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Research article

New integral inequalities using exponential type convex functions with applications

Jian Wang[1,2,*], Saad Ihsan But[3,*], Artion Kashuri[4] and Muhammad Tariq[5]

[1] Department of Basic, Quzhou College of Technology, Quzhou 324000, China
[2] College of Teacher Education, Zhejiang Normal University, Jinhua 321004, China
[3] Department of Mathematics, COMSATS University Islamabad, Lahore Campus 54000, Pakistan
[4] Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora 9400, Albania
[5] Mehran University of Engineering and Technology, Jamshoro, Pakistan

* **Correspondence:** Email: qztoby@126.com; saadihsanbutt@gmail.com.

Abstract: In this paper, we establish some new Hermite-Hadamard type inequalities for differential exponential type convex functions and discuss several special cases. Moreover, in order to give the efficient of our main results, some applications for special means and error estimations are obtain.

Keywords: Hermite-Hadamard inequality; power mean inequality; convexity; exponential type convexity; special means; error estimation
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1. Introduction

Let $\psi : I \to \Re$ be a real valued function. A function ψ is said to be convex, if

$$\psi(\chi\mu_1 + (1-\chi)\mu_2) \leq \chi\psi(\mu_1) + (1-\chi)\psi(\mu_2) \tag{1.1}$$

holds for all $\mu_1, \mu_2 \in I$ and $\chi \in [0, 1]$.

Theory of convexity has a lot of applications in pure and applied mathematics and plays an important and fundamental role in the development of various branches of engineering, financial mathematics, economics and optimization. In recent years, the concept of convex functions and their variant forms have been extended and generalized using innovative techniques to study complicated problems. It is well known that convexity is closely related to inequality theory. Many generalizations, variants and extensions for the convexity have attracted the attention of many researchers, see [1–4].

The following remarkable Hermite-Hadamard inequality states that, if $\psi : I \to \Re$ is a convex function for all $\mu_1, \mu_2 \in I$, then

$$\psi\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \psi(x)dx \leq \frac{\psi(\mu_1) + \psi(\mu_2)}{2}. \tag{1.2}$$

For interested readers, see the references [5–18]. In 1929, the notion of exponential convexity was 1^{st} time defined and investigated by Bernstein [19]. After Bernstein, Widder [20] investigated these functions as a subclass of convex functions in a given interval (a, b). The sizeable and worthwhile research on big data analysis and extensive learning has recently increased the attentiveness in information theory involving exponentially convex functions. So especially in the last few decades, different mathematicians namely Antczak (2001), Pecaric (2013), Dragomir (2015), Pal (2017), Alirezaei (2018), Awan (2018), Saima (2019), Noor (2019), Kadakal (2020), worked on the concept of exponential type convexity in different directions and contributed in the field of analysis. Due to aforesaid worked, these functions have proceeded as a remarkable and crucial new class of convex functions, which have noteworthy benefits in technology, data science, information sciences, data mining, statistics, stochastic optimization, statistical learning and sequential prediction. Some notable results and wonderful literature on the term inequalities can be found for the exponential convexity, see [21–25].

In [26], Kadakal and İşcan introduced the following class of convex functions.

Definition 1.1. A nonnegative function $\psi : I \to \Re$, is said to be exponential type convex, if

$$\psi(\chi\mu_1 + (1-\chi)\mu_2) \leq (e^{\chi} - 1)\psi(\mu_1) + \left(e^{1-\chi} - 1\right)\psi(\mu_2) \tag{1.3}$$

holds for all $\mu_1, \mu_2 \in I$ and $\chi \in [0, 1]$.

Moreover, authors in [26], proved the following Proposition 1 that will be used in Section 4.

Proposition 1. *Every nonnegative convex function is exponential type convex function.*

Functions, like $\psi_1(x) = x^s$ where $s > 1$; $\psi_2(x) = e^x$ and $\psi_3(x) = \frac{1}{x}$ for all $x > 0$, are exponential type convex.

The article consists of five sections. In Section 2 we recall two lemmas for deriving our main results. In Section 3 several new Hermite-Hadamard type integral inequalities for differential exponential type convex functions will be established and some special cases will be given as well. In Section 4, by using the main results, we will obtain some applications for special means and error estimations as well. Section 5 concludes the article finally.

2. Preliminaries

The following notations will be used in the sequel.
Let denote I° the interior of I and $L[\mu_1, \mu_2]$ the set of all integrable functions on $[\mu_1, \mu_2]$.

In order to prove our main results regarding some Hermite-Hadamard type inequalities for differential exponential type convex function, we need the following lemmas.

Lemma 2.1. *([27]) Let $\psi : I \subseteq \Re \to \Re$ be differentiable on I° and let $\rho, \sigma \in \Re, \mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. Assume that $\psi' \in L[\mu_1, \mu_2]$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&\epsilon\rho\psi(\mu_1) + (\mu_2 - \mu_1 - \epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2 - \mu_1 - \epsilon)(1-\sigma)]\psi(\mu_1 + \epsilon) \\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx \\
&= \int_0^1 [\epsilon^2(1-\rho-\chi)\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon)) \\
&+ (\mu_2 - \mu_1 - \epsilon)^2(\sigma - \chi)\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)]d\chi.
\end{aligned}$$

Lemma 2.2. *([14]) Let $m > 0$ and $0 \le s \le 1$. Then*

$$\begin{aligned}
\int_0^1 |s-\chi|^m d\chi &= \frac{s^{m+1} + (1-s)^{m+1}}{m+1}, \\
\int_0^1 \chi|s-\chi|^m d\chi &= \frac{s^{m+2} + (m+1+s)(1-s)^{m+1}}{(m+1)(m+2)}.
\end{aligned}$$

3. Main results

Using Lemma 2.1 and Lemma 2.2, we have the following new results.

Theorem 3.1. *Let $\psi : I \subseteq \Re \to \Re$ be differentiable on I° and let $\rho, \sigma \in [0,1], \mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. Assume that $\psi' \in L[\mu_1, \mu_2]$ and $0 < \epsilon < \mu_2 - \mu_1$. If $|\psi'|^q$ is exponential type convex on $[\mu_1, \mu_2]$ with $q \ge 1$, then*

$$\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1) + (\mu_2 - \mu_1 - \epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2 - \mu_1 - \epsilon)(1-\sigma)]\psi(\mu_1 + \epsilon) \\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx\Big| \\
&\le \epsilon^2\Big(\frac{(1-\rho)^2 + \rho^2}{2}\Big)^{1-\frac{1}{q}}\Big\{\frac{1}{2}\Big(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\Big)|\psi'(\mu_1)|^q \\
&+ \frac{1}{2}\Big(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2\Big(\frac{\sigma^2 + (1-\sigma)^2}{2}\Big)^{1-\frac{1}{q}} \\
&\times \Big\{\frac{1}{2}\Big(2C_3(\sigma,\epsilon) - 2\sigma^2 + 2\sigma - 1\Big)|\psi'(\mu_1)|^q + \frac{1}{2}\Big(2C_4(\sigma,\epsilon) - 2\sigma^2 + 2\sigma - 1\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}, \qquad (3.1)
\end{aligned}$$

where

$$C_1(\rho,\epsilon) := \int_0^1 |1-\rho-\chi| e^{\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi,$$

$$C_2(\rho,\epsilon) := \int_0^1 |1-\rho-\chi| e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}} d\chi,$$

$$C_3(\sigma,\epsilon) := \int_0^1 |\sigma-\chi| e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi,$$

and

$$C_4(\sigma,\epsilon) := \int_0^1 |\sigma-\chi| e^{1-\left(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\right)} d\chi.$$

Proof. Consider that $|\psi'|^q$ is exponential type convex on $[\mu_1,\mu_2]$ with $q \geq 1$. Using Lemma 2.1 and property of the modulus, we have

$$\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1) + (\mu_2-\mu_1-\epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2-\mu_1-\epsilon)(1-\mu)]\psi(\mu_1+\epsilon) \\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx\Big| \\
&\leq \epsilon^2 \int_0^1 |1-\rho-\chi||\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon))|d\chi \\
&+ (\mu_2-\mu_1-\epsilon)^2 \int_0^1 |\sigma-\chi||\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)|d\chi.
\end{aligned}$$

Case ($q = 1$). This first case doesn't need Hölder's inequality. Using exponential type convexity of $|\psi'|$ and Lemma 2.2, we get

$$\begin{aligned}
&\int_0^1 |1-\rho-\chi||\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon))|d\chi \\
&= \int_0^1 |1-\rho-\chi|\left|\psi'\left(\left(1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\right)\mu_1 + \frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\mu_2\right)\right|d\chi \\
&\leq \int_0^1 |1-\rho-\chi|\left[\left(e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}} - 1\right)|\psi'(\mu_1)| + \left(e^{\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}} - 1\right)|\psi'(\mu_2)|\right]d\chi \\
&= \int_0^1 |1-\rho-\chi| e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}|\psi'(\mu_1)|d\chi + \int_0^1 |1-\rho-\chi| e^{\frac{\epsilon(1-\chi)}{b-\mu_1}}|\psi'(\mu_2)|d\chi \\
&- \int_0^1 |1-\rho-\chi|(|\psi'(\mu_1)| + |\psi'(\mu_2)|)d\chi \\
&= e^{1-\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_1)| \int_0^1 |1-\rho-\chi| e^{\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi + e^{\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_2)| \int_0^1 |1-\rho-\chi| e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}} d\chi \\
&- \int_0^1 |1-\rho-\chi|(|\psi'(\mu_1)| + |\psi'(\mu_2)|)d\chi \\
&= C_1(\rho,\epsilon) e^{1-\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_1)| + C_2(\rho,\epsilon) e^{\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_2)| - \frac{(1-\rho)^2+\rho^2}{2}(|\psi'(\mu_1)| + |\psi'(\mu_2)|) \\
&= \frac{1}{2}\left(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\right)|\psi'(\mu_1)| + \frac{1}{2}\left(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\right)|\psi'(\mu_2)|.
\end{aligned}$$

Similarly,

$$\int_0^1 |\sigma-\chi||\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)|d\chi$$

$$
\begin{aligned}
&= \int_0^1 |\sigma-\chi|\Big|\psi'\Big(\Big(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\mu_1+\Big(1-\Big(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\Big)\mu_2\Big)\Big|d\chi \\
&\le \int_0^1 |\sigma-\chi|\Big\{(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}-1)|\psi'(\mu_1)|+(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}-1)|\psi'(\mu_2)|\Big\}d\chi \\
&= \int_0^1 |\sigma-\chi|(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}|\psi'(\mu_1)|d\chi+\int_0^1 |\sigma-\chi|(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}|\psi'(\mu_2)|d\chi \\
&- \int_0^1 |\sigma-\chi|(|\psi'(\mu_1)|+|\psi'(\mu_2)|)d\chi \\
&= C_3(\sigma,\epsilon)|\psi'(\mu_1)|+C_4(\sigma,\epsilon)|\psi'(\mu_2)|-\frac{\sigma^2+(1-\sigma)^2}{2}(|\psi'(\mu_1)|+|\psi'(\mu_2)|) \\
&= \frac{1}{2}\Big(2C_3(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_1)|+\frac{1}{2}\Big(2C_4(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_2)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\sigma\psi(\mu_2)+[\epsilon(1-\rho)+(\mu_2-\mu_1-\epsilon)(1-\mu)]\psi(\mu_1+\epsilon) \\
&-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big| \\
&\le \frac{\epsilon^2}{2}\Big\{\Big(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_1)|+\Big(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_2)|\Big\} \\
&+\frac{(\mu_2-\mu_1-\epsilon)^2}{2}\Big\{\Big(2C_3(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_1)|+\Big(2C_4(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_2)|\Big\}.
\end{aligned}
$$

Case ($q>1$). By Hölder's inequality and Lemma 2.2, we obtain

$$
\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\sigma\psi(\mu_2)+[\epsilon(1-\rho)+(\mu_2-\mu_1-\epsilon)(1-\sigma)]\psi(\mu_1+\epsilon) \\
&-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big| \\
&\le \epsilon^2\Big(\int_0^1|1-\rho-\chi|d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1|1-\rho-\chi||\psi'(\chi\mu_1+(1-\chi)(\mu_1+\epsilon))|^q d\chi\Big)^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\Big(\int_0^1|\sigma-\chi|d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\sigma-\chi||\psi'(\chi(\mu_1+\epsilon)+(1-\chi)\mu_2)|^q d\chi\Big)^{\frac{1}{q}} \\
&= \epsilon^2\Big(\frac{(1-\rho)^2+\rho^2}{2}\Big)^{1-\frac{1}{q}}\Big(\int_0^1|1-\rho-\chi||\psi'(\chi\mu_1+(1-\chi)(\mu_1+\epsilon))|^q d\chi\Big)^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\Big(\frac{\sigma^2+(1-\sigma)^2}{2}\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\sigma-\chi||\psi'(\chi(\mu_1+\epsilon)+(1-\chi)\mu_2)|^q d\chi\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Applying exponential type convexity of $|\psi'|^q$ and Lemma 2.2, we have

$$
\begin{aligned}
&\int_0^1|1-\rho-\chi||\psi'(\chi\mu_1+(1-\chi)(\mu_1+\epsilon))|^q d\chi \\
&= \int_0^1|1-\rho-\chi|\Big|\psi'\Big(\Big(1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\Big)\mu_1+\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\mu_2\Big)\Big|^q d\chi
\end{aligned}
$$

$$\leq \int_0^1 |1-\rho-\chi|\Big[(e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}-1)|\psi'(\mu_1)|^q+(e^{\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}-1)|\psi'(\mu_2)|^q\Big]d\chi$$
$$=\int_0^1 |1-\rho-\chi|e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q d\chi+\int_0^1 |1-\rho-\chi|e^{\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}|\psi'(\mu_2)|^q d\chi$$
$$-\int_0^1 |1-\rho-\chi|(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)d\chi$$
$$=e^{1-\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q\int_0^1 |1-\rho-\chi|e^{\frac{\epsilon\chi}{\mu_2-\mu_1}}d\chi+e^{\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_2)|^q\int_0^1 |1-\rho-\chi|e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}}d\chi$$
$$-\int_0^1 |1-\rho-\chi|(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)d\chi$$
$$=C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q+C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_2)|^q-\frac{(1-\rho)^2+\rho^2}{2}(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)$$
$$=\frac{1}{2}\Big(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_2)|^q.$$

Similarly,

$$\int_0^1 |\sigma-\chi||\psi'(\chi(\mu_1+\epsilon)+(1-\chi)\mu_2)|^q d\chi$$
$$=\int_0^1 |\sigma-\chi|\left|\psi'\Big(\Big(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\mu_1+\Big(1-\Big(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\Big)\mu_2\Big)\right|^q d\chi$$
$$\leq \int_0^1 |\sigma-\chi|\Big[(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}-1)|\psi'(\mu_1)|^q+(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}-1)|\psi'(\mu_2)|^q\Big]d\chi$$
$$=\int_0^1 |\sigma-\chi|(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q d\chi+\int_0^1 |\sigma-\chi|(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}|\psi'(\mu_2)|^q d\chi$$
$$-\int_0^1 |\sigma-\chi|(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)d\chi$$
$$=C_3(\sigma,\epsilon)|\psi'(\mu_1)|^q+C_4(\sigma,\epsilon)|\psi'(\mu_2)|^q-\frac{\sigma^2+(1-\sigma)^2}{2}(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)$$
$$=\frac{1}{2}\Big(2C_3(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_4(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_2)|^q.$$

Thus

$$\Big|\epsilon\rho\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\sigma\psi(\mu_2)+[\epsilon(1-\rho)+(\mu_2-\mu_1-\epsilon)(1-\sigma)]\psi(\mu_1+\epsilon)$$
$$-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|$$
$$\leq \epsilon^2\Big(\frac{(1-\rho)^2+\rho^2}{2}\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\frac{1}{2}\Big(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-2\rho^2+2\rho-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}$$
$$+(\mu_2-\mu_1-\epsilon)^2\Big(\frac{\sigma^2+(1-\sigma)^2}{2}\Big)^{1-\frac{1}{q}}$$

$$\times\left\{\frac{1}{2}\Big(2C_3(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_4(\sigma,\epsilon)-2\sigma^2+2\sigma-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}},$$

which completes the proof. □

Remark 1. We have also calculated all the coefficients $C_1(\rho,\epsilon)$, $C_2(\rho,\epsilon)$, $C_3(\sigma,\epsilon)$ and $C_4(\sigma,\epsilon)$ of Theorem 3.1 using software Maple 18. They are given as:

$$C_1(\rho,\epsilon):=\int_0^1|1-\rho-\chi|e^{\frac{\epsilon\chi}{\mu_2-\mu_1}}d\chi$$

$$=-\frac{e^{\frac{\epsilon}{\mu_2-\mu_1}}\Big[\mu_1\rho\epsilon e^{\frac{\epsilon}{\mu_1-\mu_2}}-\mu_2\rho\epsilon e^{\frac{\epsilon}{\mu_1-\mu_2}}+\mu_1^2e^{\frac{\epsilon}{\mu_1-\mu_2}}-2\mu_1\mu_2e^{\frac{\epsilon}{\mu_1-\mu_2}}-\mu_1\epsilon e^{\frac{\epsilon}{\mu_1-\mu_2}}+\mu_2^2e^{\frac{\epsilon}{\mu_1-\mu_2}}+\mu_2\epsilon e^{\frac{\epsilon}{\mu_1-\mu_2}}}{\epsilon^2}$$

$$-\frac{-2\mu_1^2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}+4\mu_1\mu_2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}-2\mu_2^2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}+\mu_1\epsilon\rho-\mu_2\epsilon\rho+(\mu_2-\mu_1)^2\Big]}{\epsilon^2};$$

$$C_2(\rho,\epsilon):=\int_0^1|1-\rho-\chi|e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}}d\chi$$

$$=\frac{e^{\frac{\epsilon\rho}{\mu_2-\mu_1}}\Big[2\mu_1\mu_2e^{\frac{\epsilon(\rho+1)}{\mu_1-\mu_2}}+\mu_1\epsilon\rho e^{\frac{\epsilon(\rho+1)}{\mu_1-\mu_2}}-\mu_2\epsilon\rho e^{\frac{\epsilon(\rho+1)}{\mu_1-\mu_2}}-\mu_1\epsilon e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}+\mu_2\epsilon e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}-\mu_1^2e^{\frac{\epsilon(\rho+1)}{\mu_1-\mu_2}}-\mu_2^2e^{\frac{\epsilon(\rho+1)}{\mu_1-\mu_2}}}{\epsilon^2}$$

$$+\frac{+\mu_1\epsilon\rho e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}-\mu_2\epsilon\rho e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}-\mu_1^2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}-\mu_2^2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}+2\mu_1\mu_2e^{\frac{\epsilon\rho}{\mu_1-\mu_2}}+2\mu_1^2e^{\frac{\epsilon}{\mu_1-\mu_2}}+2\mu_2^2e^{\frac{\epsilon}{\mu_1-\mu_2}}-4\mu_1\mu_2e^{\frac{\epsilon}{\mu_1-\mu_2}}\Big]}{\epsilon^2};$$

$$C_3(\sigma,\epsilon):=\int_0^1|\sigma-\chi|e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}d\chi$$

$$=-\frac{e^{\frac{\sigma\mu_2}{\mu_2-\mu_1}}\Big[\mu_1^2\sigma e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}-2\mu_1\mu_2\sigma e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}+\mu_1\sigma\epsilon e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}+\mu_2^2\sigma e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}-\mu_2\sigma\epsilon e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}+\mu_1^2\sigma e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$

$$+\frac{-2\mu_1\mu_2\sigma e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}+\mu_1\sigma\epsilon e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}+\mu_2^2\sigma e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}-\mu_2\sigma\epsilon e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}+\mu_1^2e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$

$$+\frac{-2\mu_1\mu_2e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}+\mu_2^2e^{\frac{\sigma\mu_2}{\mu_1-\mu_2}}-2\mu_1^2e^{\frac{\sigma(\mu_1+\epsilon)}{\mu_1-\mu_2}}+4\mu_1\mu_2e^{\frac{\sigma(\mu_1+\epsilon)}{\mu_1-\mu_2}}-2\mu_2^2e^{\frac{\sigma(\mu_1+\epsilon)}{\mu_1-\mu_2}}+(\mu_2-\mu_1)\epsilon e^{\frac{\sigma\mu_2+\mu_1-\mu_2+\epsilon}{\mu_1-\mu_2}}\Big]}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$

and

$$C_4(\sigma,\epsilon):=\int_0^1|\sigma-\chi|e^{1-\left(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}\right)}d\chi$$

$$=\frac{e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2}{\mu_2-\mu_1}}\Big[-4\mu_1\mu_2\sigma e^{\frac{\sigma\mu_2+\mu_1}{\mu_1-\mu_2}}+2\mu_1\mu_2e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_1}{\mu_1-\mu_2}}+4\mu_1\mu_2e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_2-\mu_1}}+\mu_1\epsilon\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_1}{\mu_1-\mu_2}}}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$

$$+\frac{-\mu_2\epsilon\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_1}{\mu_1-\mu_2}}+\mu_1\epsilon\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_1-\mu_2}}-\mu_2\epsilon\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_1-\mu_2}}-2(\mu_1^2+\mu_2^2)e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_1-\mu_2}}}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$
$$+\frac{-(\mu_1^2+\mu_2^2-\mu_1^2\sigma-\mu_2^2\sigma)e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_1}{\mu_1-\mu_2}}+(\mu_1^2\sigma+\mu_2^2\sigma-\mu_1\epsilon+\mu_2\epsilon)e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_1-\mu_2}}}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}$$
$$+\frac{-2\mu_1\mu_2\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_1}{\mu_1-\mu_2}}-2\mu_1\mu_2\sigma e^{\frac{\sigma\mu_1+\epsilon\sigma+\mu_2-\epsilon}{\mu_1-\mu_2}}+2(\mu_1^2+\mu_2^2)e^{\frac{\sigma\mu_2+\mu_1}{\mu_1-\mu_2}}\Bigg]}{\mu_1^2-2\mu_1\mu_2+2\mu_1\epsilon+\mu_2^2-2\mu_2\epsilon+\epsilon^2}.$$

Let us derive from Theorem 3.1 some new trapezium and midpoint type inequalities using special values of ρ, σ and suitable choices of ϵ.

Corollary 1. *Taking $\rho = \sigma = 0$ in Theorem 3.1, then*

$$\begin{aligned}
&2^{1-\frac{1}{q}}(\mu_2-\mu_1)\left|\psi(\mu_1+\epsilon)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\right| \\
&\le \epsilon^2\left\{\frac{1}{2}\Big(2C_1(0,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_2(0,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{\frac{1}{2}\Big(2C_3(0,\epsilon)-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_4(0,\epsilon)-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2. *Taking $\rho = \sigma = \frac{1}{2}$ in Theorem 3.1, then*

$$\begin{aligned}
&4^{1-\frac{1}{q}}\left|\frac{\epsilon\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\psi(\mu_2)+(\mu_2-\mu_1)\psi(\mu_1+\epsilon)}{2}-\int_{\mu_1}^{\mu_2}\psi(x)dx\right| \\
&\le \epsilon^2\left\{\frac{1}{2}\Big(2C_1\Big(\frac{1}{2},\epsilon\Big)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{2}\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_2\Big(\frac{1}{2},\epsilon\Big)e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{2}\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{\frac{1}{2}\Big(2C_3\Big(\frac{1}{2},\epsilon\Big)-\frac{1}{2}\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_4\Big(\frac{1}{2},\epsilon\Big)-\frac{1}{2}\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3. *Taking $\rho = \sigma = 1$ in Theorem 3.1, then*

$$\begin{aligned}
&2^{1-\frac{1}{q}}\left|\epsilon\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\psi(\mu_2)-\int_{\mu_1}^{\mu_2}\psi(x)dx\right| \\
&\le \epsilon^2\left\{\frac{1}{2}\Big(2C_1(1,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_2(1,\epsilon)e^{\frac{\epsilon}{\mu_2-a}}-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{\frac{1}{2}\Big(2C_3(1,\epsilon)-1\Big)|\psi'(\mu_1)|^q+\frac{1}{2}\Big(2C_4(1,\epsilon)-1\Big)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 4. *Taking $\epsilon = \frac{\mu_2-\mu_1}{2}$ in Theorem 3.1, then*

$$\begin{aligned}
&\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2}+\frac{2-\rho-\sigma}{2}\psi\Big(\frac{\mu_1+\mu_2}{2}\Big)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\right| \\
&\le \frac{(\mu_2-\mu_1)^2}{8}\Big((1-\rho)^2+\rho^2\Big)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left\{\left(2C_1\left(\rho,\frac{\mu_2-\mu_1}{2}\right)e^{\frac{1}{2}}-2\rho^2+2\rho-1\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_2\left(\rho,\frac{\mu_2-\mu_1}{2}\right)e^{\frac{1}{2}}-2\rho^2+2\rho-1\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}$$
$$+\frac{(\mu_2-\mu_1)^2}{8}\Big(\sigma^2+(1-\sigma)^2\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\left(2C_3\left(\sigma,\frac{\mu_2-\mu_1}{2}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_4\left(\sigma,\frac{\mu_2-\mu_1}{2}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.$$

Corollary 5. *Taking $\epsilon=\frac{\mu_2-\mu_1}{3}$ in Theorem 3.1, then*

$$\left|\frac{\rho\psi(\mu_1)+2\sigma\psi(\mu_2)}{3}+\Big(1-\frac{\rho}{3}-\frac{2\sigma}{3}\Big)\psi\Big(\frac{2\mu_1+\mu_2}{3}\Big)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\right|$$
$$\leq\frac{(\mu_2-\mu_1)^2}{18}\Big((1-\rho)^2+\rho^2\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\left(2C_1\left(\rho,\frac{\mu_2-\mu_1}{3}\right)e^{\frac{2}{3}}-2\lambda^2+2\lambda-1\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_2\left(\rho,\frac{\mu_2-\mu_1}{3}\right)e^{\frac{2}{3}}-2\rho^2+2\rho-1\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}$$
$$+\frac{2(\mu_2-\mu_1)^2}{9}\Big(\sigma^2+(1-\sigma)^2\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\left(2C_3\left(\sigma,\frac{\mu_2-\mu_1}{3}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_4\left(\sigma,\frac{\mu_2-\mu_1}{3}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.$$

Corollary 6. *Taking $\epsilon=\frac{2(\mu_2-\mu_1)}{3}$ in Theorem 3.1, then*

$$\left|\frac{2\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{3}+\Big(1-\frac{2\rho}{3}-\frac{\sigma}{3}\Big)\psi\Big(\frac{\mu_1+2\mu_2}{3}\Big)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\right|$$
$$\leq\frac{2(\mu_2-\mu_1)^2}{9}\Big((1-\rho)^2+\rho^2\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\left(2C_1\left(\rho,\frac{2(\mu_2-\mu_1)}{3}\right)e^{\frac{1}{3}}-2\rho^2+2\rho-1\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_2\left(\rho,\frac{2(\mu_2-\mu_1)}{3}\right)e^{\frac{1}{3}}-2\rho^2+2\rho-1\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}$$
$$+\frac{(\mu_2-\mu_1)^2}{18}\Big(\sigma^2+(1-\sigma)^2\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\left(2C_3\left(\sigma,\frac{2(\mu_2-\mu_1)}{3}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_1)|^q\right.$$
$$\left.+\left(2C_4\left(\sigma,\frac{2(\mu_2-\mu_1)}{3}\right)-2\sigma^2+2\sigma-1)\right)|\psi'(\mu_2)|^q\right\}^{\frac{1}{q}}.$$

Theorem 3.2. *Let $\psi : I \subseteq \Re \to \Re$ be differentiable on I° and let $\rho, \sigma \in [0,1], \mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. Assume that $\psi' \in L[\mu_1, \mu_2]$ and $0 < \epsilon < \mu_2 - \mu_1$. If $|\psi'|^q$ is exponential type convex on $[\mu_1, \mu_2]$ with $q \geq 1$, then*

$$\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1) + (\mu_2 - \mu_1 - \epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2 - \mu_1 - \epsilon)(1-\sigma)]\psi(\mu_1 + \epsilon) \\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx\Big| \\
&\leq \epsilon^2 \Bigg\{\Bigg(G_5(q;\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^{q+1} + \rho^{q+1}}{q+1}\Bigg)|\psi'(\mu_1)|^q \\
&+ \Bigg(G_6(q;\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^{q+1} + \rho^{q+1}}{q+1}\Bigg)|\psi'(\mu_2)|^q\Bigg\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \Bigg\{\Bigg(G_7(q;\sigma,\epsilon) - \frac{\sigma^{q+1} + (1-\sigma)^{q+1}}{q+1}\Bigg)|\psi'(\mu_1)|^q \\
&+ \Bigg(G_8(q;\sigma,\epsilon) - \frac{\sigma^{q+1} + (1-\sigma)^{q+1}}{q+1}\Bigg)|\psi'(\mu_2)|^q\Bigg\}^{\frac{1}{q}},
\end{aligned} \tag{3.2}$$

where

$$G_5(q;\rho,\epsilon) := \int_0^1 |1-\rho-\chi|^q e^{\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi, \quad G_6(q;\rho,\epsilon) := \int_0^1 |1-\rho-\chi|^q e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}} d\chi,$$
$$G_7(q;\sigma,\epsilon) := \int_0^1 |\sigma-\chi|^q e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi, \quad G_8(q;\sigma,\epsilon) := \int_0^1 |\sigma-\chi|^q e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})} d\chi.$$

Proof. Consider that $|\psi'|^q$ is exponential type convex on $[\mu_1, \mu_2]$ with $q \geq 1$. If $q = 1$, then using Theorem 3.1, we have

$$\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1) + (\mu_2 - \mu_1 - \epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2 - \mu_1 - \epsilon)(1-\sigma)]\psi(\mu_1 + \epsilon) \\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx\Big| \\
&\leq \frac{\epsilon^2}{2}\Bigg[\Bigg(2C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\Bigg)|\psi'(\mu_1)| \\
&+ \Bigg(2C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - 2\rho^2 + 2\rho - 1\Bigg)|\psi'(\mu_2)|\Bigg] \\
&+ \frac{(\mu_2 - \mu_1 - \epsilon)^2}{2} \\
&\times \Bigg[\Bigg(2C_3(\sigma,\epsilon) - 2\sigma^2 + 2\sigma - 1\Bigg)|\psi'(\mu_1)| + \Bigg(2C_4(\sigma,\epsilon) - 2\sigma^2 + 2\sigma - 1\Bigg)|\psi'(\mu_2)|\Bigg] \\
&= \epsilon^2\Bigg[\Bigg(C_1(\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^2 + \rho^2}{2}\Bigg)|\psi'(\mu_1)| \\
&+ \Bigg(C_2(\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^2 + \rho^2}{2}\Bigg)|\psi'(\mu_2)|\Bigg] \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \\
&\times \Bigg[\Bigg(C_3(\sigma,\epsilon) - \frac{\sigma^2 + (1-\sigma)^2}{2}\Bigg)|\psi'(\mu_1)| + \Bigg(C_4(\sigma,\epsilon) - \frac{\sigma^2 + (1-\sigma)^2}{2}\Bigg)|\psi'(\mu_2)|\Bigg].
\end{aligned}$$

Next, we consider that $q > 1$. Using Lemma 2.2 and the well-known power mean inequality, we get

$$\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1) + (\mu_2 - \mu_1 - \epsilon)\sigma\psi(\mu_2) + [\epsilon(1-\rho) + (\mu_2 - \mu_1 - \epsilon)(1-\sigma)]\psi(\mu_1 + \epsilon)\\
&- \int_{\mu_1}^{\mu_2} \psi(x)dx\Big|\\
&\leq \epsilon^2 \int_0^1 |1-\rho-\chi||\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon))|d\chi\\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \int_0^1 |\sigma - \chi||\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)|d\chi\\
&\leq \epsilon^2\Big(\int_0^1 d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1 |1-\rho-\chi|^q|\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon))|^q d\chi\Big)^{\frac{1}{q}}\\
&+ (\mu_2 - \mu_1 - \epsilon)^2\Big(\int_0^1 d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1 |\sigma-\chi|^q|\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)|^q d\chi\Big)^{\frac{1}{q}}.
\end{aligned}$$

Using exponential type convexity of $|\psi'|^q$ and Lemma 2.2, we obtain

$$\begin{aligned}
&\int_0^1 |1-\rho-\chi|^q|\psi'(\chi\mu_1 + (1-\chi)(\mu_1+\epsilon))|^q d\chi\\
&= \int_0^1 |1-\rho-\chi|^q\Big|\psi'\Big(\Big(1 - \frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\Big)\mu_1 + \frac{\epsilon(1-\chi)}{\mu_2-\mu_1}\mu_2\Big)\Big|^q d\chi\\
&\leq \int_0^1 |1-\rho-\chi|^q\Big[\Big(e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}} - 1\Big)|\psi'(\mu_1)|^q + \Big(e^{\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}} - 1\Big)|\psi'(\mu_2)|^q\Big]d\chi\\
&= \int_0^1 |1-\rho-\chi|^q e^{1-\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q d\chi + \int_0^1 |1-\rho-\chi|^q e^{\frac{\epsilon(1-\chi)}{\mu_2-\mu_1}}|\psi'(\mu_2)|^q d\chi\\
&- \int_0^1 |1-\rho-\chi|^q(|\psi'(\mu_1)|^q + |\psi'(\mu_2)|^q)d\chi\\
&= e^{1-\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q \int_0^1 |1-\rho-\chi|^q e^{\frac{\epsilon\chi}{\mu_2-\mu_1}} d\chi + e^{\frac{\epsilon}{\mu_2-\mu_1}}|\psi'(\mu_2)|^q \int_0^1 |1-\rho-\chi|^q e^{\frac{-\epsilon\chi}{\mu_2-\mu_1}} d\chi\\
&- \int_0^1 |1-\rho-\chi|^q(|\psi'(\mu_1)|^q + |\psi'(\mu_2)|^q)d\chi\\
&= \Big(G_5(q;\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^{q+1} + \rho^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q\\
&+ \Big(G_6(q;\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{(1-\rho)^{q+1} + \rho^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 |\sigma-\chi|^q|\psi'(\chi(\mu_1+\epsilon) + (1-\chi)\mu_2)|^q d\chi\\
&= \int_0^1 |\sigma-\chi|^q\Big|\psi'\Big(\Big(\chi - \frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\mu_1 + \Big(1 - \Big(\chi - \frac{\epsilon\chi}{\mu_2-\mu_1}\Big)\Big)\mu_2\Big)\Big|^q d\chi
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 |\sigma-\chi|^q\Big\{(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}-1)|\psi'(\mu_1)|^q+(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}-1)|\psi'(\mu_2)|^q\Big\}d\chi\\
&=\int_0^1 |\sigma-\chi|^q(e^{\chi-\frac{\epsilon\chi}{\mu_2-\mu_1}}|\psi'(\mu_1)|^q d\chi+\int_0^1 |\sigma-\chi|^q(e^{1-(\chi-\frac{\epsilon\chi}{\mu_2-\mu_1})}|\psi'(\mu_2)|^q d\chi\\
&-\int_0^1 |\sigma-\chi|^q(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)d\chi\\
&=G_7(q;\sigma,\epsilon)|\psi'(\mu_1)|^q+G_8(q;\sigma,\epsilon)|\psi'(\mu_2)|^q-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}(|\psi'(\mu_1)|^q+|\psi'(\mu_2)|^q)\\
&=\Big(G_7(q;\sigma,\epsilon)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q+\Big(G_8(q;\sigma,\epsilon)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Big|\epsilon\rho\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\sigma\psi(\mu_2)+[\epsilon(1-\rho)+(\mu_2-\mu_1-\epsilon)(1-\sigma)]\psi(\mu_1+\epsilon)\\
&-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|\\
&\leq \epsilon^2\Big\{\Big(G_5(q;\rho,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-\frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q\\
&+\Big(G_6(q;\rho,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}\\
&+(\mu_2-\mu_1-\epsilon)^2\Big\{\Big(G_7(q;\sigma,\epsilon)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q\\
&+\Big(G_8(q;\sigma,\epsilon)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof. □

Let us establish from Theorem 3.2 some new trapezium and midpoint type inequalities using special values of ρ, σ and suitable choices of ϵ.

Corollary 7. *Taking $\rho = \sigma = 0$ in Theorem 3.2, then*

$$
\begin{aligned}
&\Big|(\mu_2-\mu_1)\psi(\mu_1+\epsilon)-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|\\
&\leq \epsilon^2\Big\{\Big(G_5(q;0,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\Big)|\psi'(\mu_1)|^q+\Big(G_6(q;0,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}\\
&+(\mu_2-\mu_1-\epsilon)^2\Big\{\Big(G_7(q;0,\epsilon)-\frac{1}{q+1}\Big)|\psi'(\mu_1)|^q+\Big(G_8(q;0,\epsilon)-\frac{1}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 8. *Taking $\rho = \sigma = \frac{1}{2}$ in Theorem 3.2, then*

$$
\Big|\frac{\epsilon\psi(\mu_1)+(\mu_2-\mu_1-\epsilon)\psi(\mu_2)+(\mu_2-\mu_1)\psi(\mu_1+\epsilon)}{2}-\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|
$$

$$\le \epsilon^2\Big\{\Big(G_5\Big(q;\frac{1}{2},\epsilon\Big)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - \frac{2(\frac{1}{2})^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$
$$+\Big(G_6\Big(q;\frac{1}{2},\epsilon\Big)e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{2(\frac{1}{2})^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}$$
$$+(\mu_2-\mu_1-\epsilon)^2\Big\{\Big(G_7\Big(q;\frac{1}{2},\epsilon\Big) - \frac{2(\frac{1}{2})^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$
$$+\Big(G_8\Big(q;\frac{1}{2},\epsilon\Big) - \frac{2(\frac{1}{2})^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}.$$

Corollary 9. *Taking $\rho = \sigma = 1$ in Theorem 3.2, then*

$$\Big|\epsilon\psi(\mu_1) + (\mu_2-\mu_1-\epsilon)\psi(\mu_2) - \int_{\mu_1}^{\mu_2}\psi(x)dx\Big|$$
$$\le \epsilon^2\Big\{\Big(G_5(q;1,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - \frac{1}{q+1}\Big)|\psi'(\mu_1)|^q + \Big(G_6(q;1,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{1}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}$$
$$+(\mu_2-\mu_1-\epsilon)^2\Big\{\Big(G_7(q;1,\epsilon) - \frac{1}{q+1}\Big)|\psi'(\mu_1)|^q + \Big(G_8(q;1,\epsilon) - \frac{1}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}.$$

Corollary 10. *Taking $\epsilon = \frac{\mu_2-\mu_1}{2}$ in Theorem 3.2, then*

$$\Big|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2} + \frac{2-\rho-\sigma}{2}\psi\Big(\frac{\mu_1+\mu_2}{2}\Big) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|$$
$$\le \frac{(\mu_2-\mu_1)^2}{4}$$
$$\times\Big\{\Big(G_5\Big(q;\rho,\frac{\mu_2-\mu_1}{2}\Big)e^{\frac{1}{2}} - \frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$
$$+\Big(G_6\Big(q;\rho,\frac{\mu_2-\mu_1}{2}\Big)e^{\frac{1}{2}} - \frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q$$
$$+\Big(G_7\Big(q;\sigma,\frac{\mu_2-\mu_1}{2}\Big) - \frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$
$$+\Big(G_8\Big(q;\sigma,\frac{\mu_2-\mu_1}{2}\Big) - \frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}.$$

Corollary 11. *Taking $\epsilon = \frac{\mu_2-\mu_1}{3}$ in Theorem 3.2, then*

$$\Big|\frac{\rho\psi(\mu_1)+2\sigma\psi(\mu_2)}{3} + \Big(1-\frac{\rho}{3}-\frac{2\sigma}{3}\Big)\psi\Big(\frac{2\mu_1+\mu_2}{3}\Big) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\Big|$$
$$\le \frac{(\mu_2-\mu_1)^2}{9}\Big\{\Big(G_5\Big(q;\rho,\frac{\mu_2-\mu_1}{3}\Big)e^{\frac{2}{3}} - \frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$
$$+\Big(G_6\Big(q;\rho,\frac{\mu_2-\mu_1}{3}\Big)e^{\frac{2}{3}} - \frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\Big)|\psi'(\mu_2)|^q\Big\}^{\frac{1}{q}}$$
$$+\frac{4(\mu_2-\mu_1)^2}{9}\Big\{\Big(G_7\Big(q;\sigma,\frac{\mu_2-\mu_1}{3}\Big) - \frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\Big)|\psi'(\mu_1)|^q$$

$$+\left(G_8\left(q;\sigma,\frac{\mu_2-\mu_1}{3}\right)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\right)|\psi'(\mu_2)|^q\Bigg\}^{\frac{1}{q}}.$$

Corollary 12. *Taking $\epsilon=\frac{2(\mu_2-\mu_1)}{3}$ in Theorem 3.2, then*

$$\left|\frac{2\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{3}+\left(1-\frac{2\rho}{3}-\frac{\sigma}{3}\right)\psi\left(\frac{\mu_1+2\mu_2}{3}\right)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx\right|$$
$$\leq\frac{4(\mu_2-\mu_1)^2}{9}\Bigg\{\left(G_5\left(q;\rho,\frac{2(\mu_2-\mu_1)}{3}\right)e^{\frac{1}{3}}-\frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\right)|\psi'(\mu_1)|^q$$
$$+\left(G_6\left(q;\rho,\frac{2(\mu_2-\mu_1)}{3}\right)e^{\frac{1}{3}}-\frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\right)|\psi'(\mu_2)|^q\Bigg\}^{\frac{1}{q}}$$
$$+\frac{(\mu_2-\mu_1)^2}{9}\Bigg\{\left(G_7\left(q;\sigma,\frac{2(\mu_2-\mu_1)}{3}\right)-\frac{\sigma^{q+1}+(1-\mu)^{q+1}}{q+1}\right)|\psi'(\mu_1)|^q$$
$$+\left(G_8\left(q;\sigma,\frac{2(\mu_2-\mu_1)}{3}\right)-\frac{\mu^{q+1}+(1-\sigma)^{q+1}}{q+1}\right)|\psi'(\mu_2)|^q\Bigg\}^{\frac{1}{q}}.$$

4. Applications

In this section, we suppose that $\{\mu_1,\mu_2,w_{\mu_1},w_{\mu_2}\}\subseteq(0,\infty)$ with $\mu_1<\mu_2$ and $0<\epsilon<\mu_2-\mu_1$. The following special means will be used in the sequel:

The weighted arithmetic mean of $\{\mu_1,\mu_2\}$ with weight $\{w_{\mu_1},w_{\mu_2}\}$ is given by

$$\mathcal{A}(\mu_1,\mu_2;w_{\mu_1},w_{\mu_2})=\frac{w_{\mu_1}\mu_1+w_{\mu_2}\mu_2}{w_{\mu_1}+w_{\mu_2}}.$$

The weighted geometric mean of $\{\mu_1,\mu_2\}$ with weight $\{w_{\mu_1},w_{\mu_2}\}$ is defined as

$$\mathcal{G}(\mu_1,\mu_1;w_{\mu_1},w_{\mu_1})=\mu_1^{\frac{w_{\mu_1}}{w_{\mu_1}+w_{\mu_2}}}\mu_2^{\frac{w_{\mu_2}}{w_{\mu_1}+w_{\mu_2}}}.$$

The generalized logarithmic mean of $\{\mu_1,\mu_2\}$ is given by

$$\mathcal{L}_s(\mu_1,\mu_2)=\left(\frac{\mu_2^{s+1}-\mu_1^{s+1}}{(s+1)(\mu_2-\mu_1)}\right)^{\frac{1}{s}},\quad s\neq0,\ s\neq-1.$$

The identric mean of $\{\mu_1,\mu_2\}$ is defined as

$$\mathcal{I}(\mu_1,\mu_2)=\frac{1}{e}\left(\frac{\mu_2^{\mu_2}}{\mu_1^{\mu_1}}\right)^{\frac{1}{\mu_2-\mu_1}}.$$

Before giving our next results using above special means let investigate the following functions: $\phi_1(x)=\frac{q}{s+q}x^{\frac{s}{q}+1}$ for $s>1$, $q\geq1$ and $\phi_2(x)=\ln x$ for all $x>0$. $|\phi_1'(x)|^q=x^s$ is nonnegative convex function for $s>1$, $x>0$ and from Proposition 1 it's exponential type convex. Similarly, $|\phi_2'(x)|^q=x^{-q}$ is nonnegative convex function for $q\geq1$, $x>0$ and from Proposition 1 it's exponential type convex as well.

Proposition 2. *Suppose that $s > 1$, $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&\frac{2^{1-\frac{1}{q}} q(\mu_2 - \mu_1)}{s+q} \left| 2^{\frac{s}{q}+1} \mathcal{A}^{\frac{s}{q}+1}(\mu_1, \epsilon; 1, 1) - \mathcal{L}_{\frac{s}{q}+1}^{\frac{s}{q}+1}(\mu_1, \mu_2) \right| \\
&\leq \epsilon^2 \left\{ \frac{1}{2}\left(2C_1(0,\epsilon) e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - 1\right)\mu_1^s + \frac{1}{2}\left(2C_2(0,\epsilon) e^{\frac{\epsilon}{\mu_2-\mu_1}} - 1\right)\mu_2^s \right\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \left\{ \frac{1}{2}\left(2C_3(0,\epsilon) - 1\right)\mu_1^s + \frac{1}{2}\left(2C_4(0,\epsilon) - 1\right)\mu_2^s \right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.1}$$

Proof. Taking $\psi(x) = \frac{q}{s+q} x^{\frac{s}{q}+1}$ for $x > 0$ and applying Corollary 1, then inequality (4.1) is easily captured. □

Proposition 3. *Suppose that $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&2^{1-\frac{1}{q}}(\mu_2 - \mu_1) \left| \ln\left(\frac{\mathcal{A}(\mu_1, \mu_2; \mu_2 - \mu_1 - \epsilon, \epsilon)}{\mathcal{I}(\mu_1, \mu_2)} \right) \right| \\
&\leq \epsilon^2 \left\{ C_1(0,\epsilon)\mu_1^{-q} e^{1-\frac{\epsilon}{\mu_2-\mu_1}} + C_2(0,\epsilon)\mu_2^{-q} e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{\mu_1^{-q} + \mu_2^{-q}}{2} \right\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \left\{ C_3(0,\epsilon)\mu_1^{-q} + C_4(0,\epsilon)\mu_2^{-q} - \frac{\mu_1^{-q} + \mu_2^{-q}}{2} \right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.2}$$

Proof. Choosing $\psi(x) = \ln x$ for $x > 0$ and applying Corollary 1, then inequality (4.2) is obtained. □

Proposition 4. *Suppose that $s > 1$, $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&\frac{2^{1-\frac{1}{q}} q(\mu_2 - \mu_1)}{s+q} \left| \mathcal{A}\left(\mu_1^{\frac{s}{q}+1}, \mu_2^{\frac{s}{q}+1}; \epsilon, \mu_2 - \mu_1 - \epsilon \right) - \mathcal{L}_{\frac{s}{q}+1}^{\frac{s}{q}+1}(\mu_1, \mu_2) \right| \\
&\leq \epsilon^2 \left\{ \frac{1}{2}\left(2C_1(1,\epsilon) e^{1-\frac{\epsilon}{\mu_2-\mu_1}} - 1\right)\mu_1^s + \frac{1}{2}\left(2C_2(1,\epsilon) e^{\frac{\epsilon}{\mu_2-a}} - 1\right)\mu_2^s \right\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \left\{ \frac{1}{2}\left(2C_3(1,\epsilon) - 1\right)\mu_1^s + \frac{1}{2}\left(2C_4(1,\epsilon) - 1\right)\mu_2^s \right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.3}$$

Proof. Taking $\psi(x) = \frac{q}{s+q} x^{\frac{s}{q}+1}$ for $x > 0$ and applying Corollary 3, then inequality (4.3) is derived. □

Proposition 5. *Suppose that $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&2^{1-\frac{1}{q}}(\mu_2 - \mu_1) \left| \ln\left(\frac{\mathcal{A}(\mu_1, \mu_2; \epsilon, \mu_2 - \mu_1 - \epsilon)}{\mathcal{I}(\mu_1, \mu_2)} \right) \right| \\
&\leq \epsilon^2 \left\{ C_1(1,\epsilon)\mu_1^{-q} e^{1-\frac{\epsilon}{\mu_2-\mu_1}} + C_2(1,\epsilon)\mu_2^{-q} e^{\frac{\epsilon}{\mu_2-\mu_1}} - \frac{\mu_1^{-q} + \mu_2^{-q}}{2} \right\}^{\frac{1}{q}} \\
&+ (\mu_2 - \mu_1 - \epsilon)^2 \left\{ C_3(1,\epsilon)\mu_1^{-q} + C_4(1,\epsilon)\mu_2^{-q} - \frac{\mu_1^{-q} + \mu_2^{-q}}{2} \right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.4}$$

Proof. Choosing $\psi(x) = \ln x$ for $x > 0$ and applying Corollary 3, then inequality (4.4) is captured. □

Proposition 6. *Suppose that $s > 1$, $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&\frac{q(\mu_2-\mu_1)}{s+q}\left|2^{\frac{s}{q}+1}\mathcal{A}^{\frac{s}{q}+1}(\mu_1,\epsilon;1,1)-\mathcal{L}_{\frac{s}{q}+1}^{\frac{s}{q}+1}(\mu_1,\mu_2)\right| \\
&\leq \epsilon^2\left\{\left(G_5(q;0,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\right)\mu_1^s+\left(G_6(q;0,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\right)\mu_2^s\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{\left(G_7(q;0,\epsilon)-\frac{1}{q+1}\right)\mu_1^s+\left(G_8(q;0,\epsilon)-\frac{1}{q+1}\right)\mu_2^s\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.5}$$

Proof. Taking $\psi(x) = \frac{q}{s+q}x^{\frac{s}{q}+1}$ for $x > 0$ and applying Corollary 7, then inequality (4.5) is obtained. □

Proposition 7. *Suppose that $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&(\mu_2-\mu_1)\left|\ln\left(\frac{\mathcal{G}(\mu_1,\mu_2;\mu_2-\mu_1-\epsilon,\epsilon)}{\mathcal{I}(\mu_1,\mu_2)}\right)\right| \\
&\leq \epsilon^2\left\{G_5(q;0,\epsilon)\mu_1^{-q}e^{1-\frac{\epsilon}{\mu_2-\mu_1}}+G_6(q;0,\epsilon)\mu_2^{-q}e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{\mu_1^{-q}+\mu_2^{-q}}{q+1}\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{G_7(q;0,\epsilon)\mu_1^{-q}+G_8(q;0,\epsilon)\mu_2^{-q}-\frac{\mu_1^{-q}+\mu_2^{-q}}{q+1}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.6}$$

Proof. Choosing $\psi(x) = \ln x$ for $x > 0$ and applying Corollary 7, then inequality (4.6) is derived. □

Proposition 8. *Suppose that $s > 1$, $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&\frac{q(\mu_2-\mu_1)}{s+q}\left|\mathcal{A}\left(\mu_1^{\frac{s}{q}+1},\mu_2^{\frac{s}{q}+1};\epsilon,\mu_2-\mu_1-\epsilon\right)-\mathcal{L}_{\frac{s}{q}+1}^{\frac{s}{q}+1}(\mu_1,\mu_2)\right| \\
&\leq \epsilon^2\left\{\left(G_5(q;1,\epsilon)e^{1-\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\right)\mu_1^s+\left(G_6(q;1,\epsilon)e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{1}{q+1}\right)\mu_2^s\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{\left(G_7(q;1,\epsilon)-\frac{1}{q+1}\right)\mu_1^s+\left(G_8(q;1,\epsilon)-\frac{1}{q+1}\right)\mu_2^s\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.7}$$

Proof. Taking $\psi(x) = \frac{q}{s+q}x^{\frac{s}{q}+1}$ for $x > 0$ and applying Corollary 9, then inequality (4.7) is captured. □

Proposition 9. *Suppose that $q \geq 1$ with $0 < \mu_1 < \mu_2$ and $0 < \epsilon < \mu_2 - \mu_1$. Then*

$$\begin{aligned}
&(\mu_2-\mu_1)\left|\ln\left(\frac{\mathcal{G}(\mu_1,\mu_2;\epsilon,\mu_2-\mu_1-\epsilon)}{\mathcal{I}(\mu_1,\mu_2)}\right)\right| \\
&\leq \epsilon^2\left\{G_5(q;1,\epsilon)\mu_1^{-q}e^{1-\frac{\epsilon}{\mu_2-\mu_1}}+G_6(q;1,\epsilon)\mu_2^{-q}e^{\frac{\epsilon}{\mu_2-\mu_1}}-\frac{\mu_1^{-q}+\mu_2^{-q}}{q+1}\right\}^{\frac{1}{q}} \\
&+(\mu_2-\mu_1-\epsilon)^2\left\{G_7(q;1,\epsilon)\mu_1^{-q}+G_8(q;1,\epsilon)\mu_2^{-q}-\frac{\mu_1^{-q}+\mu_2^{-q}}{q+1}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.8}$$

Proof. Choosing $\psi(x) = \ln x$ for $x > 0$ and applying Corollary 9, then inequality (4.8) is obtained. □

Remark 2. For other suitable exponential convex functions interested reader can find several new interesting inequalities using special means from our results. We omit here their proofs.

At the end, from integral inequalities obtained above we will find some new bounds regarding error estimation for quadrature formula. For $0 < \epsilon < \mu_2 - \mu_1$ and $\rho, \sigma \in [0,1]$, let $\mathcal{P} : \mu_1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = \mu_2$ be a partition of $[\mu_1, \mu_2]$. We denote

$$\begin{aligned}
\mathcal{T}(\mathcal{P}, \psi) &:= \sum_{i=0}^{n-1} \{\epsilon_i \rho \psi(x_i) + (h_i - \epsilon_i)\sigma\psi(x_{i+1}) \\
&+ [\epsilon_i(1-\rho) + (h_i - \epsilon_i)(1-\sigma)]\psi(x_i + \epsilon_i)\}, \\
\int_{\mu_1}^{\mu_2} \psi(x)dx &= \mathcal{T}(\mathcal{P}, \psi) + \mathcal{R}(\mathcal{P}, \psi),
\end{aligned} \tag{4.9}$$

where $\mathcal{R}(\mathcal{P}, \psi)$ is the remainder term and $h_i = x_{i+1} - x_i$ for $i = 0, 1, 2, \ldots, n-1$. Using above notations, we are in position to prove the following error estimations.

Proposition 10. *Let $\psi : I \subseteq \Re \to \Re$ be differentiable on I° and let $\rho, \sigma \in [0,1], \mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. Assume that $\psi' \in L[\mu_1, \mu_2]$ and $0 < \epsilon < \mu_2 - \mu_1$. If $|\psi'|^q$ is exponential type convex on $[\mu_1, \mu_2]$ with $q \geq 1$, then*

$$\begin{aligned}
\Big|\mathcal{R}(\mathcal{P}, \psi)\Big| &\leq \sum_{i=0}^{n-1} \epsilon_i^2 \Big(\frac{(1-\rho)^2 + \rho^2}{2}\Big)^{1-\frac{1}{q}} \Big\{\frac{1}{2}\Big(2C_{1,i}(\rho, \epsilon_i)e^{1-\frac{\epsilon_i}{h_i}} - 2\rho^2 + 2\rho - 1\Big)|\psi'(x_i)|^q \\
&+ \frac{1}{2}\Big(2C_{2,i}(\rho, \epsilon_i)e^{\frac{\epsilon_i}{h_i}} - 2\rho^2 + 2\rho - 1\Big)|\psi'(x_{i+1})|^q\Big\}^{\frac{1}{q}} \\
&+ \sum_{i=0}^{n-1}(h_i - \epsilon_i)^2 \Big(\frac{\sigma^2 + (1-\sigma)^2}{2}\Big)^{1-\frac{1}{q}} \\
&\times \Big\{\frac{1}{2}\Big(2C_{3,i}(\sigma, \epsilon_i) - 2\sigma^2 + 2\sigma - 1\Big)|\psi'(x_i)|^q \\
&+ \frac{1}{2}\Big(2C_{4,i}(\sigma, \epsilon_i) - 2\sigma^2 + 2\sigma - 1\Big)|\psi'(x_{i+1})|^q\Big\}^{\frac{1}{q}},
\end{aligned} \tag{4.10}$$

where $0 < \epsilon_i < h_i$ for all $i = 0, 1, 2, \ldots, n-1$, and

$$C_{1,i}(\rho, \epsilon_i) := \int_0^1 |1 - \rho - \chi| e^{\frac{\epsilon_i \chi}{h_i}} d\chi, \quad C_{2,i}(\rho, \epsilon_i) := \int_0^1 |1 - \rho - \chi| e^{\frac{-\epsilon_i \chi}{h_i}} d\chi,$$

$$C_{3,i}(\sigma, \epsilon_i) := \int_0^1 |\sigma - \chi| e^{\chi - \frac{\epsilon_i \chi}{h_i}} d\chi, \quad C_{4,i}(\sigma, \epsilon_i) := \int_0^1 |\sigma - \chi| e^{1-\left(\chi - \frac{\epsilon_i \chi}{h_i}\right)} d\chi.$$

Proof. By applying Theorem 3.1 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \ldots, n-1$) of the partition $\mathcal{P}$ and summing the obtain inequality over i from 0 to $n-1$, we have the desired result. □

Proposition 11. *Let $\psi : I \subseteq \Re \to \Re$ be differentiable on I° and let $\rho, \sigma \in [0,1], \mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$. Assume that $\psi' \in L[\mu_1, \mu_2]$ and $0 < \epsilon < \mu_2 - \mu_1$. If $|\psi'|^q$ is exponential type convex on $[\mu_1, \mu_2]$ with $q \geq 1$, then*

$$\Big|\mathcal{R}(\mathcal{P}, \psi)\Big| \leq \sum_{i=0}^{n-1} \epsilon_i^2 \Big\{\Big(G_{5,i}(q; \rho, \epsilon_i)e^{1-\frac{\epsilon_i}{h_i}} - \frac{(1-\rho)^{q+1} + \rho^{q+1}}{q+1}\Big)|\psi'(x_i)|^q$$

$$+\left(G_{6,i}(q;\rho,\epsilon_i)e^{\frac{\epsilon_i}{h_i}}-\frac{(1-\rho)^{q+1}+\rho^{q+1}}{q+1}\right)|\psi'(x_{i+1})|^q\Big\}^{\frac{1}{q}}$$
$$+\sum_{i=0}^{n-1}(h_i-\epsilon_i)^2\Big\{\left(G_{7,i}(q;\sigma,\epsilon_i)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\right)|\psi'(x_i)|^q$$
$$+\left(G_{8,i}(q;\sigma,\epsilon_i)-\frac{\sigma^{q+1}+(1-\sigma)^{q+1}}{q+1}\right)|\psi'(x_{i+1})|^q\Big\}^{\frac{1}{q}}, \tag{4.11}$$

where $0<\epsilon_i<h_i$ *for all* $i=0,1,2,\ldots,n-1$, *and*

$$G_{5,i}(q;\rho,\epsilon_i):=\int_0^1|1-\rho-\chi|^q e^{\frac{\epsilon_i\chi}{h_i}}d\chi,\quad G_{6,i}(q;\rho,\epsilon_i):=\int_0^1|1-\rho-\chi|^q e^{\frac{-\epsilon_i\chi}{h_i}}d\chi,$$

$$G_{7,i}(q;\sigma,\epsilon_i):=\int_0^1|\sigma-\chi|^q e^{\chi-\frac{\epsilon_i\chi}{h_i}}d\chi,\quad G_{8,i}(q;\sigma,\epsilon_i):=\int_0^1|\sigma-\chi|^q e^{1-(\chi-\frac{\epsilon_i\chi}{h_i})}d\chi.$$

Proof. By applying Theorem 3.2 on the subintervals $[x_i,x_{i+1}]$ $(i=0,1,2,\ldots,n-1)$ of the partition $\mathcal{P}$ and summing the obtain inequality over i from 0 to $n-1$, we get the desired result. □

Remark 3. For suitable choices of ρ,σ and ϵ_i in Propositions 10 and 11, like that $\rho,\sigma:=0,\frac{1}{2},1$ and $\epsilon_i:=\frac{h_i}{2},\frac{h_i}{3},\frac{2h_i}{3}$, where $h_i=x_{i+1}-x_i$ $(i=0,1,2,\ldots,n-1)$, we can obtain new bounds regarding error estimation of quadrature formula given above. We omit their proofs and the details are left to the interested reader.

5. Conclusions

In this article, we have obtained some new version of Hermite-Hadamard type inequalities for differential exponential type convex functions. Moreover, several special cases are given in details. Finally, we have derived as applications from our main results several interesting inequalities using special means and some error estimations as well. This shown the efficient of our results. We believe that our results will have a very deep research in this field of inequalities and also in pure and applied sciences.

Author contribute

All authors contribute equally in this paper.

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Data availability

All data required for this paper is included within this paper.

Conflict of interest

Authors do not have any competing interests.

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