Large positive solutions to an elliptic system of competitive type with nonhomogeneous terms

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Abstract: In this paper, we study the elliptic system of competitive type with nonhomogeneous terms \( \Delta u = u^p v^q + h_1(x) \), \( \Delta v = u^r v^s + h_2(x) \) in \( \Omega \) with two types of boundary conditions: (I) \( u = v = +\infty \) and (SF) \( u = +\infty, v = f \) on \( \partial\Omega \), where \( f > 0 \), \( (p - 1)(s - 1) - qr > 0 \), and \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain. The nonhomogeneous terms \( h_1(x) \) and \( h_2(x) \) may be unbounded near the boundary and may change sign in \( \Omega \). First, for a single semilinear elliptic equation with a singular weight and nonhomogeneous term, boundary asymptotic behaviour of large positive solutions is established. Using this asymptotic behaviour, we show existence of large positive solutions for this elliptic system with the boundary condition (SF), existence of maximal solution, boundary asymptotic behaviour and uniqueness of large positive solutions for this elliptic system with (I).

Keywords: elliptic system; large solutions; nonhomogeneous term; existence; boundary asymptotic behaviour; uniqueness

Mathematics Subject Classification: 35A01, 35A02, 35B40

1. Introduction and main results

Consider the following elliptic system

\[
\begin{aligned}
\Delta u &= u^p v^q + h_1(x), \quad \text{in } \Omega, \\
\Delta v &= u^r v^s + h_2(x), \quad \text{in } \Omega,
\end{aligned}
\]

(1.1)

with two types of boundary conditions:

\[
\begin{aligned}
u &= +\infty, v = +\infty, \quad \text{on } \partial\Omega, & \text{(I)} \\
u &= +\infty, v = f, \quad \text{on } \partial\Omega, & \text{(SF)}
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain of class \( C^{2,\mu} \) for some \( 0 < \mu < 1 \) and \( h_1(x), h_2(x) \in C(\Omega), f > 0 \), the parameters \( p, s > 1 \) and \( q, r > 0 \) such that \( (p - 1)(s - 1) - qr > 0 \). The condition \( u = +\infty, v = +\infty \)
on $\partial \Omega$ is defined in the sense of that $u(x) \to +\infty$ and $v(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial \Omega) \to 0$. The system can represent the competitive model of the two populations in the environment $\Omega$. If the problem has a solution, it indicates that the two populations can coexist in $\Omega$.

The large solutions of a single equation with nonhomogeneous terms have been studied recently (see [3, 7, 8, 10, 15–17, 19]). In [10], García-Melián studied existence, uniqueness and nonexistence of boundary blow-up solutions to problem

$$\begin{cases}
\Delta u = |u|^{p-1}u + h(x), & \text{in } \Omega, \\
u = +\infty, & \text{on } \partial \Omega,
\end{cases}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. As far as we know, it was the first time that an unbounded and sign-changing inhomogeneous term $h(x)$ in this equation was studied. In [17], Wang investigated the problem

$$\begin{cases}
\Delta u = b(x)u^p - ad(x)^{-q}, & \text{in } \Omega, \\
u = +\infty, & \text{on } \partial \Omega,
\end{cases}$$

where the parameters $p > 1, a > 0, q \in \mathbb{R}$ are constants, and $b(x) \in C^{\mu}(\Omega)$ is a positive function. They showed existence, uniqueness and the first order asymptotic behaviour of positive solutions. And, in [19], Wang et al. studied the following problem

$$\begin{cases}
\Delta u = a(x)|u|^{p-1}u + h(x), & \text{in } \Omega, \\
u = +\infty, & \text{on } \partial \Omega,
\end{cases}$$

(1.2)

where $p > 1, h(x) \in C(\Omega), a(x)$ is a $C^{\mu}_{\text{loc}} (0 < \mu < 1)$ continuous nonnegative function in $\Omega$ and satisfies

$$C_1d(x)^{-\gamma} \leq a(x) \leq C_2d(x)^{-\gamma}$$

in $\{x \in \Omega | 0 < d(x) < \delta_0\}$

for some positive constants $C_1, C_2, \delta_0$ and $0 < \gamma < 2$. They studied existence, uniqueness and nonexistence of large solutions. For more results about existence and uniqueness of positive large solutions, we refer to the citation of [6].

In addition to the single equation, the study of systems is also meaningful due to its multiple applications. For example, the application of Newtonian fluids theory. Boundary blow-up solutions for elliptic systems have been studied in many papers [5, 9, 11–13, 18]. Dancer and Du [2] studied the predator-prey model. The boundary blow-up solutions to elliptic systems were studied in [11]. The existence, uniqueness and the first order boundary estimates of large solutions to the following systems $(h_1(x), h_2(x) = 0)$,

$$\begin{cases}
\Delta u = u^p v^q, & \text{in } \Omega, \\
\Delta v = u^p v^q, & \text{in } \Omega,
\end{cases}$$

with conditions: (F) $u = g, v = f$ , (I), (SF) on $\partial \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\mu}$ for some $0 < \mu < 1$ and the parameters $p, s > 1, q, r > 0, g, f > 0$, were investigated in García-Melián and Rossi [12]. In [18], Wang et al. showed the existence of positive solutions for the elliptic system (1.1) with (I). However, for large positive solutions to an elliptic system of competitive type with nonhomogeneous terms, we need to find appropriate conditions to get properties of solutions. We have to overcome this difficulty with the relevant results of a single equation.

In this paper, firstly, we explore boundary asymptotic behaviour of large positive solutions to

$$\begin{cases}
\Delta u = a(x)u^p + h(x), & \text{in } \Omega, \\
u = +\infty, & \text{on } \partial \Omega.
\end{cases}$$

(1.3)
Obviously all positive solutions of problem (1.2) are also solutions of (1.3). Then, we show boundary asymptotic behaviour and uniqueness of large solutions for (1.1) with the boundary condition (I) using the conclusions about the single equation.

The notations \( \Omega_\delta := \{ x \in \Omega : d(x) < \delta \} \), \( \Omega_\delta^c := \{ x \in \Omega : d(x) > \delta \} \) for \( \delta > 0 \), \( h^+(x):=\max \{ h(x),0 \} \) and \( h^−(x):=\min \{ h(x),0 \} \) are used through this paper.

The weight function \( a(x) \) in (1.3) satisfies the following two conditions.  

\( (a_1) \) \( a(x) \) is a \( C^\mu_{\text{loc}} \) continuous nonnegative function in \( \Omega \), \( 0 < \mu < 1 \).
\( (a_2) \) There exists a positive continuous function \( c_0(x) \) on \( \partial \Omega \) such that \( \lim_{x \to x_0} d(x)^\gamma a(x) = c_0(x_0) \) for every \( x_0 \in \partial \Omega \), \( 0 < \gamma < 2 \).

For \( h(x) \) in (1.3), here are three conditions that will be used and \( \gamma \) is the same constant in \( (a_2). \)

\( (h_1) \) \( \lim_{x \to x_0} d(x)^{\frac{2r-\gamma}{p-1}} h(x) < -\infty. \)
\( (h_2) \) \( d(x)^{2-\lambda} h^+(x) \leq C \), where \( C \) is a constant and \( 0 < \lambda < \min \{ 1, 2 - \gamma \} \).
\( (h_3) \) \( \lim_{x \to \partial \Omega} d(x)^{\frac{2r-\gamma}{p-1}} h^−(x) = 0. \)

Apparently, \( (h_3) \) implies \( (h_1) \). We use different conditions in our theorems.

The main purpose of the paper is to study the influence of nonhomogeneous terms on the properties of the solution for the elliptic systems. In fact, the results shows that the solution is stable when the nonhomogeneous terms don’t change much. It is worthwhile to mention our assumption that \( a(x) \) is nonnegative in \( \Omega \) and singular on \( \partial \Omega \). And, \( h(x) \) may be unbounded near \( \partial \Omega \) and change sign in \( \Omega \). This work can be considered as an extension of such results on the system without nonhomogeneous terms. With regard to boundary asymptotic behavior of positive solutions to (1.3), we have:

**Theorem 1.1.** Suppose that \( p > 1 \), \( a(x) \) satisfies conditions \((a_1)\) and \((a_2)\), \( h(x) \in C(\Omega) \) satisfies \((h_2)\) and \((h_3)\). The problem (1.3) admits at most one positive solution and if \( u(x) \) is the solution, then \( u(x) \) satisfies

\[
\lim_{x \to x_0} d(x)^\gamma u(x) = \left( \frac{\tau(r+1)}{c_0(x_0)} \right)^{\frac{1}{\tau}}
\]

for every \( x_0 \in \partial \Omega \), where \( \tau = \frac{2-\gamma}{p-1} \) and \( \gamma \) is given in \((a_2)\).

With regard to existence of positive solutions for the elliptic system with the boundary condition (SF), we have:

**Theorem 1.2.** Suppose that \((p-1)(s-1) > qr \) and \( \frac{1}{2}(p-1) < r < p-1 \). For \( 0 < \bar{\lambda}_1 < 1 \), \( 2 - \frac{2r}{p-1} \leq \bar{\lambda}_2 < 1 \), there exists a constant \( \bar{c} > 0 \) such that if \( \sup_{x \in \Omega} d(x)^{2-\bar{\lambda}_1} h_1(x) \leq \bar{c}, \sup_{x \in \Omega} d(x)^{2-\bar{\lambda}_2} h_2(x) \leq \bar{c} \), then system (1.1) with the boundary condition (SF) admits a positive solution.

**Definition 1.3.** If the solution \((\bar{u}^+, \bar{v}^+)\) and any other solution \((u, v)\) of system (1.1) with the boundary condition (I) is such that \( \bar{u}^+ \geq u, \bar{v}^+ \leq v \), then we call \((\bar{u}^+, \bar{v}^+)\) the maximal solution of system (1.1) with the boundary condition (I). And, if the reversing inequalities hold, then we call it is a minimal solution (denoted as \((u^+, v^+)\)).

For the elliptic system with the boundary condition (I), we have:

**Theorem 1.4.** Suppose that \((p-1)(s-1) > qr \), \( \frac{1}{2}(p-1) < r < p-1 \). For \( 0 < \bar{\lambda}_1 < 1 \), \( 2 - \frac{2r}{p-1} \leq \bar{\lambda}_2 < 1 \), there exists a constant \( \bar{c} > 0 \) such that if \( \sup_{x \in \Omega} d(x)^{2-\bar{\lambda}_1} h_1(x) \leq \bar{c}, \sup_{x \in \Omega} d(x)^{2-\bar{\lambda}_2} h_2(x) \leq \bar{c} \), then system (1.1) with the boundary condition (I) admits a maximal solution.
Theorem 1.5. Suppose that \((p-1)(s-1) > qr, h_1(x), h_2(x) \in C(\Omega)\) and
\[
\begin{align*}
    d(x)^{2-l_1}h_1^+(x) &\leq C, \lim_{x \to \partial \Omega} d(x)^{\alpha+2}h_1^+(x) = 0, \\
    d(x)^{2-l_1}h_2^+(x) &\leq C, \lim_{x \to \partial \Omega} d(x)^{\beta+2}h_2^+(x) = 0,
\end{align*}
\]
where \(\alpha = \frac{2(s-1-q)}{(p-1)(s-1)-qr}, \beta = \frac{2(p-1-r)}{(p-1)(s-1)-qr}, C > 0, 0 < \lambda_1 < \min(1, 2 - \beta q), \) and \(0 < \lambda_2 < \min(1, 2 - \alpha r)\).
Assume \((u, v)\) is a positive solution to system (1.1) with the boundary condition (I), then
\[
\begin{align*}
    \lim_{x \to x_0} d(x)^\alpha u(x) &= \left(\frac{(\alpha(\alpha + 1))^{s-1}}{\beta(\beta + 1)}\right)^{\frac{1}{q-p(n-1)+qr}}, \\
    \lim_{x \to x_0} d(x)^\beta v(x) &= \left(\frac{(\beta(\beta + 1))^{p-1}}{\alpha(\alpha + 1)}\right)^{\frac{1}{q-p(n-1)+qr}},
\end{align*}
\]
for every \(x_0 \in \partial \Omega\).

Theorem 1.6. Suppose that \((p-1)(s-1) > qr, h_1(x), h_2(x) \in C(\Omega)\) are non-positive functions and satisfy (1.4). Then system (1.1) with the boundary condition (I) admits at most one positive solution.

This paper is organized as follows. Section 2 presents some preliminaries. In Section 3, we proceed with the study of boundary asymptotic behaviour of positive solutions for the single equation. In Section 4, the existence, global estimates, boundary asymptotic behaviour and uniqueness of positive solutions to the system of competitive type with different boundary conditions are considered.

2. Preliminaries

We present some useful results about solutions to problem (1.3) in this section. The following lemma is the remark after Lemma 4.1 in [19].

Lemma 2.1 ([19]). Suppose that \(p > 1, a(x)\) satisfies conditions (a1) and (a2). And \(h(x) \in C(\Omega)\) satisfies (h1) and (h2). Then for any positive solution \(u \) of (1.3), \(Pd(x)^{\tau} \leq u \leq \tilde{P}d(x)^{-\tau} \) for some positive constants \(P, \tilde{P} \) in \(\Omega_\delta\), where \(\tau = \frac{2-\gamma}{p-1}\).

Next lemma shows the uniqueness of positive solutions for problem (1.3), even when \(h(x)\) may be unbounded near \(\partial \Omega\).

Lemma 2.2. Suppose that \(p > 1, a(x)\) satisfies conditions (a1) and (a2). And \(h(x) \in C(\Omega)\) satisfies (h1) and (h2). Then problem (1.3) has at most one positive solution.

Proof. We know that the condition (h2) implies that the function \(h(x)\) is bounded from above. Then, we have the conditions of Theorem 1.4 in [19] and omit the detail. \(\Box\)

In particular, for the following problem
\[
\begin{align*}
    \Delta u &= d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\
    u &= +\infty, & \text{on } \partial \Omega,
\end{align*}
\]
where \(p > 1, 0 < \gamma < 2 \) and \(h(x) \in C(\Omega)\) satisfies (h2), (h3), we have:

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Corollary 2.3. Assume that $p > 1$ and $\gamma < 2$. If $U_{p, \gamma, h}$ denotes the unique positive solution of problem (2.1), then

$$\lim_{x \to x_0} d(x)^p U_{p, \gamma, h}(x) = (\alpha(\alpha + 1))^{\frac{1}{p-1}}$$

for every $x_0 \in \partial \Omega$, where $\alpha = \frac{2-\gamma}{p-1}$.

Lemma 2.4. Let $u \in C^2(\Omega)$ satisfies

$$\begin{cases}
\Delta u \leq c_0 d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega,
\hline
u = +\infty, & \text{on } \partial \Omega,
\end{cases}$$

(2.2)

where $c_0$ is some positive constant and $h(x) \in C(\Omega)$ satisfies $(h_2), (h_3)$. Then $u(x) \geq c_0^{\frac{1}{p-1}} U_{p, \gamma, h}$, where

$$\tilde{h}(x) = c_0^{\frac{1}{p-1}} h(x).$$

Similarly, if

$$\begin{cases}
\Delta u \geq c_0 d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega,
\hline
u = +\infty, & \text{on } \partial \Omega,
\end{cases}$$

then $u(x) \leq c_0^{\frac{1}{p-1}} U_{p, \gamma, h}$.

Proof. Firstly, $\Delta(c_0^{\frac{1}{p-1}} u) = c_0^{\frac{1}{p-1}} \Delta u$. By (2.2), we have

$$\Delta(c_0^{\frac{1}{p-1}} u) \leq c_0^{\frac{1}{p-1}} \cdot c_0 d(x)^{-\gamma} u^p + c_0^{\frac{1}{p-1}} h(x) = d(x)^{-\gamma}(c_0^{\frac{1}{p-1}} u)^p + c_0^{\frac{1}{p-1}} h(x).$$

Note $\tilde{h}(x) = c_0^{\frac{1}{p-1}} h(x)$, then $\Delta(c_0^{\frac{1}{p-1}} u) \leq d(x)^{-\gamma}(c_0^{\frac{1}{p-1}} u)^p + \tilde{h}(x)$. It is clear that $\tilde{h}(x) \in C(\Omega)$ and satisfies the conditions $(h_2), (h_3)$. By Corollary (2.3) and the methods of sub- and supersolutions, we have $c_0^{\frac{1}{p-1}} u(x) \geq U_{p, \gamma, h}$, that is $u(x) \geq c_0^{\frac{1}{p-1}} U_{p, \gamma, h}$. The other case is proved similarly. \[\square\]

The following lemma is a straightforward extension of Lemma 2.4 that is about the case where $\Omega$ is a half-space $D = \{x \in \mathbb{R}^N : x_1 > 0\}$. We write $x = (x_1, x')$, where $x' \in \mathbb{R}^{N-1}$. This lemma will be used to deduce boundary estimates for positive solutions to system (1.1).

Lemma 2.5. Suppose that $u \in C^2(D)$ satisfies

$$\begin{cases}
\Delta u \leq C_1 x_1^{-\gamma} u^p + h(x), & \text{in } D,
\hline
u \geq K x_1^{-\alpha},
\end{cases}$$

where $C_1, K$ are some positive constants, $\alpha = \frac{2-\gamma}{p-1}$ and $h(x) \in C(\Omega)$ satisfies $(h_2), (h_3)$. Then $u \geq (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}} x_1^{-\alpha}$ in $D$. Similarly, if

$$\begin{cases}
\Delta u \geq C_1 x_1^{-\gamma} u^p + h(x), & \text{in } D,
\hline
u \leq K x_1^{-\alpha},
\end{cases}$$

then $u \leq (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}} x_1^{-\alpha}$ in $D$.

Proof. The proofs are analogous to the one of Lemma 9 in [12]. Assume that there exists $x^0 \in D$ and $l > 1$ so that $u(x^0) > l E(x^0)^{-\alpha}$, where $x^0_i$ is the first component of $x^0$ and $E = (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}}$. Set

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\[ D_0 = \{ u > lE \chi_{x_1}^{-\alpha} \} \cap B_r(x^0) \]

with \( r = \frac{d(x^0)}{2} \). By (h3), we have \( \liminf_{x \to \partial \Omega} d(x)^{2\alpha+2} h(x) > -\infty \), that is \( h(x) \geq -K_1 x_1^{-(\alpha+2)} \) in \( D_0 \) for some positive constant \( K_1 \). Then we deduce

\[ \Delta (u - lE \chi_{x_1}^{-\alpha}) > \{ E\alpha(\alpha + 1)r^p - K_1 - lE\alpha(\alpha + 1) \} x_1^{-\alpha-2} \text{ in } D_0. \]

For simplification, we note \( F := E\alpha(\alpha + 1)r^p - K_1 - lE\alpha(\alpha + 1) \).

Choose \( K_1 > 0 \) such that \( F > 0 \). Since \( x_1 \leq \frac{3r}{2} \) in \( D_0 \), and if we define \( w(x) = \frac{F x_1^{\alpha+1}}{2N^{3\alpha+2}} (r^2 - |x-x^0|^2) \), then \( \Delta (u - lE \chi_{x_1}^{-\alpha} + w) > 0 \) in \( D_0 \). By the maximum principle, we have that there exists \( x^1 \in \partial D_0 \) such that

\[ u(x^0) - lE(x^0)^{-\alpha} + w(x^0) < u(x^1) - lE(x_1^1)^{-\alpha} + w(x^1). \]

Thus \( x^1 \in \partial B_r(x^0) \). Then \( w(x^0) < u(x^1) - lE(x_1^1)^{-\alpha} \).

Now, using \( x^1 \geq \frac{r}{2} \) and the definition of \( w \), we have

\[
\begin{align*}
 u(x^1) > & \frac{2^\alpha+2 F r^{-\alpha}}{2N^{3\alpha+2}} + lE(x_1^1)^{-\alpha} \\
= & \left( \frac{2}{N^{3\alpha+2}} \left[ lE\alpha(\alpha + 1)(|p-1| - K_1) + lE \right] x_1^{-\alpha} \\
= & \left( \frac{2}{N^{3\alpha+2}} \right) \left[ lE\alpha(\alpha + 1)(|p-1| - K_1) + lE \right] x_1^{-\alpha} \\
= & \left( \frac{2}{N^{3\alpha+2}} \right) \left[ lE\alpha(\alpha + 1)(|p-1| - K_1) + 1 \right] lE(x_1^1)^{-\alpha}, \\
\end{align*}
\]

where \( \alpha(\alpha + 1)(|p-1| - K_1) > 0 \) by \( F > 0 \). Proceeding inductively, a sequence of points \( x^n \in D \) satisfies

\[ u(x^n) > \left( \frac{2}{N^{3\alpha+2}} \right) \left[ lE\alpha(\alpha + 1)(|p-1| - K_1) + 1 \right] lE(x_1^n)^{-\alpha} \]

can be obtained, which contradicts with the inequality \( u \leq Kx_1^{-\alpha} \). Similarly, we can prove the other case, and the lemma follows.

\[ \square \]

3. The single equation

Proof of Theorem 1.1. Let \( x_0 \in \partial \Omega \) and \( \{ x_n \} \subset \Omega \) be a sequence converging to \( x_0 \). Let \( W \) be an open neighborhood of \( x_0 \) such that \( \partial \Omega \) admits \( C^2,\mu \) local coordinates \( \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) : W \to \mathbb{R}^N \) with \( x \in W \cap \Omega \) if and only if \( \varphi_1(x) > 0 \). We can also suppose \( \varphi(x_0) = 0 \). If \( u(x) = \tilde{u}(\varphi(x)), h(x) = \bar{h}(\varphi(x)), \)
\( a(x) = \tilde{a}(\varphi(x)), \) then we have \( a(x) = \tilde{a}(\varphi(x)) \), then we have

\[
\sum_{i,j=1}^N a_{ij}(\varphi) \frac{\partial^2 \tilde{u}}{\partial \varphi_i \partial \varphi_j} + \sum_{i=1}^N b_{ij}(\varphi) \frac{\partial \tilde{u}}{\partial \varphi_i} = \tilde{a}(\varphi) \tilde{u}^p + \bar{h}(\varphi), \quad \text{in } \varphi(W \cap \Omega),
\]

where \( a_{ij}, b_i \) are \( C^\mu \) and \( a_{ij}(0) = \delta_{ij} \). Denote \( t_n \) be the projections onto \( \varphi(W \cap \partial \Omega) \) of \( \varphi(x_n) \), and introduce the functions

\[
u_n(y) = d_n^2 \tilde{u}(t_n + d_n y),
\]

where \( d_n = d(\varphi(x_n)), \varphi(x_n) = t_n + d_n (0, 0, \cdots, 0) \). Then the function \( u_n \) satisfies

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\[
\sum_{i,j=1}^{N} a_{ij}(t_n + d_n y_i) \frac{\partial^2 u_n}{\partial x_i \partial x_j} + d_n \sum_{i=1}^{N} b_{ij}(t_n + d_n y_i) \frac{\partial u_n}{\partial x_i} = d_n^a a(t_n + d_n y_i)u_n^p + d_n^{p+\gamma} h(t_n + d_n y_i).
\]

And, the conclusion of Lemma 2.1 implies that, for y in compact subsets \( M \) of \( D := \{ y \in \mathbb{R}^N : y_1 > 0 \} \), there exists \( n_0 = n_0(M) \) such that \( \Lambda_1 y_1^{-\gamma} \leq u_n(y) \leq \Lambda_2 y_1^{-\gamma} \) for \( n \geq n_0 \), where \( \Lambda_1, \Lambda_2 \) are positive constants. By the above estimates, the conditions \((h_2), (h_3)\) and standard methods, we obtain that for a subsequence we have \( u_n \to u_0 \) in \( C^{1, \gamma}_{loc}(M) \), where \( u_0 \) satisfies

\[
\begin{align*}
\Delta u_0 &= c_0(x_0) y_1^{-\gamma} u_0^p, & \text{in } D, \\
\Lambda_1 y_1^{-\gamma} &\leq u_0 \leq \Lambda_2 y_1^{-\gamma}.
\end{align*}
\]

By Theorem 10 and Remarks 3 b) in [1], we know this problem has a unique positive solution, that is

\[
u_0(y) = \left( \frac{r(r+1)}{c_0(x_0)} \right) \frac{1}{y_1^{\gamma}},
\]

which completes the proof. \( \Box \)

4. The system of competitive type

In this section, it is shown that existence, global estimates, asymptotic behaviour and uniqueness of positive solutions to this system with different boundary conditions.

4.1. Existence of positive solutions

In this subsection, we present existence of positive solution for the elliptic system (1.1) with (SF) and existence of maximal solution for the elliptic system (1.1) with (I).

Recall that if

\[
\begin{align*}
\Delta u &\geq u^p v^q + h_1(x), & \text{in } \Omega, \\
\Delta v &\leq u^p v^q + h_2(x), & \text{in } \Omega,
\end{align*}
\]

then \((u, v)\) is called a subsolution. And, if the reversing inequalities hold, then we call it a supersolution (denoted as \((\bar{u}, \bar{v})\)).

We shall show existence of positive solutions for the elliptic system (1.1) with (SF). First, we show some lemmas which will use to prove Theorem 1.2.

Remark 4.1. The following conclusions also hold if \( v = f(x) \) is a continuous positive function on \( \partial \Omega \).

Lemma 4.2. Let \( n \in \mathbb{N} \). Suppose that \( p > 1 \), \( a(x) \) satisfies conditions \((a_1), (a_2)\). There exists a constant \( \hat{c} > 0 \) such that if \( h(x) \in C(\Omega) \) satisfies \( \sup_{x \in \Omega} d(x)^{2-\lambda} h(x) \leq \hat{c} \) for \( 0 < \lambda < 1 \), then the following problem

\[
\begin{align*}
\Delta u &= a(x) u^p + h(x), & \text{in } \Omega, \\
u &= n, & \text{on } \partial \Omega,
\end{align*}
\]

admits a unique positive solution.

Analogous to the proofs of Corollary 1.2 in [19] and Lemma 3 in [1], we omit the proof of Lemma 4.2.

Lemma 4.3. Suppose that \((p-1)(s-1) > qr\), \( h_1(x), h_2(x) \in C(\Omega) \) satisfies (1.4). If \((u, v)\) denotes a solution to system (1.1) with (SF), then
Lemma 4.5.

\[ \lim_{x \to x_0} d(x)\theta u(x) = \left( \frac{(\theta(\theta+1))}{p'} \right)^{\frac{1}{p'-1}}, \]

where \( \theta = \frac{1}{p-1} \).

Proof. Since \( v = f \) on \( \partial \Omega \), \( u \) is a positive solution to problem

\[
\begin{align*}
\Delta u &= v^\theta u^p + h(x), & & \text{in } \Omega, \\
u &= +\infty, & & \text{on } \partial \Omega,
\end{align*}
\]

then the proof is completed by Theorem 1.1 with \( \gamma = 0 \) and \( c_0(x) \equiv f^\gamma \), for every \( x \in \Omega \). \( \square \)

Lemma 4.4 ([4]). Let \( \Omega \) be a \( C^2 \) bounded domain of \( \mathbb{R}^N \), \( g \in C(\Omega) \) is a function such that \( \sup_{x \in \Omega} d(x)^\gamma | g(x) | < +\infty \) for some \( 1 < \gamma < 2 \) and \( u \in C^2(\Omega) \) is a solution to the problem \( \Delta u = g \) in \( \Omega \) with \( u = 0 \) on \( \partial \Omega \). Then there exists \( C > 0 \) depending only on \( \Omega \) and \( \gamma \) such that

\[ \sup_{x \in \Omega} d(x)^{\gamma - 2} | u(x) | \leq C \sup_{x \in \Omega} d(x)^\gamma | g(x) |. \]

Lemma 4.5. Suppose \((\bar{u}, \bar{v})\) is a supersolution and \((u, v)\) is a subsolution to the problem (1.1) with \( u = \bar{u} = +\infty, v \geq f \geq \bar{v} \) on \( \partial \Omega \), and \( u \leq \bar{u}, v \geq \bar{v} \) in \( \Omega \). Also assume that \( \bar{u} \leq Cd(x)^{-\theta} \) for some positive constant \( C \) and \( \theta < \frac{2}{r} \), \( \frac{1}{2} < r < p - 1 \). For \( 0 < \lambda_1 < 1, 2 - \frac{2r}{p-1} \leq \lambda_2 < 1 \), there exists a constant \( \tilde{c} > 0 \) such that if \( \sup_{x \in \Omega} d(x)^{2-\lambda_1} h_1(x) \leq \tilde{c}, \sup_{x \in \Omega} d(x)^{2-\lambda_2} h_2(x) \leq \tilde{c} \), then system (1.1) admits at least a solution \((u, v)\) such that \( u \leq \bar{u}, v \geq \bar{v} \) in \( \Omega \) and \( u = +\infty, v = f \) on \( \partial \Omega \).

Proof. From Lemma 4.2, there exists a constant \( \tilde{c}_1 > 0 \) such that if \( \sup_{x \in \Omega} d(x)^{2-\lambda_2} h_2(x) \leq \tilde{c}_1 \), then \( v \) is a positive, bounded function in \( \tilde{\Omega} \). There exists a constant \( \tilde{c}_2 > 0 \) such that if \( \sup_{x \in \Omega} d(x)^{2-\lambda_2} h_2(x) \leq \tilde{c}_2 \), then the problem

\[
\begin{align*}
\Delta u &= v^\theta u^p + h_1(x), & & \text{in } \Omega, \\
u &= +\infty, & & \text{on } \partial \Omega,
\end{align*}
\]

admits a unique positive solution, which we denote by \( u_1 \). And, \( \Delta u \geq v^\theta u^p + h_1(x) \) in \( \Omega \). By uniqueness of solutions to (4.1) and the methods of sub- and supersolutions, we have \( \bar{u} \leq u_1 \). Similarly, \( \Delta \bar{u} \leq \bar{v}^\theta \bar{u}^p + h_1(x) \) in \( \Omega \), and so \( \bar{u} \geq u_1 \). By Lemma 4.2 and \( 0 < u_1 \leq C d(x)^{-\theta r} \), \( \theta r < 2 \), we let \( v_1 \) as the unique solution to

\[
\begin{align*}
\Delta v &= u_1^\gamma v^p + h_2(x), & & \text{in } \Omega, \\
v &= f, & & \text{on } \partial \Omega.
\end{align*}
\]

We see at once that \( v_1 \geq v \geq \bar{v} \) in \( \Omega \). We let \( u_2 \) as the unique solution to

\[
\begin{align*}
\Delta u &= v_1^\gamma u^p + h_1(x), & & \text{in } \Omega, \\
u &= +\infty, & & \text{on } \partial \Omega.
\end{align*}
\]

Then, we also have that \( u \leq u_2 \leq \bar{u} \) in \( \Omega \). And, \( \Delta u_1 = v_1^\theta u_1^p + h_1(x) \geq v_1^\theta u_1^p + h_1(x) \), so \( u_1 \leq u_2 \).

Recursively, we let \( v_n \) be the unique solution to (4.2), with \( u_1 \) replaced by \( u_n \), and \( u_n \) be the unique solution to (4.3), with \( v_1 \) replaced by \( v_{n-1} \). Thus, we can obtain two sequences \( \{u_n\} \) and \( \{v_n\} \) which satisfy \( \{u_n\} \) is increasing, \( \{v_n\} \) is decreasing, \( u \leq u_n \leq \bar{u} \) and \( v \geq v_n \geq \bar{v} \) in \( \Omega \). By standard methods, we conclude that there is a subsequence (still labelled by \( u_n \) and \( v_n \)) such that \( u_n \to u, v_n \to v \) in \( C_{\text{loc}}^{1,\gamma}(\Omega), \)

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where \((u, v)\) is a solution to (1.1) with the boundary condition (SF) and \(u \leq u \leq \bar{u}, v \geq v \geq \bar{v}\) in \(\Omega\), \(u = +\infty\) on \(\partial \Omega\).

Now let \(\omega_n = f - v_n\). Then \(\Delta \omega_n = -u_n'(f - \omega_n) + h_2(x)\) in \(\Omega\), \(\omega_n = 0\) on \(\partial \Omega\). Since \(u_n' \leq Cd^{-\theta r}\) and \(\theta < \frac{2}{r}\) in \(\Omega\) and \(\omega_n\) is uniformly bounded, then

\[
| \omega_n | \leq C'd^{2-\theta r} \tag{4.4}
\]

by Lemma 4.4. The inequality (4.4) also holds for \(\omega = f - v\), therefore \(\omega = 0\) on \(\partial \Omega\), that is \(v = f\) on \(\partial \Omega\).

**Proof of Theorem 1.2.** By Lemma 4.2, \(p > 1, \gamma < 2, h(x) \in C(\Omega)\) and there exist constants \(\bar{c} > 0, 0 < \lambda < 1\) such that \(\sup_{x \in \Omega} d(x)^{2-\lambda} h(x) \leq \bar{c}\), then there exists a unique positive solution to problem

\[
\begin{cases}
\Delta u = d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\
u = 1, & \text{on } \partial \Omega,
\end{cases}
\]

which is denoted by \(V_{p, \gamma}\). Let \(\varepsilon > 0\) be small, \(\delta > 0, \sigma = \frac{2r}{p-1}\) to be chosen and \((u, v) = (\varepsilon U_{p, 0}, \varepsilon^{-\delta} V_{s, \sigma})\), where \(U_{p, 0}\) is a positive solution to problem

\[
\begin{cases}
\Delta u = u^p + h(x), & \text{in } \Omega, \\
u = +\infty, & \text{on } \partial \Omega.
\end{cases}
\]

We show \((u, v)\) is a subsolution to system (1.1) with (SF) firstly. That is to show

\[
1 \geq \varepsilon^{r-\delta q-1} V_{s, \sigma}^{q} + U_{p, 0}^{-p}(\varepsilon^{-1} - 1)h_1(x),
\]

\[
1 \leq \varepsilon^{-\delta s + \delta} d(x)^{r} U_{p, 0}^{r} + d(x)^{s} V_{s, \sigma}^{-s}(\varepsilon^{-1} - 1)h_2(x). \tag{4.5}
\]

Inequalities \(p - \delta q - 1 > 0\) and \(r - \delta s + \delta < 0\) will be hold if we fix \(\delta \in \left(\frac{r}{s - 1}, \frac{p-1}{q}\right)\). We know \(V_{s, \sigma}^{q}\) and \(d(x)^{r} U_{p, 0}\) are bounded. Then, for \(\varepsilon\) small, inequality (4.5) holds because of conditions about \(h_1(x), h_2(x)\), and we get a subsolution to (1.1) with (SF).

Similarly, let \((\bar{u}, \bar{v}) = (Q U_{p, 0}, Q^{-\delta} V_{s, \sigma})\). Now we show \((\bar{u}, \bar{v})\) is a supersolution to system (1.1) with the boundary condition (SF). That is to show

\[
1 \leq Q^{p-\delta q-1} V_{s, \sigma}^{q} + U_{p, 0}^{-p}(Q^{-1} - 1)h_1(x),
\]

\[
1 \geq Q^{-\delta s + \delta} d(x)^{r} U_{p, 0}^{r} + d(x)^{s} V_{s, \sigma}^{-s}(Q^{-1} - 1)h_2(x). \tag{4.6}
\]

Then, for \(Q\) large, inequality (4.6) holds because of conditions about \(h_1(x), h_2(x)\), and we get a supersolution to (1.1) with (SF) and \(u \leq \bar{u}, v \geq \bar{v}\). Therefore, the existence of a positive solution \((u, v)\) to (1.1) with (SF) follows from Lemma 4.5 and \(r < p - 1\).

\[
\begin{cases}
\Delta u = u^{p} v^{q} + h_1(x), & \text{in } \Omega, \\
\Delta v = u^{r} v^{s} + h_2(x), & \text{in } \Omega, \\
u = k_1(x), v = k_2(x), & \text{on } \partial \Omega,
\end{cases} \tag{4.7}
\]

where \(k_1(x), k_2(x)\) are positive continuous functions on \(\partial \Omega\). By the standard method in [14], one can prove the following lemma.
Lemma 4.6 ([18]). Suppose that \((\bar{u}, \bar{v})\) is a supersolution and \((u, v)\) is a subsolution to problem (4.7) with \(u \leq k_1(x) \leq \bar{u}, v \geq k_2(x) \geq \bar{v}\) on \(\partial \Omega\). And, \(u \leq \bar{u}, v \geq \bar{v}\) in \(\Omega\). Then Problem (4.7) admits at least a solution \((u, v)\) that \(u \leq u \leq \bar{u}, v \geq v \geq \bar{v}\) in \(\Omega\) and \(u = k_1(x), v = k_2(x)\) on \(\partial \Omega\).

Proof of Theorem 1.4. Define \(\Omega_1 := \{x \in \Omega : d(x) < \frac{1}{n}\}\). First, we prove that the following problems
\[
\begin{align*}
\Delta u_n &= u_n^\prime v_n^\prime + h_1(x), &\text{in } \Omega_1^n, \\
\Delta v_n &= u_n^\prime v_n^\prime + h_2(x), &\text{in } \Omega_2^n, &n, n_0 \in \mathbb{N}, n > n_0, \\
u_n &= +\infty, v_n = f(n), &\text{on } \partial \Omega_1^n, \\
u_n^\prime &= u_n^\prime, &\text{in } \Omega_2^n.
\end{align*}
\]
we have
\[
\begin{align*}
u_n \geq u_{n+1}, &\text{ in } \Omega_1^n, \\
v_n \leq v_{n+1}, &\text{ in } \Omega_2^n.
\end{align*}
\]
Choose \(f(n) = 1 + C' (\frac{1}{n})^\varrho\), where \(C'\) is the uniform constant in (4.4), \(\varrho = 2 - \varrho r\), that is \(|v_n - f(n)| \leq C'd^\varrho\). Since \(f(n+1) = 1 + C'(\frac{1}{n+1})^\varrho\), \(f(n) = 1 + C'(\frac{1}{n})^\varrho\) on \(\partial \Omega_1^n\), then \(f(n+1) > f(n)\). That is \(v_{n+1} > v_n\) on \(\partial \Omega_1^n\). Clearly, \(u_{n+1} < u_n\) on \(\partial \Omega_2^n\). By Lemma 4.6, we have
\[
\begin{align*}
u_n &> u_{n+1}, &\text{in } \Omega_1^n, \\
v_n &< v_{n+1}, &\text{in } \Omega_2^n.
\end{align*}
\]
Thus, we get two sequences \(\{u_n\}\) and \(\{v_n\}\) which satisfy \(\{u_n\}\) is decreasing, \(\{v_n\}\) is increasing and \(u_n \to \bar{u}, v_n \to \bar{v}\) in \(\Omega\). It is easy to check that \((\bar{u}, \bar{v})\) is a solution to system (1.1) with (I) and by the construction of \(\{u_n\}, \{v_n\}, (\bar{u}, \bar{v})\) is also the maximal solution. Then, for any positive solution \((u, v)\) to system (1.1) with (I) and for any \(x \in \Omega\), we have \(u_n > u, v_n < v\). Thus, \(\bar{u}_\infty \geq u, \bar{v}_\infty \leq v\) as \(n \to +\infty\). The Theorem is proved. \(\square\)

4.2. Global estimates of solutions

Theorem 4.7. Assume that \((p-1)(s-1) > qr\) and \(r < p-1, q < s-1\), and \(h_1(x), h_2(x)\) satisfy (1.4), then positive solutions \((u, v)\) of system (1.1) with the boundary condition (I) satisfy
\[
Ad(x)^{-\alpha} \leq u \leq Bd(x)^{-\alpha}, Ad(x)^{-\beta} \leq v \leq Bd(x)^{-\beta}, \text{in } \Omega,
\]
for some \(A, B > 0\).

Proof. Let \(a_0 = \inf v > 0\). Then \(\Delta u \geq a_0^\varrho u^\varrho + h_1(x)\) in \(\Omega\). Note
\[
A_{p, q, \bar{h}} = \sup_{x \in \Omega} d(x)^\varrho U_{p, q, \bar{h}}(x), B_{p, q, \bar{h}} = \inf_{x \in \Omega} d(x)^\varrho U_{p, q, \bar{h}}(x).
\]
Clearly, they are positive and finite. Lemma 2.4 implies that \(u \leq a_0^\varrho U_{p, q, \bar{h}}(x)\), that is \(u \leq a_0^\varrho A_{p, q, \bar{h}} d^{-\alpha_0}\), where \(\alpha_0 = \frac{2}{p-1}, \bar{h}_1^{(1)} = a_0^\varrho h_1\). We use this into the right side of the latter equation in (1.1). Then,
\[
\Delta v \leq a_0^\varrho A_{p, q, \bar{h}} d^{-\alpha_0} v^\varrho + h_2(x), \text{in } \Omega.
\]
Using Lemma 2.4, we obtain
\[ v \geq (a_n^{-qr} A_{P,0,r \theta_1} B_{s,\alpha_n \theta_1} d)^{-\beta_n}, \text{ in } \Omega, \]

where \( \beta_0 = \frac{2-\alpha r}{s-1}, \ h_1^{(1)} = (a_0^{-\frac{qr}{s-1}} A_{P,0,r \theta_1} d)^{-\beta_0} h_2. \) Inductively, we have

\[ u \leq a_n^{-\frac{qr}{s-1}} A_{P,\beta_n \theta_1 q, \theta_1} d^{-\alpha_n}, \ v \geq a_n d^{-\beta_n}, \quad (4.11) \]

in \( \Omega, \) where

\[
\alpha_n = \frac{2 - \beta_{n-1} q}{p - 1}, \\
\beta_n = \frac{2 - \alpha_n r}{s - 1}, \\
a_{n+1} = a_n^{-\frac{qr}{s-1} + \frac{(p-1)(s-1)}{p-1} \beta_{n-1}} A_{P,\beta_n \theta_1 q, \theta_1} B_{s,\alpha_n \theta_1}, \\
\tilde{h}_1^{(n+1)} = a_n^{-\frac{qr}{s-1}} h_1, \\
\tilde{h}_2^{(n+1)} = (a_n^{-\frac{qr}{s-1}} A_{P,\beta_n \theta_1 q, \theta_1} d)^{-\beta_n} h_2. \quad (4.12) \]

In fact,

\[ \beta_n = \frac{2(p-1-r)}{(p-1)(s-1)} + \frac{qr}{(p-1)(s-1)} \beta_{n-1} \]

and \( \beta_1 > \beta_0, \beta_n \leq \beta. \) By elementary calculations, we have \( \beta_n \to \beta = \frac{2(p-1-r)}{(p-1)(s-1)-qr} \) as \( n \to +\infty. \) Then \( \alpha_n \to \alpha = \frac{2(s-1-q)}{(p-1)(s-1)-qr}. \)

For \( a_{n+1}, \) the third Eq in (4.12) and Lemma 7 in [12] imply that there exists a positive constant \( N \) such that \( a_{n+1} \geq N a_n^\theta, \) where \( \theta = \frac{qr}{(p-1)(s-1)} < 1. \) Iterating above inequality we deduce \( a_{n+1} \geq a_0^{\theta_{n+1}} N^{\theta_{n+1}} \).

Then \( \liminf_{n \to +\infty} a_{n+1} \geq N^{\frac{1}{\theta_{n+1}}} > 0 \) is obtained by \( n \to +\infty. \) Then, for \( \tilde{h}_2^{(n+1)} \), the fifth equation in (4.12) and the analysis of \( a_{n+1}, \) we have

\[ a_n^{-\frac{qr}{s-1} + \frac{(p-1)(s-1)}{p-1} \beta_{n-1}} A_{P,\beta_n \theta_1 q, \theta_1} \geq N a_{n-1}^\theta \geq a_0^{(-\theta)^{n+1}} N^{1-\theta+\theta^2-\ldots-\theta^{n+1}}. \]

Passing to the limit we have \( \liminf_{n \to +\infty} \left( a_n^{-\frac{qr}{s-1} A_{P,\beta_n \theta_1 q, \theta_1} d} \right)^{-\beta_n} \geq N^{1} > 0 \) when \( n \) are odd numbers, and 
\( \liminf_{n \to +\infty} \left( a_n^{-\frac{qr}{s-1} A_{P,\beta_n \theta_1 q, \theta_1} d} \right)^{-\beta_n} \geq N^{1} > 0 \) when \( n \) are even numbers.

Therefore, \( \tilde{h}_1^{(n+1)} \) and \( \tilde{h}_2^{(n+1)} \) satisfy the conditions (h2) and (h3). Thus, by (4.11), we have \( u \leq Bd^{-\alpha}, \ v \geq Ad^{-\beta} \) in \( \Omega \) as \( n \to +\infty \) for some \( A, B > 0. \) Similarly, the reversed inequalities can be proved, thus the proof is complete. \( \square \)
4.3. Boundary asymptotic behavior and Uniqueness of positive solutions

Proof of Theorem 1.5. Let \( x_0 \in \partial \Omega \) and \( \{ x_n \} \subset \Omega \) be a sequence converging to \( x_0 \). Take \( V \) be an open neighborhood of \( x_0 \) such that \( \partial \Omega \) admits \( C^{2,\mu} \) local coordinates \( \xi = (\xi_1, \xi_2, \cdots, \xi_N) : V \to \mathbb{R}^N \) with \( x \in V \cap \Omega \) if and only if \( \xi_1(x) > 0 \). We can also suppose \( \xi(x_0) = 0 \). If \( u(x) = \bar{u}(\xi(x)), \ v(x) = \bar{v}(\xi(x)) \), \( h_1(x) = \bar{h}_1(\xi(x)), \ h_2(x) = \bar{h}_2(\xi(x)) \), then we have

\[
\begin{align*}
\sum_{i,j=1}^{N} a_{ij}(\xi) \frac{\partial^2 \bar{u}}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_{i}(\xi) \frac{\partial \bar{u}}{\partial \xi_i} &= \bar{u}^{\prime \prime} \bar{v}^{\prime} + \bar{h}_1(\xi), \\
\sum_{i,j=1}^{N} a_{ij}(\xi) \frac{\partial^2 \bar{v}}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_{i}(\xi) \frac{\partial \bar{v}}{\partial \xi_i} &= \bar{u}^{\prime} \bar{v}^{\prime} + \bar{h}_2(\xi),
\end{align*}
\]

in \( \xi(V \cap \Omega) \), where \( a_{ij}, b_i \) are \( C^0 \) and \( a_{ij}(0) = \delta_{ij} \). Denote \( \zeta_n \) be the projections onto \( \xi(V \cap \partial \Omega) \) of \( \xi(x_n) \).

Let us introduce the functions

\[
u_n(y) = d_n^h \bar{u}(\zeta_n + d_n y), \quad v_n(y) = d_n^h \bar{v}(\zeta_n + d_n y)
\]

where \( d_n = d(\xi(x_n)), \xi(x_n) = \zeta_n + d_n(1, 0, \cdots, 0) \). Then the functions \((u_n, v_n)\) satisfy the equations

\[
\begin{align*}
\sum_{i,j=1}^{N} a_{ij}(\zeta_n + d_n y) \frac{\partial^2 u_n}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_{i}(\zeta_n + d_n y) \frac{\partial u_n}{\partial \xi_i} &= u_n^{\prime \prime} v_n^{\prime} + d_n^{\prime \prime} h_1(\zeta_n + d_n y), \\
\sum_{i,j=1}^{N} a_{ij}(\zeta_n + d_n y) \frac{\partial^2 v_n}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{N} b_{i}(\zeta_n + d_n y) \frac{\partial v_n}{\partial \xi_i} &= u_n^{\prime} v_n^{\prime} + d_n^{\prime} h_2(\zeta_n + d_n y).
\end{align*}
\]

On the other hand, Theorem 4.7 implies that, for \( u_n \) and \( v_n \) in compact subsets of \( D := \{ y \in \mathbb{R}^N : \ y_1 > 0 \} \), we have \( A y_1^{-\alpha} \leq u_n(y) \leq B y_1^{-\alpha}, \ A y_1^{1-\beta} \leq v_n(y) \leq B y_1^{1-\beta} \). Using standard theory we obtain that \( u_n \to u_0, \ v_n \to v_0 \) in \( C^2_{loc}(D) \), where \((u_0, v_0)\) satisfies

\[
\begin{align*}
\Delta u_0 &= u_0^{\prime \prime} v_0^{\prime}, \\
\Delta v_0 &= u_0^{\prime} v_0^{\prime}, \quad \text{in} \ D. \\
A y_1^{-\alpha} \leq u_0 &\leq B y_1^{-\alpha}, \ A y_1^{1-\beta} \leq v_0 \leq B y_1^{1-\beta}.
\end{align*}
\]

Now, we claim \( u_0 = \eta_1 y_1^{-\alpha}, v_0 = \eta_2 y_1^{1-\beta} \), where

\[
\begin{align*}
\eta_1 &= \left( \frac{(\alpha(\alpha + 1))^{\alpha - 1}}{(\alpha + 1)^{\beta + 1}} \right) \frac{1}{\int_{\eta_1^{\alpha}}^{1}} , \quad \eta_2 = \left( \frac{(\beta(\beta + 1))^{\beta - 1}}{(\alpha + 1)^{\beta + 1}} \right) \frac{1}{\int_{\eta_2^{\beta}}^{1}} .
\end{align*}
\]

(4.13)

Since \( \Delta u_0 \geq A y_1^{-\alpha} u_0^{\prime \prime} + h_1(x) \) in \( \Omega \), Lemma 2.5 implies that \( u_0 \leq B_1 y_1^{-\alpha} \) in \( \Omega \), where \( B_1 = \left( \frac{\alpha(\alpha + 1)}{A^\alpha} \right)^{\frac{1}{\alpha - 1}} \). And, since \( \Delta v_0 \leq B_1 y_1^{-\beta} v_0^{\prime} + h_2(x) \) in \( \Omega \), Lemma 2.5 implies that \( v_0 \geq A_1 y_1^{-\beta} \) in \( \Omega \), where \( A_1 = \left( \frac{\beta(\beta + 1)}{B^{1-\beta}} \right)^{\frac{1}{1-\beta}} \). Iterating this procedure, we obtain \( u_0 \leq B_n y_1^{-\alpha}, v_0 \geq A_n y_1^{-\beta} \) in \( \Omega \), where

\[
B_{n+1} = \left( \frac{\alpha(\alpha + 1)}{A_n^\alpha} \right)^{\frac{1}{\alpha - 1}}, \quad A_{n+1} = \left( \frac{\beta(\beta + 1)}{B_n^{1-\beta}} \right)^{\frac{1}{1-\beta}}.
\]

Clearly, if \( A \) is small enough, then the sequences \( \{ A_n \} \) and \( \{ B_n \} \) are convergent. And, \( A_n \to \eta_2, B_n \to \eta_1 \). Thus, \( u_0 \leq \eta_1 y_1^{-\alpha}, v_0 \geq \eta_2 y_1^{1-\beta} \) in \( \Omega \). Similarly, we can prove the reversed inequalities.

By setting \( y = e_1 \) in \( u_n \to \eta_1 y_1^{1-\beta} \) and \( v_n \to \eta_2 y_1^{1-\beta} \), and recalling \( \xi(x_n) = \zeta_n + d_{n} e_1 \), the Theorem is proved.
The uniqueness of solutions to system (1.1) can be obtained based on the fact that all positive solutions have the same boundary behavior.

**Proof of Theorem 1.6.** Let \((u_1, v_1)\) and \((u_2, v_2)\) be two positive solutions to system (1.1). By Theorem 1.5, we have \(\frac{u_1}{u_2} = 1, \frac{v_1}{v_2} = 1\) on \(\partial \Omega\), that is, \(\omega := \frac{u_1}{u_2} = 1\) on \(\partial \Omega\). Assume that \(k := \sup_{\Omega} \omega > 1\). We will prove \(k < 1\) to establish a contradiction.

On the one hand, we prove \(v_2 < k^{\frac{1}{r}} v_1\) in \(\Omega\). We argue by contradiction. Assume \(\tilde{\Omega} := \{v_2 > k^{\frac{1}{r}} v_1\}\) is nonempty. Since \(\partial \tilde{\Omega} \subset \Omega\), \(k > 1\) and \(\frac{v_1}{v_2} = 1\) on \(\partial \tilde{\Omega}\), \(v_2 = k^{\frac{1}{r}} v_1\) on \(\partial \tilde{\Omega}\). Then
\[
\Delta v_2 = u_2^r v_2^p + h_2(x) \geq k^{-r + \frac{1}{r}} u_1^r v_1^p + h_2(x) \geq k^{-r} u_1^r v_1^p + k^{\frac{1}{r}} h_2(x) = \Delta(k^{\frac{1}{r}} v_1)
\]
in \(\tilde{\Omega}\). Using the maximum principle, we have \(v_2 \leq k^{\frac{1}{r}} v_1\), which is impossible. Then, through the strong maximum principle, we have \(v_2 \leq k^{\frac{1}{r}} v_1\) in \(\Omega\) and \(v_2 < k^{\frac{1}{r}} v_1\).

On the other hand, we conclude from \(\omega = 1\) on \(\partial \Omega\) that there exists \(x_0 \in \Omega\) such that \(\omega(x_0) = 1\). Then \(\Delta \omega(x_0) \leq 0\), that is \(u_2 \Delta u_1 - u_1 \Delta u_2 \leq 0\) at \(x_0\). Therefore
\[
v_2^p(x_0) \geq k^{p-1} v_1^p(x_0) + \frac{(1-k)h(x_0)}{k h_2(x_0)} \geq k^{p-1} v_1^p(x_0),
\]
that is \(v_2(x_0) \geq k^{\frac{p-1}{p}} v_1(x_0)\).

By the above inequalities, we deduce that \(k^{\frac{1}{r}} v_1(x_0) > k^{\frac{p-1}{p}} v_1(x_0)\), and so \(k \frac{(p-1)(r-1)-gr}{gr(p-1)} < 1\). We obtain \(k < 1\) by \((p - 1)(r - 1) > gr\). This is a contradiction. Thus \(k \leq 1\) and \(u_1 < u_2\). Similarly, we can prove \(u_1 < u_2\). We obtain \(u_1 = u_2\). Since
\[
\Delta u_1 = u_1^r v_1^p + h_1(x), \quad \Delta u_2 = u_2^r v_2^p + h_1(x),
\]
we have \(v_1 = v_2\). The uniqueness of solutions is established.

\(\square\)

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**Conflict of interest**

The authors declare that there are no conflict of interest regarding the publication of this paper.

**References**


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