AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8191–8204.
DOI:10.3934/math.2021474
Received: 24 March 2021
Accepted: 19 May 2021
Published: 25 May 2021

Research article

Large positive solutions to an elliptic system of competitive type with nonhomogeneous terms

Haohao Jia, Feiyao Ma* and Weifeng Wo

Department of Mathematics, Ningbo University, Ningbo, Zhejiang 315211, China

* **Correspondence:** Email: mafeiyao@nbu.edu.cn; Tel: +8618858014083.

Abstract: In this paper, we study the elliptic system of competitive type with nonhomogeneous terms $\Delta u = u^p v^q + h_1(x)$, $\Delta v = u^r v^s + h_2(x)$ in Ω with two types of boundary conditions: (I) $u = v = +\infty$ and (SF) $u = +\infty$, $v = f$ on $\partial\Omega$, where $f > 0$, $(p-1)(s-1) - qr > 0$, and $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a smooth bounded domain. The nonhomogeneous terms $h_1(x)$ and $h_2(x)$ may be unbounded near the boundary and may change sign in Ω. First, for a single semilinear elliptic equation with a singular weight and nonhomogeneous term, boundary asymptotic behaviour of large positive solutions is established. Using this asymptotic behaviour, we show existence of large positive solutions for this elliptic system with the boundary condition (SF), existence of maximal solution, boundary asymptotic behaviour and uniqueness of large positive solutions for this elliptic system with (I).

Keywords: elliptic system; large solutions; nonhomogeneous term; existence; boundary asymptotic behaviour; uniqueness
Mathematics Subject Classification: 35A01, 35A02, 35B40

1. Introduction and main results

Consider the following elliptic system

$$\begin{cases} \Delta u = u^p v^q + h_1(x), & \text{in } \Omega, \\ \Delta v = u^r v^s + h_2(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

with two types of boundary conditions:

$$u = +\infty, v = +\infty, \ \text{on } \partial\Omega, \tag{I}$$

$$u = +\infty, v = f, \ \text{on } \partial\Omega, \tag{SF}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\mu}$ for some $0 < \mu < 1$ and $h_1(x), h_2(x) \in C(\Omega)$, $f > 0$, the parameters $p, s > 1$ and $q, r > 0$ such that $(p-1)(s-1) - qr > 0$. The condition $u = +\infty, v = +\infty$

on $\partial\Omega$ is defined in the sense of that $u(x) \to +\infty$ and $v(x) \to +\infty$ as $d(x)$:=dist$(x, \partial\Omega) \to 0$. The system can represent the competitive model of the two populations in the environment Ω. If the problem have a solution, it indicates that the two populations can coexist in Ω.

The large solutions of a single equation with nonhomogeneous terms have been studied recently (see [3, 7, 8, 10, 15–17, 19]). In [10], García-Melián studied existence, uniqueness and nonexistence of boundary blow-up solutions to problem

$$\begin{cases} \Delta u = |u|^{p-1}u + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. As far as we know, it was the first time that an unbounded and sign-changing inhomogeneous term $h(x)$ in this equation was studied. In [17], Wang investigated the problem

$$\begin{cases} \Delta u = b(x)u^p - ad(x)^{-q}, & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases}$$

where the parameters $p > 1$, $a > 0$, $q \in \mathbb{R}$ are constants, and $b(x) \in C^{\mu}(\Omega)$ is a positive function. They showed existence, uniqueness and the first order asymptotic behaviour of positive solutions. And, in [19], Wang et al. studied the following problem

$$\begin{cases} \Delta u = a(x)|u|^{p-1}u + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1$, $h(x) \in C(\Omega)$, $a(x)$ is a C^{μ}_{loc} $(0 < \mu < 1)$ continuous nonnegative function in Ω and satisfies

$$C_1 d(x)^{-\gamma} \leq a(x) \leq C_2 d(x)^{-\gamma} \ \text{ in } \{x \in \Omega | 0 < d(x) < \delta_0\}$$

for some positive constants C_1, C_2, δ_0 and $0 < \gamma < 2$. They studied existence, uniqueness and nonexistence of large solutions. For more results about existence and uniqueness of positive large solutions, we refer to the citation of [6].

In addition to the single equation, the study of systems is also meaningful due to its multiple applications. For example, the application of Newtonian fluids theory. Boundary blow-up solutions for elliptic systems have been studied in many papers [5, 9, 11–13, 18]. Dancer and Du [2] studied the predator-prey model. The boundary blow-up solutions to elliptic systems were studied in [11]. The existence, uniqueness and the first order boundary estimates of large solutions to the following systems $(h_1(x), h_2(x) = 0)$,

$$\begin{cases} \Delta u = u^p v^q, & \text{in } \Omega, \\ \Delta v = u^r v^s, & \text{in } \Omega, \end{cases}$$

with conditions: (F) $u = g$, $v = f$, (I), (SF) on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\mu}$ for some $0 < \mu < 1$ and the parameters $p, s > 1$, $q, r > 0$, $g, f > 0$, were investigated in García-Melián and Rossi [12]. In [18], Wang et al. showed the existence of positive solutions for the elliptic system (1.1) with (I). However, for large positive solutions to an elliptic system of competitive type with nonhomogeneous terms, we need to find appropriate conditions to get properties of solutions. We have to overcome this difficulty with the relevant results of a single equation.

In this paper, firstly, we explore boundary asymptotic behaviour of large positive solutions to

$$\begin{cases} \Delta u = a(x)u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Obviously all positive solutions of problem (1.2) are also solutions of (1.3). Then, we show boundary asymptotic behaviour and uniqueness of large solutions for (1.1) with the boundary condition (I) using the conclusions about the single equation.

The notations $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$, $\Omega^\delta := \{x \in \Omega : d(x) > \delta\}$ for $\delta > 0$, $h^+(x):=\max\{h(x), 0\}$ and $h^-(x):=\min\{h(x), 0\}$ are used through this paper.

The weight function $a(x)$ in (1.3) satisfies the following two conditions.

(a_1) $a(x)$ is a C^μ_{loc} continuous nonnegative function in Ω, $0 < \mu < 1$.

(a_2) There exists a positive continuous function $c_0(x)$ on $\partial\Omega$ such that $\lim_{x\to x_0} d(x)^\gamma a(x) = c_0(x_0)$ for every $x_0 \in \partial\Omega$, $0 < \gamma < 2$.

For $h(x)$ in (1.3), here are three conditions that will be used and γ is the same constant in (a_2).

(h_1) $\liminf_{x\to\partial\Omega} d(x)^{\frac{2p-\gamma}{p-1}} h(x) > -\infty$.

(h_2) $d(x)^{2-\lambda} h^+(x) \le C$, where C is a constant and $0 < \lambda < \min\{1, 2-\gamma\}$.

(h_3) $\lim_{x\to\partial\Omega} d(x)^{\frac{2p-\gamma}{p-1}} h^-(x) = 0$.

Apparently, (h_3) implies (h_1). We use different conditions in our theorems.

The main purpose of the paper is to study the influence of nonhomogeneous terms on the properties of the solution for the elliptic systems. In fact, the results shows that the solution is stable when the nonhomogeneous terms don't change much. It is worthwhile to mention our assumption that $a(x)$ is nonnegative in Ω and singular on $\partial\Omega$. And, $h(x)$ may be unbounded near $\partial\Omega$ and change sign in Ω. This work can be considered as an extension of such results on the system without nonhomogeneous terms. With regard to boundary asymptotic behavior of positive solutions to (1.3), we have:

Theorem 1.1. *Suppose that $p > 1$, $a(x)$ satisfies conditions (a_1) and (a_2), $h(x) \in C(\Omega)$ satisfies (h_2) and (h_3). The problem (1.3) admits at most one positive solution and if $u(x)$ is the solution, then $u(x)$ satisfies*

$$\lim_{x\to x_0} d(x)^\tau u(x) = \left(\frac{\tau(\tau+1)}{c_0(x_0)}\right)^{\frac{1}{p-1}}$$

for every $x_0 \in \partial\Omega$, where $\tau = \frac{2-\gamma}{p-1}$ and γ is given in (a_2).

With regard to existence of positive solutions for the elliptic system with the boundary condition (SF), we have:

Theorem 1.2. *Suppose that $(p-1)(s-1) > qr$ and $\frac{1}{2}(p-1) < r < p-1$. For $0 < \tilde\lambda_1 < 1$, $2 - \frac{2r}{p-1} \le \tilde\lambda_2 < 1$, there exists a constant $\tilde c > 0$ such that if $\sup_{x\in\Omega} d(x)^{2-\tilde\lambda_1} h_1(x) \le \tilde c$, $\sup_{x\in\Omega} d(x)^{2-\tilde\lambda_2} h_2(x) \le \tilde c$, then system (1.1) with the boundary condition (SF) admits a positive solution.*

Definition 1.3. If the solution $(\bar u^*, \bar v^*)$ and any other solution (u, v) of system (1.1) with the boundary condition (I) is such that $\bar u^* \ge u$, $\bar v^* \le v$, then we call $(\bar u^*, \bar v^*)$ the maximal solution of system (1.1) with the boundary condition (I). And, if the reversing inequalities hold, then we call it is a minimal solution (denoted as $(\underline{u}^*, \underline{v}^*)$).

For the elliptic system with the boundary condition (I), we have:

Theorem 1.4. *Suppose that $(p-1)(s-1) > qr$, $\frac{1}{2}(p-1) < r < p-1$. For $0 < \tilde\lambda_1 < 1$, $2 - \frac{2r}{p-1} \le \tilde\lambda_2 < 1$, there exists a constant $\tilde c > 0$ such that if $\sup_{x\in\Omega} d(x)^{2-\tilde\lambda_1} h_1(x) \le \tilde c$, $\sup_{x\in\Omega} d(x)^{2-\tilde\lambda_2} h_2(x) \le \tilde c$, then system (1.1) with the boundary condition (I) admits a maximal solution.*

Theorem 1.5. *Suppose that $(p-1)(s-1) > qr$, $h_1(x)$, $h_2(x) \in C(\Omega)$ and*

$$\begin{aligned} d(x)^{2-\lambda_1} h_1^+(x) \le C, \lim_{x\to\partial\Omega} d(x)^{\alpha+2} h_1^-(x) = 0, \\ d(x)^{2-\lambda_2} h_2^+(x) \le C, \lim_{x\to\partial\Omega} d(x)^{\beta+2} h_2^-(x) = 0, \end{aligned} \tag{1.4}$$

where $\alpha = \frac{2(s-1-q)}{(p-1)(s-1)-qr}$, $\beta = \frac{2(p-1-r)}{(p-1)(s-1)-qr}$, $C > 0$, $0 < \lambda_1 < \min(1, 2-\beta q)$, and $0 < \lambda_2 < \min(1, 2-\alpha r)$. Assume (u, v) is a positive solution to system (1.1) with the boundary condition (I), then

$$\begin{aligned} \lim_{x\to x_0} d(x)^{\alpha} u(x) = \left(\frac{(\alpha(\alpha+1))^{s-1}}{(\beta(\beta+1))^{q}} \right)^{\frac{1}{(p-1)(s-1)-qr}}, \\ \lim_{x\to x_0} d(x)^{\beta} v(x) = \left(\frac{(\beta(\beta+1))^{p-1}}{(\alpha(\alpha+1))^{r}} \right)^{\frac{1}{(p-1)(s-1)-qr}}, \end{aligned} \tag{1.5}$$

for every $x_0 \in \partial\Omega$.

Theorem 1.6. *Suppose that $(p-1)(s-1) > qr$, $h_1(x)$, $h_2(x) \in C(\Omega)$ are non-positive functions and satisfy (1.4). Then system (1.1) with the boundary condition (I) admits at most one positive solution.*

This paper is organized as follows. Section 2 presents some preliminaries. In Section 3, we proceed with the study of boundary asymptotic behaviour of positive solutions for the single equation. In Section 4, the existence, global estimates, boundary asymptotic behaviour and uniqueness of positive solutions to the system of competitive type with different boundary conditions are considered.

2. Preliminaries

We present some useful results about solutions to problem (1.3) in this section. The following lemma is the remark after Lemma 4.1 in [19].

Lemma 2.1 ([19])**.** *Suppose that $p > 1$, $a(x)$ satisfies conditions (a_1) and (a_2). And $h(x) \in C(\Omega)$ satisfies (h_1) and (h_2). Then for any positive solution u of (1.3), $Pd(x)^{-\tau} \le u \le \tilde{P} d(x)^{-\tau}$ for some positive constants $P, \tilde{P}$ in Ω_δ, where $\tau = \frac{2-\gamma}{p-1}$.*

Next lemma shows the uniqueness of positive solutions for problem (1.3), even when $h(x)$ may be unbounded near $\partial\Omega$.

Lemma 2.2. *Suppose that $p > 1$, $a(x)$ satisfies conditions (a_1) and (a_2). And $h(x) \in C(\Omega)$ satisfies (h_1) and (h_2). Then problem (1.3) has at most one positive solution.*

Proof. We know that the condition (h_2) implies that the function $h(x)$ is bounded from above. Then, we have the conditions of Theorem 1.4 in [19] and omit the detail. □

In particular, for the following problem

$$\begin{cases} \Delta u = d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $p > 1$, $0 < \gamma < 2$ and $h(x) \in C(\Omega)$ satisfies $(h_2), (h_3)$, we have:

Corollary 2.3. *Assume that $p > 1$ and $\gamma < 2$. If $U_{p,\gamma,h}$ denotes the unique positive solution of problem* (2.1)*, then*

$$\lim_{x\to x_0} d(x)^\alpha U_{p,\gamma,h}(x) = (\alpha(\alpha+1))^{\frac{1}{p-1}}$$

for every $x_0 \in \partial\Omega$, where $\alpha = \frac{2-\gamma}{p-1}$.

Lemma 2.4. *Let $u \in C^2(\Omega)$ satisfies*

$$\begin{cases} \Delta u \le c_0 d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where c_0 is some positive constant and $h(x) \in C(\Omega)$ satisfies $(h_2), (h_3)$. Then $u(x) \ge c_0^{-\frac{1}{p-1}} U_{p,\gamma,\tilde{h}}$, where $\tilde{h}(x) = c_0^{\frac{1}{p-1}} h(x)$. Similarly, if

$$\begin{cases} \Delta u \ge c_0 d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases}$$

then $u(x) \le c_0^{\frac{1}{p-1}} U_{p,\gamma,\tilde{h}}$.

Proof. Firstly, $\Delta(c_0^{\frac{1}{p-1}} u) = c_0^{\frac{1}{p-1}} \Delta u$. By (2.2), we have

$$\Delta(c_0^{\frac{1}{p-1}} u) \le c_0^{\frac{1}{p-1}} \cdot c_0 d(x)^{-\gamma} u^p + c_0^{\frac{1}{p-1}} h(x) = d(x)^{-\gamma} (c_0^{\frac{1}{p-1}} u)^p + c_0^{\frac{1}{p-1}} h(x).$$

Note $\tilde{h}(x) = c_0^{\frac{1}{p-1}} h(x)$, then $\Delta(c_0^{\frac{1}{p-1}} u) \le d(x)^{-\gamma} (c_0^{\frac{1}{p-1}} u)^p + \tilde{h}(x)$. It is clear that $\tilde{h}(x) \in C(\Omega)$ and satisfies the conditions $(h_2), (h_3)$. By Corollary (2.3) and the methods of sub- and supersolutions, we have $c_0^{\frac{1}{p-1}} u(x) \ge U_{p,\gamma,\tilde{h}}$, that is $u(x) \ge c_0^{-\frac{1}{p-1}} U_{p,\gamma,\tilde{h}}$. The other case is proved similarly. □

The following lemma is a straightforward extension of Lemma 2.4 that is about the case where Ω is a half-space $D = \{x \in \mathbb{R}^N : x_1 > 0\}$. We write $x = (x_1, x')$, where $x' \in \mathbb{R}^{N-1}$. This lemma will be used to deduce boundary estimates for positive solutions to system (1.1).

Lemma 2.5. *Suppose that $u \in C^2(D)$ satisfies*

$$\begin{cases} \Delta u \le C_1 x_1^{-\gamma} u^p + h(x), & \text{in } D, \\ u \ge K x_1^{-\alpha}, \end{cases}$$

where C_1, K are some positive constants, $\alpha = \frac{2-\gamma}{p-1}$ and $h(x) \in C(\Omega)$ satisfies $(h_2), (h_3)$. Then $u \ge (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}} x_1^{-\alpha}$ in D. Similarly, if

$$\begin{cases} \Delta u \ge C_1 x_1^{-\gamma} u^p + h(x), & \text{in } D, \\ u \le K x_1^{-\alpha}, \end{cases}$$

then $u \le (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}} x_1^{-\alpha}$ in D.

Proof. The proofs are analogous to the one of Lemma 9 in [12]. Assume that there exists $x^0 \in D$ and $l > 1$ so that $u(x^0) > lE(x_1^0)^{-\alpha}$, where x_1^0 is the first component of x^0 and $E = (\frac{\alpha(\alpha+1)}{C_1})^{\frac{1}{p-1}}$. Set

$$D_0 = \{u > lEx_1^{-\alpha}\} \cap B_r(x^0)$$

with $r = \frac{d(x^0)}{2}$. By (h_3), we have $\liminf_{x\to\partial\Omega} d(x)^{\frac{2p-\gamma}{p-1}} h(x) > -\infty$, that is $h(x) \geq -K_1 x_1^{-(\alpha+2)}$ in D_0 for some positive constant K_1. Then we deduce

$$\Delta(u - lEx_1^{-\alpha}) > [E\alpha(\alpha+1)l^p - K_1 - lE\alpha(\alpha+1)]x_1^{-\alpha-2} \text{ in } D_0.$$

For simplification, we note $F := E\alpha(\alpha+1)l^p - K_1 - lE\alpha(\alpha+1)$.

Choose $K_1 > 0$ such that $F > 0$. Since $x_1 \leq \frac{3r}{2}$ in D_0, and if we define $w(x) = \frac{F 2^{\alpha+2} r^{-\alpha-2}}{2N3^{\alpha+2}}(r^2 - |x - x^0|^2)$, then $\Delta(u - lEx_1^{-\alpha} + w) > 0$ in D_0. By the maximum principe, we have that there exists $x^1 \in \partial D_0$ such that

$$u(x^0) - lE(x_1^0)^{-\alpha} + w(x^0) < u(x^1) - lE(x_1^1)^{-\alpha} + w(x^1).$$

Thus $x^1 \in \partial B_r(x^0)$. Then $w(x^0) < u(x^1) - lE(x_1^1)^{-\alpha}$.

Now, using $x_1^1 \geq \frac{r}{2}$ and the definition of w, we have

$$\begin{aligned} u(x^1) >& \frac{2^{\alpha+2} F r^{-\alpha}}{2N3^{\alpha+2}} + lE(x_1^1)^{-\alpha} \\ =& \{\frac{2}{N3^{\alpha+2}}[lE\alpha(\alpha+1)(l^{p-1}-1) - K_1] + lE\}(x_1^1)^{-\alpha} \\ =& \{\frac{2}{N3^{\alpha+2}} lE[\alpha(\alpha+1)(l^{p-1}-1) - \frac{K_1}{lE}] + lE\}(x_1^1)^{-\alpha} \\ =& \{\frac{2}{N3^{\alpha+2}}[\alpha(\alpha+1)(l^{p-1}-1) - \frac{K_1}{lE}] + 1\} lE(x_1^1)^{-\alpha}, \end{aligned}$$

where $\alpha(\alpha+1)(l^{p-1}-1) - \frac{K_1}{lE} > 0$ by $F > 0$. Proceeding inductively, a sequence of points $x^n \in D$ satisfies

$$u(x^n) > \{\tfrac{2}{N3^{\alpha+2}}[\alpha(\alpha+1)(l^{p-1}-1) - \tfrac{K_1}{lE}] + 1\}^n lE(x_1^n)^{-\alpha}$$

can be obtained, which contradicts with the inequality $u \leq Kx_1^{-\alpha}$. Similarly, we can prove the other case, and the lemma follows. □

3. The single equation

Proof of Theorem 1.1. Let $x_0 \in \partial\Omega$ and $\{x_n\} \subset \Omega$ be a sequence converging to x_0. Let W be an open neighborhood of x_0 such that $\partial\Omega$ admits $C^{2,\mu}$ local coordinates $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) : W \to \mathbb{R}^N$ with $x \in W \cap \Omega$ if and only if $\varphi_1(x) > 0$. We can also suppose $\varphi(x_0) = 0$. If $u(x) = \bar{u}(\varphi(x))$, $h(x) = \bar{h}(\varphi(x))$, $a(x) = \bar{a}(\varphi(x))$, then we have

$$\sum_{i,j=1}^N a_{ij}(\varphi)\tfrac{\partial^2 \bar{u}}{\partial\varphi_i \partial\varphi_j} + \sum_{i=1}^N b_{ij}(\varphi)\tfrac{\partial \bar{u}}{\partial\varphi_i} = \bar{a}(\varphi)\bar{u}^p + \bar{h}(\varphi), \text{ in } \varphi(W \cap \Omega),$$

where a_{ij}, b_i are C^μ and $a_{ij}(0) = \delta_{ij}$. Denote t_n be the projections onto $\varphi(W \cap \partial\Omega)$ of $\varphi(x_n)$, and introduce the functions

$$u_n(y) = d_n^\tau \bar{u}(t_n + d_n y),$$

where $d_n = d(\varphi(x_n))$, $\varphi(x_n) = t_n + d_n(1, 0, \cdots, 0)$. Then the function u_n satisfies

$$\sum_{i,j=1}^{N} a_{ij}(t_n + d_n y)\frac{\partial^2 u_n}{\partial\varphi_i\partial\varphi_j} + d_n \sum_{i=1}^{N} b_{ij}(t_n + d_n y)\frac{\partial u_n}{\partial\varphi_i} = d_n^{\gamma}\bar{a}(t_n + d_n y)u_n^p + d_n^{\tau p+\gamma}\bar{h}(t_n + d_n y).$$

And, the conclusion of Lemma 2.1 implies that, for y in compact subsets M of $D := \{y \in \mathbb{R}^N : y_1 > 0\}$, there exists $n_0 = n_0(M)$ such that $\Lambda_1 y_1^{-\tau} \le u_n(y) \le \Lambda_2 y_1^{-\tau}$ for $n \ge n_0$, where Λ_1, Λ_2 are positive constants. By the above estimates, the conditions $(h_2), (h_3)$ and standard methods, we obtain that for a subsequence we have $u_n \to u_0$ in $C^{1,\mu}_{\mathrm{loc}}(M)$, where u_0 satisfies

$$\begin{cases} \Delta u_0 = c_0(x_0)y_1^{-\gamma}u_0^p, & \text{in } D,\\ \Lambda_1 y_1^{-\tau} \le u_0 \le \Lambda_2 y_1^{-\tau}. \end{cases}$$

By Theorem 10 and Remarks 3 b) in [1], we know this problem has a unique positive solution, that is

$$u_0(y) = \left(\frac{\tau(\tau+1)}{c_0(x_0)}\right)^{\frac{1}{p-1}} y_1^{-\tau},$$

which completes the proof. □

4. The system of competitive type

In this section, it is shown that existence, global estimates, asymptotic behaviour and uniqueness of positive solutions to this system with different boundary conditions.

4.1. Existence of positive solutions

In this subsection, we present existence of positive solution for the elliptic system (1.1) with (SF) and existence of maximal solution for the elliptic system (1.1) with (I).

Recall that if

$$\begin{cases} \Delta \underline{u} \ge \underline{u}^p \underline{v}^q + h_1(x), & \text{in } \Omega,\\ \Delta \underline{v} \le \underline{u}^r \underline{v}^s + h_2(x), & \text{in } \Omega, \end{cases}$$

then $(\underline{u}, \underline{v})$ is called a subsolution. And, if the reversing inequalities hold, then we call it a supersolution (denoted as $(\bar{u}, \bar{v})$).

We shall show existence of positive solutions for the elliptic system (1.1) with (SF). First, we show some lemmas which will use to prove Theorem 1.2.

Remark 4.1. The following conclusions also hold if $v = f(x)$ is a continuous positive function on $\partial\Omega$.

Lemma 4.2. *Let $n \in \mathbb{N}$. Suppose that $p > 1$, $a(x)$ satisfies conditions (a_1), (a_2). There exists a constant $\tilde{c} > 0$ such that if $h(x) \in C(\Omega)$ satisfies $\sup\limits_{x\in\Omega} d(x)^{2-\bar{\lambda}}h(x) \le \tilde{c}$ for $0 < \bar{\lambda} < 1$, then the following problem*

$$\begin{cases} \Delta u = a(x)u^p + h(x), & \text{in } \Omega,\\ u = n, & \text{on } \partial\Omega, \end{cases}$$

admits a unique positive solution.

Analogous to the proofs of Corollary 1.2 in [19] and Lemma 3 in [1], we omit the proof of Lemma 4.2.

Lemma 4.3. *Suppose that $(p-1)(s-1) > qr$, $h_1(x)$, $h_2(x) \in C(\Omega)$ satisfies (1.4). If (u, v) denotes a solution to system (1.1) with (SF), then*

$$\lim_{x\to x_0} d(x)^{\vartheta}u(x) = \left(\frac{(\vartheta(\vartheta+1))}{f^q}\right)^{\frac{1}{(p-1)}},$$

where $\vartheta = \frac{2}{p-1}$.

Proof. Since $v = f$ on $\partial\Omega$, u is a positive solution to problem

$$\begin{cases} \Delta u = v^q u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases}$$

then the proof is completed by Theorem 1.1 with $\gamma = 0$ and $c_0(x) \equiv f^q$, for every $x \in \Omega$. □

Lemma 4.4 ([4]). *Let* Ω *be a* C^2 *bounded domain of* $\mathbb{R}^N$, $g \in C(\Omega)$ *is a function such that* $\sup_{x\in\Omega} d(x)^{\gamma} \mid g(x) \mid< +\infty$ *for some* $1 < \gamma < 2$ *and* $u \in C^2(\Omega)$ *is a solution to the problem* $\Delta u = g$ *in* Ω *with* $u = 0$ *on* $\partial\Omega$. *Then there exists* $C > 0$ *depending only on* Ω *and* γ *such that*

$$\sup_{x\in\Omega} d(x)^{\gamma-2} \mid u(x) \mid\leq C \sup_{x\in\Omega} d(x)^{\gamma} \mid g(x) \mid.$$

Lemma 4.5. *Suppose* $(\bar{u}, \bar{v})$ *is a supersolution and* $(\underline{u}, \underline{v})$ *is a subsolution to the problem (1.1) with* $\underline{u} = \bar{u} = +\infty, \underline{v} \geq f \geq \bar{v}$ *on* $\partial\Omega$, *and* $\underline{u} \leq \bar{u}$, $\underline{v} \geq \bar{v}$ *in* Ω. *Also assume that* $\bar{u} \leq Cd(x)^{-\vartheta}$ *for some positive constant* C *and* $\vartheta < \frac{2}{r}$, $\frac{1}{2}(p-1) < r < p-1$. *For* $0 < \tilde{\lambda}_1 < 1$, $2 - \frac{2r}{p-1} \leq \tilde{\lambda}_2 < 1$, *there exists a constant* $\tilde{c} > 0$ *such that if* $\sup_{x\in\Omega} d(x)^{2-\tilde{\lambda}_1} h_1(x) \leq \tilde{c}, \sup_{x\in\Omega} d(x)^{2-\tilde{\lambda}_2} h_2(x) \leq \tilde{c}$, *then system (1.1) admits at least a solution* (u, v) *such that* $\underline{u} \leq u \leq \bar{u}$, $\underline{v} \geq v \geq \bar{v}$ *in* Ω *and* $u = +\infty, v = f$ *on* $\partial\Omega$.

Proof. From Lemma 4.2, there exists a constant $\tilde{c}_1 > 0$ such that if $\sup_{x\in\Omega} d(x)^{2-\tilde{\lambda}_2} h_2(x) \leq \tilde{c}_1$, then $\underline{v}$ is a positive, bounded function in $\bar{\Omega}$. There exists a constant $\tilde{c}_2 > 0$ such that if $\sup_{x\in\Omega} d(x)^{2-\tilde{\lambda}_2} h_2(x) \leq \tilde{c}_2$, then the problem

$$\begin{cases} \Delta u = \underline{v}^q u^p + h_1(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

admits a unique positive solution, which we denote by u_1. And, $\Delta\underline{u} \geq \underline{v}^q\underline{u}^p + h_1(x)$ in Ω. By uniqueness of solutions to (4.1) and the methods of sub- and supersolutions, we have $\underline{u} \leq u_1$. Similarly, $\Delta\bar{u} \leq \bar{v}^q\bar{u}^p + h_1(x) \leq \underline{v}^q\bar{u}^p + h_1(x)$ in Ω, and so $\bar{u} \geq u_1$. By Lemma 4.2 and $0 < u_1^r \leq Cd(x)^{-\vartheta r}$, $\vartheta r < 2$, we let v_1 as the unique solution to

$$\begin{cases} \Delta v = u_1^r v^s + h_2(x), & \text{in } \Omega, \\ v = f, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We see at once that $\underline{v} \geq v_1 \geq \bar{v}$ in Ω. We let u_2 as the unique solution to

$$\begin{cases} \Delta u = v_1^q u^p + h_1(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then, we also have that $\underline{u} \leq u_2 \leq \bar{u}$ in Ω. And, $\Delta u_1 = \underline{v}^q u_1^p + h_1(x) \geq v_1^q u_1^p + h_1(x)$, so $u_1 \leq u_2$.

Recursively, we let v_n be the unique solution to (4.2), with u_1 replaced by u_n, and u_n be the unique solution to (4.3), with v_1 replaced by v_{n-1}. Thus, we can obtain two sequences $\{u_n\}$ and $\{v_n\}$ which satisfy $\{u_n\}$ is increasing, $\{v_n\}$ is decreasing, $\underline{u} \leq u_n \leq \bar{u}$ and $\underline{v} \geq v_n \geq \bar{v}$ in Ω. By standard methods, we conclude that there is a subsequence (still labelled by u_n and v_n) such that $u_n \to u$, $v_n \to v$ in $C^{1,\mu}_{\text{loc}}(\Omega)$, where (u, v) is a solution to (1.1) with the boundary condition (SF) and $\underline{u} \le u \le \bar{u}$, $\underline{v} \ge v \ge \bar{v}$ in Ω, $u = +\infty$ on $\partial\Omega$.

Now let $\omega_n = f - v_n$. Then $\Delta\omega_n = -u_n^r(f - \omega_n)^s + h_2(x)$ in Ω, $\omega_n = 0$ on $\partial\Omega$. Since $u_n^r \le Cd^{-\vartheta r}$ and $\vartheta < \frac{2}{r}$ in Ω and ω_n is uniformly bounded, then

$$| \omega_n | \le C' d^{2-\vartheta r} \tag{4.4}$$

by Lemma 4.4. The inequality (4.4) also holds for $\omega = f - v$, therefore $\omega = 0$ on $\partial\Omega$, that is $v = f$ on $\partial\Omega$. □

Proof of Theorem 1.2. By Lemma 4.2, $p > 1, \gamma < 2$, $h(x) \in C(\Omega)$ and there exist constants $\tilde{c} > 0$, $0 < \bar{\lambda} < 1$ such that $\sup\limits_{x\in\Omega} d(x)^{2-\bar{\lambda}} h(x) \le \tilde{c}$, then there exists a unique positive solution to problem

$$\begin{cases} \Delta u = d(x)^{-\gamma} u^p + h(x), & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega, \end{cases}$$

which is denoted by $V_{p,\gamma}$. Let $\varepsilon > 0$ be small, $\delta > 0$, $\sigma = \frac{2r}{p-1}$ to be chosen and $(\underline{u}, \underline{v}) = (\varepsilon U_{p,0}, \varepsilon^{-\delta} V_{s,\sigma})$, where $U_{p,0}$ is a positive solution to problem

$$\begin{cases} \Delta u = u^p + h(x), & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega. \end{cases}$$

We show $(\underline{u}, \underline{v})$ is a subsolution to system (1.1) with (SF) firstly. That is to show

$$\begin{aligned} &1 \ge \varepsilon^{p-\delta q-1} V_{s,\sigma}^q + U_{p,0}^{-p}(\varepsilon^{-1} - 1)h_1(x), \\ &1 \le \varepsilon^{r-\delta s+\delta} d(x)^\sigma U_{p,0}^r + d(x)^\sigma V_{s,\sigma}^{-s}(\varepsilon^\delta - 1)h_2(x). \end{aligned} \tag{4.5}$$

Inequalities $p - \delta q - 1 > 0$ and $r - \delta s + \delta < 0$ will be hold if we fix $\delta \in \left(\frac{r}{s-1}, \frac{p-1}{q}\right)$. We know $V_{s,\sigma}^q$ and $d(x)^\sigma U_{p,0}^r$ are bounded. Then, for ε small, inequality (4.5) holds because of conditions about $h_1(x), h_2(x)$, and we get a subsolution to (1.1) with (SF).

Similarly, let $(\bar{u}, \bar{v}) = (QU_{p,0}, Q^{-\delta} V_{s,\sigma})$. Now we show $(\bar{u}, \bar{v})$ is a supersolution to system (1.1) with the boundary condition (SF). That is to show

$$\begin{aligned} &1 \le Q^{p-\delta q-1} V_{s,\sigma}^q + U_{p,0}^{-p}(Q^{-1} - 1)h_1(x), \\ &1 \ge Q^{r-\delta s+\delta} d(x)^\sigma U_{p,0}^r + d(x)^\sigma V_{s,\sigma}^{-s}(Q^\delta - 1)h_2(x). \end{aligned} \tag{4.6}$$

Then, for Q large, inequality (4.6) holds because of conditions about $h_1(x), h_2(x)$, and we get a supersolution to (1.1) with (SF) and $\underline{u} \le \bar{u}, \underline{v} \ge \bar{v}$. Therefore, the existence of a positive solution (u, v) to (1.1) with (SF) follows from Lemma 4.5 and $r < p - 1$. □

Next, we show existence of solutions to the following problem

$$\begin{cases} \Delta u = u^p v^q + h_1(x), & \text{in } \Omega, \\ \Delta v = u^r v^s + h_2(x), & \text{in } \Omega, \\ u = k_1(x), v = k_2(x), & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $k_1(x), k_2(x)$ are positive continuous functions on $\partial\Omega$. By the standard method in [14], one can prove the following lemma.

Lemma 4.6 ([18]). *Suppose that $(\bar{u}, \bar{v})$ is a supersolution and $(\underline{u}, \underline{v})$ is a subsolution to problem (4.7) with $\underline{u} \le k_1(x) \le \bar{u}$, $\underline{v} \ge k_2(x) \ge \bar{v}$ on $\partial\Omega$. And, $\underline{u} \le \bar{u}$, $\underline{v} \ge \bar{v}$ in Ω. Then Problem (4.7) admits at least a solution (u, v) that $\underline{u} \le u \le \bar{u}$, $\underline{v} \ge v \ge \bar{v}$ in Ω and $u = k_1(x)$, $v = k_2(x)$ on $\partial\Omega$.*

Proof of Theorem 1.4. Define $\Omega_{\frac{1}{n}} := \{x \in \Omega : d(x) < \frac{1}{n}\}$. First, we prove that for the following problems

$$\begin{cases} \Delta u_n = u_n^p v_n^q + h_1(x), & \text{in } \Omega_{\frac{1}{n}}, \\ \Delta v_n = u_n^r v_n^s + h_2(x), & \text{in } \Omega_{\frac{1}{n}}, \quad n, n_0 \in \mathbb{N}, n > n_0, \\ u_n = +\infty, v_n = f(n), & \text{on } \partial\Omega_{\frac{1}{n}}, \end{cases} \tag{4.8}$$

we have

$$\begin{cases} u_n \ge u_{n+1}, & \text{in } \Omega_{\frac{1}{n}}, \\ v_n \le v_{n+1}, & \text{in } \Omega_{\frac{1}{n}}. \end{cases}$$

Choose $f(n) = 1 + C'(\frac{1}{n^2})^{\varrho}$, where C' is the uniform constant in (4.4), $\varrho = 2 - \vartheta r$, that is $| v_n - f(n) | \le C' d^{\varrho}$. Since $f(n+1) = 1 + C'(\frac{1}{(n+1)^2})^{\varrho}$, $f(n) = 1 + C'(\frac{1}{n^2})^{\varrho}$ on $\partial\Omega_{\frac{1}{n}}$, then $f(n+1) > f(n)$. That is $v_{n+1} > v_n$ on $\partial\Omega_{\frac{1}{n}}$. Clearly, $u_{n+1} < u_n$ on $\partial\Omega_{\frac{1}{n}}$. By Lemma 4.6, we have

$$\begin{cases} u_n > u_{n+1}, & \text{in } \Omega_{\frac{1}{n}}, \\ v_n < v_{n+1}, & \text{in } \Omega_{\frac{1}{n}}. \end{cases}$$

Thus, we get two sequences $\{u_n\}$ and $\{v_n\}$ which satisfy $\{u_n\}$ is decreasing, $\{v_n\}$ is increasing and $u_n \to \bar{u}^*, v_n \to \bar{v}^*$ in Ω. It is easy to check that $(\bar{u}^*, \bar{v}^*)$ is a solution to system (1.1) with (I) and by the construction of $\{u_n\}$, $\{v_n\}$, $(\bar{u}^*, \bar{v}^*)$ is also the maximal solution. Then, for any positive solution (u, v) to system (1.1) with (I) and for any $x \in \Omega$, we have $u_n > u$, $v_n < v$. Thus, $\bar{u}_\infty \ge u$, $\bar{v}_\infty \le v$ as $n \to +\infty$. The Theorem is proved. □

4.2. Global estimates of solutions

Theorem 4.7. *Assume that $(p-1)(s-1) > qr$ and $r < p-1$, $q < s-1$, and $h_1(x)$, $h_2(x)$ satisfy (1.4), then positive solutions (u, v) of system (1.1) with the boundary condition (I) satisfy*

$$Ad(x)^{-\alpha} \le u \le Bd(x)^{-\alpha}, Ad(x)^{-\beta} \le v \le Bd(x)^{-\beta}, \text{in } \Omega, \tag{4.9}$$

for some $A, B > 0$.

Proof. Let $a_0 = \inf v > 0$. Then $\Delta u \ge a_0^q u^p + h_1(x)$ in Ω. Note

$$A_{p,\gamma,\tilde{h}} = \sup_{x\in\Omega} d(x)^{\alpha} U_{p,\gamma,\tilde{h}}(x), B_{p,\gamma,\tilde{h}} = \inf_{x\in\Omega} d(x)^{\alpha} U_{p,\gamma,\tilde{h}}(x). \tag{4.10}$$

Clearly, they are positive and finite. Lemma 2.4 implies that $u \le a_0^{-\frac{q}{p-1}} U_{p,0,\tilde{h}_1^{(1)}}$, that is $u \le a_0^{-\frac{q}{p-1}} A_{p,0,\tilde{h}_1^{(1)}} d^{-\alpha_0}$, where $\alpha_0 = \frac{2}{p-1}$, $\tilde{h}_1^{(1)} = a_0^{\frac{q}{p-1}} h_1$. We use this into the right side of the latter equation in (1.1). Then,

$$\Delta v \le a_0^{-\frac{qr}{p-1}} A^r_{p,0,\tilde{h}_1^{(1)}} d^{-\alpha_0 r} v^s + h_2(x), \text{ in } \Omega.$$

Using Lemma 2.4, we obtain

$$v \geq (a_0^{-\frac{qr}{p-1}} A^r_{p,0,\tilde{h}_1^{(1)}})^{-\frac{1}{s-1}} B_{s,\alpha_0 r,\tilde{h}_2^{(1)}} d^{-\beta_0}, \text{ in } \Omega,$$

where $\beta_0 = \frac{2-\alpha_0 r}{s-1}$, $\tilde{h}_2^{(1)} = (a_0^{-\frac{q}{p-1}} A_{p,0,\tilde{h}_1^{(1)}})^{\frac{r}{s-1}} h_2$. Inductively, we have

$$u \leq a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q,\tilde{h}_1^{(n+1)}} d^{-\alpha_n}, v \geq a_{n+1} d^{-\beta_n}, \tag{4.11}$$

in Ω, where

$$\begin{aligned}
\alpha_n &= \frac{2-\beta_{n-1}q}{p-1},\\
\beta_n &= \frac{2-\alpha_n r}{s-1},\\
a_{n+1} &= a_n^{\frac{qr}{(p-1)(s-1)-qr}} A^{-\frac{r}{s-1}}_{p,\beta_{n-1}q} B_{s,\alpha_n r},\\
\tilde{h}_1^{(n+1)} &= a_n^{\frac{q}{p-1}} h_1,\\
\tilde{h}_2^{(n+1)} &= (a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q,\tilde{h}_1^{(n+1)}})^{\frac{r}{s-1}} h_2.
\end{aligned} \tag{4.12}$$

In fact,

$$\beta_n = \frac{2(p-1-r)}{(p-1)(s-1)} + \frac{qr}{(p-1)(s-1)} \beta_{n-1}$$

and $\beta_1 > \beta_0$, $\beta_n \leq \beta$. By elementary calculations, we have $\beta_n \to \beta = \frac{2(p-1-r)}{(p-1)(s-1)-qr}$ as $n \to +\infty$. Then $\alpha_n \to \alpha = \frac{2(s-1-q)}{(p-1)(s-1)-qr}$.

For a_{n+1}, the third Eq in (4.12) and Lemma 7 in [12] imply that there exists a positive constant N such that $a_{n+1} \geq N a_n^{\theta}$, where $\theta = \frac{qr}{(p-1)(s-1)} < 1$. Iterating above inequality we deduce

$$a_{n+1} \geq a_0^{\theta^{n+1}} N^{\theta^n+\theta^{n-1}+\cdots+\theta+1}.$$

Then $\liminf_{n\to+\infty} a_{n+1} \geq N^{\frac{1}{1-\theta}} > 0$ is obtained by $n \to +\infty$. Then, for $\tilde{h}_2^{(n+1)}$, the fifth equation in (4.12) and the analysis of a_{n+1}, we have

$$a_n^{-\frac{qr}{(p-1)(s-1)}} A^{\frac{r}{s-1}}_{p,\beta_{n-1}q,\tilde{h}_1^{(n+1)}} \geq N a_{n-1}^{-\theta} \geq a_0^{(-\theta)^n} N^{1-\theta+\theta^2-\theta^3+\cdots+(-\theta)^{n-1}}.$$

Passing to the limit we have $\liminf_{n\to+\infty} (a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q,\tilde{h}_1^{(n+1)}})^{\frac{r}{s-1}} \geq N^1 > 0$ when n are odd numbers, and $\liminf_{n\to+\infty} (a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q,\tilde{h}_1^{(n+1)}})^{\frac{r}{s-1}} \geq N^{\frac{1-\theta}{1+\theta}} > 0$ when n are even numbers.

Therefore, $\tilde{h}_1^{(n+1)}$ and $\tilde{h}_2^{(n+1)}$ satisfy the conditions (h_2) and (h_3). Thus, by (4.11), we have $u \leq B d^{-\alpha}$, $v \geq A d^{-\beta}$ in Ω as $n \to +\infty$ for some $A, B > 0$. Similarly, the reversed inequalities can be proved, thus the proof is complete. □

4.3. Boundary asymptotic behavior and Uniqueness of positive solutions

Proof of Theorem 1.5. Let $x_0 \in \partial\Omega$ and $\{x_n\} \subset \Omega$ be a sequence converging to x_0. Take V be an open neighborhood of x_0 such that $\partial\Omega$ admits $C^{2,\mu}$ local coordinates $\xi = (\xi_1, \xi_2, \cdots, \xi_N) : V \to \mathbb{R}^N$ with $x \in V \cap \Omega$ if and only if $\xi_1(x) > 0$. We can also suppose $\xi(x_0) = 0$. If $u(x) = \bar{u}(\xi(x))$, $v(x) = \bar{v}(\xi(x))$, $h_1(x) = \bar{h}_1(\xi(x))$, $h_2(x) = \bar{h}_2(\xi(x))$, then we have

$$\begin{cases} \sum\limits_{i,j=1}^{N} a_{ij}(\xi)\frac{\partial^2 \bar{u}}{\partial\xi_i\partial\xi_j} + \sum\limits_{i=1}^{N} b_{ij}(\xi)\frac{\partial\bar{u}}{\partial\xi_i} = \bar{u}^p\bar{v}^q + \bar{h}_1(\xi), \\ \sum\limits_{i,j=1}^{N} a_{ij}(\xi)\frac{\partial^2 \bar{v}}{\partial\xi_i\partial\xi_j} + \sum\limits_{i=1}^{N} b_{ij}(\xi)\frac{\partial\bar{v}}{\partial\xi_i} = \bar{u}^r\bar{v}^s + \bar{h}_2(\xi), \end{cases}$$

in $\xi(V \cap \Omega)$, where a_{ij}, b_i are C^μ and $a_{ij}(0) = \delta_{ij}$. Denote ζ_n be the projections onto $\xi(V \cap \partial\Omega)$ of $\xi(x_n)$. Let us introduce the functions

$$u_n(y) = d_n^\alpha \bar{u}(\zeta_n + d_n y),\ v_n(y) = d_n^\beta \bar{v}(\zeta_n + d_n y)$$

where $d_n = d(\xi(x_n))$, $\xi(x_n) = \zeta_n + d_n(1, 0, \cdots, 0)$. Then the functions (u_n, v_n) satisfy the equations

$$\begin{cases} \sum\limits_{i,j=1}^{N} a_{ij}(\zeta_n + d_n y)\frac{\partial^2 u_n}{\partial\xi_i\partial\xi_j} + d_n \sum\limits_{i=1}^{N} b_{ij}(\zeta_n + d_n y)\frac{\partial u_n}{\partial\xi_i} = u_n^p v_n^q + d_n^{\alpha+2}\bar{h}_1(\zeta_n + d_n y), \\ \sum\limits_{i,j=1}^{N} a_{ij}(\zeta_n + d_n y)\frac{\partial^2 v_n}{\partial\xi_i\partial\xi_j} + d_n \sum\limits_{i=1}^{N} b_{ij}(\zeta_n + d_n y)\frac{\partial v_n}{\partial\xi_i} = u_n^r v_n^s + d_n^{\beta+2}\bar{h}_2(\zeta_n + d_n y) \end{cases}$$

On the other hand, Theorem 4.7 implies that, for u_n and v_n in compact subsets of $D := \{y \in \mathbb{R}^N : y_1 > 0\}$, we have $Ay_1^{-\alpha} \le u_n(y) \le By_1^{-\alpha}$, $Ay_1^{-\beta} \le v_n(y) \le By_1^{-\beta}$. Using standard theory we obtain that $u_n \to u_0$, $v_n \to v_0$ in $C^2_{\text{loc}}(D)$, where (u_0, v_0) satisfies

$$\begin{cases} \Delta u_0 = u_0^p v_0^q, & \\ \Delta v_0 = u_0^r v_0^s, & \text{in } D. \\ Ay_1^{-\alpha} \le u_0 \le By_1^{-\alpha}, Ay_1^{-\beta} \le v_0 \le By_1^{-\beta} & , \end{cases}$$

Now, we claim $u_0 = \eta_1 y_1^{-\alpha}$, $v_0 = \eta_2 y_1^{-\beta}$, where

$$\eta_1 = \left(\frac{(\alpha(\alpha+1))^{s-1}}{(\beta(\beta+1))^q}\right)^{\frac{1}{(p-1)(s-1)-qr}}, \eta_2 = \left(\frac{(\beta(\beta+1))^{p-1}}{(\alpha(\alpha+1))^r}\right)^{\frac{1}{(p-1)(s-1)-qr}}. \tag{4.13}$$

Since $\Delta u_0 \ge A^q y_1^{-\beta q} u_0^p + h_1(x)$ in Ω, Lemma 2.5 implies that $u_0 \le B_1 y_1^{-\alpha}$ in Ω, where $B_1 = \left(\frac{\alpha(\alpha+1)}{A^q}\right)^{\frac{1}{p-1}}$. And, since $\Delta v_0 \le B_1^r y_1^{-\alpha r} v_0^s + h_2(x)$ in Ω, Lemma 2.5 implies that $v_0 \ge A_1 y_1^{-\beta}$ in Ω, where $A_1 = \left(\frac{\beta(\beta+1)}{B_1^r}\right)^{\frac{1}{s-1}}$. Iterating this procedure, we obtain $u_0 \le B_n y_1^{-\alpha}$, $v_0 \ge A_n y_1^{-\beta}$ in Ω, where

$$B_{n+1} = \left(\frac{\alpha(\alpha+1)}{A_n^q}\right)^{\frac{1}{p-1}}, A_{n+1} = \left(\frac{\beta(\beta+1)}{B_{n+1}^r}\right)^{\frac{1}{s-1}}.$$

Clearly, if A is small enough, then the sequences $\{A_n\}$ and $\{B_n\}$ are convergent. And, $A_n \to \eta_2$, $B_n \to \eta_1$. Thus, $u_0 \le \eta_1 y_1^{-\alpha}$, $v_0 \ge \eta_2 y_1^{-\beta}$ in Ω. Similarly, we can prove the reversed inequalities.

By setting $y = e_1$ in $u_n \to \eta_1 y_1^{-\beta}$ and $v_n \to \eta_2 y_1^{-\beta}$, and recalling $\xi(x_n) = \zeta_n + d_n e_1$, the Theorem is proved. $\square$

The uniqueness of solutions to system (1.1) can be obtained based on the fact that all positive solutions have the same boundary behavior.

Proof of Theorem 1.6. Let (u_1, v_1) and (u_2, v_2) be two positive solutions to system (1.1). By Theorem 1.5, we have $\frac{u_1}{u_2} = 1$, $\frac{v_1}{v_2} = 1$ on $\partial\Omega$, that is, $\omega := \frac{u_1}{u_2} = 1$ on $\partial\Omega$. Assume that $k := \sup_{\Omega} \omega > 1$. We will prove $k < 1$ to establish a contradiction.

On the one hand, we prove $v_2 < k^{\frac{r}{s-1}} v_1$ in Ω. We argue by contradiction. Assume $\tilde{\Omega} := \{v_2 > k^{\frac{r}{s-1}} v_1\}$ is nonempty. Since $\partial\tilde{\Omega} \subset \Omega$, $k > 1$ and $\frac{v_1}{v_2} = 1$ on $\partial\Omega$, $v_2 = k^{\frac{r}{s-1}} v_1$ on $\partial\tilde{\Omega}$. Then

$$\Delta v_2 = u_2^r v_2^s + h_2(x) \geq k^{-r+\frac{rs}{s-1}} u_1^r v_1^s + h_2(x) \geq k^{\frac{r}{s-1}} u_1^r v_1^s + k^{\frac{r}{s-1}} h_2(x) = \Delta(k^{\frac{r}{s-1}} v_1)$$

in $\tilde{\Omega}$. Using the maximum principle, we have $v_2 \leq k^{\frac{r}{s-1}} v_1$, which is impossible. Then, through the strong maximum principle, we have $v_2 \leq k^{\frac{r}{s-1}} v_1$ in Ω and $v_2 < k^{\frac{r}{s-1}} v_1$.

On the other hand, we conclude from $\omega = 1$ on $\partial\Omega$ that there exists $x_0 \in \Omega$ such that $\omega(x_0) = k$. Then $\Delta\omega(x_0) \leq 0$, that is $u_2\Delta u_1 - u_1\Delta u_2 \leq 0$ at x_0. Therefore

$$v_2^q(x_0) \geq k^{p-1} v_1^q(x_0) + \frac{(1-k)h_1(x_0)}{k u_2^p(x_0)} \geq k^{p-1} v_1^q(x_0),$$

that is $v_2(x_0) \geq k^{\frac{p-1}{q}} v_1(x_0)$.

By the above inequalities, we deduce that $k^{\frac{r}{s-1}} v_1(x_0) > k^{\frac{p-1}{q}} v_1(x_0)$, and so $k^{\frac{(p-1)(s-1)-qr}{q(s-1)}} < 1$. We obtain $k < 1$ by $(p-1)(s-1) > qr$. This is a contradiction. Thus $k \leq 1$ and $u_1 \leq u_2$. Similarly, we can prove $u_1 \geq u_2$. We obtain $u_1 = u_2$. Since

$$\Delta u_1 = u_1^p v_1^q + h_1(x), \Delta u_2 = u_2^p v_2^q + h_1(x),$$

we have $v_1 = v_2$. The uniqueness of solutions is established. □

Acknowledgments

This work is supported by the Zhejiang Provincial Natural Science Foundation of China No. LY20A010010, LY20A010011, the National Natural Science Foundation of China No. 11971251, and K. C. Wong Magna Fund in Ningbo University.

Conflict of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

References

1. M. Chuaqui, C. Cortázar, M. Elgueta, J. García-Melián, Uniqueness and boundary behaviour of large solutions to elliptic problems with singular weights, *Commun. Pure Appl. Anal.*, **3** (2004), 653–662.
2. E. N. Dancer, Y. Du, Effects of certain degeneracies in the predator-prey model, *SIAM J. Math. Anal.*, **34** (2001), 292–314.
3. G. Díaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: Existence and uniqueness, *Nonlinear Anal. Theor.*, **20** (1993), 97–125.

4. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1998.
5. J. López-Gómez, L. Maire, Boundary blow-up rate and uniqueness of the large solution for an elliptic cooperative system of logistic type, *Nonlinear Anal. Real*, **33** (2017), 298–316.
6. J. López-Gómez, *Metasolutions of parabolic equations in population dynamics*, CRC Press, 2016.
7. A. Mohammed, G. Porru, Large solutions to non-divergence structure semilinear elliptic equations with inhomogeneous term, *Adv. Nonlinear Anal.*, **8** (2017), 517–532.
8. A. Mohammed, V. D. Rădulescu, A. Vitolo, Blow-up solutions for fully nonlinear equations: Existence, asymptotic estimates and uniqueness, *Adv. Nonlinear Anal.*, **9** (2018), 39–64.
9. C. Mu, S. Huang, Q. Tian, L. Liu, Large solutions for an elliptic system of competitive type: Existence, uniqueness and asymptotic behavior, *Nonlinear Anal. Theor.*, **71** (2009), 4544–4552.
10. J. García-Melián, Large solutions for an elliptic equation with a nonhomogeneous term, *J. Math. Anal. Appl.*, **434** (2016), 872–881.
11. J. García-Melián, J. S. De Lis, R. Letelier-Albornoz, The solvability of an elliptic system under a singular boundary condition, *P. Roy. Soc. A-Math. phy.*, **136** (2006), 509–546.
12. J. García-Melián, J. D. Rossi, Boundary blow-up solutions to elliptic systems of competitive type, *J. Differ. Equations*, **206** (2004), 156–181.
13. J. García-Melián, J. D. Rossi, J. S. De Lis, Elliptic systems with boundary blow-up: Existence, uniqueness and applications to removability of singularities, *Commun. Pure Appl. Math.*, **15** 2016, 549–562.
14. C. V. Pao, *Nonlinear parabolic and elliptic equations*, Springer, 1992.
15. S. Qi, P. Zhao, Large solutions for a nonhomogeneous quasilinear elliptic problem, *Nonlinear Anal. Real*, **41** (2018), 428–442.
16. L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, *J. Anal. Math.*, **59** (1992), 231–250.
17. Y. Wang, Explosive solutions of elliptic equation with singular weight, *Appl. Anal.*, **100** (2021), 1744–1751.
18. Z. Wang, F. Ma, Boundary blow-up solutions for elliptic systems with nonhomogeneous term, *Journal of Ningbo University (Natural Science & Engineering Edition)*, **33** (2020), 69–71.
19. Z. Wang, F. Ma, W. Wo, Large solutions of a semilinear elliptic equation with singular weights and nonhomogeneous term, *Arch. Math.*, **115** (2020), 99–110.

AIMS Press