Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8239–8255.
DOI:10.3934/math.2021477
Received: 15 March 2021
Accepted: 18 May 2021
Published: 26 May 2021

Research article

Existence and uniqueness results for sequential ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions

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Abstract: In this paper, we discuss the existence and uniqueness of boundary value problems for sequential ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions. The existence results are obtained via the well known Krasnoselskii's fixed point theorem while the uniqueness is demonstrated by using the Banach's contraction mapping principle. Some examples are also given to demonstrate the application of the main results.

Keywords: existence; uniqueness; ψ-Hilfer Fractional derivative; pantograph equations; fixed point theory
Mathematics Subject Classification: 26A33, 34A08, 34A12, 34B15

1. Introduction

Fractional calculus is a powerful tool to investigate several complex problems in numerous scientific and engineering disciplines such as physics, chemistry, biology, economics, and control theory. Differential equations of fractional order describe many real world processes more accurately compared to the classical order differential equations. For more details about the theory of fractional differential equations and applications, see [1–6].

In the literature, the most used derivatives of fractional order are the Caputo and the Riemann-Liouville derivatives. A generalization of these derivatives was introduced by R. Hilfer in [7], and this derivative is called the Hilfer fractional derivative. For more details we give the following references [8, 9].

In [10], the authors began the study of nonlocal boundary value problems involving the Hilfer fractional derivatives, by studying the following problem

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t, x(t)), t \in [a, b], 1 < \alpha < 2, 0 \leq \beta \leq 1, \\ x(a) = 0, x(b) = \sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i), \varphi_i > 0, \delta_i \in \mathbb{R}, \xi_i \in [a, b], \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, and parameter β, I^{φ_i} is the Riemann-Liouville fractional integral of order $\varphi_i > 0$, several fixed point theorems were used to prove the existence and uniqueness results.

In [11], the authors considered the existence and uniqueness for a class of system of Hilfer-Hadamard fractional differential equations with two point boundary conditions

$$\begin{cases} \left({}^{H}D_{1^+}^{\alpha_1,\beta_1} + k_1\, {}^{H}D_{1^+}^{\alpha_1-1,\beta_1}\right) u(t) = f(t, u(t), v(t)), t \in [1, e], \\ \left({}^{H}D_{1^+}^{\alpha_2,\beta_2} + k_2\, {}^{H}D_{1^+}^{\alpha_2-1,\beta_2}\right) v(t) = g(t, u(t), v(t)), t \in [1, e], \\ u(1) = 0, u(e) = A_1, \\ v(1) = 0, v(e) = A_2, \end{cases} \tag{1.2}$$

where ${}^{H}D^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order $1 < \alpha_i \leq 2$, and type $0 \leq \beta_i \leq 1$ for $i \in \{1, 2\}$, $k_1, k_2, A_1, A_2 \in \mathbb{R}^+$, and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Another fractional derivative, which is a derivative with respect to another function, is the ψ-Hilfer fractional derivative, it was introduced in [12]. A lot of papers studied the existence and uniqueness of fractional differential equations using the ψ-Hilfer fractional derivatives, please see [13–20] and references therein.

On the other hand, another important class of fractional differential equations are the pantograph equations. The pantograph equations are an important class of delay equations and they are used in deterministic situations. Initial value problems for pantograph equations with the Hilfer fractional derivative were studied in [21–24].

Recently in [25], the authors studied the existence and uniqueness of solutions for a new class of boundary value problems of sequential ψ-Hilfer fractional differential equations with multi-point boundary conditions of the form

$$\begin{cases} \left({}^{H}D_{0^+}^{\alpha,\beta;\psi} + k\, {}^{H}D_{0^+}^{\alpha-1,\beta;\psi}\right) x(t) = f(t, x(t)), t \in (a, b] \\ x(a) = 0, x(b) = \sum_{i=1}^{m} \lambda_i x(\theta_i). \end{cases} \tag{1.3}$$

In this paper, we consider a new class of sequential ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions as follows

$$\begin{cases} \left({}^{H}D_{0^+}^{\alpha,\beta;\psi} + p\, {}^{H}D_{0^+}^{\alpha-1,\beta;\psi}\right) x(t) = f(t, x(t), x(\sigma t)), t \in (0, T], 0 < \sigma < 1 \\ x(0) = 0, \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{j=1}^{n} \omega_j I_{0^+}^{\beta_j;\psi} x(\theta_j) + \sum_{k=1}^{r} \lambda_k\, {}^{H}D_{0^+}^{\mu_k,\beta} x(\xi_k) = A, \end{cases} \tag{1.4}$$

where ${}^{H}D_{0^+}^{u,\beta;\psi}$ are the ψ-Hilfer derivatives of order $u = \{\alpha, \mu_k\}$, $1 < \mu_k < \alpha \le 2$, $0 \le \beta \le 1$, $I_{0^+}^{\beta_j;\psi}$ are the ψ-Riemann Liouville fractional integrals of order β_j, with $\beta_j > 0$, for $j = 1, 2, ..., n$, $p, A, \delta_i, \omega_j, \lambda_k \in \mathbb{R}$ are given constants, the points η_i, θ_j, ξ_k are in J, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$, $k = 1, 2, ..., r$ and the function $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, $J = [0, T]$, $T > 0$.

It is important for us to note that the problem considered in this paper provide more insight in the study of sequential ψ-Hilfer-type fractional differential equations, this paper can be viewed as a generalization of some existing papers in the literature. Our nonlocal boundary conditions are more useful and more general. We note that the mixed nonlocal boundary conditions include multi-point, fractional derivative of multi-order and fractional integral of multi-order boundary conditions.

This research paper is organized as follows, in section 2, we provide some definitions and lemmas that will be used throughout the paper, in section 3, we establish the existence and uniqueness results by means of the fixed point theorems, and last but not least, in section 4, we give some examples to illustrate the applicability of the results.

2. Preliminaries and auxiliary results

In this section, we introduce some definitions, lemmas and useful notations that will be used throughout the paper.

Let $C(J, \mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm defined by $\|f\| = \sup_{t \in J}\{|f(t)|\}$.

We also define the n-times absolutely continuous functions given by

$$\mathcal{AC}^n(J, \mathbb{R}) = \{f : J \longrightarrow \mathbb{R}; f^{(n-1)} \in \mathcal{AC}(J, \mathbb{R})\}.$$

Definition 2.1. *(see [1]) Let (a, b), $(-\infty \le a < b \le \infty)$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b). The ψ-Riemann-Liouville fractional integral of a function f with respect to other function ψ is defined by*

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds, \; t > a > 0,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *(see [1]) Let $\psi'(t) \neq 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to another function ψ of order α, is defined by*

$$\begin{aligned} D_{a^+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number α.

Definition 2.3. *(see [12]) Let $n - 1 < \alpha < n$ with $n \in \mathbb{N}$, $[a, b]$ is the interval such that $-\infty \le a < b \le \infty$ and $f, \psi \in C^n([a, b], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all*

$t \in [a,b]$. *The ψ-Hilfer fractional derivative of a function f of order α and type $0 \le \beta \le 1$, is defined by*

$$^{H}D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t) = I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f(t),$$

where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number α with $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 2.4. *(see [1]) Let $\alpha, \beta > 0$. Then we have the following semigroup property given by*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi} f(t),\ t > a.$$

Proposition 2.5. *(see [1,12]) Let $\alpha \ge 0$, $\nu > 0$ and $t > a$. Then, the ψ-fractional integral and derivative of a power function are given by*

(i) $I_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\nu-1}(t) = \dfrac{\Gamma(\nu)}{\Gamma(\nu+\alpha)} (\psi(t) - \psi(a))^{\nu+\alpha-1}$,

(ii) $D_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\nu-1}(t) = \dfrac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} (\psi(t) - \psi(a))^{\nu-\alpha-1}$,

(iii) $^{H}D_{a^+}^{\alpha,\beta;\psi} (\psi(t) - \psi(a))^{\nu-1}(t) = \dfrac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} (\psi(t) - \psi(a))^{\nu-\alpha-1}$, $\nu > \gamma = \alpha + \beta(2 - \alpha)$.

Lemma 2.6. *(see [12]) Let $m - 1 < \alpha < m$, $n - 1 < \beta < n$, $n, m \in \mathbb{N}$, $n \le m$, $0 \le \rho \le 1$ and $\alpha \ge \beta + \rho(n - \beta)$. If $h \in C^n(J, \mathbb{R})$, then*

$$^{H}D_{a^+}^{\beta,\rho;\psi} I_{a^+}^{\alpha,\psi} h(t) = I_{a^+}^{\alpha-\beta;\psi} h(t).$$

Lemma 2.7. *(see [12]) If $f \in C^n(J, \mathbb{R})$, $n - 1 < \alpha < n$, $0 \le \beta \le 1$ and $\gamma = \alpha + \beta(n - \alpha)$ then*

$$I_{a^+}^{\alpha;\psi}\, {}^{H}D_{a^+}^{\alpha,\beta;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_{\psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a),$$

for all $t \in J$, where $f_{\psi}^{[n]} f(t) := \left(\dfrac{1}{\psi'(t)} \dfrac{d}{dt} \right)^n f(t)$.

Fixed point theorems play an important role in our study, we will give in this next part some well-known fixed point theorems that we have used in this paper.

Lemma 2.8. *(Banach contraction principle, see [26]) Let D be a non-empty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Lemma 2.9. *(Krasnoselskii's fixed point theorem, see [27]) Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that*

(a) $Ax + By \in M$ *whenever* $x, y \in M$,
(b) *A is compact and continuous,*
(c) *B is a contraction mapping.*
Then, there exists $z \in M$ such that $z = Az + Bz$.

In order to convert the problem (1.4) into a fixed point problem, we must transform it into an equivalent integral equation. We provide the following Lemma, which is a linear variant of the boundary value problem (1.4).

Lemma 2.10. *Let $1 < \mu_k < \alpha \le 2$, $\gamma = \alpha + \beta(2-\alpha)$, $k = 1, 2, ..., r$, and $\Lambda \neq 0$. Suppose that $h \in C$. Then $x \in C^2$ is a solution of the problem*

$$\begin{cases} \left({}^H D_{0^+}^{\alpha,\beta;\psi} + p\,{}^H D_{0^+}^{\alpha-1,\beta;\psi}\right) x(t) = h(t), t \in (0, T], \\ x(0) = 0, \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{j=1}^{n} \omega_j I_{0^+}^{\beta_j;\psi} x(\theta_j) + \sum_{k=1}^{r} \lambda_k {}^H D_{0^+}^{\mu_k,\beta} x(\xi_k) = A, \end{cases} \tag{2.1}$$

if and only if x satisfies the integral equation

$$\begin{aligned} x(t) &= I^{\alpha;\psi} h(t) - p I^{1;\psi} x(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \Big[A + p\Big(\sum_{i=1}^{m} \delta_i I_{0^+}^{1;\psi} x(\eta_i) \\ &\quad + \sum_{k=1}^{r} \lambda_k I_{0^+}^{1-\mu_k;\psi} x(\xi_k) + \sum_{j=1}^{n} \omega_j I_{0^+}^{1+\beta_j;\psi} x(\theta_j) \Big) - \Big(\sum_{i=1}^{m} \delta_i I_{0^+}^{\alpha;\psi} h(\eta_i) \\ &\quad + \sum_{k=1}^{r} \lambda_k I_{0^+}^{\alpha-\mu_k;\psi} h(\xi_k) + \sum_{j=1}^{n} \omega_j I_{0^+}^{\alpha+\beta_j;\psi} h(\theta_j) \Big) \Big], \end{aligned}$$

where

$$\Lambda = \sum_{i=1}^{m} \frac{\delta_i (\psi(\eta_i) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + \sum_{k=1}^{r} \frac{\lambda_k (\psi(\xi_k) - \psi(0))^{\gamma-\mu_k-1}}{\Gamma(\gamma - \mu_k)} + \sum_{j=1}^{n} \frac{\omega_j (\psi(\theta_j) - \psi(0))^{\gamma+\beta_j-1}}{\Gamma(\gamma + \beta_j)}.$$

Proof. Let x be a solution of the problem (2.1). By using Lemma 2.7, and operating $I_{0^+}^{\alpha;\psi}$ on both sides of Eq (2.1) we obtain

$$x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + c_2 \frac{(\psi(t) - \psi(0))^{\gamma-2}}{\Gamma(\gamma - 1)} - p I_{0^+}^{1;\psi} x(t) + I_{0^+}^{\alpha;\psi} h(t),$$

where c_1, c_2 are real constants.
For $t = 0$, we get $c_2 = 0$, and thus

$$x(t) = c_1 \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + I_{0^+}^{\alpha;\psi} h(t) - p I_{0^+}^{1;\psi} x(t). \tag{2.2}$$

Applying the operators ${}^H D_{0^+}^{\mu_k,\rho;\psi}$ and $I_{0^+}^{\beta_j;\psi}$ to (2.2), we obtain

$$\begin{aligned} {}^H D_{0^+}^{\mu_k,\rho;\psi} x(t) &= c_1 \frac{(\psi(t) - \psi(0))^{\gamma-\mu_k-1}}{\Gamma(\gamma - \mu_k)} - p I_{0^+}^{1-\mu_k;\psi} x(t) + I_{0^+}^{\alpha-\mu_k;\psi} h(t), \\ I_{0^+}^{\beta_j;\psi} x(t) &= c_1 \frac{(\psi(t) - \psi(0))^{\gamma+\beta_j-1}}{\Gamma(\gamma + \beta_j)} - p I_{0^+}^{1+\beta_j;\psi} x(t) + I_{0^+}^{\alpha+\beta_j;\psi} h(t). \end{aligned}$$

By using the second boundary condition in (2.1), we obtain

$$c_1\Big[\sum_{i=1}^{m}\frac{\delta_i(\psi(\eta_i)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}+\sum_{k=1}^{r}\frac{\lambda_k(\psi(\xi_k)-\psi(0))^{\gamma-\mu_k-1}}{\Gamma(\gamma-\mu_k)}+\sum_{j=1}^{n}\frac{\omega_j(\psi(\theta_j)-\psi(0))^{\gamma+\beta_j-1}}{\Gamma(\gamma+\beta_j)}\Big]$$
$$-p\Big(\sum_{i=1}^{m}\delta_i I_{0^+}^{1;\psi}x(\eta_i)+\sum_{k=1}^{r}\lambda_k I_{0^+}^{1-\mu_k;\psi}x(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{1+\beta_j;\psi}x(\theta_j)\Big)+\sum_{i=1}^{m}\delta_i I_{0^+}^{\alpha;\psi}h(\eta_i)$$
$$+\sum_{k=1}^{r}\lambda_k I_{0^+}^{\alpha-\mu_k;\psi}h(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{\alpha+\beta_j;\psi}h(\theta_j)=A,$$

from which we can get

$$c_1=\frac{1}{\Lambda}\Big[A+p\Big(\sum_{i=1}^{m}\delta_i I_{0^+}^{1;\psi}x(\eta_i)+\sum_{k=1}^{r}\lambda_k I_{0^+}^{1-\mu_k;\psi}x(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{1+\beta_j;\psi}x(\theta_j)\Big)$$
$$-\Big(\sum_{i=1}^{m}\delta_i I_{0^+}^{\alpha;\psi}h(\eta_i)+\sum_{k=1}^{r}\lambda_k I_{0^+}^{\alpha-\mu_k;\psi}h(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{\alpha+\beta_j;\psi}h(\theta_j)\Big)\Big],$$

where Λ is defined in Lemma 2.10. By substituting the value of c_1 in (2.2), we obtain the solution.

Conversly, it is easy to show that the solution x given in Lemma 2.10 satisfies the problem (2.1). The proof is now completed. □

3. Existence and uniqueness results

In this section, we present the existence and uniqueness results to the problem (1.4).

For convenience, we are going to use the following expressions:

$$Q(\chi,\epsilon)=\frac{(\psi(\chi)-\psi(0))^{\epsilon}}{\Gamma(\epsilon+1)},\tag{3.1}$$

$$\Omega_1=\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0))+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)},\tag{3.2}$$

$$\Omega_2=\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}.\tag{3.3}$$

In view of Lemma 2.10, we define the operator $T: C\longrightarrow C$ by

$$\begin{aligned}(Tx)(t) &= I_{0^+}^{\alpha;\psi}F_x(s)(t)-pI_{0^+}^{1;\psi}x(t)+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big[A+p\Big(\sum_{i=1}^{m}\delta_i I_{0^+}^{1;\psi}x(\eta_i)\\ &\quad+\sum_{k=1}^{r}\lambda_k I_{0^+}^{1-\mu_k;\psi}x(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{1+\beta_j;\psi}x(\theta_j)\Big)-\Big(\sum_{i=1}^{m}\delta_i I_{0^+}^{\alpha;\psi}F_x(s)(\eta_i)\\ &\quad+\sum_{k=1}^{r}\lambda_k I_{0^+}^{\alpha-\mu_k;\psi}F_x(s)(\xi_k)+\sum_{j=1}^{n}\omega_j I_{0^+}^{\alpha+\beta_j;\psi}F_x(s)(\theta_j)\Big)\Big],\end{aligned}$$

where $F_x(t) = f(t, x(t), x(\sigma t))$, $0 < \sigma < 1$.

It should be mentionned here that the problem (1.4) has solutions if and only if the operator T has fixed points.

3.1. Uniqueness results via the Banach's contraction principle

By applying the Banach's contraction principle, we establish the existence and uniqueness of solutions for the problem (1.4).

Theorem 3.1. *We consider the following hypotheses:*

(H_1) *The functions $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and there exists a constant $L > 0$ such that*

$$|f(t, x, y) - f(t, z, w)| \leq L\,(|x - z| + |y - w|),\ \textit{for all } t \in J,\ \textit{and } x, y, z, w \in \mathbb{R}.$$

If we have

$$2L\Big(Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|}\Big) + \Big(\,|p|\,\Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p|\,(\psi(T) - \psi(0))\Big) < 1,$$

where Ω_1 and Ω_2 are given by (3.2) *and* (3.3) *respectively, then the problem* (1.4) *has a unique solution on J.*

Proof. First of all, we transform the problem (1.4) into a fixed point problem, $x = Tx$, where the operator T is defined in the previous section. By applying the Banach's contraction principle, we show that the operator T has a unique fixed point, which is the unique solution of problem (1.4).

Let $\sup_{t \in J} |f(t, 0, 0)| = M < \infty$, and we set $B_r := \{x \in C : \|x\| \leq r\}$ with

$$r \geq \frac{\Big[Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|}\Big]M + \frac{|A|\,Q(T, \gamma - 1)}{|\Lambda|}}{1 - \Big[\,|p|\,\Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p|\,(\psi(T) - \psi(0))\Big] - 2L\Big[Q(T, \alpha) + \Omega_2 \frac{Q(T, \gamma - 1)}{|\Lambda|}\Big]},$$

where $Q(\chi, \epsilon)$, Ω_1, Ω_2 are given by (3.1),(3.2),(3.3) respectively. It is clear that B_r is a bounded, closed and convex subset of C.

Step I. We first show that $TB_r \subset B_r$.

We have from the hypothesis (H_1) that

$$\begin{aligned} |F_x(t)| &\leq |f(t, x(t), x(\sigma t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq L(|x(t)| + |x(\sigma t)|) + M \\ &\leq 2L\,\|x\| + M \end{aligned}$$

Then we have

$$
\begin{aligned}
|(Tx)(t)| &= I_{0^+}^{\alpha;\psi}\,|F_x(s)|\,(T) + |p|\,I_{0^+}^{1;\psi}\,|x(t)| \\
&\quad + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big[\,|A| + |p|\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{1;\psi}\,|x(\eta_i)| + \sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{1-\mu_k;\psi}\,|x(\xi_k)| \\
&\quad + \sum_{j=1}^{n}\big|\omega_j\big|\,I_{0^+}^{\alpha+\beta_j;\psi}\,\big|F_x(s)(\theta_j)\big|\Big)\Big] \\
&\le \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}(2L\,\|x\|+M) + |p|\,(\psi(T)-\psi(0))\,\|x\| + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\,|A| \\
&\quad + |p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0)) + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&\quad + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)\|x\| + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)} + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)(2L\,\|x\|+M) \\
&\le \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}(2Lr+M) + |p|\,(\psi(T)-\psi(0))r + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\,|A| \\
&\quad + |p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0)) + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&\quad + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)r + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)} + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)(2Lr+M) \\
&\le \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\,|A| + \Big[\,|p|\,(\psi(T)-\psi(0)) \\
&\quad + |p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0)) + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&\quad + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)\Big]r \\
&\quad + \Big[\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)} + \sum_{j=1}^{n}\big|\omega_j\big|\,\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)\Big](2Lr+M) \\
&\le \Big[Q(T,\alpha) + \Omega_2\frac{Q(T,\gamma-1)}{|\Lambda|}\Big](2Lr+M) + \Big[\,|p|\,\Omega_1\frac{Q(T,\gamma-1)}{|\Lambda|} + |p|\,(\psi(T)-\psi(0))\Big]r \\
&\quad + \frac{|A|\,Q(T,\gamma-1)}{|\Lambda|} \\
&\le r,
\end{aligned}
$$

which implies that $TB_r \subset B_r$.

Step II. We show that the operator $T : C \longrightarrow C$ is a contraction.

For any $x, y \in C$ and for each $t \in J$, we have

$$
\begin{aligned}
|(Tx)(t)-(Ty)(t)| \;\leq\;& I_{0^+}^{\alpha;\psi}\left|F_x(s)-F_y(s)\right|(T)+|p|\,I_{0^+}^{1;\psi}\,|x(t)-y(t)| \\
&+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big[\,|A|+|p|\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{1;\psi}\,|x(\eta_i)-y(\eta_i)| \\
&+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{1-\mu_k;\psi}\,|x(\xi_k)-y(\xi_k)| \\
&+\sum_{j=1}^{n}\left|\omega_j\right|I_{0^+}^{1+\beta_j;\psi}\left|x(\theta_j)-y(\theta_j)\right|\Big)+\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{\alpha;\psi}\left|F_x(s)(\eta_i)-F_y(s)(\eta_i)\right| \\
&+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{\alpha-\mu_k;\psi}\left|F_x(s)(\xi_k)-F_y(s)(\xi_k)\right| \\
&+\sum_{j=1}^{n}\left|\omega_j\right|I_{0^+}^{\alpha+\beta_j;\psi}\left|F_x(s)(\theta_j)-F_y(s)(\theta_j)\right|\Big)\Big] \\
\leq\;& \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}.2L\,\|x-y\|+|p|\,(\psi(T)-\psi(0))\,\|x-y\| \\
&+|p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0))+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)\|x-y\|+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big).2L\,\|x-y\| \\
\leq\;& \Big[2L\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}+|p|\,(\psi(T)-\psi(0)) \\
&+|p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0))+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)+2L\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)\Big]\,\|x-y\| \\
=\;& \Big[2L\Big(Q(T,\alpha)+\Omega_2\frac{Q(T,\gamma-1)}{|\Lambda|}\Big)+\Big(|p|\,\Omega_1\frac{Q(T,\gamma-1)}{|\Lambda|}+|p|\,(\psi(T)-\psi(0))\Big)\Big]\,\|x-y\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|(Tx)(t)-(Ty)(t)| \;\leq\;& \Big[2L\Big(Q(T,\alpha)+\Omega_2\frac{Q(T,\gamma-1)}{|\Lambda|}\Big) \\
&+\Big(|p|\,\Omega_1\frac{Q(T,\gamma-1)}{|\Lambda|}+|p|\,(\psi(T)-\psi(0))\Big)\Big]\,\|x-y\|
\end{aligned}
$$

And as

$$
2L\Big(Q(T,\alpha)+\Omega_2\frac{Q(T,\gamma-1)}{|\Lambda|}\Big)+\Big(|p|\,\Omega_1\frac{Q(T,\gamma-1)}{|\Lambda|}+|p|\,(\psi(T)-\psi(0))\Big)<1,
$$

we get that the operator T is a contraction.

Therefore, by the Banach's contraction mapping principle, the operator T has a unique fixed point, and hence the problem (1.4) has a unique solution on J. The proof is now completed. □

3.2. Existence results via the Krasnoselskii's fixed point theorem

Now, we present an existence result based on the Krasnoselskii's fixed point theorem.

Theorem 3.2. *Let us assume that $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function satisfying:*

(H_2) $|f(t, u, v)| \leq \phi(t)$, $\forall (t, u, v) \in J \times \mathbb{R}^2$, and $\phi(t) \in C(J, \mathbb{R}^+)$.

In addition if:

$$\Big(|p|\, \Omega_1 \frac{Q(T, \gamma - 1)}{|\Lambda|} + |p|\, (\psi(T) - \psi(0)) \Big) < 1, \tag{3.4}$$

where $Q(T, \gamma - 1)$ and Ω_1 are defined by (3.1) *and* (3.2) *respectively, then the problem* (1.4) *has at least one solution on J.*

Proof. Let $\sup\limits_{t \in J} |\phi(t)| = \|\phi\|$ and $B_r := \{x \in C : \|x\| \leq r\}$, where

$$r \quad \geq \quad \frac{\Big[Q(T, \alpha) + \Omega_2 \dfrac{Q(T, \gamma - 1)}{|\Lambda|} \Big] \|\phi\| + \dfrac{|A|\, Q(T, \gamma - 1)}{|\Lambda|}}{1 - \Big[|p|\, \Omega_1 \dfrac{Q(T, \gamma - 1)}{|\Lambda|} + |p|\, (\psi(T) - \psi(0)) \Big]}.$$

We define the operators T_1 and T_2 on B_r by

$$\begin{aligned} (T_1 x)(t) &= I_{0^+}^{\alpha;\psi} F_x(s)(t) \\ &\quad - \frac{(\psi(t) - \psi(0))^{\gamma - 1}}{\Lambda \Gamma(\gamma)} \Big(\sum_{i=1}^{m} \delta_i I_{0^+}^{\alpha;\psi} F_x(s)(\eta_i) + \sum_{k=1}^{r} \lambda_k I_{0^+}^{\alpha - \mu_k;\psi} F_x(s)(\xi_k) + \sum_{j=1}^{n} \omega_j I_{0^+}^{\alpha + \beta_j;\psi} F_x(s)(\theta_j) \Big), \\ (T_2 x)(t) &= -p I_{0^+}^{1;\psi} x(t) + \frac{(\psi(t) - \psi(0))^{\gamma - 1}}{\Lambda \Gamma(\gamma)} \Big[A + p \Big(\sum_{i=1}^{m} \delta_i I_{0^+}^{1;\psi} x(\eta_i) + \sum_{k=1}^{r} \lambda_k I_{0^+}^{1 - \mu_k;\psi} x(\xi_k) + \sum_{j=1}^{n} \omega_j I_{0^+}^{1 + \beta_j;\psi} x(\theta_j) \Big) \Big]. \end{aligned}$$

We note that $T = T_1 + T_2$.

For any $x, y \in B_r$, we have:

$$
\begin{aligned}
|(T_1x)(t)+(T_2y)(t)| &\leq I_{0^+}^{\alpha;\psi}|F_x(s)|(T)+|p|\,I_{0^+}^{1;\psi}|y(t)| \\
&\quad+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big[\,|A| \\
&\quad+|p|\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{1;\psi}|y(\eta_i)|+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{1-\mu_k;\psi}|y(\xi_k)| \\
&\quad+\sum_{j=1}^{n}\left|\omega_j\right| I_{0^+}^{1+\beta_j;\psi}\left|y(\theta_j)\right|\Big) \\
&\quad+\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{\alpha;\psi}|F_x(s)(\eta_i)|+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{\alpha-\mu_k;\psi}|F_x(s)(\xi_k)| \\
&\quad+\sum_{j=1}^{n}\left|\omega_j\right| I_{0^+}^{\alpha+\beta_j;\psi}\left|F_x(s)(\theta_j)\right|\Big)\Big] \\
&\leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|\phi\|+|p|\,(\psi(T)-\psi(0))\,\|y\| \\
&\quad+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}|A| \\
&\quad+|p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0)) \\
&\quad+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&\quad+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)\|y\| \\
&\quad+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)\|\phi\| \\
&\leq \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}|A|+\Big[\,|p|\,(\psi(T)-\psi(0)) \\
&\quad+|p|\,\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,(\psi(\eta_i)-\psi(0)) \\
&\quad+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)}\Big)\Big]r \\
&\quad+\Big[\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad+\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big)\Big]\|\phi\| \\
&\leq \Big[Q(T,\alpha)+\Omega_2\frac{Q(T,\gamma-1)}{|\Lambda|}\Big]\|\phi\|+\Big[\,|p|\,\Omega_1\frac{Q(T,\gamma-1)}{|\Lambda|}+|p|\,(\psi(T)-\psi(0))\Big]r \\
&\quad+\frac{|A|\,Q(T,\gamma-1)}{|\Lambda|} \\
&\leq r.
\end{aligned}
$$

This implies that $T_1x + T_2y \in B_r$, which satisfies the assumption (a) of Lemma 2.9.

We show now that the second assumption (b) of Lemma 2.9 is satisfied.

Let x_n be a sequence such that $x_n \longrightarrow x$ in C. Then for each $t \in J$, we have

$$\begin{aligned}
|(T_1x_n)(t) - (T_1x)(t)| &\leq I_{0^+}^{\alpha;\psi} \left|F_{x_n}(s) - F_x(s)\right|(T) \\
&\quad + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \Big(\sum_{i=1}^{m} |\delta_i|\, I_{0^+}^{\alpha;\psi} \left|F_{x_n}(s)(\eta_i) - F_x(s)(\eta_i)\right| \\
&\quad + \sum_{k=1}^{r} |\lambda_k|\, I_{0^+}^{\alpha-\mu_k;\psi} \left|F_{x_n}(s)(\xi_k) - F_x(s)(\xi_k)\right| \\
&\quad + \sum_{j=1}^{n} \left|\omega_j\right| I_{0^+}^{\alpha+\beta_j;\psi} \left|F_{x_n}(s)(\theta_j) - F_x(s)(\theta_j)\right| \Big)\Big] \\
&\leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \left\|F_{x_n} - F_x\right\| \\
&\quad + \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \Big(\sum_{i=1}^{m} |\delta_i| \frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \sum_{k=1}^{r} |\lambda_k| \frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)} \\
&\quad + \sum_{j=1}^{n} \left|\omega_j\right| \frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)} \Big) \left\|F_{x_n} - F_x\right\|.
\end{aligned}$$

Since f is continuous, this implies that the operator F_x is also continuous. Hence, we obtain

$$\left\|F_{x_n} - F_x\right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus, this shows that the operator T_1x is continuous. Also the set T_1B_r is uniformly bounded on B_r as

$$\begin{aligned}
\|T_1x\| &\leq I_{0^+}^{\alpha;\psi} |F_x(s)|(T) \\
&\quad + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \Big(\sum_{i=1}^{m} |\delta_i|\, I_{0^+}^{\alpha;\psi} |F_x(s)(\eta_i)| + \sum_{k=1}^{r} |\lambda_k|\, I_{0^+}^{\alpha-\mu_k;\psi} |F_x(s)(\xi_k)| \\
&\quad + \sum_{j=1}^{n} \left|\omega_j\right| I_{0^+}^{\alpha+\beta_j;\psi} \left|F_x(s)(\theta_j)\right| \Big) \\
&\leq \left[Q(T,\alpha) + \Omega_2 \frac{Q(T,\gamma-1)}{|\Lambda|} \right] \|\phi\|.
\end{aligned}$$

Next, we prove the compactness of the operator T_1. Let $\sup_{(t,u,v)\in J\times B_r\times B_r} |f(t,u,v)| = \widehat{f} < \infty$, then for each $t_1, t_2 \in J$ with $0 \leq t_1 \leq t_2 \leq T$, we obtain

$$
\begin{aligned}
|(T_1x)(t_2)-(T_1x)(t_1)| &= \left|I_{0^+}^{\alpha;\psi}F_x(s)(t_2)-I_{0^+}^{\alpha;\psi}F_x(s)(t_1)\right| \\
&\quad +\frac{(\psi(t_2)-\psi(0))^{\gamma-1}-(\psi(t_1)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \\
&\quad \times\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{\alpha;\psi}\,|F_x(s)(\eta_i)|+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{\alpha-\mu_k;\psi}\,|F_x(s)(\xi_k)| \\
&\quad +\sum_{j=1}^{n}\left|\omega_j\right|I_{0^+}^{\alpha+\beta_j;\psi}\left|F_x(s)(\theta_j)\right|\Big) \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}\psi'(s)\Big[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}\Big]F_x(s)ds \\
&\quad +\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}F_x(s)ds\Big| \\
&\quad +\frac{(\psi(t_2)-\psi(0))^{\gamma-1}-(\psi(t_1)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \\
&\quad \times\Big(\sum_{i=1}^{m}|\delta_i|\,I_{0^+}^{\alpha;\psi}\,|F_x(s)(\eta_i)|+\sum_{k=1}^{r}|\lambda_k|\,I_{0^+}^{\alpha-\mu_k;\psi}\,|F_x(s)(\xi_k)| \\
&\quad +\sum_{j=1}^{n}\left|\omega_j\right|I_{0^+}^{\alpha+\beta_j;\psi}\left|F_x(s)(\theta_j)\right|\Big) \\
&\le \frac{\widehat{f}}{\Gamma(\alpha+1)}\Big[2(\psi(t_2)-\psi(t_1))^{\alpha}+|(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}|\Big] \\
&\quad +\widehat{f}\frac{(\psi(t_2)-\psi(0))^{\gamma-1}-(\psi(t_1)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\Big(\sum_{i=1}^{m}|\delta_i|\,\frac{(\psi(\eta_i)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad +\sum_{k=1}^{r}|\lambda_k|\,\frac{(\psi(\xi_k)-\psi(0))^{\alpha-\mu_k}}{\Gamma(\alpha-\mu_k+1)}+\sum_{j=1}^{n}\left|\omega_j\right|\frac{(\psi(\theta_j)-\psi(0))^{\alpha+\beta_j}}{\Gamma(\alpha+\beta_j+1)}\Big).
\end{aligned}
$$

The right hand side of the inequality above is independant of x and tends to 0 as $t_2 \longrightarrow t_1$.

Therefore, the operator T_1 is equicontinuous. Thus, T_1 is relatively compact on B_r. Then, by the well-known Arzela-Ascoli theorem, T_1 is a compact operator on B_r.

Now we show that the operator T_2 is a contraction, which is the third and last condition of Lemma 2.9.

For any $x, y \in C$ and for each $t \in J$, we have

$$
\begin{aligned}
|(T_2x)(t)-(T_2y)(t)| \leq\ & |p|\, I_{0^+}^{1;\psi}\, |x(t)-y(t)| \\
&+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)}\, |p| \Big(\sum_{i=1}^{m} |\delta_i|\, I_{0^+}^{1;\psi}\, |x(\eta_i)-y(\eta_i)| \\
&+\sum_{k=1}^{r} |\lambda_k|\, I_{0^+}^{1-\mu_k;\psi}\, |x(\xi_k)-y(\xi_k)| \\
&+\sum_{j=1}^{n} \left|\omega_j\right| I_{0^+}^{1+\beta_j;\psi} \left|x(\theta_j)-y(\theta_j)\right| \Big) \\
\leq\ & |p|\,(\psi(T)-\psi(0))\, \|x-y\| \\
&+|p|\, \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \Big(\sum_{i=1}^{m} |\delta_i|\, (\psi(\eta_i)-\psi(0)) \\
&+\sum_{k=1}^{r} |\lambda_k|\, \frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&+\sum_{j=1}^{n} \left|\omega_j\right| \frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)} \Big) \|x-y\| \\
\leq\ & \Big[|p|\,(\psi(T)-\psi(0)) \\
&+|p|\, \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \Big(\sum_{i=1}^{m} |\delta_i|\, (\psi(\eta_i)-\psi(0)) \\
&+\sum_{k=1}^{r} |\lambda_k|\, \frac{(\psi(\xi_k)-\psi(0))^{1-\mu_k}}{\Gamma(2-\mu_k)} \\
&+\sum_{j=1}^{n} \left|\omega_j\right| \frac{(\psi(\theta_j)-\psi(0))^{1+\beta_j}}{\Gamma(2+\beta_j)} \Big) \Big] \|x-y\| \\
=\ & \Big(|p|\, \Omega_1 \frac{Q(T,\gamma-1)}{|\Lambda|} + |p|\,(\psi(T)-\psi(0)) \Big) \|x-y\|\,.
\end{aligned}
$$

Using (3.4), we conclude that the operator T_2 is a contraction. Thus, all assumptions of Lemma 2.9 are satisfied. So we conclude that the problem (1.4) has at least one solution on J. The proof is completed. □

4. Examples

This section presents some examples which illustrate the validity of the main results.

Consider the following sequential ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions:

$$
\begin{cases}
\left({}^{H}D_{0^+}^{\frac{8}{5},\frac{1}{4};e^{\frac{t}{2}}} + \frac{1}{7}\, {}^{H}D_{0^+}^{\frac{3}{5},\frac{1}{4};e^{\frac{t}{2}}} \right) x(t) = f(t, x(t), x(\sigma t)), t \in (0,1], \\
x(0)=0, \sum_{i=1}^{3} \left(\frac{-i}{i+5} \right)^{i+1} x\left(\frac{i}{3}\right) + \sum_{j=1}^{2} \left(\frac{j+1}{j+2} \right) I_{0^+}^{\frac{j}{3};e^{\frac{t}{2}}} x\left(\frac{j}{2}\right) + \sum_{k=1}^{4} \left(\frac{-k}{k+2} \right)^{k} {}^{H}D_{0^+}^{\frac{k+8}{8};\frac{1}{4};e^{\frac{t}{2}}} x\left(\frac{k}{4}\right) = \frac{1}{2},
\end{cases}
\tag{4.1}
$$

Here we have: $\alpha = \frac{8}{5}$, $\beta = \frac{1}{4}$, $p = \frac{1}{7}$, $T = 1$, $\sigma = \frac{1}{3}$, $A = \frac{1}{2}$, $m = 3$,$n = 2$,$r = 4$, $\psi(t) = e^{\frac{t}{2}}$,

$\delta_i = \left(\dfrac{-i}{i+5}\right)^{i+1}$, $\omega_j = \left(\dfrac{j+1}{j+2}\right)$, $\lambda_k = \left(\dfrac{-k}{k+2}\right)^k$, $\eta_i = \frac{i}{3}$,$\theta_j = \frac{j}{2}$, $\xi_k = \frac{k}{4}$, $\beta_j = \frac{j}{3}$, $\mu_k = \frac{k+8}{8}$, for $i = 1,2,3$, $j = 1,2$ and $k = 1,2,3,4$.

After doing some calculations we find that: $\Lambda \simeq 0.5377547 \neq 0$, $\Omega_1 \simeq 1.8265034$ and $\Omega_2 \simeq 0.9099$.

Example 4.1. *Consider the function:*

$$f(t, x(t), x(\sigma t)) = \frac{cos\,|x(t) + x(\sigma t)|}{50} + \frac{|x(\sigma t)|}{(t^3 + 5)^4},$$

hence, f satisfies the hypothesis (H_1) as
for any $x, y \in \mathbb{R}$, $t \in J$, we have:

$$|f(t, x(t), x(\sigma t)) - f(t, y(t), y(\sigma t))| \leq 0.0222\,|x - y|.$$

We set $L = 0.0222$, therefore we obtain:

$$2L\Big(Q(T,\alpha) + \Omega_2 \frac{Q(T,\gamma-1)}{|\Lambda|}\Big) + \Big(|p|\,\Omega_1 \frac{Q(T,\gamma-1)}{|\Lambda|} + |p|\,(\psi(T) - \psi(0))\Big) \simeq 0.500778 < 1.$$

It follows from Therorem 3.1 that the problem (4.1) *has a unique solution x on* $[0, 1]$.

Example 4.2. *By considering the function*

$$f(t, x(t), x(\sigma t)) = \frac{2cos\,|x(t)|}{9 + 2t} + \frac{2sin\,|x(\sigma t))|}{4 + 2t} + e^{-2t},$$

it is easy to see that f satisfies the hypothesis (H_2) as

$$|f(t, x(t), x(\sigma t))| \leq \frac{2}{9 + 2t} + \frac{2}{4 + 2t} + e^{-2t},$$

and we have:

$$\Big(|p|\,\Omega_1 \frac{Q(T,\gamma-1)}{|\Lambda|} + |p|\,(\psi(T) - \psi(0))\Big) \simeq 0.36879 < 1.$$

It follows from Theorem 3.2 that the problem (4.1) *has at least one solution x on* $[0, 1]$.

5. Conclusions

This paper studied a new class of sequential ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions. Existence and uniqueness results are established, we first proved the uniqueness results by using the Banach's contraction mapping principle, followed by the existence results using the Krasnoselskii's fixed point theorem. Our results are not only original and new, but also for example by taking $\omega_j = 0$ and $\lambda_k = 0$, for $j = 1, 2, ..., n$, $k = 1, 2, ..., r$, our results correspond to the ones for boundary value problems for sequential ψ-Hilfer pantograph differential equations supplemented with multi-point boundary conditions, and by taking $\delta_i = 0$ and $\lambda_k = 0$, for $i = 1, 2, ..., m$, $k = 1, 2, ..., r$, our results correspond to the ones for boundary value problems for sequential ψ-Hilfer pantograph differential equations supplemented with multi-term integral boundary conditions. In the end, we have given two examples to strenghten our theoretical findings. The work established in this paper is new and contributes in the developpement of the literature on boundary value problems for nonlinear ψ-Hilfer fractional differential equations.

Acknowledgments

The authors received no specific funding for this work.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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