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Research article

Airplane designing using Quadratic Trigonometric B-spline with shape parameters

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Abstract: The B-spline curves have been grasped tremendous achievements inside the widely identified field of Computer Aided Geometric Design (CAGD). In CAGD, spline functions have been used for the designing of various objects. In this paper, new Quadratic Trigonometric B-spline (QTBS) functions with two shape parameters are introduced. The proposed QTBS functions inherit the basic properties of classical B-spline and have been proved in this paper. The proposed scheme associated with two shape parameters where the classical B-spline functions do not have. The QTBS has been used for designing of different parts of airplane like winglet, airfoil, turbo-machinery blades and vertical stabilizer. The designed part can be controlled or changed using free parameters. The effect of shape parameters is also expressed.

Keywords: Quadratic Trigonometric B-spline functions; shape parameters; airplane parts; uniform knots; curve designing; open curves; closed curves
Mathematics Subject Classification: 65D10, 65D17, 65D18, 68U07

1. Introduction

It has been a fascinating task/exercise for the designers, researcher, as well as for the air craft manufacturers to advance the geometrical techniques of airplane designing. Different splines have been used in the designing of airplane models and the purpose behind this designing is to generate flexible and realistic shapes. Splines are used for airplane modeling due to numerically stability, accuracy and consistency. It also attracted the aircraft designers because of easy control and editing the shape of geometrical models [1]. Different optimization techniques are brought into practice to develop designing problems in order to improve the geometry of the required designing and to reduce the time and cost [2]. In curve modeling, we deal with trigonometric Bez'ier-like [3–7], Q-Bez'ier [8], H-Bez'ier [9], Ball Bez'ier-like [10], S-λ Bezier-like [11], classical Bez'ier, B-spline and NURBS curves etc.

Recently, Kulfan [12] introduced an algorithm for surface fitting using splines that fulfils some specific boundary conditions in which knots are generated automatically but one shape parameter is compulsory which provides the closeness and smoothness to fit a curve. The assignment to improve the geometry of the curves motivated to the many researchers. Reconstruction of curves and images is a compelling area in geometrical modeling [13]. The scheme of image reconstruction is employed in the field of medical. It is an amazing acquirement of fractured parts [14]. An effort has been brought by Henderson [15] to describe the effect of canard on the aerodynamic characteristics of a designed model. Winglet yield an airplane fuel efficiency and cruising range. It decreases the aerodynamic drag. This quality reduces the consumption of fuel and drag efficiency [16]. Various technologies [17] came into practice to improve performance and geometry of structure of airplane. An attempt is introduced to decrease induced drag. The process timing if winglet has been reduced with the association of CAD. It focused on decreasing design time while designing is at initial process see [18]. A scheme has been proposed using B-spline collocation method based on cubic B-spline functions [19]. A new type of cubic B-spline functions with one shape parameter has been presented [20] and this work is extended to rational cubic B-spline curves for geometrical modeling of the curves [21].The QT curves have been designed using one free parameters and this work is extended to cubic curves see for more details [22, 23].

Lin and Reutskiy [24] used cubic B-spline for the numerical solution of 3D-steady state convection-diffusion reaction problems. Non-linear generalized telegraph equation in irregular domain has been solved numerically using B-spline by [25]. The novel B-spline method has proposed by [26] for the modeling of transport problems in anisotropic inhomogeneous media. Hu *et al.* [27] proposed a novel model based on Bezier for surface modeling. Hu *et al.* [28] proposed generalized quartic H-Bezier curve for the construction of developable surfaces. Shape adjustable generalized Bezier surface is proposed by [29].

This paper presents a distinct QTBS basis functions with two shape parameters. The proposed QTBS functions in addition of two shape parameters have all the characteristics of simple B-spline like convex hull, positivity, local support, continuity and partition of unity and proved in section 2.

The proposed QTBS curves yield tight envelop in the geometrical modeling which is a demanding feature in shape designing. It generates more suitable and elastic curves for shape designing. The presence of shape parameters enables the designer to control the shape of the curve easily. The proposed scheme approximates the curves in a batter way as it saves cost and time.

For test purpose open and close curves have been designed and the effect of shape parameters has also been checked. Different components of jet transport like airfoil, winglet, turbo machinery blade and vertical blades have been designed using proposed scheme.

2. Quadratic Trigonometric polynomial basis functions

Suppose $a_0 < a_1 < a_2 < a_3 < \ldots < a_{n+3}$ be the knots, the shape parameters v_i,$a_i \in R$, and $\Delta a_i = a_{i+1} - a_1, \beta_i = a_i a_{i+1} a_i$, the QTBS polynomial basis functions are defined by

$$C_i(a) = \begin{cases} g_i(u_i(a)), & a \in [a_i, a_{i+1}), \\ 1 - f_{i+1}(u_{i+1}(a)) - g_{i+1}(u_{i+1}(a)) & a \in [a_{i+1}, a_{i+2}) \\ f_{i+2}(u_{i+2}(a)), & a \in [a_{i+2}, a_{i+3}) \\ 0, & \text{Otherwise} \end{cases} \tag{2.1}$$

where,
$f_i(u) = \gamma_i\left(1 - (1 + v_{i-1})\sin u + v_{i-1}\sin^2 u\right) - \beta_{i-1}\cos^2 u,$
$g_i(u) = \omega_i\left(1 - (1 + v_i)\cos u + v_i\cos^2 u\right) + \beta_i\sin^2 u,$
$\gamma_i = \dfrac{\Delta a_i}{\Delta a_{i-1} + \Delta a_i},$
$w_i = \dfrac{\Delta a_i}{\Delta a_i + \Delta a_{i+1}},$
$u(a) = \dfrac{\pi}{2}\dfrac{a - a_i}{\Delta a_i},$

Remark 2.1. *If we use $v_i = v_{i+1} = \beta_i = \beta_{i+1} = 0$, then the basis function $C_i(a)$ will convert into linear trigonometric basis functions. Figure 1 shows the graphical behavior of the QTBS basis function.*

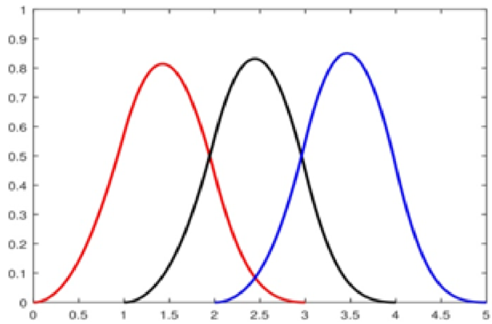


Figure 1. Trigonometric basis functions.

3. Geometric properties of the QTBS functions

In this section, some geometric properties of the proposed basis functions are discussed.

3.1. Positivity

If $-1 < v_i, v_{i+1} \leq 1$, $\beta_i \geq -\min\{\gamma_{i+2}, \frac{1}{2}\omega_i(1 - v_i)$ and $\beta_{i+1} \leq \min\{\frac{1}{2}\gamma_{i+2}(1 - v_{i+1}), \omega_i\}$, then $C_i(a) > 0$, for all $a_i < a < a_{i+3}$.

Proof. See Appendix 1. □

3.2. *Local support*

$C_i\,(a) = 0,$ for $a_0 < a < a_{i+3},\ a_{i+3} < a < a_{n+3}.$

3.3. *Piecewise polynomial*

The QTBS are the piecewise polynomial functions as defined in section 2.

3.4. *Partition of unity*

$\sum_{i=1}^{k} C_i(u) = 1,\ a \in [a_2, a_{n+1})$

Proof. See Appendix 2. □

3.5. *Continuity*

The basis functions satisfy the continuity property at knot points. The QTBS $C_i(a)$ has C^1 continuity.

Proof. See Appendix 3. □

Remark 3.1. *It is important that we can only discuss the continuity of the basis function at knot points, continuity cannot be discussed at first and last point as it is obvious.*

4. Quadratic Trigonometric B-spline curve

For the points p_i (i = 0, 1, 2,..., n) in R^2 or R^3 and A = (u_0, u_1, u_2, ..., u_{n+3}). The

$$S\ (u) = \sum_{i=0}^{n} C_i\,(u)\,p_i \tag{4.1}$$

is known as QTBS polynomial curve with shape parameters.

4.1. *Continuity between two uurve segments*

If $u_i \neq u_{i+1} (2 \leq i \leq n)$, the representation of the curve segment *S(u)* can be written as:

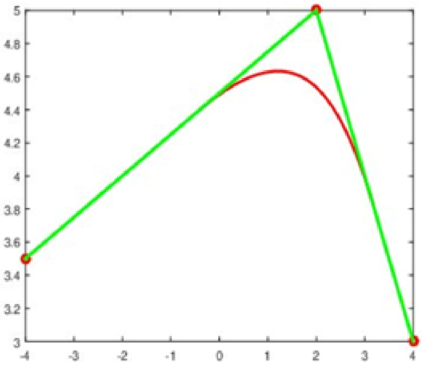
$$S\,(u) = C_{i-2}(u)p_{i-2} + C_{i-1}(u)p_{i-1} + C_i(u)p_i \tag{4.2}$$

Moreover,

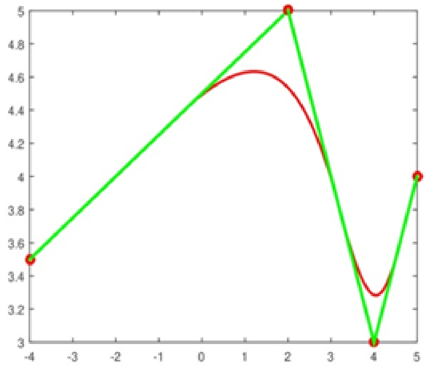
$$\begin{cases} S\,(u_i) \ = \ (\gamma_i - \beta_i - 1)p_{i-2} + (\omega_{i-1} + \beta_{i-1})p_{i-1} \\ S'\,(u_i) \ = \frac{\pi(1-v_{i-1})}{2(u_{i-1}+u_i)}\,(p_{i-1} - p_{i-2}) \end{cases} \tag{4.3}$$

4.2. *Construction of the curve*

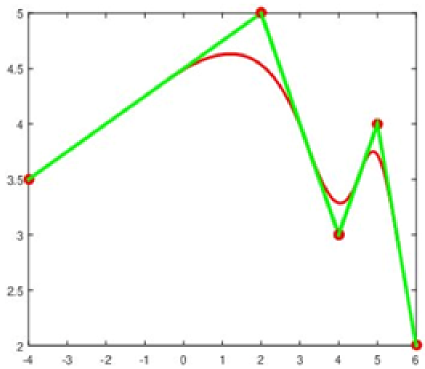
Proposed scheme has been used to construct different curve segments in this section which can be depicted in Figure 2. After constructing the different curve segments using proposed scheme. It has been used to construct the open curve as shown in Figure 3. Figure 4 represents the effect of shape parameters. The curve changes its behavior as we change the shape parameters. To check the applicability of proposed scheme we have also constructed the closed curve as shown Figure 5.



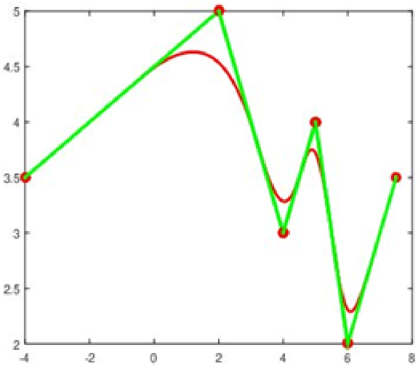
(a) One segment curve



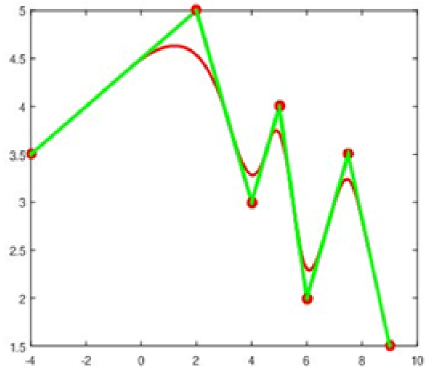
(b) Two segments curve



(c) Three segments curve



(d) Four segments curve



(e) Five segments curve

Figure 2. Curve designing using QTBS

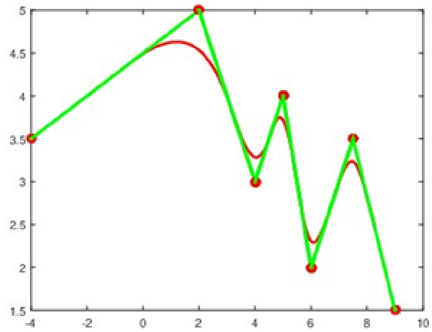


Figure 3. Open curve using proposed QTBS.

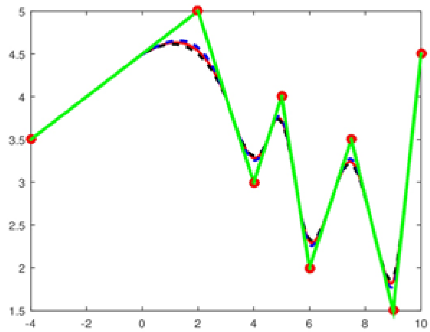


Figure 4. Open curve with different values of shape parameter using QTBS.

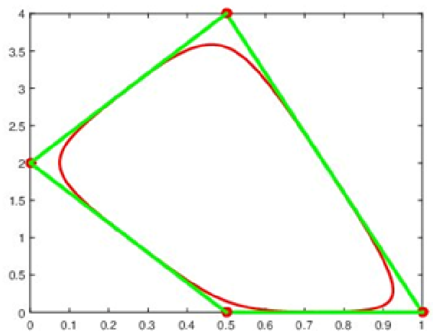


Figure 5. Closed curve.

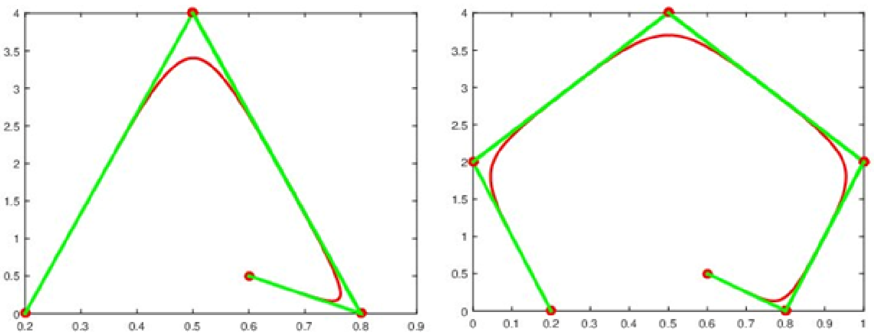
4.3. Shape of the curve

The curve designing can be controlled in two ways i.e

1. Using more control points
2. By shape parameters

4.3.1. Using more control points

In this section, the QTBS curve has been designed by different control points like in Figure 6(a) we used 4 control points. By adding the number of control points the curve changed its behavior as shown in Figure 6(b).



(a) Curve with 4 control points (b) Curve with 6 control point

Figure 6. Construction of curves using QTBS

4.3.2. Effect of shape parameter

Curve can also be controlled by shape parameters. Different values of shape parameters ν and β are tabulated in Table 1 have been used for the construction of Quadratic trigonometric B-spline curve as shown in the Figures 7 and 8. It is observed that by assigning different vales of shape parameters both open and closed curves change its behavior.

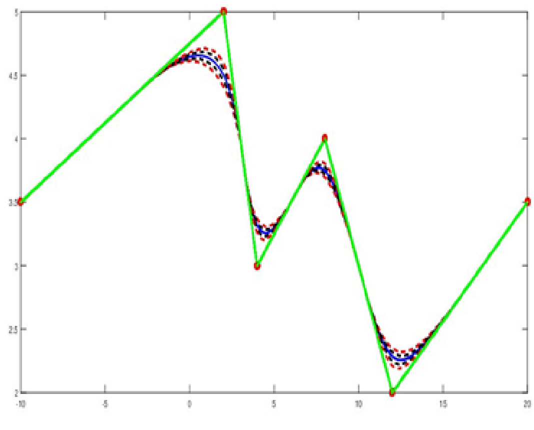


Figure 7. Open curve with different values of shape parameter.

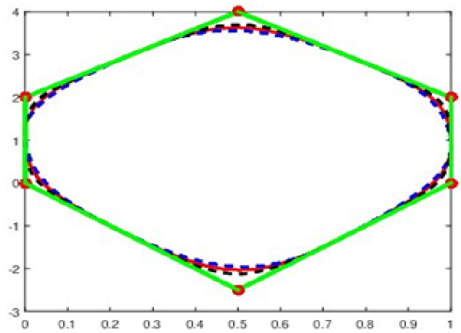


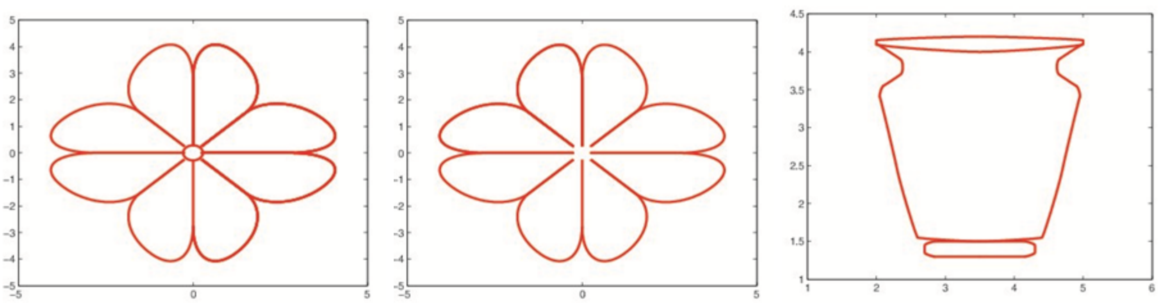
Figure 8. Closed curve with different values of shape parameter.

Table 1. Behavior of the curve with different values of shape parameters.

Sr. no.	value of ν	value of β
1	0.1	0.225
2	0.2	0.200
3	0.3	0.175
4	0.4	0.150
5	0.5	0.125
6	0.6	0.100
7	0.7	0.075
8	0.8	0.050
9	0.9	0.025

4.4. Geometrical modeling using QTBS curves

To test the proposed scheme we have constructed the different models like flowers and vase as shown in Figure 9. When the curve $S(a)$ is generated in the interval$[a_2,\ a_{n+1}]$, we are free from the choice of first and last two knot. These can be adjusted to the given boundary behavior of the curve. The choice of knot vector for open trigonometric curve is $A = (a_0 = a_1 = a_2,\ a_n,\ a_{n+1} = a_{n+2} = a_{n+3})$.



(a) Close petals (b) Open petals (c) Vase

Figure 9. Designing of different objects using QTBS curves.

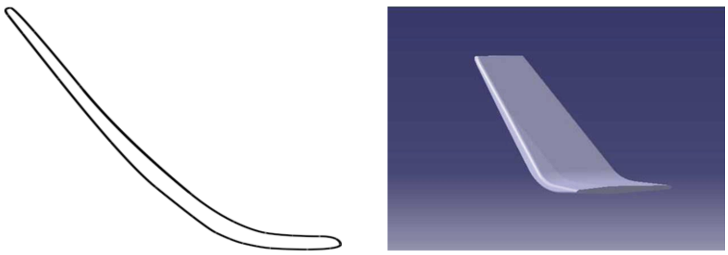
5. Implementation of the proposed method

In this section, we have constructed different parts of an airplane i.e. airfoil, winglet and vertical stabilizer.

5.1. Winglet

In aircraft industry winglets have become one of the distinguish fuel saving technology and their use keep increasing. Winglets also increase the operating efficiency of airplane.

Here the winglet of an airplane has been designed see Figure 10. In the designing of the winglet different values of shape parameters which they have been tabulated in Table 2, makes the designing more appropriate and smooth.



(a) Designing of Winglet (b) 3D image of Winglet

Figure 10. Winglet of airplane.

Table 2. Different values of shape parameters.

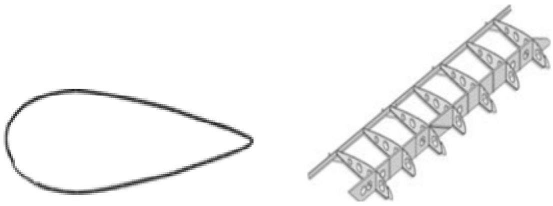
Sr. no	1	2	3	4	5	6	7	8
ν	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
β	0.2425	0.245	0.2425	0.24	0.2375	0.235	0.2325	0.23
Sr. no	9	10	11	12	13	14	15	16
ν	0.09	0.1	0.11	0.12	0.13	0.14	0.15	0.16
β	0.2275	0.225	0.2225	0.22	0.2175	0.215	0.2125	0.21

6. Airfoils

They are used for the description of cross sectional structure of airplane when passed through a fluid such as aerodynamic force. Airfoils are employed on planes wings to provide elevate. Here, an airfoil has been constructed using QTBS applying various valued of two shape parameter ν and β as shown in Table 2. The 2D designing is shown in Figure 11(a) where in Figure 11(b) presents the 3D image of airfoil.

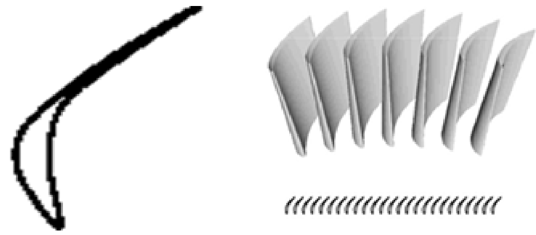
7. Turbomachinery blade

Turbomachinery blade has been designed using QTBS curves as shown in Figure 12.



(a) Designing of airfoil (b) 3D image of airfoil

Figure 11. Airfoil of airplane.

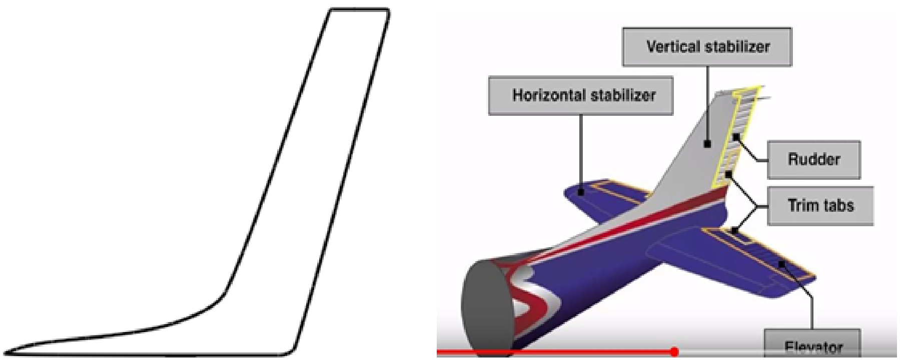


(a) Designing of turbomachinery blade (b) 3D image of turbomachinery blade

Figure 12. Airfoil of airplane.

8. Vertical blade/Stabilizer

The most important function of vertical blade/stabilizer is to maintain the aircraft in straight position and to balance level flight of aircraft. It also keeps the stability of airplane about its vertical axis. Here, the designing of a vertical stabilizer has been constructed using QTBS curves with the association of two shape parameters ν and β as shown in Figure 13. The 2D and 3D designed vertical blades are shown in Figure 13(a) and 13(b), respectively.



(a) Designing of vertical blades (b) 3D image of vertical blades

Figure 13. Vertical blade of airplane.

9. Approximate to the quadratic NURBS curves

For $a\in [a_i, a_{i+1}]$, and $u = \frac{\pi}{2}\frac{a-a_i}{\Delta a_i}$, the QTBS can be rewrite as:

$$S_i(u) = P_{i-1} + (f_i(u) + g_i(u))\left(\frac{f_i(u)P_{i-2} + g_i(u)P_i}{f_i(u) + g_i(u)} - P_{i-1}\right). \tag{9.1}$$

According to this, the NURBS curve can be approximated as:

$$T_i(v) = P_{i-1} + (b_{i0}(v) + b_{i2}(v))\left(\frac{b_{i0}(v)P_{i-2} + b_{i2}(v)p_i}{b_{i0}(v) + b_{i2}(v)} - P_{i-1}\right), \tag{9.2}$$

where, $b_{i0}(v) = \alpha_{i2}\gamma_i(1-v)^2/\alpha(v)$, $b_{i0}(v) = \alpha_i\omega_i(v)^2/\alpha(v)$, $\alpha(v) = \alpha_{i-2}\gamma_i(1-v)^2 + \alpha_i\omega_i(v)^2 + \alpha_{i-1}(1-\gamma_i)(1-v)^2 + 2(1-v)v + (1-\omega_i)v^2$, and α_{i-1}, α_{i-2}, $\alpha - i$ are weight numbers. The boundaries of the curve are:

$$\begin{cases} T_i(0) = \left(\frac{a_{i-2}\gamma_i P_{i-2} + \alpha_{i-1}\omega_{i-1} P_{i-1}}{\alpha_{i-1}\gamma_{i+1} + \alpha_i\omega_i}\right), \\ T_i(1) = \left(\frac{a_{i-1}\gamma_i + P_{i-1} + \alpha_i\omega_i P_i}{\alpha_{i-1}\gamma_{i+1} + \alpha_i\omega_i}\right). \end{cases} \tag{9.3}$$

10. Conclusions

In this paper, we have constructed QTBS with two shape parameters. Our proposed scheme satisfies all the basic properties and has been proved. The proposed quadratic trigonometric B-spline is more suitable for designing due to the presence of shape parameters than ordinary quadratic B-spline. With the use of shape parameters we obtain more flexible curve which is according to a designing technology. It is an important feature in the construction of curves that it should be controlled easily and within the convex hull. The proposed QTBS can control the modeling of the geometric curves in a decorous manner because it can construct more appropriate and flexible curves. It is numerically stable. It is also useful from economical point of view as it reduces computational cost as compare to inserting more control points. Different parts of an airplane have been designed using proposed technique.

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Conflicts of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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Appendix 1

Proof. For $a \in (a_i, a_{i+1}), u = \frac{a-a_i}{2\Delta a_i} \in \left(0, \frac{\pi}{2}\right),\ c_1(a) = g_i(u)$, we have $\omega_i + \beta_i - \omega_i(1 - v_i \cos u) + (\omega_i v_i - \beta_i) > 0$ and $\omega_i + \beta_i - \omega_i(1 - v_i \cos u) + \cos^2 u(\omega_i v_i - \beta_i) > 0$, for $a \in (a_{i+1}, a_{i+2}), u = \frac{a-a_i}{2\Delta a_{i+1}} \in \left(0, \frac{\pi}{2}\right),\ c_i(a) = 1 - f_{i+1}(u) - g_{i+1}(u)$.
when $u = 0$,

$$c_i(a) = 1 - y_{i+1} + \beta_i$$

$$= \omega_i + \beta_i > 0$$

when $a \in (a_i, a_{i+1})$, let, $\xi_{i+1} = \max\{\gamma_{i+1} + \omega_{i+1}\}$,

$$f_{i+1}(u) + g_{i+1}(u) < \xi_{i+1}[\left\{1 - (1 - v_i)\sin u + v_i \sin^2 u\right\} + \{1 - (1 - v_{i+1})\cos u + v_i \cos^2 u\}$$

$$= \xi_{i+1} - \beta_i \cos^2 u + \beta_i \sin^2 u$$
$$= \xi_{i+1} - \beta_i + (\beta_i + \beta_i + 1)\sin^2 u.$$

Since $\beta_i \geq -\min\{\gamma_{i+2}, \omega_i\}$,$\beta_{i+1} \leq \min\{\gamma_{i+2}, \omega_i\}$, we have $y_{i+1} - \beta_i \leq 1,\ y_{i+1} \leq 1$.
Thus, $f_{i+1}(u) + g_{i+1}(u) < 1$ and $C_i(a) > 0$. For $a \in (a_{i+2}, a_{i+3})$, $u = \frac{a-a_i}{2\Delta a_{i+2}} \in \left(0, \frac{\pi}{2}\right)$, $c_1(a) = f_{i+2}(u)$ and

$$f_{i+2}(u) < \gamma_{i+2}\left\{1 - (1 - \nu_{i+1})\sin u + \nu_{i+1} sin^2 u\right\} - \beta_{i+1} cos^2 u,$$

$$\beta_{i+2} \leq \gamma_{i+2}(1 - \nu_{i+1})\, c < \gamma_{i+1},$$

$$\gamma_{i+2} - \beta_{i+1} - (\gamma_{i+2} + \gamma_{i+2}\nu_{i+1})\sin u + (\gamma_{i+2}\nu_{i+1} + \beta_{i+1})\sin^2 u > 0.$$

Thus, we can say$-1 < \nu_i \leq 1,\ \ i = 0, 1, \ldots, n + 1,$

$$\beta_i \geq -\min\left\{\gamma_{i+1}, \frac{1}{2}\omega_i(1 - \nu_i)\right\},\ i = 0, 1, 2, \ldots, n$$

$$\beta_i \geq -\min\left\{\frac{1}{2}\gamma_{i+1}(1 - \nu_i),\ \omega_{i+1}\right\}, i = 0, 1, 2, \ldots, n + 1$$

Hence, for all $c_i(u) > 0$, for$a \in (a_{i+2}, a_{i+3})$, $i = 0, 1, 2, \ldots, n$ □

Appendix 2

Proof. The QTBS basis functions (2.1) can also be written as:

$$\begin{cases} C_{i-2}(a) = f(u_i(a)), \\ C_{i-1}(a) = 1 - f(u_i(a)) - g(u_i(a)) \\ C_i(a) = g(u_i(a)) \end{cases}$$

Let $C_k(a) = 0, k \neq i - 2,\ i - 1,\ i$
Thus, $\sum_{i=0}^{n} C_i(a) = C_{i-2}(a) + C_{i-1}(a) + C_i(a)$.
Hence, $\sum_{i=0}^{n} C(a) = 1$ □

Appendix 3

Continuity at first knot

Proof.

$$\begin{cases} C_i(a^-_{?+1}) = \omega_i + \beta_i, \\ C_i(a^-_{?+1}) = 1 + \lambda_i. \end{cases}$$

L.H.S continuity, consider $C_i(a) = g(u_i(a))$. Putting values, we have

$$g(u_i(a)) = \begin{cases} \omega_i\left(1 - (1 + v)\cos(u) + v cos^2(u)\right) + \beta_i sin^2 u \\ \omega_i(1 - (1 + v)\, cos\left(\frac{\pi}{2}\frac{a-a_i}{\Delta a_i}\right) + v cos^2\left(\frac{\pi}{2}\frac{a-a_i}{\Delta a_i}\right)) + \beta_i sin^2\left(\frac{\pi}{2}\frac{a-a_i}{\Delta a_i}\right) \end{cases}$$

Replacing a by a_{i+1}, we have

$$C_i(a^-_{\mathrm{i+1}}) = \omega i\ \ 1 - (1 + v)\cos\left(\frac{\pi}{2}\frac{a_i + 1 - a_i}{\Delta a_i}\right) + v\cos 2\left(\frac{\pi}{2}\frac{a - a_i}{\Delta a_i}\right) + \beta i sin2\left(\frac{\pi}{2}\frac{a - a_i}{\Delta a_i}\right)$$

$$
\begin{aligned}
&= \omega i \; 1-(1+\nu)\cos\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) + \nu\cos 2\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) + \beta i\sin 2\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) \\
&= \omega i \; 1-(1+\nu)\cos\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) + \nu\cos 2\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) + \beta i\sin 2\left(\frac{\pi}{2}\frac{\Delta a_i}{\Delta a_i}\right) \\
&= \omega i + \beta i
\end{aligned}
$$

Hence, proved.

R.H.S continuity, consider $C_i(a^+_{i+1}) = 1-\gamma_i$, let $1-f(u_{i+1}(a))-g(u_{i+1}(a))$, putting value of u_{i+1}, we have

$1-f\,(u_{i+1}(a))-g(u_{i+1}(a)) = 1-\{\gamma_{i+1}\{1-(1-\nu)sin(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}}) + \nu sin^2(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}})\} - \beta_i\cos^2(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}})\}$ $-\{\omega_{i+1}\{1-(1-\nu)\,cos(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}})\;\nu\,cos^2\,(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}})\} + \beta_{i+1}\,sin(\frac{\pi}{2}\frac{a-a_{i+1}}{\Delta a_{i+1}})\}$. Replacing a by a_{i+1}, we obtain $1-f(u_{i+1}(a))-g(u_{i+1}(a)) = 1-\{\gamma_{i+1}\{1-(1-\nu)\;sin(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}}) + \nu sin^2(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}}) - \beta_i\cos^2(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}})\} - \{\omega_{i+1}\;1-(1-\nu)cos\,(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}}) + \nu cos^2(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}})\} + \beta_{i+1}sin\,(\frac{\pi}{2}\frac{a_{i+1}-a_{i+1}}{\Delta a_{i+1}})\}$.

After some simplification,

$1-f(u_{i+1}(a))-g(u_{i+1}(a)) = 1-\{\gamma_{i+1}\{1-(1-\nu)\;sin\,(0)+\nu sin^2\,(0)\}-\beta_i cos^2\,(0)\}$
$-\{\omega_{i+1}\{1-(1-\nu)\;cos(0)+\nu cos^2(0)\}-\beta_{i+1}sin^2\,(0)\}$,
$= 1-\{\gamma_{i+1}\{1+0+0\}-\beta_i\}-\{\omega_{i+1}\{1-1+\nu-\nu\}-0\}$.

Thus,
$c_i\,(a^+_{i+1}) = 1-\gamma_{i+1}+\beta_i$.

Similarly,

$$
\begin{cases}
c_i\,(a^-_{i+2}) = 1-\omega_{i+1}-\beta_{i+1}, \\
c_i\,(a^+_{i+2}) = \gamma_{i+2}-\beta_{i+1},
\end{cases}
$$

and

$$
\begin{cases}
c'_i\,\left(a^-_{i+1}\right) = \frac{\pi}{2}\frac{(1+\nu)}{\Delta a_i}\omega_i, \quad c'_i\,\left(a^+_{i+1}\right) = \frac{\pi}{2}\frac{(1+\nu)}{\Delta a_i}\gamma_{i+1}, \\
c'_i\,\left(a^-_{i+2}\right) = \frac{\pi}{2}\frac{(1+\nu)}{\Delta a_{i+1}}\omega_{i+1}, \quad c'_i\,\left(a^+_{i+2}\right) = \frac{\pi}{2}\frac{(1+\nu)}{\Delta a_{i+2}}\gamma_{i+2}, \\
\gamma_{j+1} = 1-\omega_{j+1}\;,\gamma_{j+1}/\Delta a_{j+1} = \omega_i/\Delta a_i\;,\; 0 \le i \le n+1.
\end{cases}
$$

Thus, we obtain the following results

$$
\begin{cases}
c_i^{(k)}\,\left(a^-_{i+1}\right) = c_i^{(k)}\,\left(a^+_{i+1}\right), \\
c_i^{(k)}\,\left(a^-_{i+2}\right) = c_i^{(k)}\,\left(a^+_{i+2}\right),
\end{cases}
\quad \text{for } k = 0, 1
$$

If $\nu = 0$, then our QTBS basis functions will become the linear trigonometric basis functions. □

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