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Research article

New generalized conformable fractional impulsive delay differential equations with some illustrative examples

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Abstract: This article is concerned to develop a novel generalized class of fractional impulsive delay differential equations. These equations are defined using newly discovered generalized conformable fractional operators which unify various previously-defined operators into a single form. The successive approximation method is employed and a sufficient criterion for the existence and uniqueness of the solution is developed. For the sake of illustration, three examples are provided at the end of the main results.

Keywords: fractional impulsive delay differential equations; conformable operators; successive approximation method; Banach spaces
Mathematics Subject Classification: 26D15

1. Introduction

Fractional Differential Equations (FDE) are of immense significance as they are great contributors to research fields of applied sciences [1]. They have gained substantial popularity and importance due to their attractive applications in extensive areas of science and engineering [2, 3]. In addition to this, Impulsive FDE (I-FDE) have also played an influential role in describing phenomena, particularly in modeling dynamics of populations subject to abrupt changes [4, 5]. They provide a realistic framework of modeling systems in fields like control theory, population dynamics, biology, physics, and medicine [6, 7]. Similarly, Delay Differential Equations (DDE) are significant because they have the ability to describe processes with retarded time. The importance of DDE in various sciences like biology, physics, economics, medical science, and social sciences has been acknowledged [8, 9]. When the above-mentioned classes of equations come to a single platform, and when they are studied combined, they are then called Impulsive Delay FDE (ID-FDE). Such type of equations have been getting worthwhile attention from researchers in the present age. For the theory of ID-FDE and recent development on this topic, one can see [10–22] and the references therein.
Recently, Khan *et al.* have defined fractional integral and derivative operators [23]. Unlike other fractional operators, they satisfy properties like continuity, boundedness, linearity and unify some previously-presented operators into a single form. They are defined as under:

Definition 1 ([23]). Let ϕ be a function that is conformable integrable on the interval $[p, q] \subseteq [0, \infty)$. The left-sided and right-sided Generalized Conformable Fractional (GCF) integral operators ${}^{\sigma}_{\theta}K^{\nu}_{p^+}$ and ${}^{\sigma}_{\theta}K^{\nu}_{q^-}$ of order $\nu > 0$ with $\theta \in (0, 1]$, $\sigma \in \mathbb{R}$, and $\theta + \sigma \neq 0$ are defined by:

$$ {}^{\sigma}_{\theta}K^{\nu}_{p^+}\phi(\tau) = \frac{1}{\Gamma(\nu)} \int_{p}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} \phi(w) w^{\sigma} d_{\theta}w, \quad \tau > p, \tag{1.1} $$

and

$$ {}^{\sigma}_{\theta}K^{\nu}_{q^-}\phi(\tau) = \frac{1}{\Gamma(\nu)} \int_{\tau}^{q} \left(\frac{w^{\sigma+\theta} - \tau^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} \phi(w) w^{\sigma} d_{\theta}w, \quad q > \tau, \tag{1.2} $$

respectively, and ${}^{\sigma}_{\theta}K^{0}_{p^+}\phi(\tau) = {}^{\sigma}_{\theta}K^{0}_{q^-}\phi(\tau) = \phi(\tau)$.

The integral $\int_{p}^{q} d_{\theta}w$, in the Definition 1, represents the conformable integration defined as [24]:

$$\int_p^q \phi(w)d_\theta w := \int_p^q \phi(w)w^{\theta-1}dw. \tag{1.3}$$

The associated left- and right-sided GCF derivative operators are defined as follows [23]:

Definition 2. Let ϕ be a function that is conformable integrable on the interval $[p,q] \subseteq [0,\infty)$ such that $\theta \in (0,1]$, $\sigma \in \mathbb{R}$ and $\theta + \sigma \neq 0$. The left- and right-sided GCF derivative operators ${}^{\sigma}_{\theta}T^{\nu}_{p^+}$ and ${}^{\sigma}_{\theta}T^{\nu}_{q^-}$ of order $\nu \in (0,1)$ are defined, respectively, by:

$${}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(\tau) = \frac{\tau^{-\sigma}}{\Gamma(1-\nu)}T_\theta \int_p^\tau \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{-\nu} \phi(w)w^{\sigma}d_\theta w, \quad \tau > p, \tag{1.4}$$

$${}^{\sigma}_{\theta}T^{\nu}_{q^-}\phi(\tau) = \frac{\tau^{-\sigma}}{\Gamma(1-\nu)}T_\theta \int_\tau^q \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{-\nu} \phi(w)w^{\sigma}d_\theta w, \quad q > \tau, \tag{1.5}$$

and ${}^{\sigma}_{\theta}T^{0}_{p^+}\phi(\tau) = {}^{\sigma}_{\theta}T^{0}_{q^-}\phi(\tau) = \phi(\tau)$. Here T_θ represents the θth-order conformable derivative with respect to τ, and it is defined as in the following definition.

Definition 3 ([24])**.** Consider the real-valued function ϕ defined on the interval $[0,\infty)$. The θth-order conformable derivative T_θ of the function ϕ, where $\theta \in (0,1]$, is defined as:

$$T_\theta\phi(w) = \begin{cases} \lim\limits_{\epsilon\to 0} \frac{\phi(w+\epsilon w^{1-\theta})-\phi(w)}{\epsilon}, & w \in (0,\infty); \\ \lim\limits_{w\to 0^+} T_\theta\phi(w), & w = 0. \end{cases} \tag{1.6}$$

The relation between ordinary derivative $\phi'(w)$ and the conformable derivative $T_\theta\phi(w)$, is given as follows [24]:

$$T_\theta\phi(w) = w^{1-\theta}\phi'(w). \tag{1.7}$$

The conformable operators have gained a considerable attention of many researchers in a very short span of time. Due to their classical properties, they have been used in various fields, for example one can see [25–36] and the references therein. The following Remark 1 highlights that how the GCF operators unify various early-defined operators. Here, the left-sided operators are only taking into account. A similar methodology can be carried out also for the right-sided operator.

Remark 1. **(1)** The following well-known Katugampula fractional derivative operator is obtained when $\theta = 1$ is put in the Definition 2 [37]:

$${}^{\sigma}_{1}T^{\nu}_{p^+}\phi(\tau) = \frac{\tau^{-\sigma}}{\Gamma(1-\nu)}T_1 \int_p^\tau \left(\frac{\tau^{\sigma+1} - w^{\sigma+1}}{\sigma+1}\right)^{-\nu} \phi(w)w^{\sigma}dw, \quad \tau > p. \tag{1.8}$$

(2) For $\sigma = 0$ in the Definition 2, the New Riemann-Liouville type conformable fractional derivative operator is obtained as given below [23]:

$$ {}^{0}_{\theta}T^{\nu}_{p^+}\phi(\tau) = \frac{1}{\Gamma(1-\nu)} T_\theta \int_p^\tau \left(\frac{\tau^\theta - w^\theta}{\theta} \right)^{\nu-1} \phi(w) d_\theta w, \quad \tau > p. \tag{1.9} $$

(3) Using the definition of conformable integral given in the Eq (1.3) and the L'Hospital rule, it is straightforward that when $\theta \to 0$ in Eq (1.9), we get the Hadamard fractional derivative operator as under [23]:

$$ {}^{0}_{0^+}T^{\nu}_{p^+}\phi(\tau) = \frac{1}{\Gamma(1-\nu)} T_{0^+} \int_p^\tau \left(\log \frac{\tau}{w} \right)^{\nu-1} \phi(w) \frac{dw}{w}, \quad \tau > p. \tag{1.10} $$

(4) For $\theta = 1$ in Eq (1.9), the well-known Riemann-Liouville fractional derivative operator is obtained as under [23]:

$$ {}^{0}_{1}T^{\nu}_{p^+}\phi(\tau) = \frac{1}{\Gamma(1-\nu)} T_1 \int_p^\tau (\tau - w)^{\nu-1} \phi(w) dw, \quad \tau > p. \tag{1.11} $$

(5) For the case $\nu = 1, \sigma = 0$ in Definition 2, we get the conformable fractional derivatives. And when $\theta = \nu = 1$, $\sigma = 0$ we get ordinary derivatives [23].

The inverse property of the newly introduced GCF derivative operators is given below, which will be used in the proofs of our results.

Theorem 1 ([23]). *Let $\sigma \in \mathbb{R}$, $\theta \in (0, 1]$ such that $\sigma + \theta \neq 0$ and $\nu \in (0, 1)$. For any continuous function $\phi : [p, q] \subseteq [0, \infty) \to \mathbb{R}$, in the domain of ${}^{\sigma}_{\theta}K^{\nu}_{p^+}$ and ${}^{\sigma}_{\theta}K^{\nu}_{q^-}$ we have:*

$$ {}^{\sigma}_{\theta}T^{\nu}_{p^+} {}^{\sigma}_{\theta}K^{\nu}_{p^+}\phi(r) = \phi(r), \tag{1.12} $$

Similarly

$$ {}^{\sigma}_{\theta}T^{\nu}_{q^-} {}^{\sigma}_{\theta}K^{\nu}_{q^-}\phi(r) = \phi(r). \tag{1.13} $$

One of the fundamental theorems in mathematical analysis, in the theory of double integrals, is Fubini's theorem. This theorem allows the order of integration to be changed in certain iterated integrals. This is stated as follows [38]:

Theorem 2 (**Fubini's Theorem**). *Let $\phi : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a continuous function on $S := [p_1, q_1] \times [p_2, q_2]$. Then*

$$\int_{p_1}^{q_1}\int_{p_2}^{q_2} \phi(r,s)drds = \int_{p_2}^{q_2}\int_{p_1}^{q_1} \phi(r,s)dsdr, \tag{1.14}$$

where $s \in [p_1, q_1]$, $r \in [p_2, q_2]$.

The scope and novelty of the present paper is that it addresses a new class of generalized fractional impulsive delay differential equations. This class is defined using newly introduced GCF operators which are the generalizations of fractional operators of the types Katugampola, Riemann-Lioville, Hadamard, Riemann-Lioville's type, conformable and ordinary or classical operators [23]. That is, while considering the generalized problem containing GCF operators, we work with various (above-mentioned) operators at the same time. Therefore, the paper combines various previously defined operators (or work) into a single form and is expected to provide a unique platform for the researchers working with different operators in this field. Moreover, since researchers commonly face the problem of choosing a convenient approach or a suitable operator to solve a problem, thus this kind of study, in which one can work with several operators at a time, is helpful in this regard.

2. Main results

In the present work, we start by stating the GCFDE with delay and impulse terms as under:

$$\begin{cases} {}^{\sigma}_{\theta}T^{\nu}_{\tau_k^+}\phi(\tau) = f(\tau, \phi_\tau), \ k = 0, 1, 2, ...m, \ \tau \in \aleph'; \\ \Delta\phi(\tau_k) = I_k(\phi(\tau_k)), \quad k = 1, 2, 3...m; \\ \phi(\tau) = \psi(\tau), \ \ \tau \in [-\omega, 0], \end{cases} \tag{2.1}$$

where ${}^{\sigma}_{\theta}T^{\nu}_{\tau_k^+}$ is the GCF derivative of order $\nu \in (0, 1)$, $\theta \in (0, 1]$ and $\sigma \in \mathbb{R}$ where $\sigma + \theta \neq 0$, ω is a non-negative real number and $0 = \tau_0 < \tau_1 < \tau_2 ... < \tau_{m+1} = \mathtt{T}$. Also $f : \aleph' \times \mathbb{R} \to \mathbb{R}$ (i.e $f \in \mathbb{C}(\aleph' \times \mathbb{R}, \mathbb{R})$), where $\aleph' := [0, \mathtt{T}] \setminus \{\tau_1, \tau_2, ...\tau_m\}$. Moreover, we fix $\aleph_0 = [\tau_0, \tau_1]$ and $\aleph_k = (\tau_k, \tau_k]$ for $k = 1, 2, 3...m$. Further, $I_k : \mathbb{R} \to \mathbb{R}$, $\Delta\phi(\tau_k) = \phi(\tau_k^+) - \phi(\tau_k^-)$, where $\phi(\tau_k^-)$ and $\phi(\tau_k^+)$ denotes the left and right hand limits of the function ϕ at the point τ_k respectively such that $\phi(\tau_k) = \phi(\tau_k^-)$. If $\phi : [-\omega, \mathtt{T}] \to \mathbb{R}$, then for any $\tau \in \aleph := [0, \mathtt{T}]$, define ϕ_τ by $\phi_\tau(\eta) = \phi(\tau + \eta)$, where $\eta \in [-\omega, 0]$. Also $\psi : [-\omega, 0] \to \mathbb{R}$ is such that $\psi(0) = 0$.

Also we define $\mathtt{BC(A)} = \{\phi : \mathtt{A} := [-\omega, \mathtt{T}] \to \mathbb{R}\}$, it is easy to show that $\mathtt{BC(A)}$ is a Banach space with the norm defined by $\|\phi(\tau)\| = \sup_{\tau\in\mathtt{A}} |\phi(\tau)|$.

A motivation to study the system in Eq (2.1), as compare to other systems in the literature, is that it contains fractional operators having such properties which are not satisfied by those obtained earlier [24]. These operators are simple, friendly (while dealing with them) and have properties analogous to ordinary derivative and integral operators [23]. Moreover, there are some classes of differential equations which cannot be solved easily using previous definitions of fractional derivatives. An example of such type of equation has been given by Khalil *et. al.*, in the paper [24]. Therefore, to establish results related to the solution of such type of problems, we have chosen a generalized problem in the form of Eq (2.1), which covers equations of the type given in [24] as well others in the literature.

To establish our main results, first we need to prove the following lemma.

Lemma 1. *Let* $\phi : [p,q] \subseteq [0,\infty) \to \mathbb{R}$ *be a conformable integrable function. Then for* $0 < \nu < 1$*,* $\theta \in (0,1]$*,* $\sigma \in \mathbb{R}$ *such that* $\sigma + \theta \neq 0$*, we have:*

$$ {}^{\sigma}_{\theta}K^{\nu}_{p^+}\,{}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(r) = \phi(r) - \phi(p), \quad r \in [p,q]. \tag{2.2}$$

Similarly

$$ {}^{\sigma}_{\theta}K^{\nu}_{q^-}\,{}^{\sigma}_{\theta}T^{\nu}_{q^-}\phi(r) = \phi(q) - \phi(r), \quad r \in [p,q]. \tag{2.3}$$

Proof. First using definition of the integral operator ${}^{\sigma}_{\theta}K^{\nu}_{p^+}$ (Eq (1.1)), then of the derivative operator ${}^{\sigma}_{\theta}T^{\nu}_{p^+}$ (Eq (1.4)), and then definition of conformable derivative and integral operators (Eq (1.7) and Eq (1.3)) in sequence, we have:

$$\begin{aligned}
&{}^{\sigma}_{\theta}K^{\nu}_{p^+}\,{}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(r)\\
&= \frac{1}{\Gamma(\nu)}\int_p^r \left(\frac{r^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} {}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(w) w^{\sigma} d_{\theta}w\\
&= \frac{1}{\Gamma(1-\nu)\Gamma(\nu)}\int_p^r \left(\frac{r^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} T_{\theta}\int_p^w \left(\frac{w^{\sigma+\theta} - s^{\sigma+\theta}}{\sigma+\theta}\right)^{-\nu} \phi(s) s^{\sigma} d_{\theta}s d_{\theta}w\\
&= \frac{1}{\Gamma(1-\nu)\Gamma(\nu)}\int_p^r \left(\frac{r^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} \frac{d}{dw}\int_p^w \left(\frac{w^{\sigma+\theta} - s^{\sigma+\theta}}{\sigma+\theta}\right)^{-\nu} \phi(s) s^{\sigma+\theta-1} ds dw\\
&= \frac{\sigma+\theta}{\Gamma(1-\nu)\Gamma(\nu)}\int_p^r \left(r^{\sigma+\theta} - w^{\sigma+\theta}\right)^{\nu-1} \mathbf{I}(w)\, dw, \qquad (2.4)
\end{aligned}$$

where

$$\mathbf{I}(w) = \frac{d}{dw}\int_p^w \left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu} s^{\sigma+\theta-1}\phi(s)ds.$$

To find the value of $\mathbf{I}(w)$, Let $\frac{du}{ds} = u'(s) = s^{\sigma+\theta-1}\left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu}$, $v(s) = \phi(s)$. Using integration by parts formula, we get:

$$\begin{aligned}\int_p^w u'(s)v(s)ds &= -u(p)v(p) - \int_p^w u(s)v'(s)ds \\ &= -u(p)v(p) + \int_p^w \frac{\left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu+1}}{(\sigma+\theta)(-\nu+1)}\phi'(s)ds.\end{aligned} \tag{2.5}$$

This means that:

$$\mathbf{I}(w) = \frac{d}{dw}\int_p^w u'(s)v(s)ds = \frac{d}{dw}\int_p^w \frac{\left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu+1}}{(\sigma+\theta)(-\nu+1)}\phi'(s)ds. \tag{2.6}$$

Thanks to Lebnitz rule of differentiating integral:

$$\frac{\partial}{\partial w}\int_p^w \phi(w,s)ds = \phi(w,w) + \int_p^w \frac{\partial}{\partial w}\phi(w,s)ds.$$

We have from Eq (2.6):

$$\mathbf{I}(w) = \int_p^w \left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu} w^{\sigma+\theta-1}\phi'(s)ds. \tag{2.7}$$

Putting value of $\mathbf{I}(w)$ in Eq (2.4), we get:

$$\begin{aligned}&{}^{\sigma}_{\theta}K^{\nu}_{p^+}\,{}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(r) \\ &= \frac{\sigma+\theta}{\Gamma(1-\nu)\Gamma(\nu)}\int_p^r\int_p^w \left(r^{\sigma+\theta} - w^{\sigma+\theta}\right)^{\nu-1}\left(w^{\sigma+\theta} - s^{\sigma+\theta}\right)^{-\nu} w^{\sigma+\theta-1}\phi'(s)ds\,dw.\end{aligned} \tag{2.8}$$

Switching the order of integration (using Fubini's Theorem) and changing variables to u by defining $w^{\sigma+\theta} = s^{\sigma+\theta} + (r^{\sigma+\theta} - s^{\sigma+\theta})u$, we have:

$$\begin{aligned}
{}^{\sigma}_{\theta}K^{\nu}_{p^+}{}^{\sigma}_{\theta}T^{\nu}_{p^+}\phi(r) &= \frac{1}{\Gamma(1-\nu)\Gamma(\nu)}\int_p^r\int_0^1 u^{-\nu}(1-u)^{\nu-1}\phi'(s)du d_\theta s \\
&= \int_p^r \phi'(s)ds \\
&= \phi(r) - \phi(p), \qquad (2.9)
\end{aligned}$$

where in the last step fundamental theorem of calculus has been used and in the second last step definition of Euler Beta function **B** and its relation with Gamma function have been used as under:

$$\int_0^1 u^{-\nu}(1-u)^{\nu-1}du = \mathbf{B}(\nu,\ 1-\nu) = \Gamma(\nu)\Gamma(1-\nu). \qquad (2.10)$$

The proof of the Eq (2.3) is same to the procedure developed for the proof of Eq (2.2). It can easily be obtained by first applying the definition of ${}^{\sigma}_{\theta}K^{\nu}_{q^-}$ and then of ${}^{\sigma}_{\theta}T^{\nu}_{q^-}$. The rest of the process is same as above. This completes our proof. □

To proceed further, we need to prove another lemma, which transforms our proposed generalized problem to an integral equation as under:

Lemma 2. *Let $f \in \mathbb{C}(\aleph', \mathbb{R})$ and ${}^{\sigma}_{\theta}T^{\nu}_{\tau_k^+}\phi(\tau)$ denotes the νth-order GCF derivative of the function $\phi \in \mathtt{BC(A)}$. Then ϕ is a solution of the problem:*

$$\begin{cases} {}^{\sigma}_{\theta}T^{\nu}_{\tau_k^+}\phi(\tau) = f(\tau), \ \ k = 0,1,2,...m, \ \ \tau \in \aleph'; \\ \Delta\phi(\tau_k) = I_k(\phi(\tau_k)), \ \ k = 1,2,3...m; \\ \phi(\tau) = \psi(\tau), \ \ \tau \in [-\omega, 0], \end{cases} \qquad (2.11)$$

if and only if ϕ satisfies the following integral equation:

$$\phi(\tau) = \begin{cases} \psi(\tau), \ \ \tau \in [-\omega, 0]; \\ \frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta})^{\nu-1}f(w)w^{\sigma}d_\theta w + \sum_{j=1}^{k} I_j\left(\phi(\tau_j)\right) \\ + \sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta})^{\nu-1}f(w)w^{\sigma}d_\theta w, \ \ \tau \in \aleph_k, \ k = 0,1,2,...m. \end{cases} \qquad (2.12)$$

Proof. Firstly, suppose that ϕ satisfies Eq (2.11). Then for $\tau \in [-\omega, 0]$ the result follows directly. For $k = 0$, we have from Eq (2.11) (taking into account $\tau_0 = 0$) that:

$$ {}^{\sigma}_{\theta}T^{\nu}_{0^+}\phi(\tau) = f(\tau), \ \ \tau \in \aleph_0 = [\tau_0, \tau_1]. \tag{2.13}$$

Now applying the operator ${}^{\sigma}_{\theta}K^{\nu}_{0^+}$ from the left to both sides of the Eq (2.13) and using Eq (2.2) (keeping in mind that $\phi(0) = 0$) we have:

$$\phi(\tau) = \frac{1}{\Gamma(\nu)} \int_0^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w, \ \ \tau \in \aleph_0, \tag{2.14}$$

which is Eq (2.12) for $k = 0$.

Now when $k = 1$ in Eq (2.11), we have $I_1(\phi(\tau_1)) = \phi(\tau_1^+) - \phi(\tau_1^-)$, using Eq (2.14) and also we know that $\phi(\tau_1) = \phi(\tau_1^-)$, thus:

$$\phi(\tau_1^+) = I_1(\phi(\tau_1)) + \frac{1}{\Gamma(\nu)} \int_0^{\tau_1} \left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w. \tag{2.15}$$

Also we have

$$ {}^{\sigma}_{\theta}T^{\nu}_{\tau_1^+}\phi(\tau) = f(\tau), \ \ \tau \in (\tau_1, \tau_2]. \tag{2.16}$$

Applying the operator ${}^{\sigma}_{\theta}K^{\nu}_{\tau_1^+}$ to both sides of the Eq (2.16), using Eq (2.2) and then putting values from Eq (2.15), we get for $\tau \in \aleph_1 = (\tau_1, \tau_2]$:

$$\begin{aligned} \phi(\tau) =& \phi(\tau_1^+) + \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w \\ =& I_1(\phi(\tau_1)) + \frac{1}{\Gamma(\nu)} \int_0^{\tau_1} \left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w \\ &+ \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w, \end{aligned} \tag{2.17}$$

which is Eq (2.12) for $k = 1$.

Similarly for $k = 2$, we have from Eq (2.11), $\phi(\tau_2^+) = \phi(\tau_2^-) + I_2(\phi(\tau_2)) = \phi(\tau_2) +$

$I_2(\phi(\tau_2))$. Using Eq (2.17):

$$\begin{aligned}\phi(\tau_2^+) =&\phi(\tau_2) + I_2(\phi(\tau_2))\\ =&I_1(\phi(\tau_1)) + \frac{1}{\Gamma(\nu)}\int_0^{\tau_1}\left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w\\ &+ \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\frac{\tau_2^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w + I_2(\phi(\tau_2))\\ =&\sum_{j=1}^{2} I_j\left(\phi(\tau_j)\right) + \sum_{i=0}^{1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w.\end{aligned} \tag{2.18}$$

Also from Eq (2.11):

$$ {}^{\sigma}_{\theta}T^{\nu}_{\tau_2^+}\phi(\tau) = f(\tau),\ \tau \in (\tau_2, \tau_3]. \tag{2.19}$$

Applying the operator ${}^{\sigma}_{\theta}K^{\nu}_{\tau_2^+}$ to both sides of the Eq (2.19), using Eq (2.2) and then using Eq (2.18) we have for $\tau \in \aleph_2 = (\tau_2, \tau_3]$:

$$\begin{aligned}\phi(\tau) =&{}^{\sigma}_{\theta}K^{\nu}_{\tau_2^+}f(\tau) + \phi(\tau_2^+)\\ =&\frac{1}{\Gamma(\nu)}\int_{\tau_2}^{\tau}\left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w + \sum_{j=1}^{2} I_j\left(\phi(\tau_j)\right)\\ &+ \sum_{i=0}^{1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w,\end{aligned} \tag{2.20}$$

which is Eq (2.12) for $k = 2$.

Continuing in the same way, the solution $\phi(\tau)$ for $\tau \in \aleph_k$, $k = 0, 1, 2, 3...m$, can be generally written as:

$$\begin{aligned}\phi(\tau) =&\frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w + \sum_{j=1}^{k} I_j\left(\phi(\tau_j)\right)\\ &+ \sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w)w^{\sigma}d_\theta w.\end{aligned} \tag{2.21}$$

Conversely, if ϕ satisfies Eq (2.12), the proof is easy and it can be obtained by direct computations. Suppose for $\tau \in [-\omega, 0]$, once again the result follows directly. Now we check for $\tau \in \aleph_k$. For this when $k = 0$, the Eq (2.12) implies that for $\tau \in \aleph_0 = [\tau_0, \tau_1]$:

$$\begin{aligned}\phi(\tau) =& \frac{1}{\Gamma(\nu)} \int\limits_{\tau_0=0}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w \\ =& {}_{\theta}^{\sigma} K_{\tau_0^+}^{\nu} f(\tau).\end{aligned} \tag{2.22}$$

Applying the operator ${}_{\theta}^{\sigma} T_{\tau_0^+}^{\nu}$ to both sides of Eq (2.22) using Eq (1.12) we have:

$$ {}_{\theta}^{\sigma} T_{\tau_0^+}^{\nu} \phi(\tau) = f(\tau), \quad \tau \in [\tau_0, \tau_1]. \tag{2.23}$$

Similarly when $k = 1$, we have from Eq (2.12):

$$\begin{aligned}\phi(\tau) =& \frac{1}{\Gamma(\nu)} \int\limits_{\tau_1}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w + I_1(\phi(\tau_1)) \\ &+ \frac{1}{\Gamma(\nu)} \int\limits_{\tau_0}^{\tau_1} \left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w, \quad \tau \in (\tau_1, \tau_2].\end{aligned} \tag{2.24}$$

Applying the operator ${}_{\theta}^{\sigma} T_{\tau_1^+}^{\nu}$ to both sides of Eq (2.24) using the Eq (1.12) we have:

$$ {}_{\theta}^{\sigma} T_{\tau_1^+}^{\nu} \phi(\tau) = f(\tau), \quad \tau \in (\tau_1, \tau_2]. \tag{2.25}$$

Since $\phi(\tau_1^+)$ denotes right hand limit of the function ϕ at the point τ_1, thus from the Eq (2.24), we get:

$$\phi(\tau_1^+) = I_1(\phi(\tau_1)) + \frac{1}{\Gamma(\nu)} \int\limits_{\tau_0}^{\tau_1} \left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w. \tag{2.26}$$

Also for $\tau = \tau_1$, we can write from Eq (2.22):

$$\phi(\tau_1^-) = \phi(\tau_1) = \frac{1}{\Gamma(\nu)} \int\limits_{\tau_0=0}^{\tau_1} \left(\frac{\tau_1^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma + \theta} \right)^{\nu-1} f(w) w^{\sigma} d_{\theta} w. \tag{2.27}$$

Combining Eq (2.27) and Eq (2.26):

$$I_1(\phi(\tau_1)) = \phi(\tau_1^+) - \phi(\tau_1^-). \tag{2.28}$$

Similarly for $k = 2, 3, 4 ... m$, the procedure is same as above. This completes the proof. □

2.1. Existence and Uniqueness of the Solution

We set $S_0 = \{z/z \in \mathbb{C}(\aleph, \mathbb{R}), z(0) = 0\}$. For each $z \in S_0$, we define the function $\bar{z}$ by:

$$\bar{z}(\tau) = \begin{cases} z(\tau), \ \tau \in \aleph; \\ 0, \ \tau \in [-\omega, 0]. \end{cases} \tag{2.29}$$

If ϕ is a solution of Eq (2.1), then $\phi(.)$ can be decomposed as $\phi(\tau) = \bar{z}(\tau) + g(\tau)$ for $\tau \in [-\omega, \mathrm{T}]$, which implies that $\phi_\tau = \bar{z}_\tau + g_\tau$, for $\tau \in [0, \mathrm{T}]$, where:

$$g(\tau) = \begin{cases} 0, \ \tau \in \aleph; \\ \psi(\tau), \ \tau \in [-\omega, 0]. \end{cases} \tag{2.30}$$

Therefore taking into account the above Lemma 2 and the definition of ϕ_τ, we may say that the problem in Eq (2.1) can be transformed into the following fixed point operator, $\Theta = S_0 \to \mathbb{R}$,

$$\begin{aligned} \Theta z(\tau) =& \frac{1}{\Gamma(\nu)} \int_{\tau_k}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} f(w, \bar{z}_w + g_w) w^\sigma d_\theta w + \sum_{j=1}^{k} I_j \left(\bar{z}(\tau_j) \right) \\ &+ \sum_{i=0}^{k-1} \frac{1}{\Gamma(\nu)} \int_{\tau_i}^{\tau_{i+1}} \left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} f(w, \bar{z}_w + g_w) w^\sigma d_\theta w, \end{aligned} \tag{2.31}$$

where $\tau \in \aleph_k$, $k = 0, 1, 2, 3...m$.

In the following theorem we prove our main result.

Theorem 3. *Consider the functions $f : \aleph \times \mathbb{R} \to \mathbb{R}$ and $I_k : \mathbb{R} \to \mathbb{R}$, and let the following conditions hold:*
(1) *There exists a continuous function, $h : [0, T] \to \mathbb{R}^+$, such that*

$$|f(\tau, a_\tau) - f(\tau, b_\tau)| \le h(\tau) \sup_{w \in [0,\tau]} |a(w) - b(w)|, \ a, b \in \mathbb{R}, \ \tau \in [0, T]. \tag{2.32}$$

(2) *There exists a constant $M_k > 0$, such that*

$$\begin{gathered} |I_k(a) - I_k(b)| \le M_k |a - b|, \ k = 1, 2...m; \\ \sum_{i=1}^{m+1} \frac{T^{\nu(\sigma+\theta)} h_i}{(\sigma+\theta)^\nu \Gamma(\nu+1)} + \sum_{j=1}^{m} M_j < 1, \quad h_k = \sup_{\tau \in [0,T]} h(\tau). \end{gathered} \tag{2.33}$$

(3) *There exists a constant $L > 0$, such that $|f(\tau, g_\tau)| \le L$, where g is defined as in Eq. (2.30).*

Then there exist a unique solution of the Eq (2.1) in the set S_0.

Proof. Using the method of successive approximations, we define a sequence of functions $z_n : [0, \mathtt{T}] \to \mathbb{R}$, $n = 0, 1, 2...$ as follows:

$$z_0(\tau) = 0,\ z_n(\tau) = \Theta z_{n-1}(\tau). \tag{2.34}$$

Since $z_0(\tau) = 0$, we can say from Eq (2.29) that $\overline{z_0}(w) = z_0(w) = 0$, $w \in \aleph = [0, \mathtt{T}]$, then we have:

$$\begin{aligned}
|z_1(\tau) - z_0(\tau)| =& |\Theta z_0(\tau) - z_0(\tau)| \\
\leq& \frac{1}{\Gamma(\nu)} \int_{\tau_k}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} |f(w, g(w))| w^{\sigma} d_\theta w + \sum_{j=1}^{k} |I_j(0)| \\
&+ \sum_{i=0}^{k-1} \frac{1}{\Gamma(\nu)} \int_{\tau_i}^{\tau_{i+1}} \left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} |f(w, g(w))| w^{\sigma} d_\theta w \\
\leq& \frac{L(\tau^{\sigma+\theta} - \tau_k^{\sigma+\theta})^{\nu}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{i=1}^{k} \frac{L(\tau_i^{\sigma+\theta} - \tau_{i-1}^{\sigma+\theta})^{\nu}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{j=1}^{k} |I_j(0)| \\
\leq& \sum_{i=1}^{k+1} \frac{L(\tau_i^{\sigma+\theta} - \tau_{i-1}^{\sigma+\theta})^{\nu}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{j=1}^{k} |I_j(0)| := \Theta_0.
\end{aligned} \tag{2.35}$$

Moreover

$$\begin{aligned}
&|z_n(\tau) - z_{n-1}(\tau)| \\
=& |\Theta z_{n-1}(\tau) - \Theta z_{n-2}(\tau)| \\
\leq& \frac{1}{\Gamma(\nu)} \int_{\tau_k}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} |f(w, (\overline{z}_{n-1})_w + g_w) - f(w, (\overline{z}_{n-2})_w + g_w)| w^{\sigma} d_\theta w \\
&+ \sum_{i=0}^{k-1} \frac{1}{\Gamma(\nu)} \int_{\tau_i}^{\tau_{i+1}} \left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} |f(w, (\overline{z}_{n-1})_w + g_w) - f(w, (\overline{z}_{n-2})_w + g_w)| w^{\sigma} d_\theta w \\
&+ \sum_{j=1}^{k} |I_j(\overline{z}_{n-1})(\tau_j) - I_j(\overline{z}_{n-2})(\tau_j)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\nu)} \int_{\tau_k}^{\tau} \left(\frac{\tau^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} h(w) \sup_{x\in[0,w]} |\bar{z}_{n-1}(x) - \bar{z}_{n-2}(x)| w^{\sigma} d_{\theta}w \\
&+ \sum_{i=0}^{k-1} \frac{1}{\Gamma(\nu)} \int_{\tau_i}^{\tau_{i+1}} \left(\frac{\tau_{i+1}^{\sigma+\theta} - w^{\sigma+\theta}}{\sigma+\theta} \right)^{\nu-1} h(w) \sup_{x\in[0,w]} |\bar{z}_{n-1}(x) - \bar{z}_{n-2}(x)| w^{\sigma} d_{\theta}w \\
&+ \sum_{j=1}^{k} |I_j(\bar{z}_{n-1})(\tau_j) - I_j(\bar{z}_{n-2})(\tau_j)| \qquad (2.36) \\
&\leq \left(\frac{h_k(\tau^{\sigma+\theta} - \tau_k^{\sigma+\theta})^{\nu}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{i=1}^{k} \frac{h_i(\tau_i^{\sigma+\theta} - \tau_{i-1}^{\sigma+\theta})^{\nu}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{j=1}^{k} M_j \right) \|z_{n-1} - z_{n-2}\| \\
&\leq \left(\sum_{i=1}^{m+1} \frac{\mathrm{T}^{\nu(\sigma+\theta)} h_i}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)} + \sum_{j=1}^{m} M_j \right) \|z_{n-1} - z_{n-2}\| \\
&:= \Theta_1 \|z_{n-1} - z_{n-2}\|,
\end{aligned}
$$

which shows that $\|z_n - z_{n-1}\| \leq \Theta_1 \|z_{n-1} - z_{n-2}\|$, with $\Theta_1 < 1$. It can be seen that for any $0 < n < t$, we have:

$$
\begin{aligned}
\|z_t - z_n\| &\leq \|z_{n+1} - z_n\| + \|z_{n+2} - z_{n+1}\| + \|z_{n+3} - z_{n+2}\| ... + \|z_t - z_{t-1}\| \\
&\leq \left(\Theta_1^n + \Theta_1^{n+1} + \Theta_1^{n+2} ... + \Theta_1^{t-1} \right) \|z_1 - z_0\| \qquad (2.37) \\
&= \frac{\Theta_1^n}{1 - \Theta_1} \|z_1 - z_0\|.
\end{aligned}
$$

For large values of n, t, when $n \to \infty$, then from above inequality Eq (2.37), it is clear that $\|z_t - z_n\| \to 0$. This implies that z_n is a cauchy sequence in the Banach space $\mathtt{BC}(\aleph)$. By definition of the Banach space, since it is a complete normed linear space, where every cauchy sequence converges to a limit in it (in our case say z) so $\|z_n - z\| \to 0$, as $n \to \infty$. Which shows that $z_n(\tau)$ is uniformly convergent to $z(\tau)$.

Next we will show that $z(\tau)$ is a solution of the Eq (2.1). Keeping the Eq (2.29) and Eq (2.30) in mind, we proceed:

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,(\overline{z}_n)_w+g_w)w^{\sigma}d_{\theta}w\\
&\quad-\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,\overline{z}_w+g_w)w^{\sigma}d_{\theta}w\Big|\\
\leq&\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} |f(w,(\overline{z}_n)_w+g_w)-f(w,\overline{z}_w+g_w)|w^{\sigma}d_{\theta}w \qquad (2.38)\\
\leq&\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} h(w)\sup_{x\in[0,w]}|\overline{z}_n(x)-\overline{z}(x)|w^{\sigma}d_{\theta}w\\
=&\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} h(w)\sup_{x\in[0,w]}|z_n(x)-z(x)|w^{\sigma}d_{\theta}w.
\end{aligned}
$$

Since $z_n(\tau)\to z(\tau)$, as $n\to\infty$. By definition of convergence, for any $\epsilon>0$, there exists a sufficiently large number $p_0>0$, such that for $n>p_0$, we have

$$
|z_n(x)-z(x)|<\min\left\{\frac{(\sigma+\theta)^{\nu}\Gamma(\nu+1)\epsilon}{\sum\limits_{i=0}^{m}h_i\mathrm{T}^{(\sigma+\theta)\nu}},\frac{\epsilon}{\sum\limits_{j=1}^{m}M_j}\right\}. \qquad (2.39)
$$

Therefore, using Eq (2.38) we get:

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\overline{z}_n)_w+g_w)w^{\sigma}d_{\theta}w\\
-&\frac{1}{\Gamma(\nu)}\int\limits_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,\overline{z}_w+g_w)w^{\sigma}d_{\theta}w\Big|<\epsilon.
\end{aligned} \qquad (2.40)
$$

And also

$$\begin{aligned}
&\Big|\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\overline{z}_n)_w+g_w)w^{\sigma}d_\theta w\\
&-\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,\overline{z}_w+g_w)w^{\sigma}d_\theta w\Big|\\
&\leq\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} |f(w,(\overline{z}_n)_w+g_w)-f(w,\overline{z}_w+g_w)|w^{\sigma}d_\theta w\\
&\leq\sum_{i=0}^{k-1}h(\tau_i)\frac{(\tau_i^{(\sigma+\theta)\nu}-\tau_{i-1}^{(\sigma+\theta)\nu})}{\Gamma(\nu+1)(\sigma+\theta)^{\nu}}\sup_{x\in[0,w]}|z_n(x)-z(x)|d_\theta w<\epsilon.
\end{aligned}\tag{2.41}$$

Also

$$\begin{aligned}
\Big|\sum_{j=1}^{k}I_j(\overline{z}_n(\tau_j))-\sum_{j=1}^{k}I_j(\overline{z}(\tau_j))\Big| &\leq\sum_{j=1}^{k}M_j|\overline{z}_n(\tau_j)-\overline{z}(\tau_j)|\\
&=\sum_{j=1}^{k}M_j|z_n(\tau_j)-z(\tau_j)|<\epsilon.
\end{aligned}\tag{2.42}$$

In consequence, we can see that for a sufficiently large number $n>p_0$:

$$\begin{aligned}
&|z(\tau)-\Theta z(\tau)|\\
\leq&|z(\tau)-z_{n+1}(\tau)|+|z_{n+1}(\tau)-\Theta z_n(\tau)|+|\Theta z_n(\tau)-\Theta z(\tau)|\\
\leq&|z(\tau)-z_{n+1}(\tau)|+\bigg|z_{n+1}(\tau)-\Big[\frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\overline{z}_n)_w+g_w)w^{\sigma}d_\theta w\\
&+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\overline{z}_n)_w+g_w)w^{\sigma}d_\theta w+\sum_{j=1}^{k}I_j(\overline{z}_n(\tau_j))\Big]\bigg|
\end{aligned}\tag{2.43}$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\bar{z}_n)_w+g_w)w^{\sigma}d_{\theta}w\right.\\
&+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\bar{z}_n)_w+g_w)w^{\sigma}d_{\theta}w\\
&+\sum_{j=1}^{k}I_j(\bar{z}_n(\tau_j))-\left[\frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\bar{z})_w+g_w)w^{\sigma}d_{\theta}w\right.\\
&\left.\left.+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} f(w,((\bar{z})_w+g_w)w^{\sigma}d_{\theta}w+\sum_{j=1}^{k}I_j(\bar{z}(\tau_j))\right]\right|
\end{aligned}
\tag{2.44}
$$

using Eq. (2.40), Eq (2.41) and Eq (2.42) we get that $|z(\tau)-\Theta z(\tau)|\to 0$. This shows that $z(\tau)$ is the solution of Eq (2.1).

Now we show that the solution is unique. On contrary suppose that there exists two solutions z_1 and z_2 of Eq (2.1). Then

$$
\begin{aligned}
|z_1(\tau)-z_2(\tau)|\leq&\frac{1}{\Gamma(\nu)}\int_{\tau_k}^{\tau}\left(\frac{\tau^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} h(w)\sup_{x\in[0,w]}|\bar{z}_1(x)-\bar{z}_2(x)|w^{\sigma}d_{\theta}w\\
&+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\nu)}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{\tau_{i+1}^{\sigma+\theta}-w^{\sigma+\theta}}{\sigma+\theta}\right)^{\nu-1} h(w)\sup_{x\in[0,w]}|\bar{z}_1(x)-\bar{z}_2(x)|w^{\sigma}d_{\theta}w\\
&+\sum_{j=1}^{k}I_j|\bar{z}_1(\tau_j)-\bar{z}_2(\tau_j)|\\
\leq&\left(\sum_{p=1}^{\nu+1}\frac{h_pT^{\nu(\sigma+\theta)}}{(\sigma+\theta)^{\nu}\Gamma(\nu+1)}+\sum_{q=1}^{\nu}M_q\right)\|z_1-z_2\|.
\end{aligned}
\tag{2.45}
$$

Using the condition 2 in the theorem hypothesis, the uniqueness of the solution of Eq (2.1) follows immediately, which completes the proof. □

3. Illustrative examples

To illustrate the obtained results, some examples are presented in this section.

Example 1. A particular GCF differential equation with delay and impulse is considered as follows:

$$\begin{cases} {}_{1}^{1}T_{\tau_k^+}^{\frac{1}{2}}\phi(\tau) = \frac{1}{(10+\tau)^2}\frac{|\phi_\tau|}{(1+|\phi_\tau|)}, \ \tau \in [0,2], \ \tau \neq \frac{3}{4}, \ k = 0, 1; \\ \Delta\phi(\frac{3}{4}) = \frac{|\phi(\frac{3}{4})|}{12+|\phi(\frac{3}{4})|}; \\ \phi(\tau) = \psi(\tau) = \frac{e^{-\tau}-1}{2}, \ \tau \in [-\omega, 0], \end{cases} \tag{3.1}$$

where ω is a non-negative constant. Here, $\sigma = 1$, $\theta = 1$, $\nu = \frac{1}{2}$, $\tau_0 = 0$, $\tau_1 = \frac{3}{4}$, $\tau_2 = \mathtt{T} = 2$, $f(\tau, \phi_\tau) = \frac{|\phi_\tau|}{(10+\tau)^2(1+|\phi_\tau|)}$ and $I(\phi) = \frac{|\phi|}{12+|\phi|}$ are fixed in Eq (2.1). Moreover $\phi_\tau(s) = \phi(\tau + s)$, for $s \in [-\omega, 0]$, $\tau \in [0, 2]$, $\phi \in [0, \infty)$.

To check whether a unique solution of the problem in Eq (3.1) exists or not, we have to verify all the three conditions of the Theorem 3. We consider:

$$\begin{aligned} |f(\tau, a_\tau) - f(\tau, b_\tau)| =& \frac{1}{(10+\tau)^2}\left|\frac{|a_\tau|}{1+|a_\tau|} - \frac{|b_\tau|}{1+|b_\tau|}\right| \\ \leq& \frac{|a_\tau - b_\tau|}{(10+\tau)^2} \\ \leq& h(\tau) \sup_{w\in[0,\tau]} |a(w) - b(w)|, \end{aligned} \tag{3.2}$$

where $h(\tau) = \frac{1}{(10+\tau)^2}$, which shows that the condition 1 of Theorem 3 is satisfied. Also we have:

$$\begin{aligned} |I(a) - I(b)| =& \frac{12|a-b|}{(12+a)(12+b)} \\ \leq& \frac{1}{12}|a-b|, \quad a, \ b > 0, \end{aligned} \tag{3.3}$$

where $M_1 = \frac{1}{12}$, also from above $h_1 = \sup_{\tau\in[0,2]} h(\tau) = \sup_{\tau\in[0,2]} \frac{1}{(10+\tau)^2} = \frac{1}{100}$. Now we can see by putting values of all the parameters that:

$$\sum_{i=1}^{m+1} \frac{h_i \mathrm{T}^{\nu(\sigma+\theta)}}{(\sigma+\theta)^\nu \Gamma(\nu+1)} + \sum_{i=1}^{m} M_i < 1, \tag{3.4}$$

which shows that the condition 2 of Theorem 3 is also satisfied.

Finally:

$$f(\tau, \phi_\tau) = \frac{|\phi_\tau|}{(10+\tau)^2(1+|\phi_\tau|)} \leq \frac{1}{(10+\tau)^2} \leq \frac{1}{100}, \quad \tau \in [0, 2]. \tag{3.5}$$

So the condition 3 of Theorem 3 is also satisfied.

Now using the Theorem 3, it is concluded that the solution of the Eq (3.1) exists and it is unique.

Example 2. Consider the following ID-FDE containing GCF derivative operator of order $\frac{2}{3}$, with the parameters $\sigma = 100$, $\theta = 50$.

$$\begin{cases} {}^{100}_{50}T^{\frac{2}{3}}_{k^+}\phi(\tau) = \frac{e^{-\tau}}{(50+|\phi_\tau|)(9+e^\tau)}, \ \tau \in [0,1], \ \tau \neq \frac{1}{2}; \\ \Delta\phi(\frac{1}{2}) = \frac{|\phi(\frac{1}{2})|}{300+|\phi(\frac{1}{2})|}; \\ \phi(\tau) = \psi(\tau) = \frac{e^{-\tau}-1}{2}, \ \tau \in [-\omega, 0]. \end{cases} \tag{3.6}$$

To check whether there exist a unique solution for the equation (3.6), we proceed as follows. Since the Eq (3.6) is a special case of the Eq (2.1) with $f(\tau, \phi) = \frac{e^{-\tau}}{(9+e^\tau)(50+|\phi_\tau|)}$, for $\tau \in [0, 1]$, $\tau \neq \frac{1}{2}$, also $I(\phi) = \frac{|\phi|}{300+|\phi|}$, and $\psi(\tau) = \frac{e^{-\tau}-1}{2}$, for $\tau \in [-\omega, 0]$. All we have to do is to verify the three conditions of Theorem 3. To check this, we first consider:

$$\begin{aligned} |f(\tau, a_\tau) - f(\tau, b_\tau)| &\leq \frac{e^{-\tau}|a_\tau - b_\tau|}{9 + e^\tau} \\ &\leq h(\tau) \sup_{w\in[0,\tau]} |a(w) - b(w)|, \end{aligned} \tag{3.7}$$

where $h(\tau) = \frac{e^{-\tau}}{9+e^\tau}$, and $h = \sup_{\tau\in[0,1]} h(\tau) = \frac{1}{10}$, which shows that the condition 1 of Theorem 3 is satisfied.

Also we have:

$$\begin{aligned} |I(a) - I(b)| &= \frac{300|a - b|}{(300 + a)(300 + b)} \\ &\leq \frac{1}{300}|a - b|, \quad a, b > 0, \end{aligned} \tag{3.8}$$

where:

$$\sum_{p=1}^{\nu+1} \frac{h_p \mathrm{T}^{\nu(\sigma+\theta)}}{(\sigma + \theta)^\nu \Gamma(\nu + 1)} + \sum_{q=1}^{\nu} M_q = \frac{1}{300(150)^{\frac{2}{3}}\Gamma(\frac{5}{3})} + \frac{1}{300} < 1. \tag{3.9}$$

Thus the condition 2 of Theorem 3 also holds true.

Finally:

$$f(\tau, \phi_\tau) = \frac{e^{-\tau}}{(50 + |\phi_\tau|)(9 + e^\tau)} \leq \frac{e^{-\tau}}{9 + e^\tau} \leq \frac{1}{10}, \tag{3.10}$$

where $\tau \in [0, 1]$. So the condition 3 of Theorem 3 is also satisfied.

Thus using Theorem 3, it is established that solution of the Eq (3.6) exists and it will be unique.

Example 3. Consider another ID-FDE containing GCF derivative operator of order $\frac{3}{7}$, with the parameters $\sigma = 2$, $\theta = 5$.

$$\begin{cases} {}^{2}_{5}T^{\frac{3}{7}}_{k^+}\phi(\tau) = \frac{\tau(\tau^2-\frac{1}{8})^2}{100}\left(\frac{1}{5+|\phi_\tau|} + \frac{1}{2\cos\tau}\right), \ \tau \in [0,2], \ \tau \neq \frac{3}{2}; \\ \Delta\phi(\frac{3}{2}) = \frac{1}{4}\arctan\phi(\frac{3}{2}); \\ \phi(\tau) = \psi(\tau) = e^{\sqrt[3]{\tau}} - e^{\frac{\sqrt[3]{\tau}}{3}}, \ \tau \in [-\omega, 0]. \end{cases} \tag{3.11}$$

To verify existence of a unique solution for the Eq (3.11), we proceed as follows. Since the Eq (3.11) is a special case of the Eq (2.1) with $f(\tau,\phi) = \frac{\tau(\tau^2-\frac{1}{8})^2}{100}\left(\frac{1}{5+|\phi_\tau|} + \frac{1}{2\cos\tau}\right)$, for $\tau \in [0,2]$, $\tau \neq \frac{3}{2}$, also $I(\phi) = \frac{1}{4}\arctan\phi(\frac{3}{2})$, and $\psi(\tau) = e^{\sqrt[3]{\tau}} - e^{\frac{\sqrt[3]{\tau}}{3}}$, for $\tau \in [-\omega, 0]$. We have to verify the three conditions of Theorem 3. To check this, we first consider:

$$\begin{aligned} |f(\tau,a_\tau) - f(\tau,b_\tau)| &\leq \frac{\tau(\tau^2-\frac{1}{8})^2}{100}|a_\tau - b_\tau| \\ &\leq h(\tau)\sup_{w\in[0,\tau]}|a(w)-b(w)|, \end{aligned} \tag{3.12}$$

where $h(\tau) = \frac{\tau(\tau^2-\frac{1}{8})^2}{100}$, and $h = \sup_{\tau\in[0,2]} h(\tau) = \frac{961}{3200}$, which shows that the condition 1 of Theorem 3 is satisfied.

Also we have:

$$\begin{aligned} |I(a) - I(b)| &= \frac{1}{4}|\arctan a - \arctan b| \\ &= \frac{1}{4}\left|\arctan\left(\frac{a-b}{1+ab}\right)\right| \\ &\leq \frac{\pi}{8}|a-b|, \quad a,\ b > 0, \end{aligned} \tag{3.13}$$

where:

$$\sum_{p=1}^{\nu+1}\frac{h_p \mathrm{T}^{\nu(\sigma+\theta)}}{(\sigma+\theta)^\nu\Gamma(\nu+1)} + \sum_{q=1}^{\nu} M_q = \frac{961\pi}{3200\times(150)^{\frac{2}{3}}\Gamma(\frac{5}{3})} + \frac{\pi}{8} < 1. \tag{3.14}$$

Thus the condition 2 of Theorem 3 also holds true.

Finally:

$$f(\tau,\phi_\tau) = f(\tau,\phi) = \frac{\tau(\tau^2-\frac{1}{8})^2}{100}\left(\frac{1}{5+|\phi_\tau|} + \frac{1}{2\cos\tau}\right) < \frac{\tau(\tau^2-\frac{1}{8})^2}{100} \leq \frac{961}{3200}, \tag{3.15}$$

where $\tau \in [0,2]$. So the condition 3 of Theorem 3 is also satisfied.

Hence in the light of Theorem 3, it can be claimed that solution of the Eq (3.11) will exist and it will be unique.

4. Conclusions

A new generalized class of ID-FDE has been constructed successfully. A sufficient criterion for the existence and uniqueness of the solution of this type of systems have been developed. The results have been supported by the successive approximation method. All the results have been given in terms of newly introduced GCF operators. To illustrate the obtained results, some particular examples have been presented. The present attempt also allows direct applications of the obtained results to FDE of the types Katugampola, Riemann-Liovilles, Hadamard, New Riemann-Lioville's, conformable and ordinary differential equations, which can be considered as special cases of our established results.

Since there exist many fractional derivative and integral operators, which have been defined with the passage of time. Each operator satisfies some useful properties and also has some flaws. In most of the cases there arises a confusion regarding selection of a suitable fractional operator for solving a given mathematical problem. In this context, there is a need for such operators that combine most of the previously defined operators into a single form. In this regard, GCF operators nicely fulfill this criterion using which one can work with multiple number of operators at the same time.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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