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Research article

Simultaneous and non-simultaneous quenching for a coupled semilinear parabolic system in a n-dimensional ball with singular localized sources

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Abstract: In this paper, we investigate a coupled semilinear parabolic system with singular localized sources at the point x_0: $u_t - \Delta u = af(v(x_0,t))$, $v_t - \Delta v = bg(u(x_0,t))$ for $x \in B_1(x_0)$ and $t \in (0,T)$ with the Dirichlet boundary condition, where a and b are positive real numbers, $B_1(x_0)$ is a n-dimensional ball with the center and radius being x_0 and 1, and the nonlinear sources f and g are positive functions such that they are unbounded when u and v tend to a positive constant c, respectively. We prove that the solution (u,v) quenches simultaneously and non-simultaneously under some sufficient conditions.

Keywords: quenching; parabolic system; localized source
Mathematics Subject Classification: 35K51, 35K57, 35K58, 35K61, 35K67

1. Introduction

Let a and b be positive real numbers, c be a positive constant, x_0 be a fixed point in a n-dimensional space $\mathbb{R}^n$ with $n = 1, 2, ...,$ and $B_1(x_0)$ be a n-dimensional open ball with the center x_0 and radius 1 such that $B_1(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < 1\}$ where $\|x - x_0\|$ represents the Euclidean distance between x and x_0. We also let $\overline{B_1(x_0)}$ and $\partial B_1(x_0)$ denote the closure and boundary of $B_1(x_0)$, respectively. Let L be the parabolic operator such that $Lu = u_t - \Delta u$. In this paper, we deal with the quenching problem of a coupled semilinear parabolic system with nonlinear singular localized sources at x_0. This problem is described below:

$$\begin{cases} Lu(x,t) = af(v(x_0,t)) \text{ for } x \in B_1(x_0) \text{ and } t > 0, \\ Lv(x,t) = bg(u(x_0,t)) \text{ for } x \in B_1(x_0) \text{ and } t > 0, \end{cases} \tag{1.1}$$

$$\begin{cases} u(x,0) = 0 \text{ for } x \in \overline{B_1(x_0)},\ u(x,t) = 0 \text{ for } x \in \partial B_1(x_0) \text{ and } t > 0, \\ v(x,0) = 0 \text{ for } x \in \overline{B_1(x_0)},\ v(x,t) = 0 \text{ for } x \in \partial B_1(x_0) \text{ and } t > 0. \end{cases} \tag{1.2}$$

In the problem (1.1)–(1.2), we assume that the source functions f and g are differentiable over the interval $[0,c)$ and satisfy the following hypotheses:

(H_1) $f > 0$, $f' > 0$, $f'' > 0$, $g > 0$, $g' > 0$, $g'' > 0$;

(H_2) both f and g being unbounded when u and v tend to c, that is, $f(v) \to \infty$ when $v \to c^-$ (that is, v approaches c from the left) and $g(u) \to \infty$ when $u \to c^-$.

The problem (1.1)–(1.2) describes the instabilities in some dynamic systems of certain reactions that have localized electrodes immersed in a bulk medium at the point x_0, see [1, 12]. Li and Wang [10] used the equation (1.1) to explore a thermal ignition driven by the temperature at a single point. Chadam et al. [2] examined the blow-up set of solutions.

The quenching problem is able to illustrate the polarization phenomena in ionic conductors and the phase transition between liquids and solids, see [11]. We say that the solution (u, v) quenches at a point in $\overline{B_1(x_0)}$ if there exists a finite time T (> 0) such that

$$\max\{u(x,t) : x \in \overline{B_1(x_0)}\} \to c^- \text{ and } \max\{v(x,t) : x \in \overline{B_1(x_0)}\} \to c^- \text{ as } t \to T^-,$$

where $t \to T^-$ represents t approaching T from the left. T is called the quenching time. Quenching and blow-up problems are related. Under some transformations, quenching problems are able to change to blow-up problems, see [5, 6].

Ji et al. [7] studied simultaneous and non-simultaneous quenching of one-dimensional coupled system with the singular nonlinear reaction sources on the boundary. They used this model to describe heat propagations between two different materials. The multi-dimensional quenching problem of coupled semilinear parabolic systems describes non-Newtonian filtration systems incorporated with the effect of singular nonlinear reaction sources inside the domain, see Jia et al. [8]. Their model is

$$\begin{aligned} Lu(x,t) &= (1-u(x,t))^{-p_1} + (1-v(x,t))^{-q_1}, \ x \in \Omega, \ t > 0, \\ Lv(x,t) &= (1-u(x,t))^{-p_2} + (1-v(x,t))^{-q_2}, \ x \in \Omega, \ t > 0, \\ u(x,0) &= u_0(x), \ v(x,0) = v_0(x), \ x \in \bar{\Omega}, \\ u(x,t) &= 0, \ v(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \end{aligned}$$

where p_1, p_2, q_1, and q_2 are positive real numbers, and Ω is a bounded domain in $\mathbb{R}^n$. When $\Omega = B_R(x_0)$, they proved that the solution (u, v) quenches simultaneously if $p_2 \geq p_1 + 1$ and $q_1 \geq q_2 + 1$. Depending on the initial data u_0 and v_0, they also showed that both simultaneous and non-simultaneous quenching may occur when $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$. Zheng and Wang [16] studied simultaneous and non-simultaneous quenching for the coupled system: $Lu = v^{-p}$, $Lv = u^{-q}$ in $B_R(x_0) \times (0, T)$ subject to the Dirichlet boundary condition. When Ω is a square domain in $\mathbb{R}^2$, Chan [3] studied the simultaneous quenching for the coupled system: $Lu = a/(1 - v(0,0,t))$, $Lv = b/(1 - u(0,0,t))$ in $\Omega \times (0, T)$ with the homogeneous first boundary condition. He also computed an approximated critical value of a and b by a numerical method.

The main goals of this paper are to study (a) simultaneous quenching and (b) non-simultaneous quenching of the solution (u, v) under some conditions on $\int_0^c f(\omega)\,d\omega$ and $\int_0^c g(\omega)\,d\omega$. In this article, simultaneous quenching means that the maximum of u and v tends to c in the same finite time. Non-simultaneous quenching means that either the maximum of u or v tends to c in a finite time, but the other remains bounded by c. We are going to study cases (a) and (b) of the problem (1.1)–(1.2) when these two integrals are either infinite or finite. Without loss of generality, let us assume x_0 being the origin 0. The problem (1.1)–(1.2) becomes

$$\begin{cases} Lu = af(v(0,t)) \text{ in } B_1(0) \times (0,T), \\ Lv = bg(u(0,t)) \text{ in } B_1(0) \times (0,T), \end{cases} \tag{1.3}$$

$$\begin{cases} u(x,0)=0 \text{ for } x\in\overline{B_1(0)},\ u(x,t)=0 \text{ for } (x,t)\in\partial B_1(0)\times(0,T), \\ v(x,0)=0 \text{ for } x\in\overline{B_1(0)},\ v(x,t)=0 \text{ for } (x,t)\in\partial B_1(0)\times(0,T). \end{cases} \tag{1.4}$$

Similar consideration is also available in [4, 8, 16]. In section 2, we provide some properties of the solution (u,v). The results of simultaneous and non-simultaneous quenching are going to illustrate in section 3.

2. Properties of the solution

In this section, we are going to show some properties of the solution (u,v). One of the main results is to prove that u and v attain their maximum at $x=0$, and they both quench only at $x=0$. In the sequel, we assume that k_j are positive constants for $j=1,\ 2,...,\ 19$. We also let $Y(x,t)$ and $Z(x,t)$ be nontrivial and nonnegative bounded functions on $\overline{B_1(0)}\times[0,\infty)$. Here is the comparison theorem.

Lemma 2.1. *Assume that (u,v) is the solution to the problem below:*

$$\begin{cases} Lu\geq Y(x,t)\,v(0,t) \ \textit{in}\ B_1(0)\times(0,T), \\ Lv\geq Z(x,t)\,u(0,t) \ \textit{in}\ B_1(0)\times(0,T), \end{cases}$$

$$\begin{cases} u(x,0)=0\,\textit{for}\ x\in\overline{B_1(0)},\ u(x,t)=0\,\textit{for}\ (x,t)\in\partial B_1(0)\times(0,T), \\ v(x,0)=0\,\textit{for}\ x\in\overline{B_1(0)},\ v(x,t)=0\,\textit{for}\ (x,t)\in\partial B_1(0)\times(0,T), \end{cases}$$

then $u(x,t)\geq 0$ and $v(x,t)\geq 0$ on $\overline{B_1(0)}\times[0,T)$.

Proof. Let ε be a positive real number, and

$$\begin{aligned} \Phi(x,t) &= u(x,t)+\varepsilon\hat{\phi}_1(x)\,e^{\gamma t}, \\ \Psi(x,t) &= v(x,t)+\varepsilon\hat{\phi}_1(x)\,e^{\gamma t}, \end{aligned}$$

where γ is a positive real number to be determined and $\hat{\phi}_1$ is the first eigenfunction of the following eigenvalue problem:

$$\Delta\hat{\phi}+\lambda\hat{\phi}=0 \text{ in } B_1(0) \text{ and } \frac{\partial\hat{\phi}}{\partial\nu}+\hat{\phi}=0 \text{ on } \partial B_1(0),$$

where $\partial/\partial\nu$ is the outward normal derivative on $\partial B_1(0)$. Let $\hat{\lambda}_1$ be the corresponding eigenvalue. By Theorem 3.1.2 of [13], $\hat{\phi}_1$ exists and $\hat{\phi}_1>0$ on $\overline{B_1(0)}$ and $\hat{\lambda}_1>0$. Based on the construction, we know that $\Phi(x,0)>0$ and $\Psi(x,0)>0$ on $\overline{B_1(0)}$. By a direct calculation, we obtain the inequality below

$$\begin{aligned} & L\Phi-Y\Psi(0,t) \\ =\ & u_t+\varepsilon\gamma\hat{\phi}_1e^{\gamma t}-\left(\Delta u+\varepsilon\Delta\hat{\phi}_1e^{\gamma t}\right)-Y\left(v(0,t)+\varepsilon\hat{\phi}_1(0)\,e^{\gamma t}\right) \\ \geq\ & \varepsilon e^{\gamma t}\left(\gamma\hat{\phi}_1+\hat{\lambda}_1\hat{\phi}_1-Y\hat{\phi}_1(0)\right). \end{aligned}$$

Since $\hat{\phi}_1>0$ on $\overline{B_1(0)}$, Y is nonnegative and bounded, and $\hat{\lambda}_1>0$, we are able to choose γ such that $\gamma>Y\hat{\phi}_1(0)/\hat{\phi}_1-\hat{\lambda}_1$ in $B_1(0)$. Thus,

$$L\Phi-Y\Psi(0,t)>0 \text{ in } B_1(0)\times(0,T).$$

Suppose $\Phi(x,t)\leq 0$ somewhere in $B_1(0)\times(0,T)$. Then, the set $\{t:\Phi(x,t)\leq 0 \text{ for some } x\in\overline{B_1(0)}\}$ is non-empty. Let $\tilde{t}$ denote the infimum of this set. Then, $0<\tilde{t}<T$ because $\Phi(x,0)>0$ on $\overline{B_1(0)}$.

Thus, there exists some point $x_1 \in B_1(0)$ such that $\Phi(x_1, \tilde{t}) = 0$ and $\Phi_t(x_1, \tilde{t}) \le 0$. On the other hand, Φ attains its local minimum at $(x_1, \tilde{t})$. Then, $\Delta\Phi(x_1, \tilde{t}) \ge 0$. Let us consider $t = \tilde{t}$, we get

$$\Phi_t(x_1, \tilde{t}) - Y(x_1, \tilde{t})\Psi(0, \tilde{t}) \ge L\Phi(x_1, \tilde{t}) - Y(x_1, \tilde{t})\Psi(0, \tilde{t}) > 0. \tag{2.1}$$

Follow a similar argument, if we assume that $\Psi(x, t) \le 0$ somewhere in $B_1(0) \times (0, T)$, then there exist some $\hat{t} \in (0, T)$ and $x_2 \in B_1(0)$ such that $\Psi(x_2, \hat{t}) = 0$, $\Psi_t(x_2, \hat{t}) \le 0$, and Ψ attains its local minimum at $(x_2, \hat{t})$. Then, at $t = \hat{t}$

$$\Psi_t(x_2, \hat{t}) - Z(x_2, \hat{t})\Phi(0, \hat{t}) \ge L\Psi(x_2, \hat{t}) - Z(x_2, \hat{t})\Phi(0, \hat{t}) > 0. \tag{2.2}$$

Let us assume that $\hat{t} < \tilde{t}$. As Φ attains its local minimum at $(x_1, \tilde{t})$, we have $\Phi(0, \hat{t}) > 0$. From the expression (2.2) and Z is nonnegative and bounded, we have the inequality below:

$$0 \ge \Psi_t(x_2, \hat{t}) \ge \Psi_t(x_2, \hat{t}) - Z(x_2, \hat{t})\Phi(0, \hat{t}) > 0.$$

This is a contradiction. Hence, $\Psi(x, t) > 0$ in $B_1(0) \times (0, T)$. Then by (2.1), we show that $\Phi(x, t) > 0$ in $B_1(0) \times (0, T)$. Through a similar calculation, we obtain the same result when $\hat{t} \ge \tilde{t}$. Let $\varepsilon \to 0$, we have $u(x, t) \ge 0$ and $v(x, t) \ge 0$ in $B_1(0) \times (0, T)$. Following the homogeneous initial-boundary conditions, we conclude that u and v are non-negative on $\overline{B_1(0)} \times [0, T)$. The proof is complete. □

By Lemma 2.1, $(0, 0)$ is a lower solution of the problem (1.3)–(1.4). On the other side, $u < c$ and $v < c$ on $\overline{B_1(0)} \times [0, T)$. Since u and v stop to exist for $u \ge c$ and $v \ge c$, it follows from Theorem 2.1 of [2] that the problem (1.3)–(1.4) has the unique classical solution $(u, v) \in C\left(\overline{B_1(0)} \times [0, T)\right) \cap C^{2+\alpha, 1+\alpha/2}(B_1(0) \times [0, T))$ for some $\alpha \in (0, 1)$ such that $0 \le u < c$ and $0 \le v < c$ on $\overline{B_1(0)} \times [0, T)$. As f and g are differentiable, it follows from Theorem 8.9.2 of Pao [13] that the solution (u, v) exists either in a finite time or globally.

Based on the result of Lemma 2.1, we prove u_t and v_t being positive over the domain.

Lemma 2.2. *The solution (u, v) has the properties: (i) $u_t \ge 0$ and $v_t \ge 0$ on $\overline{B_1(0)} \times [0, T)$, and (ii) $u_t > 0$ and $v_t > 0$ in $B_1(0) \times (0, T)$.*

Proof. (i) For $\theta_1 > 0$, let us consider the first equation of the problem (1.3) at $t + \theta_1$. We have $Lu(x, t + \theta_1) = af(v(0, t + \theta_1))$ in $B_1(0) \times (0, T - \theta_1)$. Subtract the first equation of the problem (1.3) from this equation, and based on the mean value theorem, there exists some ζ_1 where ζ_1 is between $v(0, t + \theta_1)$ and $v(0, t)$ such that

$$Lu(x, t + \theta_1) - Lu(x, t) = af'(\zeta_1)[v(0, t + \theta_1) - v(0, t)] \text{ in } B_1(0) \times (0, T - \theta_1).$$

Since $u \ge 0$ on $\overline{B_1(0)} \times [0, T)$, we have $u(x, \theta_1) - u(x, 0) \ge 0$ for $x \in \overline{B_1(0)}$. From the boundary condition, $u(x, t + \theta_1) - u(x, t) = 0$ for $x \in \partial B_1(0)$ and $t > 0$. By Lemma 2.1, $(u(x, t + \theta_1) - u(x, t))/\theta_1 \ge 0$ on $\overline{B_1(0)} \times [0, T - \theta_1)$. As $\theta_1 \to 0^+$, $u_t \ge 0$ on $\overline{B_1(0)} \times [0, T)$. Similarly, we obtain $v_t \ge 0$ on $\overline{B_1(0)} \times [0, T)$.

(ii) To show that u_t is positive, we differentiate the first equation of the problem (1.3) with respect to t to get

$$Lu_t = af'(v(0, t))v_t(0, t) \text{ in } B_1(0) \times (0, T).$$

From (i), we know $v_t \ge 0$ on $\overline{B_1(0)} \times [0, T)$. By (H_1) (see section 1) and the strong maximum principle, we have $u_t > 0$ in $B_1(0) \times (0, T)$. We follow the similar procedure to conclude $v_t > 0$ in $B_1(0) \times (0, T)$. □

By the symmetry of $B_1(0)$, we represent the problem (1.3)–(1.4) in the polar coordinate system

$$\begin{cases} u_t(r,t) - u_{rr}(r,t) - \dfrac{n-1}{r} u_r(r,t) = af(v(0,t)) \text{ in } (0,1)\times(0,T), \\ v_t(r,t) - v_{rr}(r,t) - \dfrac{n-1}{r} v_r(r,t) = bg(u(0,t)) \text{ in } (0,1)\times(0,T), \\ u(r,0) = 0 \text{ for } r\in[0,1],\, u_r(0,t) = 0 \text{ and } u(1,t) = 0 \text{ for } t\in(0,T), \\ v(r,0) = 0 \text{ for } r\in[0,1],\, v_r(0,t) = 0 \text{ and } v(1,t) = 0 \text{ for } t\in(0,T). \end{cases} \tag{2.4}$$

Lemma 2.3. *The solution (u,v) to the problem (2.4) attains its maximum at $r = 0$ for $t \in (0,T)$.*
Proof. It is noticed that the solution to the problem (2.4) is radial symmetric with respect to $r = 0$. To show u and v attaining their maximum at $r = 0$, we are going to prove $u_r < 0$ and $v_r < 0$ for $r \in (0,1]$. We let $H(r,t) = u_r(r,t)$. Differentiating the first equation of the problem (2.4) with respect to r, we have

$$H_t - H_{rr} - \frac{n-1}{r} H_r + \frac{n-1}{r^2} H = 0 \text{ in } (0,1)\times(0,T).$$

At $t = 0$, $H(r,0) = 0$ for $r \in [0,1]$. By Lemma 2.2(ii), $u_t > 0$ in $B_1(0)\times(0,T)$. By Hopf's Lemma, $H(1,t) < 0$ for $t \in (0,T)$. Also, $H(0,t) = u_r(0,t) = 0$ for $t \in [0,T)$. By the maximum principle [13], $H < 0$ for $(r,t) \in (0,1]\times(0,T)$. Therefore, $u(0,t) \ge u(r,t)$ for $(r,t) \in [0,1]\times(0,T)$. Similarly, we prove that $v_r < 0$ for $(r,t) \in (0,1]\times(0,T)$. Hence, u and v achieve their maximum at $r = 0$ for $t \in (0,T)$. □

Let ϕ_1 be the eigenfunction corresponding to the first eigenvalue $\lambda_1\,(>0)$ of the eigenvalue problem below:

$$\Delta\phi + \lambda\phi = 0 \text{ in } B_1(0),\ \phi = 0 \text{ on } \partial B_1(0).$$

This eigenfunction has the properties: $0 < \phi_1 \le 1$ in $B_1(0)$ and $\int_{B_1(0)} \phi_1 dx = 1$ [15]. Let $k_1 = abf''(0)g''(0)/[2(af''(0) + bg''(0))]$ and $k_2 = af(0) + bg(0)$. By (H_1), k_1 and k_2 are positive. We show that either u or v quenches in a finite time.
Lemma 2.4. *If $2\sqrt{k_1k_2} > \lambda_1$, then either u or v quenches on $\overline{B_1(0)}$ in a finite time $\tilde{T}$.*
Proof. By Lemma 2.3, $u(0,t) \ge u(x,t)$ and $v(0,t) \ge v(x,t)$ on $\overline{B_1(0)}\times(0,T)$. Let $\hat{u}(x,t)$ and $\hat{v}(x,t)$ be the solutions to the following auxiliary parabolic system:

$$\begin{cases} L\hat{u} = af(\hat{v}(x,t)) \text{ in } B_1(0)\times(0,T), \\ L\hat{v} = bg(\hat{u}(x,t)) \text{ in } B_1(0)\times(0,T), \end{cases} \tag{2.5}$$

$$\begin{cases} \hat{u}(x,0) = 0 \text{ and } \hat{v}(x,0) = 0 \text{ on } \overline{B_1(0)}, \\ \hat{u}(x,t) = 0 \text{ and } \hat{v}(x,t) = 0 \text{ on } \partial B_1(0)\times(0,T). \end{cases} \tag{2.6}$$

By the comparison theorem [13], $\hat{u}(x,t) \ge 0$ and $\hat{v}(x,t) \ge 0$ on $\overline{B_1(0)}\times(0,T)$. Further, $u - \hat{u}$ and $v - \hat{v}$ satisfy the expression below:

$$\begin{aligned} L(u-\hat{u}) &= af(v(0,t)) - af(\hat{v}(x,t)) \ge af(v(x,t)) - af(\hat{v}(x,t)), \\ L(v-\hat{v}) &= bg(u(0,t)) - bg(\hat{u}(x,t)) \ge bg(u(x,t)) - bg(\hat{u}(x,t)). \end{aligned}$$

By $u - \hat{u} = 0$ and $v - \hat{v} = 0$ on $\overline{B_1(0)}$ and $\partial B_1(0)\times(0,T)$, and the comparison theorem, we have $u \ge \hat{u}$ and $v \ge \hat{v}$ on $\overline{B_1(0)}\times(0,T)$. It suffices to prove either $\hat{u}$ or $\hat{v}$ to quench over $\overline{B_1(0)}$ in a finite time.

Multiplying ϕ_1 on both sides of (2.5) and integrating expressions over the domain $B_1(0)$, we obtain

$$\int_{B_1(0)} \hat{u}_t\phi_1 dx - \int_{B_1(0)} \Delta\hat{u}\phi_1 dx = a\int_{B_1(0)} \phi_1 f(\hat{v}(x,t))\,dx,$$

$$\int_{B_1(0)} \hat{v}_t\phi_1 dx - \int_{B_1(0)} \Delta\hat{v}\phi_1 dx = b\int_{B_1(0)} \phi_1 g(\hat{u}(x,t))\,dx.$$

Using the Green's second identity and (2.6), it gives

$$\left(\int_{B_1(0)} \hat{u}\phi_1 dx\right)_t = -\lambda_1 \int_{B_1(0)} \hat{u}\phi_1 dx + a\int_{B_1(0)} \phi_1 f(\hat{v})\,dx,$$

$$\left(\int_{B_1(0)} \hat{v}\phi_1 dx\right)_t = -\lambda_1 \int_{B_1(0)} \hat{v}\phi_1 dx + b\int_{B_1(0)} \phi_1 g(\hat{u})\,dx.$$

Applying the Maclaurin's series on the functions f and g, we have

$$\left(\int_{B_1(0)} \hat{u}\phi_1 dx\right)_t \geq -\lambda_1 \int_{B_1(0)} \hat{u}\phi_1 dx + a\int_{B_1(0)} \frac{f''(0)}{2}(\hat{v})^2\phi_1 dx + a\int_{B_1(0)} f(0)\,\phi_1 dx,$$

$$\left(\int_{B_1(0)} \hat{v}\phi_1 dx\right)_t \geq -\lambda_1 \int_{B_1(0)} \hat{v}\phi_1 dx + b\int_{B_1(0)} \frac{g''(0)}{2}(\hat{u})^2\phi_1 dx + b\int_{B_1(0)} g(0)\,\phi_1 dx.$$

By $0 < \phi_1 \leq 1$ in $B_1(0)$ and the Jensen's inequality [15], we have

$$\begin{aligned}\int_{B_1(0)} (\hat{v})^2\phi_1 dx &\geq \int_{B_1(0)} (\hat{v})^2(\phi_1)^2 dx \geq \left(\int_{B_1(0)} \hat{v}\phi_1 dx\right)^2,\\ \int_{B_1(0)} (\hat{u})^2\phi_1 dx &\geq \int_{B_1(0)} (\hat{u})^2(\phi_1)^2 dx \geq \left(\int_{B_1(0)} \hat{u}\phi_1 dx\right)^2.\end{aligned}$$

Let $R(t) = \int_{B_1(0)} \hat{u}\phi_1 dx$ and $P(t) = \int_{B_1(0)} \hat{v}\phi_1 dx$. From these two inequalities above, we have the following inequality:

$$\frac{d}{dt}(P+R) \geq -\lambda_1(P+R) + \frac{af''(0)}{2}P^2 + \frac{bg''(0)}{2}R^2 + af(0) + bg(0). \tag{2.7}$$

Then, by the inequality below:

$$\begin{aligned}&\frac{\left(\frac{af''(0)}{2} - k_1\right)P^2 + \left(\frac{bg''(0)}{2} - k_1\right)R^2}{2}\\ \geq\ &\sqrt{\left(\frac{af''(0)}{2} + \frac{bg''(0)}{2}\right)\left[\frac{abf''(0)\,g''(0)}{2\,(af''(0)+bg''(0))} - k_1\right] + k_1^2}\,PR\\ =\ &k_1 PR,\end{aligned}$$

we obtain this expression

$$\frac{af''(0)}{2}P^2 + \frac{bg''(0)}{2}R^2 \geq k_1(P+R)^2.$$

Then, the differential inequality (2.7) becomes

$$\frac{d}{dt}(P+R) \geq -\lambda_1 (P+R) + k_1(P+R)^2 + k_2.$$

Let $E(t) = P(t) + R(t)$. Then, $E(t) \geq 0$ in $[0,T)$ and

$$\frac{d}{dt}E \geq -\lambda_1 E + k_1 E^2 + k_2.$$

Using separation of variables and integrating both sides over $(0,t)$, we obtain

$$t \leq \frac{2}{\sqrt{4k_1k_2 - \lambda_1^2}}\left[\tan^{-1}\left(\frac{2k_1E(t) - \lambda_1}{\sqrt{4k_1k_2 - \lambda_1^2}}\right) + \tan^{-1}\left(\frac{\lambda_1}{\sqrt{4k_1k_2 - \lambda_1^2}}\right)\right].$$

Suppose that $E(t)$ exists for all $t > 0$. By the assumption $2\sqrt{k_1k_2} > \lambda_1$, we have

$$\tan^{-1}\left(\frac{2k_1E(t) - \lambda_1}{\sqrt{4k_1k_2 - \lambda_1^2}}\right) \to \infty \text{ if } t \to \infty.$$

But, $\tan^{-1}\left[(2k_1E(t) - \lambda_1)/\sqrt{4k_1k_2 - \lambda_1^2}\right]$ is bounded above by $\pi/2$. This is a contradiction. It implies that $E(t)$ ceases to exist in a finite time $\hat{T}$. This shows that either $P(t)$ or $R(t)$ does not exist when t tends to $\hat{T}$. Thus, either $\hat{u}$ or $\hat{v}$ quenches on $\overline{B_1(0)}$ at $\hat{T}$. Since $u \geq \hat{u}$ and $v \geq \hat{v}$, we then have either u or v quenches on $\overline{B_1(0)}$ in a finite time $\tilde{T}$ where $\tilde{T} \leq \hat{T}$. □

Let M_1 and M_2 be positive constants such that $M_1/(2n) < c$ and $M_2/(2n) < c$. We are going to prove the global existence of solutions when a and b are sufficiently small. Our method is to construct a global-existed upper solution of the problem (1.1)–(1.2).

Lemma 2.5. *If a and b are sufficiently small, then the solution (u,v) exists globally.*

Proof. It suffices to construct an upper solution which exists all time. Let $\bar{u}(x) = M_1\left(1 - \|x\|^2\right)/(2n)$ and $\bar{v}(x) = M_2\left(1 - \|x\|^2\right)/(2n)$. Clearly, $0 \leq \bar{u}, \bar{v} < c$ for all $x \in \overline{B_1(0)}$. Let us consider the following problem:

$$L\bar{u} - af(\bar{v}(0)) = M_1 - af(M_2/(2n)),$$
$$L\bar{v} - bg(\bar{u}(0)) = M_2 - bg(M_1/(2n)).$$

If a and b are sufficiently small, then

$$L\bar{u} - af(\bar{v}(0)) = M_1 - af(M_2/(2n)) \geq 0 \text{ in } B_1(0) \times (0,\infty),$$
$$L\bar{v} - bg(\bar{u}(0)) = M_2 - bg(M_1/(2n)) \geq 0 \text{ in } B_1(0) \times (0,\infty).$$

Then, we subtract Eq (1.3) from above inequalities and by the mean value theorem to obtain

$$L(\bar{u} - u) \geq a[f(\bar{v}(0)) - f(v(0,t))] = af'(\chi_1)[\bar{v}(0) - v(0,t)] \text{ in } B_1(0) \times (0,\infty),$$
$$L(\bar{v} - v) \geq b[g(\bar{u}(0)) - g(u(0,t))] = bg'(\chi_2)[\bar{u}(0) - u(0,t)] \text{ in } B_1(0) \times (0,\infty),$$

where χ_1 is between $\bar{v}(0)$ and $v(0,t)$ and χ_2 is between $\bar{u}(0)$ and $u(0,t)$. On $\partial B_1(0)$, $\bar{u} - u = 0$ and $\bar{v} - v = 0$. By Lemma 2.1, $u(x,t) \leq \bar{u}(x)$ and $v(x,t) \leq \bar{v}(x)$ on $\overline{B_1(0)} \times [0,\infty)$. Thus, the solution (u,v) exists globally. The proof is complete. □

From the result of Lemma 2.3, we know that $x = 0$ is a quenching point of u and v if they quench. Let T^* be the supremum of the time T for which the problem (1.3)–(1.4) has the unique solution (u, v).

Theorem 2.6. *If $T^* < \infty$, then either $u(0,t)$ or $v(0,t)$ quenches at T^*.*

Proof. Suppose that both u and v do not quench at $x = 0$ when $t = T^*$. Then, there exist k_3 and k_4 such that $u(0,t) \le k_3 < c$ and $v(0,t) \le k_4 < c$ for $t \in [0, T^*]$. This shows that $af(v(0,t)) < k_5$ and $bg(u(0,t)) < k_6$ for $t \in [0, T^*]$. Then, by Theorem 4.2.1 of [9], u and $v \in C^{2+\alpha,1+\alpha/2}\left(\overline{B_1(0)} \times [0, T^*]\right)$. This implies that there exist k_7 and k_8 such that $u(x,t) \le k_7 < c$ and $v(x,t) \le k_8 < c$ for $(x,t) \in \overline{B_1(0)} \times [0, T^*]$. In order to arrive at a contradiction, we need to show that u and v can continue to exist in a longer time interval $[0, T^* + t_1)$ for some positive t_1. This can be accomplished by extending the upper bound of u and v. Let us construct upper solutions $\psi(x,t) = k_7 h(t)$ and $\sigma(x,t) = k_8 i(t)$, where $h(t)$ and $i(t)$ are solutions to the following system:

$$\begin{aligned} \frac{d}{dt} k_7 h(t) &= af(k_8 i(t)) \text{ for } t > T^*,\ h(T^*) = 1, \\ \frac{d}{dt} k_8 i(t) &= bg(k_7 h(t)) \text{ for } t > T^*,\ i(T^*) = 1. \end{aligned}$$

From $af(k_8 i(t)) > 0$ and $bg(k_7 h(t)) > 0$, this implies that $h(t)$ and $i(t)$ are increasing functions of t. Let t_1 be a positive real number determined by $k_7 h(T^* + t_1) = k_9 < c$ and $k_8 i(T^* + t_1) = k_{10} < c$ for some $k_9 (> k_7)$ and $k_{10} (> k_8)$. By our construction, $\psi(x,t) = \psi(0,t)$ and $\sigma(x,t) = \sigma(0,t)$ satisfy

$$\begin{aligned} L\psi(x,t) &= af(\sigma(0,t)) \text{ in } B_1(0) \times (T^*, T^* + t_1), \\ L\sigma(x,t) &= bg(\psi(0,t)) \text{ in } B_1(0) \times (T^*, T^* + t_1), \\ \psi(x,T^*) &= k_7 h(T^*) \ge u(x,T^*) \text{ and } \sigma(x,T^*) = k_8 i(T^*) \ge v(x,T^*) \text{ on } \overline{B_1(0)}, \\ \psi(x,t) &= k_7 h(t) > 0 \text{ and } \sigma(x,t) = k_8 i(t) > 0 \text{ on } \partial B_1(0) \times (T^*, T^* + t_1). \end{aligned}$$

By Lemma 2.1, $\psi(x,t) \ge u(x,t)$ and $\sigma(x,t) \ge v(x,t)$ on $\overline{B_1(0)} \times [T^*, T^* + t_1)$. Therefore, we find the solution (u, v) to the problem (1.3)–(1.4) on $\overline{B_1(0)} \times [T^*, T^* + t_1)$. This contradicts the definition of T^*. Hence, either $u(0,t)$ or $v(0,t)$ quenches at T^*. □

Let $y = u_t$ and $z = v_t$. We differentiate the problem (2.4) with respect to t to obtain the following system

$$\begin{cases} y_t(r,t) - y_{rr}(r,t) - \dfrac{(n-1)}{r} y_r(r,t) = af'(v(0,t))\, z(0,t) \text{ in } (0,1) \times (0,T), \\ z_t(r,t) - z_{rr}(r,t) - \dfrac{(n-1)}{r} z_r(r,t) = ag'(u(0,t))\, y(0,t) \text{ in } (0,1) \times (0,T), \\ y(r,0) \ge 0 \text{ for } r \in [0,1) \text{ and } y(1,0) = 0,\ y(0,t) > 0 \text{ and } y(1,t) = 0 \text{ for } t \in (0,T), \\ z(r,0) \ge 0 \text{ for } r \in [0,1) \text{ and } z(1,0) = 0,\ z(0,t) > 0 \text{ and } z(1,t) = 0 \text{ for } t \in (0,T). \end{cases} \tag{2.8}$$

The result below shows that $u_t(r,t)$ and $v_t(r,t)$ are decreasing functions in r.

Lemma 2.7. *$u_t(r_2,t) < u_t(r_1,t)$ and $v_t(r_2,t) < v_t(r_1,t)$ for $0 < r_1 < r_2 < 1$ and $t \in (0,T)$.*

Proof. We differentiate the first equation of problem (2.8) with respect to r to obtain the following differential equation

$$y_{tr} - y_{rrr} - \frac{(n-1)}{r} y_{rr} + \frac{(n-1)}{r^2} y_r = 0.$$

For $r \in [0,1)$, $u_{rt}(r,0)$ is given by

$$u_{rt}(r,0) = \lim_{\theta_1 \to 0} \frac{u_r(r,\theta_1) - u_r(r,0)}{\theta_1}.$$

Using $u_r(r,0) = 0$ and Lemma 2.3, we have $u_{rt}(r,0) \leq 0$. Thus, $y_r(r,0) \leq 0$ for $r \in [0,1)$. By Lemma 2.2(i),

$$\frac{\partial y(1,0)}{\partial r} = \lim_{\theta_2 \to 0} \frac{y(1,0) - y(1-\theta_2,0)}{\theta_2} \leq 0.$$

By the Hopf's lemma, $\partial y(1,t)/\partial r < 0$ for $t > 0$. By the symmetry of $B_1(0)$ with respect to 0, $\partial y(0,t)/\partial r = 0$ for $t \geq 0$. Let $U = y_r (= u_{rt})$. U satisfies the following initial-boundary value problem:

$$\begin{cases} U_t - U_{rr} - \dfrac{(n-1)}{r} U_r + \dfrac{(n-1)}{r^2} U = 0 \text{ in } (0,1) \times (0,T), \\ U(r,0) \leq 0 \text{ for } r \in [0,1],\ U(0,t) = 0 \text{ and } U(1,t) < 0 \text{ for } t \in (0,T). \end{cases} \tag{2.9}$$

By the maximum principle, $U(r,t) < 0$ for $(0,1] \times (0,T)$. We integrate $U(r,t) < 0$ with respect to r over (r_1, r_2) to yield $y(r_2,t) < y(r_1,t)$. That is, $u_t(r_2,t) < u_t(r_1,t)$ for $0 < r_1 < r_2 < 1$ and $t \in (0,T)$. We follow a similar procedure to obtain $v_t(r_2,t) < v_t(r_1,t)$ for $0 < r_1 < r_2 < 1$ and $t \in (0,T)$. □

Here is the corollary of above lemma. It illustrates that u_t and v_t attain their maximum value at $r = 0$ for $t \in (0,T)$.

Corollary 2.8. *$u_t(r,t) < u_t(0,t)$ and $v_t(r,t) < v_t(0,t)$ for $(r,t) \in (0,1) \times (0,T)$.*

Now, we are going to prove that the solution (u,v) quenches at $x = 0$ only.

Theorem 2.9. *The solution (u,v) quenches only at $x = 0$.*

Proof. To establish this result, we let $V = v_{rt} (= z_r)$ and $t_2 \in (0,T)$. V satisfies the problem (2.9) with U substituting by V. By Lemma 2.7, $U(r_2,t) < 0$ and $V(r_2,t) < 0$ for $r_2 \in (0,1)$ and $t \in [t_2, s)$ where $s \leq T$. Also, $U(r,t_2) < 0$ and $V(r,t_2) < 0$ for $r \in (0, r_2]$. Let J be the parabolic operator such that $JW = W_t - W_{rr} - (n-1) W_r/r + (n-1) W/r^2$. Let us consider the following auxiliary problem below:

$$\begin{cases} JW = 0 \text{ for } (r,t) \in (0,1) \times (t_2, T), \\ W(r,t_2) (= U(r,t_2)) < 0 \text{ for } r \in (0,1),\ W(0,t) = 0 \text{ and } W(1,t) = 0 \text{ for } t \in [t_2, T). \end{cases}$$

By the maximum principle, $W(r,t) < 0$ for $(0,1) \times (t_2,T)$. For $(r,t) \in [0,1] \times [t_2,T)$, the integral representation form of W is given by

$$W(r,t) = \int_0^1 K(r,\xi,t-t_2) W(\xi,t_2) d\xi,$$

where K is the Green's function of the parabolic operator J. K is able to determine using the method of separation of variables and it would be represented in the form of infinite series, see [14]. Since W is negative in $(0,1) \times (t_2,T)$ and K is positive in the set $\{(r,\xi,t) : r$ and ξ are in $(0,1)$, and $t > t_2\}$, there exists a positive constant ρ such that $W(r,t) < -\rho$ for $(r,t) \in (0,1) \times (t_2,T)$. By $U(1,t) < 0$ for $t \in (0,T)$ and the comparison theorem, $U(r,t) \leq W(r,t)$ for $(r,t) \in [0,1] \times [t_2,T)$. Thus, $U(r,t) \leq W(r,t) < -\rho$ for $(r,t) \in (0,1) \times (t_2,T)$. Now, we integrate $U(r,t) (= u_{rt}(r,t)) < -\rho$ with respect to r over (r_3,r_4) and then with respect to t over (t,t_3) where $r_3, r_4 \in (0, r_2]$ to obtain

$$u(r_4,t_3) - u(r_4,t) < u(r_3,t_3) - u(r_3,t) - \rho(r_4 - r_3)(t_3 - t).$$

Since $u_r < 0$ in $(0,1] \times (0,T)$, u has no maximum except $r = 0$. Suppose that u quenches for $r \in (0, 1 - r_2)$. Let us assume that $u(r_3, t)$ and $u(r_4, t)$ both quench at T. Therefore, $u(r_3, t_3) \to c^-$ and $u(r_4, t_3) \to c^-$ as $t_3 \to T^-$. From the above inequality, we have

$$\begin{aligned} \lim_{t_3 \to T^-} u(r_4, t_3) - u(r_4, t) &\leq \lim_{t_3 \to T^-} u(r_3, t_3) - u(r_3, t) - \rho(r_4 - r_3)(T - t) \\ -u(r_4, t) &\leq -u(r_3, t) - \rho(r_4 - r_3)(T - t). \end{aligned}$$

Equivalently,

$$u(r_4, t) > u(r_3, t).$$

This contradicts $u_r(r,t) < 0$ for $(r,t) \in (0,1] \times (0,T)$. Hence, u quenches only at $x = 0$. Similarly, v quenches only at $x = 0$ also. □

3. Simultaneous and non-simultaneous quenching

In this section, we prove the solution (u, v) to quench either (i) simultaneously or (ii) non-simultaneously under some conditions. Let $\varphi_0(x) \in C\left(\overline{B_1(0)}\right) \cap C^2(B_1(0))$ such that $\Delta\varphi_0(x) < 0$, $\varphi_0(x) > 0$ in $B_1(0)$, and $\varphi_0(x) = 0$ on $\partial B_1(0)$ and $\max_{x \in \overline{B_1(0)}} \varphi_0(x) \leq 1$. Let $\varphi(x,t)$ be the solution to the following first initial-boundary value problem:

$$\begin{aligned} Lw &= 0 \text{ in } B_1(0) \times (0, \infty), \\ w(x, 0) &= \varphi_0(x) \text{ on } \overline{B_1(0)},\ w(x,t) = 0 \text{ on } \partial B_1(0) \times (0, \infty). \end{aligned}$$

By the maximum principle, $\varphi(x,t) > 0$ in $B_1(0) \times [0, \infty)$ and is bounded above by $\varphi_0(x)$, and $\varphi(x,t)$ satisfies

$$\max_{(x,t) \in \overline{B_1(0)} \times [0,\infty)} \varphi(x,t) \leq 1.$$

Let $t_4 \in (0,T)$ such that $v(0, t_4) \leq k_{11} < c$. Then,

$$a\varphi(x, t_4) f(k_{11}) \geq a\varphi(x, t_4) f(v(0, t_4)). \tag{3.1}$$

By Lemma 2.2(ii), $u_t(x,t) > 0$ in $B_1(0) \times (0,T)$. Since $u_t(x, t_4) > 0$ and $\varphi(x, t_4) > 0$ in $B_1(0)$, and $u_t(x, t_4) = \varphi(x, t_4) = 0$ on $\partial B_1(0)$, we choose a positive real number η_1 (< 1) such that

$$u_t(x, t_4) \geq a\eta_1\varphi(x, t_4) f(k_{11}) \text{ on } \overline{B_1(0)}. \tag{3.2}$$

Clearly, $u_t(x,t) = a\eta_1\varphi(x,t) f(v(0,t))$ for $(x,t) \in \partial B_1(0) \times [0,T)$. Let $I(x,t) = u_t(x,t) - a\eta_1\varphi(x,t) f(v(0,t))$. By inequalities (3.1) and (3.2), $I(x, t_4) \geq 0$ on $\overline{B_1(0)}$. Let $Q(x,t) = v_t(x,t) - b\eta_2\varphi(x,t) g(u(0,t))$ for some positive η_2 less than 1. We follow a similar computation to get $Q(x, t_4) \geq 0$ on $\overline{B_1(0)}$. We modify the proof of Lemma 3.4 of [4] to obtain the result below.

Lemma 3.1. *$I(x,t) \geq 0$ and $Q(x,t) \geq 0$ on $\overline{B_1(0)} \times [t_4, T)$.*

Proof. By a direct computation,

$$I_t = u_{tt} - a\eta_1\varphi f'(v(0,t)) v_t(0,t) - a\eta_1 f(v(0,t))\varphi_t,$$

$$\Delta I = \Delta u_t - a\eta_1 f(v(0,t))\Delta\varphi.$$

Then, we have

$$LI = af'(v(0,t))\, v_t(0,t)(1-\eta_1\varphi) \text{ in } B_1(0)\times(0,T).$$

By $\varphi \le 1$ on $\overline{B_1(0)}\times[0,\infty)$, $\eta_1 < 1$, and $v_t(0,t) > 0$ for $t \in (0,T)$, it gives $LI \ge 0$ in $B_1(0)\times(0,T)$. In addition, $I(x,t_4) \ge 0$ on $\overline{B_1(0)}$, and $I(x,t) = 0$ on $\partial B_1(0)\times(t_4,T)$. By the maximum principle, $I(x,t) \ge 0$ on $\overline{B_1(0)}\times[t_4,T)$. Similarly, we have $Q(x,t) \ge 0$ on $\overline{B_1(0)}\times[t_4,T)$. □

Now, we provide the result of simultaneous quenching of the solution (u,v) when $\int_0^c f(\omega)\,d\omega = \infty$ and $\int_0^c g(\omega)\,d\omega = \infty$. With these two integrals and (H_2) (see section 1), we know that $\int_m^c f(\omega)\,d\omega = \infty$ and $\int_m^c g(\omega)\,d\omega = \infty$, and $\int_0^m f(\omega)\,d\omega < \infty$ and $\int_0^m g(\omega)\,d\omega < \infty$ for $m \in [0,c)$.

Theorem 3.2. *If $\int_0^c f(\omega)\,d\omega = \infty$ and $\int_0^c g(\omega)\,d\omega = \infty$, and either u or v quenches at $x = 0$ in T, then u and v both quench at $x = 0$ in the same time T.*

Proof. Suppose not, let us assume that $v(0,t)$ quenches at T but $u(0,t)$ remains bounded on $[0,T]$. Then, $0 \le u(0,t) \le k_{12} < c$ for $t \in [0,T]$. From Lemma 3.1, we have

$$u_t(x,t) \ge a\eta_1\varphi(x,t) f(v(0,t)) \text{ on } \overline{B_1(0)}\times[t_4,T),$$

$$v_t(x,t) \ge b\eta_2\varphi(x,t) g(u(0,t)) \text{ on } \overline{B_1(0)}\times[t_4,T).$$

By Lemma 2.3, u and v both attain the maximum at $x = 0$ for $t \in (0,T)$. Then, $\Delta u(0,t) < 0$ and $\Delta v(0,t) < 0$ over $(0,T)$. From the equation (1.3), we obtain the following inequalities:

$$\begin{cases} a\eta_1\varphi(0,t) f(v(0,t)) \le u_t(0,t) < af(v(0,t)), \\ b\eta_2\varphi(0,t) g(u(0,t)) \le v_t(0,t) < bg(u(0,t)). \end{cases} \tag{3.3}$$

By $g > 0$ and $\varphi(0,t) > 0$ for $t \in [0,\infty)$, we divide the first inequality by the second one to achieve

$$\frac{a\eta_1\varphi(0,t) f(v(0,t))}{bg(u(0,t))} \le \frac{du(0,t)}{dv(0,t)} \le \frac{af(v(0,t))}{b\eta_2\varphi(0,t) g(u(0,t))}. \tag{3.4}$$

From the first-half inequality, it yields the expression below:

$$a\eta_1\varphi(0,t) f(v(0,t))\, dv(0,t) \le bg(u(0,t))\, du(0,t).$$

Let δ be a positive real number such that $\delta = \min_{[0,T]}\varphi(0,t)$. Then, we integrate both sides over $[t_4,s)$ for $s \in (t_4,T]$ to attain

$$a\eta_1\delta \int_{v(0,t_4)}^{v(0,s)} f(v(0,t))\, dv(0,t) \le b\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\, du(0,t).$$

When $s \to T^-$, $v(0,s) \to c^-$. By assumption $\int_0^c f(\omega)\,d\omega = \infty$, $\lim_{s\to T^-}\int_{v(0,t_4)}^{v(0,s)} f(v(0,t))\, dv(0,t) = \infty$. If $u(0,s) \le k_{12} < c$ as $s \to T^-$, then there exists k_{13} such that

$$\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\, du(0,t) \le \int_{u(0,t_4)}^{k_{12}} g(u(0,t))\, du(0,t) \le k_{13}.$$

Therefore,

$$a\eta_1\delta \lim_{s\to T^-}\int_{v(0,t_4)}^{v(0,s)} f(v(0,t))\, dv(0,t) \le bk_{13}.$$

It leads to a contradiction. Hence, $u(0,t)$ quenches at T. From the second-half of inequality (3.4) and $\int_0^c g(\omega)\,d\omega = \infty$, we prove that $v(0,t)$ quenches at $t = T$ if $u(0,t)$ quenches. This completes the proof. □

Theorem 3.3. *Suppose that* $\int_0^c f(\omega)\,d\omega < \infty$ *and* $\int_0^c g(\omega)\,d\omega < \infty$*, and depending on a and b, then the following three cases could happen: (i) u and v both quench in* T *at* $x = 0$*, (ii) either u or v quenches in* T *at* $x = 0$*, or (iii) both u and v do not quench.*
Proof. From (3.3), we have the inequality below:

$$b\eta_2\varphi(0,t)\,g(u(0,t))\,u_t(0,t) \le u_t(0,t)\,v_t(0,t) < av_t(0,t)\,f(v(0,t)). \tag{3.5}$$

Thus,

$$b\eta_2\varphi(0,t)\,g(u(0,t))\,u_t(0,t) < av_t(0,t)\,f(v(0,t)).$$

We integrate both sides with respect to t over $[t_4, s)$ for $s \in (t_4, T]$ to obtain

$$b\eta_2\delta\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\,du(0,t) < a\int_{v(0,t_4)}^{v(0,s)} f(v(0,t))\,dv(0,t) < \infty. \tag{3.6}$$

(i) In this case, we prove simultaneous quenching of u and v in T at $x = 0$.

Let us assume that $v(0,t)$ quenches at $t = T$ but $u(0,t)$ remains bounded on $[0,T]$. We integrate the inequality (3.5) with respect to t over $[t_4, s)$ to obtain

$$b\eta_2\int_{t_4}^{s} \varphi(0,t)\,g(u(0,t))\,u_t(0,t)\,dt \le \int_{t_4}^{s} u_t(0,t)\,v_t(0,t)\,dt < a\int_{t_4}^{s} v_t(0,t)\,f(v(0,t))\,dt.$$

By the mean value theorem for definite integrals, there exists $t_5 \in (t_4, s)$ such that $\int_{t_4}^{s} u_t(0,t)\,v_t(0,t)\,dt = v_t(0,t_5)\int_{t_4}^{s} u_t(0,t)\,dt$. This gives

$$b\eta_2\int_{t_4}^{s} \varphi(0,t)\,g(u(0,t))\,u_t(0,t)\,dt \le v_t(0,t_5)\int_{t_4}^{s} u_t(0,t)\,dt < a\int_{v(0,t_4)}^{v(0,s)} f(v(0,t))\,dv(0,t).$$

We evaluate the integral of middle expression to yield

$$b\eta_2\delta\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\,du(0,t) \le v_t(0,t_5)\left[u(0,s) - u(0,t_4)\right].$$

As $v_t(0,t_5) > 0$, it is equivalent to

$$\frac{b\eta_2\delta\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\,du(0,t)}{v_t(0,t_5)} \le u(0,s) - u(0,t_4).$$

By $v_t(0,t) \le bg(u(0,t))$ and $u(0,t)$ remains bounded on $[0,T]$, then there exists k_{14} such that $v_t(0,t_5) \le k_{14}$ for $t_5 \in [t_4, s]$ for $s \in (t_4, T]$. This implies

$$\frac{b\eta_2\delta\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g(u(0,t))\,du(0,t)}{k_{14}} \le \lim_{s\to T^-} u(0,s) - u(0,t_4).$$

If we choose b being sufficiently large such that $b\eta_2\delta\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g\left(u\left(0,t\right)\right)du\left(0,t\right)/k_{14} \ge c$, then we have

$$c \le u\left(0,T\right) - u\left(0,t_4\right).$$

This leads to a contradiction. Therefore, u quenches in T at $x = 0$ also when b is sufficient large. Hence, u and v quench simultaneously in T at $x = 0$.

(ii) We prove non-simultaneous quenching.

Let us assume that both $v\left(0,t\right)$ and $u\left(0,t\right)$ do not quench in any finite time. From the inequality (3.6),

$$b\eta_2\delta\int_{u(0,t_4)}^{u(0,s)} g\left(u\left(0,t\right)\right)du\left(0,t\right) < a\int_{v(0,t_4)}^{v(0,s)} f\left(v\left(0,t\right)\right)dv\left(0,t\right) < \infty.$$

Then, there exists k_{15} such that

$$b\eta_2\delta\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g\left(u\left(0,t\right)\right)du\left(0,t\right) \le ak_{15}.$$

Since $\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g\left(u\left(0,t\right)\right)du\left(0,t\right) < \infty$, we choose a sufficiently large b such that

$$ak_{15} < b\eta_2\delta\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} g\left(u\left(0,t\right)\right)du\left(0,t\right).$$

This leads to a contradiction. Therefore, either u or v quenches in T at $x = 0$, or u and v quench simultaneously at $x = 0$.

Now, let us assume that the solution (u, v) quenches simultaneously at $x = 0$. By Lemma 2.2(ii), $v_t\left(0,t\right) > 0$ for $t > 0$. Then, there exists k_{16} such that $v_t\left(0,t\right) > k_{16}$ for $t \in [t_4, s)$ where $s \in (t_4, T]$. From the inequality (3.5), we have

$$u_t\left(0,t\right)v_t\left(0,t\right) < av_t\left(0,t\right)f\left(v\left(0,t\right)\right).$$

We integrate this expression with respect to t over (t_4, s) to achieve

$$\int_{t_4}^{s} u_t\left(0,t\right)v_t\left(0,t\right)dt < \int_{t_4}^{s} av_t\left(0,t\right)f\left(v\left(0,t\right)\right)dt.$$

We take the limit s to T on both sides and by $v_t\left(0,t\right) > k_{16}$ to get

$$k_{16}\lim_{s\to T^-}\int_{u(0,t_4)}^{u(0,s)} du\left(0,t\right) \le a\int_0^c f\left(\omega\right)d\omega.$$

Evaluating the integration on the left side of the above expression, we have

$$\lim_{s\to T^-} u\left(0,s\right) \le u\left(0,t_4\right) + \frac{a}{k_{16}}\int_0^c f\left(\omega\right)d\omega.$$

Let us assume that $u\left(0,t_4\right) = k_{17}\,(< c)$ and $u\left(0,T\right) = c$. We choose a being small enough so that $\left(a\int_0^c f\left(\omega\right)d\omega\right)/k_{16} < c - k_{17}$. Then,

$$c = u\left(0,T\right) \le u\left(0,t_4\right) + \frac{a}{k_{16}}\int_0^c f\left(\omega\right)d\omega < c.$$

It leads to a contradiction. Hence, u and v quench non-simultaneously at $x = 0$.

(iii) By Lemma 2.5, the solution (u, v) exists globally if a and b are sufficiently small. Thus, both u and v do not quench. □

Theorem 3.4. *Suppose that* $\int_0^c f(\omega)\, d\omega < \infty$ *and* $\int_0^c g(\omega)\, d\omega = \infty$*, then any quenching in the problem (1.3)–(1.4) is non-simultaneous with* $\lim_{s\to T^-} u(0, s) \leq k_{18} < c$*. That is, u does not quench in T at $x = 0$.*

Proof. From the expression (3.4)

$$\frac{a\eta_1\varphi(0,t) f(v(0,t))}{bg(u(0,t))} \leq \frac{du(0,t)}{dv(0,t)} \leq \frac{af(v(0,t))}{b\eta_2\varphi(0,t) g(u(0,t))},$$

we have

$$a\eta_1\varphi(0,t) f(v(0,t))\, dv(0,t) \leq bg(u(0,t))\, du(0,t) \leq \frac{af(v(0,t))}{\eta_2\varphi(0,t)} dv(0,t).$$

Then, we integrate the expression over the time interval $[0, s)$ for $s \in (0, T]$ and by the mean value theorem for definite integrals to give

$$a\eta_1\delta \int_0^{v(0,s)} f(\omega)\, d\omega \leq b \int_0^{u(0,s)} g(\omega)\, d\omega \leq \frac{a}{\eta_2\varphi(0,t_6)} \int_0^{v(0,s)} f(\omega)\, d\omega$$

for some $t_6 \in (0, s)$ with $\varphi(0, t_6) > 0$. Suppose that $v(0, s) \to c^-$ as $s \to T^-$. By assumption $\int_0^c f(\omega)\, d\omega < \infty$, it implies that $\lim_{s\to T^-} \int_0^{u(0,s)} g(u(0,t))\, du(0,t) < \infty$. Thus, $\lim_{s\to T^-} u(0, s) \leq k_{18} < c$. Hence, u does not quench in T at $x = 0$. □

Based on a similar proof of Theorem 3.4, we also prove that any quenching in the problem (1.3)–(1.4) is non-simultaneous with $\lim_{s\to T^-} v(0, s) \leq k_{19} < c$ when $\int_0^c g(\omega)\, d\omega < \infty$ and $\int_0^c f(\omega)\, d\omega = \infty$.

4. Conclusions

In this article, we prove that the solution (u, v) to the problem (1.3)–(1.4) attains its maximum value at the center $x = 0$ over the domain $B_1(0)$. Further, we obtain the main result that $x = 0$ is the only quenching point. Then, we show that the solution (u, v) quenches simultaneously at $x = 0$ when $\int_0^c f(\omega)\, d\omega = \infty$ and $\int_0^c g(\omega)\, d\omega = \infty$. When the integrals $\int_0^c f(\omega)\, d\omega$ and $\int_0^c g(\omega)\, d\omega$ are both finite, the solution (u, v) could quench simultaneously or non-simultaneously, or (u, v) exists globally. When one of the integrals is finite and the other is unbounded, we show that (u, v) quenches non-simultaneously.

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Conflict of interest

The author declares that there are no conflicts of interest in this paper.

References

1. K. Bimpong-Bota, P. Ortoleva, J. Ross, Far-from-equilibrium phenomena at local sites of reaction, *J. Chem. Phys.*, **60** (1974), 3124–3133.
2. J. M. Chadam, A. Peirce, H. M. Yin, The blowup property of solutions to some diffusion equations with localized nonlinear reactions, *J. Math. Anal. Appl.*, **169** (1992), 313–328.
3. W. Y. Chan, Simultaneous quenching for semilinear parabolic system with localized sources in a square domain, *J. Appl. Math. Phys.*, **7** (2019), 1473–1487.
4. C. Chang, Y. Hsu, H. T. Liu, Quenching behavior of parabolic problems with localized reaction term, *Math. Stat.*, **2** (2014), 48–53.
5. K. Deng, H. A. Levine, On the blowup of $u_t$at quenching, *Proc. Amer. Math. Soc.*, **106** (1989), 1049–1056.
6. J. S. Guo, On the quenching behavior and the solution of a semilinear parabolic equation, *J. Math. Anal. Appl.*, **151** (1990), 58–79.
7. R. H. Ji, C. Y. Qu, L. D. Wang, Simultaneous and non-simultaneous quenching for coupled parabolic system, *Appl. Anal.*, **94** (2015), 233–250.
8. Z. Jia, Z. Yang, C. Wang, Non-simultaneous quenching in a semilinear parabolic system with multi-singular reaction terms, *Electron. J. Differ. Equ.*, **100** (2019), 1–13.
9. G. S. Ladde, V. Lakshmikantham, A. S. Vatsala, *Monotone iterative techniques for nonlinear differential equations*, Pitman, 1985, 139.
10. H. Li, M. Wang, Blow-up properties for parabolic systems with localized nonlinear sources, *Appl. Math. Lett.*, **17** (2004), 771–778.
11. N. Nouaili, A Liouville theorem for a heat equation and applications for quenching, *Nonlinearity*, **24** (2011), 797–832.
12. P. Ortoleva, J. Ross, Local structures in chemical reactions with heterogeneous catalysis, *J. Chem. Phys.*, **56** (1972), 4397–4400.
13. C. V. Pao, *Nonlinear parabolic and elliptic equations*, New York: Plenum Press, 1992, pp. 54, 55, 97, and 436.
14. G. F. Roach, *Green's functions*, New York: Cambridge University Press, 1982, 267–268.
15. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, New York: Walter de Gruyter, 1995, 10–11.
16. S. Zheng, W. Wang, Non-simultaneous versus simultaneous quenching in a coupled nonlinear parabolic system, *Nonlinear Anal.*, **69** (2008), 2274–2285.