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Research article

On a class of inverse palindromic eigenvalue problem

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Abstract: In this paper we first consider the following inverse palindromic eigenvalue problem (IPEP): Given matrices $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $i, j = 1, \cdots, p$, $X = [x_1, \cdots, x_p] \in \mathbb{C}^{n\times p}$ with $\mathrm{rank}(X) = p$, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2i} = \bar{\lambda}_{2i-1} \in \mathbb{C}$, $x_{2i} = \bar{x}_{2i-1} \in \mathbb{C}^n$ for $i = 1, \cdots, m$, and $\lambda_j \in \mathbb{R}$, $x_j \in \mathbb{R}^n$ for $j = 2m + 1, \cdots, p$, find a matrix $A \in \mathbb{R}^{n\times n}$ such that $AX = A^\top X\Lambda$. We then consider a best approximation problem (BAP): Given $\tilde{A} \in \mathbb{R}^{n\times n}$, find $\hat{A} \in \mathcal{S}_A$ such that $\|\hat{A} - \tilde{A}\| = \min_{A\in\mathcal{S}_A} \|A - \tilde{A}\|$, where $\|\cdot\|$ is the Frobenius norm and $\mathcal{S}_A$ is the solution set of IPEP. By partitioning the matrix Λ and using the QR-decomposition, the expression of the general solution of Problem IPEP is derived. Also, we show that the best approximation solution $\hat{A}$ is unique and derive an explicit formula for it.

Keywords: inverse palindromic eigenvalue problem; best approximation problem; QR-decomposition
Mathematics Subject Classification: 15A09, 15A24

1. Introduction

Since the 1960s, the rapid development of high-speed rail has made it a very important means of transportation. However, the vibration will be caused because of the contact between the wheels of the train and the train tracks during the operation of the high-speed train. Therefore, the analytical vibration model can be mathematically summarized as a quadratic palindromic eigenvalue problem (QPEP) (see [1, 2])

$$(\lambda^2 A_1 + \lambda A_0 + A_1^\top)x = 0,$$

with $A_i \in \mathbb{R}^{n\times n}$, $i = 0, 1$ and $A_0^\top = A_0$. The eigenvalues λ, the corresponding eigenvectors x are relevant to the vibration frequencies and the shapes of the vibration, respectively. Many scholars have put forward many effective methods to solve QPEP [3–7]. In addition, under mild assumptions, the quadratic palindromic eigenvalue problem can be converted to the following linear palindromic eigenvalue problem (see [8])

$$Ax = \lambda A^\top x, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ is a given matrix, $\lambda \in \mathbb{C}$ and nonzero vectors $x \in \mathbb{C}^n$ are the wanted eigenvalues and eigenvectors of the vibration model. We can obtain $\frac{1}{\lambda}x^\top A^\top = x^\top A$ by transposing the equation (1). Thus, λ and $\frac{1}{\lambda}$ always come in pairs. Many methods have been proposed to solve the palindromic eigenvalue problem such as URV-decomposition based structured method [9], QR-like algorithm [10], structure-preserving methods [11], and palindromic doubling algorithm [12].

On the other hand, the modal data obtained by the mathematical model are often evidently different from the relevant experimental ones because of the complexity of the structure and inevitable factors of the actual model. Therefore, the coefficient matrices need to be modified so that the updated model satisfies the dynamic equation and closely matches the experimental data. Al-Ammari [13] considered the inverse quadratic palindromic eigenvalue problem. Batzke and Mehl [14] studied the inverse eigenvalue problem for T-palindromic matrix polynomials excluding the case that both +1 and −1 are eigenvalues. Zhao et al. [15] updated ∗-palindromic quadratic systems with no spill-over. However, the linear inverse palindromic eigenvalue problem has not been extensively considered in recent years.

In this work, we just consider the linear inverse palindromic eigenvalue problem (IPEP). It can be stated as the following problem:

Problem IPEP. Given a pair of matrices (Λ, X) in the form

$$\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p},$$

and

$$X = [x_1, \cdots, x_p] \in \mathbb{C}^{n\times p},$$

where diagonal elements of Λ are all distinct, X is of full column rank p, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2i} = \bar{\lambda}_{2i-1} \in \mathbb{C}$, $x_{2i} = \bar{x}_{2i-1} \in \mathbb{C}^n$ for $i = 1, \cdots, m$, and $\lambda_j \in \mathbb{R}$, $x_j \in \mathbb{R}^n$ for $j = 2m+1, \cdots, p$, find a real-valued matrix A that satisfy the equation

$$AX = A^\top X\Lambda. \tag{2}$$

Namely, each pair (λ_t, x_t), $t = 1, \cdots, p$, is an eigenpair of the matrix pencil

$$P(\lambda) = Ax - \lambda A^\top x.$$

It is known that the mathematical model is a "good" representation of the system, we hope to find a model that is closest to the original model. Therefore, we consider the following best approximation problem:

Problem BAP. Given $\tilde{A} \in \mathbb{R}^{n\times n}$, find $\hat{A} \in \mathcal{S}_A$ such that

$$\|\hat{A} - \tilde{A}\| = \min_{A\in\mathcal{S}_A} \|A - \tilde{A}\|, \tag{3}$$

where $\|\cdot\|$ is the Frobenius norm, and $\mathcal{S}_A$ is the solution set of Problem IPEP.

In this paper, we will put forward a new direct method to solve Problem IPEP and Problem BAP. By partitioning the matrix Λ and using the QR-decomposition, the expression of the general solution of Problem IPEP is derived. Also, we show that the best approximation solution $\hat{A}$ of Problem BAP is unique and derive an explicit formula for it.

2. The solution of Problem IPEP

We first rearrange the matrix Λ as

$$\Lambda = \begin{array}{c} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & \Lambda_2 \end{array}\right] \begin{array}{c} t \\ 2s \\ 2(k+2l) \end{array} \\ \begin{array}{ccc} t & 2s & 2(k+2l) \end{array} \end{array}, \tag{4}$$

where $t + 2s + 2(k + 2l) = p$, $t = 0$ *or* 1,

$$\Lambda_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{2s-1}, \lambda_{2s}\}, \lambda_i \in \mathbb{R},\ \lambda_{2i-1}^{-1} = \lambda_{2i},\ 1 \le i \le s,$$
$$\Lambda_2 = \mathrm{diag}\{\delta_1, \cdots, \delta_k, \delta_{k+1}, \delta_{k+2}, \cdots, \delta_{k+2l-1}, \delta_{k+2l}\},\ \delta_j \in \mathbb{C}^{2\times 2},$$

with

$$\delta_j = \left[\begin{array}{cc} \alpha_j + \beta_j i & 0 \\ 0 & \alpha_j - \beta_j i \end{array}\right],\ i = \sqrt{-1},\ 1 \le j \le k + 2l,$$
$$\delta_j^{-1} = \bar{\delta}_j,\ 1 \le j \le k,$$
$$\delta_{k+2j-1}^{-1} = \delta_{k+2j},\ 1 \le j \le l,$$

and the adjustment of the column vectors of X corresponds to those of Λ.

Define T_p as

$$T_p = \mathrm{diag}\left\{I_{t+2s}, \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & -i \\ 1 & i \end{array}\right], \cdots, \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & -i \\ 1 & i \end{array}\right]\right\} \in \mathbb{C}^{p\times p}, \tag{5}$$

where $i = \sqrt{-1}$. It is easy to verify that $T_p^{\mathrm{H}} T_p = I_p$. Using this matrix of (5), we obtain

$$\tilde{\Lambda} = T_p^{\mathrm{H}} \Lambda T_p = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & \tilde{\Lambda}_2 \end{array}\right], \tag{6}$$

$$\tilde{X} = X T_p = [x_t, \cdots, x_{t+2s}, \sqrt{2} y_{t+2s+1}, \sqrt{2} z_{t+2s+1}, \cdots, \sqrt{2} y_{p-1}, \sqrt{2} z_{p-1}], \tag{7}$$

where

$$\tilde{\Lambda}_2 = \mathrm{diag}\left\{\left[\begin{array}{cc} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{array}\right], \cdots, \left[\begin{array}{cc} \alpha_{k+2l} & \beta_{k+2l} \\ -\beta_{k+2l} & \alpha_{k+2l} \end{array}\right]\right\} \triangleq \mathrm{diag}\{\tilde{\delta}_1, \cdots, \tilde{\delta}_{k+2l}\},$$

and $\tilde{\Lambda}_2 \in \mathbb{R}^{2(k+2l)\times 2(k+2l)}$, $\tilde{X} \in \mathbb{R}^{n\times p}$. y_{t+2s+j} and z_{t+2s+j} are, respectively, the real part and imaginary part of the complex vector x_{t+2s+j} for $j = 1, 3, \cdots, 2(k + 2l) - 1$. Using (6) and (7), the matrix equation (2) is equivalent to

$$A\tilde{X} = A^{\top} \tilde{X} \tilde{\Lambda}. \tag{8}$$

Since $\mathrm{rank}(X) = \mathrm{rank}(\tilde{X}) = p$. Now, let the QR-decomposition of $\tilde{X}$ be

$$\tilde{X} = Q\left[\begin{array}{c} R \\ 0 \end{array}\right], \tag{9}$$

where $Q = [Q_1, Q_2] \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $R \in \mathbb{R}^{p\times p}$ is nonsingular. Let

$$Q^\top AQ = \begin{array}{c} \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \begin{array}{c} p \\ n-p \end{array} \\ \begin{array}{cc} p & n-p \end{array} \end{array}. \tag{10}$$

Using (9) and (10), then the equation of (8) is equivalent to

$$A_{11}R = A_{11}^\top R\tilde{\Lambda}, \tag{11}$$
$$A_{21}R = A_{12}^\top R\tilde{\Lambda}. \tag{12}$$

Write

$$R^\top A_{11}R \triangleq F = \begin{array}{c} \left[\begin{array}{ccc} f_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{array}\right] \begin{array}{c} t \\ 2s \\ 2(k+2l) \end{array} \\ \begin{array}{ccc} t & 2s & 2(k+2l) \end{array} \end{array}, \tag{13}$$

then the equation of (11) is equivalent to

$$F_{12} = F_{21}^\top\Lambda_1,\ F_{21} = F_{12}^\top, \tag{14}$$
$$F_{13} = F_{31}^\top\tilde{\Lambda}_2,\ F_{31} = F_{13}^\top, \tag{15}$$
$$F_{23} = F_{32}^\top\tilde{\Lambda}_2,\ F_{32} = F_{23}^\top\Lambda_1, \tag{16}$$
$$F_{22} = F_{22}^\top\Lambda_1, \tag{17}$$
$$F_{33} = F_{33}^\top\tilde{\Lambda}_2. \tag{18}$$

Because the elements of $\Lambda_1, \tilde{\Lambda}_2$ are distinct, we can obtain the following relations by Eqs (14)-(18)

$$F_{12} = 0,\ F_{21} = 0,\ F_{13} = 0,\ F_{31} = 0,\ F_{23} = 0,\ F_{32} = 0, \tag{19}$$
$$F_{22} = \operatorname{diag}\left\{\left[\begin{array}{cc} 0 & h_1 \\ \lambda_1 h_1 & 0 \end{array}\right], \cdots, \left[\begin{array}{cc} 0 & h_s \\ \lambda_{2s-1}h_s & 0 \end{array}\right]\right\}, \tag{20}$$
$$F_{33} = \operatorname{diag}\left\{G_1, \cdots, G_k, \left[\begin{array}{cc} 0 & G_{k+1} \\ G_{k+1}^\top\tilde{\delta}_{k+1} & 0 \end{array}\right], \cdots, \left[\begin{array}{cc} 0 & G_{k+l} \\ G_{k+l}^\top\tilde{\delta}_{k+2l-1} & 0 \end{array}\right]\right\}, \tag{21}$$

where

$$G_i = a_iB_i,\ G_{k+j} = a_{k+2j-1}D_1 + a_{k+2j}D_2,\ G_{k+j}^\top = G_{k+j},$$
$$B_i = \left[\begin{array}{cc} 1 & \frac{1-\alpha_i}{\beta_i} \\ -\frac{1-\alpha_i}{\beta_i} & 1 \end{array}\right],\ D_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right],\ D_2 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right],$$

and $1 \le i \le k, 1 \le j \le l$. $h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}$ are arbitrary real numbers. It follows from Eq (12) that

$$A_{21} = A_{12}^\top E, \tag{22}$$

where $E = R\tilde{\Lambda}R^{-1}$.

Theorem 1. *Suppose that* $\Lambda = diag\{\lambda_1, \cdots, \lambda_p\} \in \mathbb{C}^{p\times p}$, $X = [x_1, \cdots, x_p] \in \mathbb{C}^{n\times p}$, *where diagonal elements of* Λ *are all distinct,* X *is of full column rank* p, *and both* Λ *and* X *are closed under complex conjugation in the sense that* $\lambda_{2i} = \bar{\lambda}_{2i-1} \in \mathbb{C}$, $x_{2i} = \bar{x}_{2i-1} \in \mathbb{C}^n$ *for* $i = 1, \cdots, m$, *and* $\lambda_j \in \mathbb{R}$, $x_j \in \mathbb{R}^n$ *for* $j = 2m + 1, \cdots, p$. *Rearrange the matrix* Λ *as* (4), *and adjust the column vectors of* X *with corresponding to those of* Λ. *Let* Λ, X *transform into* $\tilde{\Lambda}, \tilde{X}$ *by* (6) − (7) *and QR-decomposition of the matrix* $\tilde{X}$ *be given by* (9). *Then the general solution of* (2) *can be expressed as*

$$\mathcal{S}_A = \left\{ A \,\middle|\, A = Q \begin{bmatrix} R^{-\top} \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} R^{-1} & A_{12} \\ A_{12}^{\top} E & A_{22} \end{bmatrix} Q^{\top} \right\}, \tag{23}$$

where $E = R\tilde{\Lambda}R^{-1}$, f_{11} *is arbitrary real number,* $A_{12} \in \mathbb{R}^{p\times(n-p)}, A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ *are arbitrary real-valued matrices and* F_{22}, F_{33} *are given by* (20) − (21).

3. The solution of Problem BAP

In order to solve Problem BAP, we need the following lemma.

Lemma 1. *[16] Let* A, B *be two real matrices, and* X *be an unknown variable matrix. Then*

$$\frac{\partial tr(BX)}{\partial X} = B^{\top}, \ \frac{\partial tr(X^{\top}B^{\top})}{\partial X} = B^{\top}, \ \frac{\partial tr(AXBX)}{\partial X} = (BXA + AXB)^{\top},$$
$$\frac{\partial tr(AX^{\top}BX^{\top})}{\partial X} = BX^{\top}A + AX^{\top}B, \ \frac{\partial tr(AXBX^{\top})}{\partial X} = AXB + A^{\top}XB^{\top}.$$

By Theorem 1, we can obtain the explicit representation of the solution set $\mathcal{S}_A$. It is easy to verify that $\mathcal{S}_A$ is a closed convex subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. By the best approximation theorem (see Ref. [17]), we know that there exists a unique solution of Problem BAP. In the following we will seek the unique solution $\hat{A}$ in $\mathcal{S}_A$. For the given matrix $\tilde{A} \in \mathbb{R}^{n\times n}$, write

$$Q^{\top}\tilde{A}Q = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{matrix} p \\ n-p \end{matrix}, \quad \begin{matrix} p & n-p \end{matrix} \tag{24}$$

then

$$\begin{aligned} \|A - \tilde{A}\|^2 &= \left\| \begin{bmatrix} R^{-\top} \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} R^{-1} - \tilde{A}_{11} & A_{12} - \tilde{A}_{12} \\ A_{12}^{\top}E - \tilde{A}_{21} & A_{22} - \tilde{A}_{22} \end{bmatrix} \right\|^2 \\ &= \left\| R^{-\top} \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} R^{-1} - \tilde{A}_{11} \right\|^2 \\ &+ \|A_{12} - \tilde{A}_{12}\|^2 + \|A_{12}^{\top}E - \tilde{A}_{21}\|^2 + \|A_{22} - \tilde{A}_{22}\|^2. \end{aligned}$$

Therefore, $\|A - \tilde{A}\| = \min$ if and only if

$$\left\| R^{-\top} \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} R^{-1} - \tilde{A}_{11} \right\|^2 = \min, \tag{25}$$

$$\|A_{12} - \tilde{A}_{12}\|^2 + \|A_{12}^\top E - \tilde{A}_{21}\|^2 = \min, \tag{26}$$

$$A_{22} = \tilde{A}_{22}. \tag{27}$$

Let

$$R^{-1} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \tag{28}$$

then the relation of (25) is equivalent to

$$\|R_1^\top f_{11} R_1 + R_2^\top F_{22} R_2 + R_3^\top F_{33} R_3 - \tilde{A}_{11}\|^2 = \min. \tag{29}$$

Write

$$R_1 = [r_{1,t}], \ R_2 = \begin{bmatrix} r_{2,1} \\ \vdots \\ r_{2,2s} \end{bmatrix}, \ R_3 = \begin{bmatrix} r_{3,1} \\ \vdots \\ r_{3,k+2l} \end{bmatrix}, \tag{30}$$

where $r_{1,t} \in \mathbb{R}^{t\times p}, r_{2,i} \in \mathbb{R}^{1\times p}, r_{3,j} \in \mathbb{R}^{2\times p}, \ i = 1, \cdots, 2s, \ j = 1, \cdots, k + 2l.$
Let

$$\begin{cases} J_t = r_{1,t}^\top r_{1,t}, \\ J_{t+i} = \lambda_{2i-1} r_{2,2i}^\top r_{2,2i-1} + r_{2,2i-1}^\top r_{2,2i} \ (1 \le i \le s), \\ J_{r+i} = r_{3,i}^\top B_i r_{3,i} \ (1 \le i \le k), \\ J_{r+k+2i-1} = r_{3,k+2i}^\top D_1 \tilde{\delta}_{k+2i-1} r_{3,k+2i-1} + r_{3,k+2i-1}^\top D_1 r_{3,k+2i} \ (1 \le i \le l), \\ J_{r+k+2i} = r_{3,k+2i}^\top D_2 \tilde{\delta}_{k+2i-1} r_{3,k+2i-1} + r_{3,k+2i-1}^\top D_2 r_{3,k+2i} \ (1 \le i \le l), \end{cases} \tag{31}$$

with $r = t + s, q = t + s + k + 2l$. Then the relation of (29) is equivalent to

$$\begin{aligned} & g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}) = \\ & \|f_{11} J_t + h_1 J_{t+1} + \cdots + h_s J_r + a_1 J_{r+1} + \cdots + a_{k+2l} J_q - \tilde{A}_{11}\|^2 = \min, \end{aligned}$$

that is,

$$\begin{aligned} & g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}) \\ & = \mathrm{tr}[(f_{11} J_t + h_1 J_{t+1} + \cdots + h_s J_r + a_1 J_{r+1} + \cdots + a_{k+2l} J_q - \tilde{A}_{11})^\top \\ & (f_{11} J_t + h_1 J_{t+1} + \cdots + h_s J_r + a_1 J_{r+1} + \cdots + a_{k+2l} J_q - \tilde{A}_{11})] \\ & = f_{11}^2 c_{t,t} + 2 f_{11} h_1 c_{t,t+1} + \cdots + 2 f_{11} h_s c_{t,r} + 2 f_{11} a_1 c_{t,r+1} + \cdots + 2 f_{11} a_{k+2l} c_{t,q} - 2 f_{11} e_t \\ & + h_1^2 c_{t+1,t+1} + \cdots + 2 h_1 h_s c_{t+1,r} + 2 h_1 a_1 c_{t+1,r+1} + \cdots + 2 h_1 a_{k+2l} c_{t+1,q} - 2 h_1 e_{t+1} \\ & + \cdots \\ & + h_s^2 c_{r,r} + 2 h_s a_1 c_{r,r+1} + \cdots + 2 h_s a_{k+2l} c_{r,q} - 2 h_s e_r \\ & + a_1^2 c_{r+1,r+1} + \cdots + 2 a_1 a_{k+2l} c_{r+1,q} - 2 a_1 e_{r+1} \end{aligned}$$

$$+\cdots$$
$$+a_{k+2l}^2 c_{q,q} - 2a_{k+2l}e_q + \mathrm{tr}(\tilde{A}_{11}^\top \tilde{A}_{11}),$$

where $c_{i,j} = \mathrm{tr}(J_i^\top J_j), e_i = \mathrm{tr}(J_i^\top \tilde{A}_{11})(i, j = t, \cdots, t+s+k+2l)$ and $c_{i,j} = c_{j,i}$.
Consequently,

$$\begin{aligned}
\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial f_{11}} &= 2f_{11}c_{t,t} + 2h_1 c_{t,t+1} + \cdots + 2h_s c_{t,r} + 2a_1 c_{t,r+1} \\
&+ \cdots + 2a_{k+2l}c_{t,q} - 2e_t, \\
\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial h_1} &= 2f_{11}c_{t+1,t} + 2h_1 c_{t+1,t+1} + \cdots + 2h_s c_{t+1,r} + 2a_1 c_{t+1,r+1} \\
&+ \cdots + 2a_{k+2l}c_{t+1,q} - 2e_{t+1}, \\
&\cdots \\
\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial h_s} &= 2f_{11}c_{r,t} + 2h_1 c_{r,t+1} + \cdots + 2h_s c_{r,r} + 2a_1 c_{r,r+1} \\
&+ \cdots + 2a_{k+2l}c_{r,q} - 2e_r, \\
\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial a_1} &= 2f_{11}c_{r+1,t} + 2h_1 c_{r+1,t+1} + \cdots + 2h_s c_{r+1,r} + 2a_1 c_{r+1,r+1} \\
&+ \cdots + 2a_{k+2l}c_{r+1,q} - 2e_{r+1}, \\
&\cdots \\
\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial a_{k+2l}} &= 2f_{11}c_{q,t} + 2h_1 c_{q,t+1} + \cdots + 2h_s c_{q,r} + 2a_1 c_{q,r+1} \\
&+ \cdots + 2a_{k+2l}c_{q,q} - 2e_q.
\end{aligned}$$

Clearly, $g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}) = \min$ if and only if

$$\frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial f_{11}} = 0, \cdots, \frac{\partial g(f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l})}{\partial a_{k+2l}} = 0.$$

Therefore,

$$\begin{aligned}
&f_{11}c_{t,t} + h_1 c_{t,t+1} + \cdots + h_s c_{t,r} + a_1 c_{t,r+1} + \cdots + a_{k+2l}c_{t,q} = e_t, \\
&f_{11}c_{t+1,t} + h_1 c_{t+1,t+1} + \cdots + h_s c_{t+1,r} + a_1 c_{t+1,r+1} + \cdots + a_{k+2l}c_{t+1,q} = e_{t+1}, \\
&\cdots \\
&f_{11}c_{r,t} + h_1 c_{r,t+1} + \cdots + h_s c_{r,r} + a_1 c_{r,r+1} + \cdots + a_{k+2l}c_{r,q} = e_r, \\
&f_{11}c_{r+1,t} + h_1 c_{r+1,t+1} + \cdots + h_s c_{r+1,r} + a_1 c_{r+1,r+1} + \cdots + a_{k+2l}c_{r+1,q} = e_{r+1}, \\
&\cdots \\
&f_{11}c_{q,t} + h_1 c_{q,t+1} + \cdots + h_s c_{q,r} + a_1 c_{q,r+1} + \cdots + a_{k+2l}c_{q,q} = e_q.
\end{aligned} \tag{32}$$

If let

$$C=\begin{bmatrix} c_{t,t} & c_{t,t+1} & \cdots & c_{t,r} & c_{t,r+1} & \cdots & c_{t,q} \\ c_{t+1,t} & c_{t+1,t+1} & \cdots & c_{t+1,r} & c_{t+1,r+1} & \cdots & c_{t+1,q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{r,t} & c_{r,t+1} & \cdots & c_{r,r} & c_{r,r+1} & \cdots & c_{r,q} \\ c_{r+1,t} & c_{r+1,t+1} & \cdots & c_{r+1,r} & c_{r+1,r+1} & \cdots & c_{r+1,q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{q,t} & c_{q,t+1} & \cdots & c_{q,r} & c_{q,r+1} & \cdots & c_{q,q} \end{bmatrix}, \ h=\begin{bmatrix} f_{11} \\ h_1 \\ \vdots \\ h_s \\ a_1 \\ \vdots \\ a_{k+2l} \end{bmatrix}, \ e=\begin{bmatrix} e_t \\ e_{t+1} \\ \vdots \\ e_r \\ e_{r+1} \\ \vdots \\ e_q \end{bmatrix},$$

where C is symmetric matrix. Then the equation (32) is equivalent to

$$Ch = e, \tag{33}$$

and the solution of the equation (33) is

$$h = C^{-1}e. \tag{34}$$

Substituting (34) into (20)-(21), we can obtain f_{11}, F_{22} and F_{33} explicitly. Similarly, the equation of (26) is equivalent to

$$g(A_{12}) = \mathrm{tr}(A_{12}^\top A_{12}) + \mathrm{tr}(\tilde{A}_{12}^\top \tilde{A}_{12}) - 2\mathrm{tr}(A_{12}^\top \tilde{A}_{12})$$
$$+\mathrm{tr}(E^\top A_{12}A_{12}^\top E) + \mathrm{tr}(\tilde{A}_{21}^\top \tilde{A}_{21}) - 2\mathrm{tr}(E^\top A_{12}\tilde{A}_{21}).$$

Applying Lemma 1, we obtain

$$\frac{\partial g(A_{12})}{\partial A_{12}} = 2A_{12} - 2\tilde{A}_{12} + 2EE^\top A_{12} - 2E\tilde{A}_{21}^\top,$$

setting $\frac{\partial g(A_{12})}{\partial A_{12}} = 0$, we obtain

$$A_{12} = (I_p + EE^\top)^{-1}(\tilde{A}_{12} + E\tilde{A}_{21}^\top), \tag{35}$$

Theorem 2. *Given $\tilde{A} \in \mathbb{R}^{n\times n}$, then the Problem BAP has a unique solution and the unique solution of Problem BAP is*

$$\hat{A} = Q\begin{bmatrix} R^{-\top}\begin{bmatrix} f_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix}R^{-1} & A_{12} \\ A_{12}^\top E & \tilde{A}_{22} \end{bmatrix}Q^\top, \tag{36}$$

where $E = R\tilde{\Lambda}R^{-1}$, $F_{22}, F_{33}, A_{12}, \tilde{A}_{22}$ are given by (20), (21), (35), (24) *and $f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}$ are given by* (34).

4. A numerical example

Based on Theorems 1 and 2, we can describe an algorithm for solving Problem BAP as follows.
Algorithm 1.

1) Input matrices Λ, X and $\tilde{A}$;

2) Rearrange Λ as (4), and adjust the column vectors of X with corresponding to those of Λ;

3) Form the unitary transformation matrix T_p by (5);

4) Compute real-valued matrices $\tilde{\Lambda}, \tilde{X}$ by (6) and (7);

5) Compute the QR-decomposition of $\tilde{X}$ by (9);

6) $F_{12} = 0, F_{21} = 0, F_{13} = 0, F_{31} = 0, F_{23} = 0, F_{32} = 0$ by (19) and $E = R\tilde{\Lambda}R^{-1}$;

7) Compute $\tilde{A}_{ij} = Q_i^\top \tilde{A} Q_j, i, j = 1, 2$;

8) Compute R^{-1} by (28) to form R_1, R_2, R_3;

9) Divide matrices R_1, R_2, R_3 by (30) to form $r_{1,t}, r_{2,i}, r_{3,j}, i = 1, \cdots, 2s, j = 1, \cdots, k + 2l$;

10) Compute $J_i, i = t, \cdots, t + s + k + 2l$, by (31);

11) Compute $c_{i,j} = \mathrm{tr}(J_i^\top J_j), e_i = \mathrm{tr}(J_i^\top \tilde{A}_{11}), i, j = t, \cdots, t + s + k + 2l$;

12) Compute $f_{11}, h_1, \cdots, h_s, a_1, \cdots, a_{k+2l}$ by (34);

13) Compute F_{22}, F_{33} by (20), (21) and $A_{22} = \tilde{A}_{22}$;

14) Compute A_{12} by (35) and A_{21} by (22);

15) Compute the matrix $\hat{A}$ by (36).

Example 1. Consider a 11-DOF system, where

$$\tilde{A} = \begin{bmatrix} 96.1898 & 18.1847 & 51.3250 & 49.0864 & 13.1973 & 64.9115 & 62.5619 & 81.7628 & 58.7045 & 31.1102 & 26.2212 \\ 0.4634 & 26.3803 & 40.1808 & 48.9253 & 94.2051 & 73.1722 & 78.0227 & 79.4831 & 20.7742 & 92.3380 & 60.2843 \\ 77.4910 & 14.5539 & 7.5967 & 33.7719 & 95.6135 & 64.7746 & 8.1126 & 64.4318 & 30.1246 & 43.0207 & 71.1216 \\ 81.7303 & 13.6069 & 23.9916 & 90.0054 & 57.5209 & 45.0924 & 92.9386 & 37.8609 & 47.0923 & 18.4816 & 22.1747 \\ 86.8695 & 86.9292 & 12.3319 & 36.9247 & 5.9780 & 54.7009 & 77.5713 & 81.1580 & 23.0488 & 90.4881 & 11.7418 \\ 8.4436 & 57.9705 & 18.3908 & 11.1203 & 23.4780 & 29.6321 & 48.6792 & 53.2826 & 84.4309 & 97.9748 & 29.6676 \\ 39.9783 & 54.9860 & 23.9953 & 78.0252 & 35.3159 & 74.4693 & 43.5859 & 35.0727 & 19.4764 & 43.8870 & 31.8778 \\ 25.9870 & 14.4955 & 41.7267 & 38.9739 & 82.1194 & 18.8955 & 44.6784 & 93.9002 & 22.5922 & 11.1119 & 42.4167 \\ 80.0068 & 85.3031 & 4.9654 & 24.1691 & 1.5403 & 68.6775 & 30.6349 & 87.5943 & 17.0708 & 25.8065 & 50.7858 \\ 43.1414 & 62.2055 & 90.2716 & 40.3912 & 4.3024 & 18.3511 & 50.8509 & 55.0156 & 22.7664 & 40.8720 & 8.5516 \\ 91.0648 & 35.0952 & 94.4787 & 9.6455 & 16.8990 & 36.8485 & 51.0772 & 62.2475 & 43.5699 & 59.4896 & 26.2482 \end{bmatrix},$$

the measured eigenvalue and eigenvector matrices Λ and X are given by

$$\Lambda = \mathrm{diag}\{1.0000,\ -1.8969,\ -0.5272,\ -0.1131 + 0.9936i, -0.1131 - 0.9936i,$$
$$1.9228 + 2.7256i,\ 1.9228 - 2.7256i,\ 0.1728 - 0.2450i,\ 0.1728 + 0.2450i\},$$

and

$$X = \begin{bmatrix} -0.0132 & -1.0000 & 0.1753 & 0.0840 + 0.4722i & 0.0840 - 0.4722i & 0.6669 + 0.2418i & 0.6669 - 0.2418i & 0.2556 - 0.1080i & 0.2556 + 0.1080i \\ -0.0955 & 0.3937 & 0.1196 & -0.3302 - 0.1892i & -0.3302 + 0.1892i & -0.1172 - 0.0674i & -0.1172 + 0.0674i & -0.5506 - 0.1209i & -0.5506 + 0.1209i \\ -0.1992 & 0.5220 & -0.0401 & 0.3930 - 0.2908i & 0.3930 + 0.2908i & 0.5597 - 0.2765i & 0.5597 + 0.2765i & -0.3308 + 0.1936i & -0.3308 - 0.1936i \\ 0.0740 & 0.0287 & 0.6295 & -0.3587 - 0.3507i & -0.3587 + 0.3507i & -0.7217 - 0.0566i & -0.7217 + 0.0566i & -0.7306 - 0.2136i & -0.7306 + 0.2136i \\ 0.4425 & -0.3609 & -0.5745 & 0.4544 - 0.3119i & 0.4544 + 0.3119i & 0.0909 + 0.0713i & 0.0909 - 0.0713i & 0.5577 + 0.1291i & 0.5577 - 0.1291i \\ 0.4544 & -0.3192 & -0.2461 & -0.3002 - 0.1267i & -0.3002 + 0.1267i & 0.1867 + 0.0254i & 0.1867 - 0.0254i & 0.2866 + 0.1427i & 0.2866 - 0.1427i \\ 0.2597 & 0.3363 & 0.9046 & -0.2398 - 0.0134i & -0.2398 + 0.0134i & -0.5311 - 0.1165i & -0.5311 + 0.1165i & -0.3873 - 0.1096i & -0.3873 + 0.1096i \\ 0.1140 & 0.0966 & 0.0871 & 0.1508 + 0.0275i & 0.1508 - 0.0275i & 0.2624 + 0.0114i & 0.2624 - 0.0114i & -0.6438 + 0.2188i & -0.6438 - 0.2188i \\ -0.0914 & -0.0356 & -0.2387 & -0.1890 - 0.0492i & -0.1890 + 0.0492i & -0.0619 - 0.1504i & -0.0619 + 0.1504i & 0.2787 - 0.2166i & 0.2787 + 0.2166i \\ 0.2431 & 0.5428 & -1.0000 & 0.6652 + 0.3348i & 0.6652 - 0.3348i & 0.3294 - 0.1718i & 0.3294 + 0.1718i & 0.9333 + 0.0667i & 0.9333 - 0.0667i \\ 1.0000 & -0.2458 & 0.2430 & -0.2434 + 0.6061i & -0.2434 - 0.6061i & -0.4812 + 0.5188i & -0.4812 - 0.5188i & 0.6483 - 0.1950i & 0.6483 + 0.1950i \end{bmatrix}.$$

Using Algorithm 1, we obtain the unique solution of Problem BAP as follows:

$$\hat{A} = \begin{bmatrix} 34.2563 & 41.7824 & 33.3573 & 33.6298 & 23.8064 & 42.0770 & 50.0641 & 37.5705 & 31.0908 & 48.6169 & 19.0972 \\ 18.8561 & 35.2252 & 35.9592 & 44.3502 & 31.9918 & 55.2920 & 55.3052 & 54.3793 & 31.3909 & 60.8345 & 16.9540 \\ 29.6359 & 7.6805 & 19.1249 & 17.7183 & 16.7082 & 40.0636 & 18.2916 & 49.9437 & 37.6913 & 15.6027 & 4.9603 \\ 58.8782 & 51.4906 & 47.8974 & 35.6985 & 45.6889 & 56.0434 & 53.0908 & 56.5402 & 55.5120 & 38.3447 & 35.8894 \\ 33.4087 & 46.9635 & 9.7767 & 41.4215 & 51.4466 & 52.1058 & 65.6724 & 60.1293 & 5.8061 & 62.0139 & 16.5231 \\ 31.6580 & 51.2359 & 24.7978 & 65.5567 & 61.7840 & 62.5494 & 58.9363 & 74.7099 & 52.2105 & 55.8532 & 44.3925 \\ 19.2961 & 51.2333 & 22.4280 & 56.9340 & 42.6348 & 45.8453 & 56.3729 & 61.5555 & 31.6836 & 67.9525 & 40.2012 \\ 41.2796 & 71.3821 & 34.4140 & 33.2817 & 77.4393 & 60.8944 & 32.1411 & 108.5056 & 49.6078 & 19.8351 & 85.7434 \\ 64.0890 & 57.6524 & 19.1280 & 25.0394 & 39.0524 & 66.7740 & 20.9023 & 48.8512 & 14.4695 & 18.9284 & 24.8348 \\ 37.2550 & 32.3254 & 38.3534 & 59.7358 & 33.5902 & 54.0265 & 50.7770 & 70.2011 & 65.4159 & 58.0720 & 40.0652 \\ 28.1301 & 14.7638 & 8.9507 & 20.0963 & 25.5907 & 59.6940 & 30.8558 & 66.8781 & 30.4807 & 23.6107 & 12.9984 \end{bmatrix},$$

and

$$\|\hat{A}X - \hat{A}^\top X\Lambda\| = 8.2431 \times 10^{-13}.$$

Therefore, the new model $\hat{A}X = \hat{A}^\top X\Lambda$ reproduces the prescribed eigenvalues (the diagonal elements of the matrix Λ) and eigenvectors (the column vectors of the matrix X).

Example 2. (Example 4.1 of [12]) Given $\alpha = \cos(\theta)$, $\beta = \sin(\theta)$ with $\theta = 0.62$ and $\lambda_1 = 0.2, \lambda_2 = 0.3, \lambda_3 = 0.4$. Let

$$J_0 = \begin{bmatrix} 0_2 & \Gamma \\ I_2 & I_2 \end{bmatrix}, \; J_s = \begin{bmatrix} 0_3 & \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\} \\ I_3 & 0_3 \end{bmatrix},$$

where $\Gamma = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$. We construct

$$\tilde{A} = \begin{bmatrix} J_0 & 0 \\ 0 & J_s \end{bmatrix},$$

the measured eigenvalue and eigenvector matrices Λ and X are given by

$$\Lambda = \text{diag}\{5, 0.2, 0.8139 + 0.5810i, 0.8139 - 0.5810i\},$$

and

$$X = \begin{bmatrix} -0.4155 & 0.6875 & -0.2157 - 0.4824i & -0.2157 + 0.4824i \\ -0.4224 & -0.3148 & -0.3752 + 0.1610i & -0.3752 - 0.1610i \\ -0.0703 & -0.6302 & -0.5950 - 0.4050i & -0.5950 + 0.4050i \\ -1.0000 & -0.4667 & 0.2293 - 0.1045i & 0.2293 + 0.1045i \\ 0.2650 & 0.3051 & -0.2253 + 0.7115i & -0.2253 - 0.7115i \\ 0.9030 & -0.2327 & 0.4862 - 0.3311i & 0.4862 + 0.3311i \\ -0.6742 & 0.3132 & 0.5521 - 0.0430i & 0.5521 + 0.0430i \\ 0.6358 & 0.1172 & -0.0623 - 0.0341i & -0.0623 + 0.0341i \\ -0.4119 & -0.2768 & 0.1575 + 0.4333i & 0.1575 - 0.4333i \\ -0.2062 & 1.0000 & -0.1779 - 0.0784i & -0.1779 + 0.0784i \end{bmatrix}.$$

Using Algorithm 1, we obtain the unique solution of Problem BAP as follows:

$$\hat{A} = \begin{bmatrix} -0.1169 & -0.2366 & 0.6172 & -0.7195 & -0.0836 & 0.2884 & 0.0092 & -0.0490 & -0.0202 & 0.0171 \\ -0.0114 & -0.0957 & 0.1462 & 0.6194 & 0.3738 & -0.1637 & 0.1291 & -0.0071 & 0.0972 & 0.1247 \\ 0.7607 & -0.0497 & 0.5803 & -0.0346 & 0.0979 & 0.2959 & 0.0937 & -0.1060 & 0.1323 & -0.0339 \\ -0.0109 & 0.6740 & -0.3013 & 0.7340 & 0.1942 & -0.0872 & 0.0054 & 0.0051 & 0.0297 & 0.0814 \\ 0.1783 & 0.2283 & 0.2643 & 0.0387 & 0.0986 & -0.3125 & -0.0292 & 0.2926 & -0.0717 & -0.0546 \\ 0.0953 & 0.1027 & 0.0360 & 0.2668 & -0.2418 & 0.1206 & 0.1406 & -0.0551 & 0.3071 & 0.2097 \\ -0.0106 & -0.2319 & 0.1946 & -0.0298 & -0.1935 & 0.0158 & -0.0886 & 0.0216 & -0.0560 & 0.2484 \\ 0.1044 & 0.1285 & 0.1902 & 0.2277 & 0.6961 & 0.1657 & 0.0728 & -0.0262 & -0.0831 & -0.0001 \\ 0.0906 & 0.0021 & 0.0764 & -0.1264 & 0.2144 & 0.6703 & -0.0850 & 0.0764 & -0.0104 & -0.0149 \\ -0.1245 & 0.0813 & 0.1952 & -0.0784 & 0.0760 & -0.0875 & 0.7978 & -0.0093 & 0.0206 & -0.1182 \end{bmatrix},$$

and

$$\|\hat{A}X - \hat{A}^\top X\Lambda\| = 1.7538 \times 10^{-8}.$$

Therefore, the new model $\hat{A}X = \hat{A}^\top X\Lambda$ reproduces the prescribed eigenvalues (the diagonal elements of the matrix Λ) and eigenvectors (the column vectors of the matrix X).

5. Concluding remarks

In this paper, we have developed a direct method to solve the linear inverse palindromic eigenvalue problem by partitioning the matrix Λ and using the QR-decomposition. The explicit best approximation solution is given. The numerical examples show that the proposed method is straightforward and easy to implement.

Conflict of interest

The authors declare no conflict of interest.

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