_i) | + \sum_{k=1}^{r} |\lambda_k| I_0^{1, \psi} | F_s(s)(\xi_k) | \right) \\
+ \sum_{j=1}^{n} |\omega_j| I_0^{1, \psi} \left| F_s(s)(\theta_j) \right| \right) 
\]
\[
\leq \frac{\left( \psi(T) - \psi(0) \right)^\alpha |\phi|| + |p| (\psi(T) - \psi(0)) y||}{\Gamma(\alpha + 1)} + \frac{|p| \left( \psi(T) - \psi(0) \right)^{\alpha-1} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right)}{|A| \Gamma(\gamma)} \\
+ \sum_{k=1}^{r} |\lambda_k| \left( \frac{\psi(\xi_k) - \psi(0)}{\Gamma(2 - \mu_k)} \right)^{1-\mu_k} \\
+ \sum_{j=1}^{n} |\omega_j| \left( \frac{\psi(\theta_j) - \psi(0)}{\Gamma(2 + \beta_j)} \right) \left| y \right| \\
+ \frac{\left( \psi(T) - \psi(0) \right)^{\alpha-1} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right)}{|A| \Gamma(\gamma)} \\
+ \sum_{k=1}^{r} |\lambda_k| \left( \frac{\psi(\xi_k) - \psi(0)}{\Gamma(\alpha - \mu_k + 1)} \right)^{\alpha-\mu_k} + \sum_{j=1}^{n} |\omega_j| \left( \frac{\psi(\theta_j) - \psi(0)}{\Gamma(\alpha + \beta_j + 1)} \right) \left| y \right| 
\]
\[
\leq \frac{\left( \psi(T) - \psi(0) \right)^{\alpha-1} |A| + \left| p \right| (\psi(T) - \psi(0))}{\Gamma(\alpha + 1)} + \frac{|p| \left( \psi(T) - \psi(0) \right)^{\alpha-1} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right)}{|A| \Gamma(\gamma)} \\
+ \sum_{k=1}^{r} |\lambda_k| \left( \frac{\psi(\xi_k) - \psi(0)}{\Gamma(2 - \mu_k)} \right)^{1-\mu_k} + \sum_{j=1}^{n} |\omega_j| \left( \frac{\psi(\theta_j) - \psi(0)}{\Gamma(2 + \beta_j)} \right) \left| y \right| \\
+ \left[ \frac{\left( \psi(T) - \psi(0) \right)^\alpha}{\Gamma(\alpha + 1)} + \frac{\left( \psi(T) - \psi(0) \right)^{\alpha-1} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right)}{|A| \Gamma(\gamma)} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right)^\alpha \right] \left| y \right| \\
+ \sum_{k=1}^{r} |\lambda_k| \left( \frac{\psi(\xi_k) - \psi(0)}{\Gamma(\alpha - \mu_k + 1)} \right)^{\alpha-\mu_k} + \sum_{j=1}^{n} |\omega_j| \left( \frac{\psi(\theta_j) - \psi(0)}{\Gamma(\alpha + \beta_j + 1)} \right) \left| y \right| 
\]
\[
\leq \left[ Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|A|} \right] \left| \phi \right| + \left| p \right| \Omega_1 \frac{Q(T, \gamma - 1)}{|A|} + \left| p \right| (\psi(T) - \psi(0)) \right] \left| y \right| \\
+ \frac{|A| \Omega_1 Q(T, \gamma - 1)}{|A|} \left| \phi \right| + \frac{|A| \Omega_1 Q(T, \gamma - 1)}{|A|} \left| \phi \right| 
\]
\[
\leq r. 
\]
Thus, this shows that the operator \( T \) is continuous. Hence, we obtain

\[
\|F_{x_n} - F_x\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

This implies that \( T_{1,x} + T_{2,y} \in B_r \), which satisfies the assumption (a) of Lemma 2.9.

We show now that the second assumption (b) of Lemma 2.9 is satisfied.

Let \( x_n \) be a sequence such that \( x_n \rightarrow x \) in \( C \). Then for each \( t \in J \), we have

\[
|\langle T_1 x_n \rangle(t) - \langle T_1 x \rangle(t)\rangle \leq \frac{I_{0+}^{\alpha,\psi} |F_{x_n}(s) - F_x(s)|}{\|\psi\|} (T)  
+ \frac{(\psi(T) - \psi(0))^{\gamma - 1}}{|\Lambda| \Gamma(\gamma)} \left( \sum_{i=1}^{m} |\delta_i| I_{0+}^{\alpha,\psi} |F_{x_n}(s)(\eta_i) - F_x(s)(\eta_i)| \right) 
+ \sum_{k=1}^{r} |\lambda_k| I_{0+}^{\alpha-\mu_k,\psi} |F_{x_n}(s)(\xi_k) - F_x(s)(\xi_k)| 
+ \sum_{j=1}^{n} |\omega_j| I_{0+}^{\alpha+\beta_j,\psi} |F_{x_n}(s)(\theta_j) - F_x(s)(\theta_j)| 
\]

Since \( f \) is continuous, this implies that the operator \( F_x \) is also continuous. Hence, we obtain

\[
\|F_{x_n} - F_x\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Thus, this shows that the operator \( T_{1,x} \) is continuous. Also the set \( T_{1,B_r} \) is uniformly bounded on \( B_r \) as

\[
\|T_{1,x}\| \leq \frac{I_{0+}^{\alpha,\psi} |F_x(s)|}{\|\psi\|} (T)  
+ \frac{(\psi(T) - \psi(0))^{\gamma - 1}}{|\Lambda| \Gamma(\gamma)} \left( \sum_{i=1}^{m} |\delta_i| I_{0+}^{\alpha,\psi} |F_x(s)(\eta_i)| + \sum_{k=1}^{r} |\lambda_k| I_{0+}^{\alpha-\mu_k,\psi} |F_x(s)(\xi_k)| \right) 
+ \sum_{j=1}^{n} |\omega_j| I_{0+}^{\alpha+\beta_j,\psi} |F_x(s)(\theta_j)| 
\]

Next, we prove the compactness of the operator \( T_1 \). Let \( \sup_{(t,u,v) \in J \times B \times B_j} |f(t,u,v)| = \tilde{f} < \infty \), then for each \( t_1, t_2 \in J \) with \( 0 \leq t_1 \leq t_2 \leq T \), we obtain
\[
| (T_1 x)(t_2) - (T_1 x)(t_1) | = \frac{| \int_0^{t_2} I_{0}^{\alpha, \psi} F_x(t_2) - \int_0^{t_1} I_{0}^{\alpha, \psi} F_x(t_1) |}{(\psi(t_2) - \psi(0))^{\alpha - 1} - (\psi(t_1) - \psi(0))^{\alpha - 1}} + \frac{\sum_{i=1}^{m} |\delta_i| | I_{0}^{\alpha, \psi} F_x(\eta_i) | + \sum_{k=1}^{r} |\lambda_k| | I_{0}^{\alpha-\mu_k, \psi} F_x(\xi_k) |}{|\Lambda| \Gamma(\gamma)} \\
\times \left( \sum_{i=1}^{n} |\omega_i| | I_{0}^{\alpha+\beta_i, \psi} F_x(\theta_j) | \right) \\
\leq \frac{\left( \sum_{i=1}^{m} |\delta_i| | I_{0}^{\alpha, \psi} F_x(\eta_i) | + \sum_{k=1}^{r} |\lambda_k| | I_{0}^{\alpha-\mu_k, \psi} F_x(\xi_k) | \right)}{|\Lambda| \Gamma(\gamma)} \\
\times \left( \sum_{i=1}^{n} |\omega_i| | I_{0}^{\alpha+\beta_i, \psi} F_x(\theta_j) | \right) \\
\leq \frac{\int_{\alpha}^{\alpha + 1} \left[ 2(\psi(t_2) - \psi(t_1))^{\alpha} + (\psi(t_2) - \psi(0))^{\alpha} - (\psi(t_1) - \psi(0))^{\alpha} \right]}{\Gamma(\alpha + 1)} + \frac{\int_{\alpha}^{\alpha + 1} (\psi(t_2) - \psi(0))^{\alpha - 1} - (\psi(t_1) - \psi(0))^{\alpha - 1}}{\Gamma(\alpha + 1)} \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0))^{\alpha} \right) \\
+ \frac{\sum_{k=1}^{r} |\lambda_k| (\psi(\xi_k) - \psi(0))^{\alpha - \mu_k}}{\Gamma(\alpha - \mu_k + 1)} + \sum_{j=1}^{n} |\omega_j| (\psi(\theta_j) - \psi(0))^{\alpha + \beta_j} \Gamma(\alpha + \beta_j + 1) \right). 
\]

The right hand side of the inequality above is independant of \( x \) and tends to 0 as \( t_2 \rightarrow t_1 \).

Therefore, the operator \( T_1 \) is equicontinuous. Thus, \( T_1 \) is relatively compact on \( B_r \). Then, by the well-known Arzela-Ascoli theorem, \( T_1 \) is a compact operator on \( B_r \).

Now we show that the operator \( T_2 \) is a contraction, which is the third and last condition of Lemma 2.9.

For any \( x, y \in C \) and for each \( t \in J \), we have
\[ |(T_2 x)(t) - (T_2 y)(t)| \leq |p| \left| \frac{1^\psi}{0^\psi} |x(t) - y(t)| \right| + \frac{1}{|\Lambda| \Gamma(\gamma)} |p| \left| \left( \sum_{i=1}^{m} |\delta_i| \frac{1^\psi}{0^\psi} |x(\eta_i) - y(\eta_i)| \right) \right| \\
+ \sum_{k=1}^{r} |\lambda_k| \frac{1^{-\mu_k \psi}}{0^{-\mu_k \psi}} |x(\xi_k) - y(\xi_k)| \\
+ \sum_{j=1}^{n} |\omega_j| \frac{1^{1+\beta_j \psi}}{0^{1+\beta_j \psi}} |x(\theta_j) - y(\theta_j)| \right) 
\]

\[ \leq |p| \left| (\psi(T) - \psi(0)) \right| |x - y| + \frac{1}{|\Lambda| \Gamma(\gamma)} \left| \left( \sum_{i=1}^{m} |\delta_i| (\psi(\eta_i) - \psi(0)) \right) \right| \\
+ \sum_{k=1}^{r} |\lambda_k| \frac{1^{-\mu_k \psi}}{0^{-\mu_k \psi}} \frac{(\psi(\xi_k) - \psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
+ \sum_{j=1}^{n} |\omega_j| \frac{1^{1+\beta_j \psi}}{0^{1+\beta_j \psi}} \frac{(\psi(\theta_j) - \psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)} \right| |x - y| \\
= \left| p \right| \Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p| (\psi(T) - \psi(0)) |x - y|. \]

Using (3.4), we conclude that the operator \( T_2 \) is a contraction. Thus, all assumptions of Lemma 2.9 are satisfied. So we conclude that the problem (1.4) has at least one solution on \( J \). The proof is completed. \( \square \)

4. Examples

This section presents some examples which illustrate the validity of the main results.

Consider the following sequential \( \psi \)-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions:

\[
\begin{cases}
\left( \frac{1^\psi}{0^\psi} + \frac{1}{7} \frac{1^\psi}{0^\psi} \right) x(t) = f(t, x(t), x(\sigma t)), t \in (0, 1], \\
x(0) = 0, \sum_{i=1}^{3} \left( \left( \frac{-i}{i+5} \right)^{i+1} \right) x \left( \frac{i}{3} \right) + \frac{1}{7} \sum_{j=1}^{2} \left( \frac{j+1}{j+2} \right) x \left( \frac{j}{2} \right) + \frac{1}{7} \sum_{k=1}^{4} \left( \frac{-k}{k+2} \right)^{k} x \left( \frac{k}{4} \right) = \frac{1}{2},
\end{cases}
\]

Here we have: \( \alpha = \frac{8}{3}, \beta = \frac{1}{4}, p = \frac{1}{7}, T = 1, \sigma = \frac{1}{3}, A = \frac{1}{2}, m = 3, n = 2, r = 4, \psi(t) = e^t, \)
\[
\delta_i = \left(\frac{-i}{i + 5}\right)^{i+1}, \quad \omega_j = \left(\frac{j + 1}{j + 2}\right), \quad \lambda_k = \left(\frac{-k}{k + 2}\right)^k, \quad \eta_i = \frac{i}{3}, \quad \theta_j = \frac{j}{2}, \quad \xi_k = \frac{k}{4}, \quad \beta_j = \frac{j}{3}, \quad \mu_k = \frac{k + 8}{8}, \text{ for } i = 1, 2, 3, j = 1, 2 \text{ and } k = 1, 2, 3, 4.
\]

After doing some calculations we find that: \( \Lambda \approx 0.5377547 \neq 0, \Omega_1 \approx 1.8265034 \) and \( \Omega_2 \approx 0.9099. \)

**Example 4.1.** Consider the function:

\[
f(t, x(t), x(\sigma t)) = \frac{\cos |x(t) + x(\sigma t)|}{50} + \frac{|x(\sigma t)|}{(t^3 + 5)^4}
\]

hence, \( f \) satisfies the hypothesis \((H_1)\) as for any \( x, y \in \mathbb{R}, t \in J, \) we have:

\[
|f(t, x(t), x(\sigma t)) - f(t, y(t), y(\sigma t))| \leq 0.0222 |x - y|
\]

We set \( L = 0.0222, \) therefore we obtain:

\[
2L \left( Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|A|} \right) + \left( |p| \Omega_1 \frac{Q(T, \gamma - 1)}{|A|} + |p| (\psi(T) - \psi(0)) \right) \approx 0.500778 < 1.
\]

It follows from Theorem 3.1 that the problem (4.1) has a unique solution \( x \) on \([0, 1] \).

**Example 4.2.** By considering the function

\[
f(t, x(t), x(\sigma t)) = \frac{2 \cos |x(t)|}{9 + 2t} + \frac{2 \sin |x(\sigma t)|}{4 + 2t} + e^{-2t},
\]

it is easy to see that \( f \) satisfies the hypothesis \((H_2)\) as

\[
|f(t, x(t), x(\sigma t))| \leq \frac{2}{9 + 2t} + \frac{2}{4 + 2t} + e^{-2t},
\]

and we have:

\[
\left( |p| \Omega_1 \frac{Q(T, \gamma - 1)}{|A|} + |p| (\psi(T) - \psi(0)) \right) \approx 0.36879 < 1.
\]

It follows from Theorem 3.2 that the problem (4.1) has at least one solution \( x \) on \([0, 1] \).

**5. Conclusions**

This paper studied a new class of sequential \( \psi\)-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions. Existence and uniqueness results are established, we first proved the uniqueness results by using the Banach’s contraction mapping principle, followed by the existence results using the Krasnoselskii’s fixed point theorem. Our results are not only original and new, but also for example by taking \( \omega_j = 0 \) and \( \lambda_k = 0, \) for \( j = 1, 2, ..., n, k = 1, 2, ..., r, \) our results correspond to the ones for boundary value problems for sequential \( \psi\)-Hilfer pantograph differential equations supplemented with multi-point boundary conditions, and by taking \( \delta_i = 0 \) and \( \lambda_k = 0, \)
for \( i = 1, 2, ..., m, \ k = 1, 2, ..., r, \) our results correspond to the ones for boundary value problems for sequential \( \psi \)-Hilfer pantograph differential equations supplemented with multi-term integral boundary conditions. In the end, we have given two examples to strengthen our theoretical findings. The work established in this paper is new and contributes in the development of the literature on boundary value problems for nonlinear \( \psi \)-Hilfer fractional differential equations.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


