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Research article

A space-time spectral method for the 1-D Maxwell equation

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Abstract: A Legendre-tau space-time spectral method is established for the 1-D Maxwell equation. The polynomials of different degrees are used to approximate the electric and magnetic fields, respectively, so that they can be decoupled in computation. Also, the time multi-interval Legendre-tau space-time spectral method is considered to keep the long-time computation stable. Error estimates for the method of single and multi-internal are given, respectively. Moreover, the space-time spectral method is applied to the numerical solutions of the 1-D nonlinear Maxwell equation and describes its implicit-explicit iteration scheme. Numerical examples are compared with some other methods, which verifies the effectiveness of the methods for the 1-D Maxwell equation.

Keywords: Maxwell equation; space-time spectral method; Legendre polynomial; interval decomposition; error estimate
Mathematics Subject Classification: 65M12, 65M70

1. Introduction

It is well known that the spectral method has high-order accuracy for smooth problems. The spectral method together with the difference method and the finite element method has become an important method for the numerical solution of partial differential equations (PDEs), and has been successfully applied to solve many practical problems. In recent years, with regard to the differential equations of time evolution, the high-order discrete scheme in time has received widespread attention and has become one of the hot spots in the field of numerical computing. The discontinuous Galerkin method in time is constantly developing, and a better higher-order discrete scheme in time is established [1–3]. The explicit, implicit and implicit-explicit Runge-Kutta methods have also made great progress: a local discontinuous Galerkin method with implicit-explicit time-marching is used to solve the multi-dimensional convection-diffusion problems and time-dependent incompressible fluid flow in [4–6]. In [7–9], the spectral method in time and the time multi-interval spectral method are also proposed. The single interval and multi-interval Legendre spectral methods in time are established for the parabolic equations, in which the L^2-optimal error estimate in space is obtained in [10].

The Maxwell equation is a set of important PDEs that describes electromagnetic field phenomena, and some effective numerical methods have been established for the Maxwell equation by scholars [11–13]. The finite-difference time-domain method (also called Yee's scheme) for the Maxwell equation is proposed in [14]. In [15, 16], an energy-conserved splitting spectral method for solving the Maxwell equation is given. For the 2-D Maxwell equation, a Legendre-Galerkin method in space and the energy-conserved splitting spectral method in time is constructed [17]. In previous work, the different method is used in the time direction. For the 1-D Maxwell equation of inhomogeneous media with discontinuous solutions, the multidomain Legendre-Galerkin and the multidomain Legendre-tau method are established in [18, 19], and the optimal error estimates of the semi-discrete schemes are given.

Consider the following 1-D Maxwell equation [20]

$$\begin{cases} \epsilon\partial_t E_z = \partial_x H_y, & (x,t) \in \Omega, \\ \mu\partial_t H_y = \partial_x E_z, & (x,t) \in \Omega, \\ E_z(-1,t) = E_z(1,t) = 0, & t \in I_t, \\ E_z(x,0) = E_{z0}(x),\ H_y(x,0) = H_{y0}(x), & x \in I_x, \end{cases} \tag{1.1}$$

where $I_x = (-1, 1)$, $I_t = (0, T]$, and $\Omega = I_x \times I_t$. E_z and H_y stand for the electric field and the magnetic field, respectively. The positive constants ϵ and μ stand for the electric permeability and the magnetic permeability, respectively.

In [21,22], an h-p version of the Petrov-Galerkin time stepping method is used to solve the nonlinear initial value problems by transforming the second-order problem into a first-order system. For the linear second-order wave equation, it is often transformed into the first-order system similar to equation (1.1) by using the substitution $v = \frac{\partial u}{\partial t}$, $w = \frac{\partial u}{\partial x}$ [23]. It is interesting to note that some methods use the derivative as the main unknown function, and u is expressed as the integral of w.

In this paper, a Legendre-tau space-time (LT-ST) spectral method is developed to solve the 1-D Maxwell equation (1.1) and a time multi-interval Legendre-tau spectral method is considered. The scheme is based on the Legendre-tau method, which uses polynomials of different degrees are used to approximate the electric field E_z and magnetic field H_y, respectively, so that they can be decoupled in computation. After decoupling, it is an equation only about E_z, which can be solved by the method in [10]. The method is also applied to the numerical solutions of the 1-D nonlinear Maxwell equation.

The paper is organized as follows. In Section 2, a Legendre-tau space-time spectral method for (1.1) is presented, and stability analysis and error estimate are given. In Section 3, a time multi-interval Legendre-tau spectral method is developed, and its error estimate is also obtained. Some numerical results are given in Section 4. Finally, the method is applied to the numerical solution of the 1-D nonlinear Maxwell equation in Section 5.

2. Legendre-tau space-time spectral method

In this section, a Legendre-tau space-time spectral method is presented for the problem (1.1). Moreover, the stability and the error estimate of this method are given.

2.1. Preliminaries

Let $(\cdot,\cdot)_Q$ and $\|\cdot\|_Q$ be the inner product and the norm of $L^2(Q)$, where Q stands for Ω, I_x and I_t, respectively. For a nonnegative integer m, let $\|\cdot\|_{m,I}$ and $|\cdot|_{m,I}$ be the norm and the semi-norm of the classical Sobolev space $H^m(I)$, where I stands for I_x or I_t, respectively. Define

$$H_0^1(I) = \{v \in H^1(I) : v(-1) = v(1) = 0\}.$$

For a pair of positive integers N and M, define $L = (N, M)$. Let $\mathbb{P}_N(I_x)$ be the space of polynomials of degree at most N on I_x. Define the polynomial space

$$V_N = \{v \in \mathbb{P}_N(I_x)\},$$

and the approximation space in space

$$V_N^0 = H_0^1(I_x) \cap V_N, \quad V_{N-1} = \{v \in \mathbb{P}_{N-1}(I_x)\}. \tag{2.1}$$

Let $\mathbb{P}_M(I_t)$ be the space of polynomials of degree at most M on I_t, we define the approximation space in time

$$V_M = \{v \in \mathbb{P}_M(I_t)\}, \quad V_{M-1} = \{v \in \mathbb{P}_{M-1}(I_t)\}. \tag{2.2}$$

Let x_j^C and $\omega_j^C (0 \le j \le N)$ be the Chebyshev-Gauss-Lobatto (CGL) points and the corresponding weights on I_x. We define the CGL interpolation operator $I_N^C v \in V_N$:

$$I_N^C v(x_j^C) = v(x_j^C), \quad 0 \le j \le N.$$

Similarly, let x_j^L and $\omega_j^L (0 \le j \le N)$ be the Legendre-Gauss-Lobatto (LGL) points and the corresponding weights on I_x. $I_N^L v \in V_N$ denotes the LGL interpolation operator, and

$$I_N^L v(x_j^L) = v(x_j^L), \quad 0 \le j \le N.$$

We denote by $P_N : L^2(I_x) \to V_N$ the $L^2(I_x)$-Legendre projection operator and define $P_N^1 : H^1(I_x) \to V_N$ by

$$P_N^1 u(x) = u(-1) + \int_{-1}^{x} P_{N-1}\partial_x u(y)\mathrm{d}y, \quad x \in I_x. \tag{2.3}$$

It is easy to see that

$$P_N^1 u(-1) = u(-1), \qquad P_N^1 u(1) = u(1), \tag{2.4}$$

$$(\partial_x P_N^1 u - \partial_x u, v) = (P_{N-1}\partial_x u - \partial_x u, v) = 0, \quad \forall v \in V_{N-1}. \tag{2.5}$$

Let C be a generic positive constant independent of N, and the following approximation results can be found in [10, 24].

Lemma 2.1. *If $u \in H^r(I_x)$, then*

$$\begin{aligned}
&\|P_N u - u\|_{I_x} \le CN^{-r}|u|_{r,I_x}, && r \ge 0,\\
&\|I_N^L u - u\|_{I_x} \le CN^{l-r}|u|_{r,I_x}, && r \ge 1, \quad l = 0, 1.\\
&|P_N^1 u - u|_{l,I_x} \le CN^{l-r}|u|_{r,I_x}, && r \ge 1, \quad l = 0, 1.
\end{aligned}$$

Let t_j^C and $\omega_j^C (0 \le j \le M)$ be the CGL points and the corresponding weights on I_t, and let t_j^L and $\omega_j^L (0 \le j \le M)$ be the LGL points and the corresponding weights on I_t. We denote by $P_M : L^2(I_t) \to V_M$ the $L^2(I_t)$-Legendre projection operator and define $P_M^1 : H^1(I_t) \to V_M$ as

$$P_M^1 v(t) = v(0) + \int_0^t P_{M-1}\partial_t v(s)\mathrm{ds}, \quad t \in I_t. \tag{2.6}$$

It is easy to find that

$$P_M^1 u(-1) = u(-1), \qquad P_M^1 u(1) = u(1), \tag{2.7}$$

$$(\partial_t P_M^1 u - \partial_t u, v) = (P_{M-1}\partial_t u - \partial_t u, v) = 0, \ \ \forall v \in V_{M-1}. \tag{2.8}$$

The following approximation result can be found in [10].

Lemma 2.2. *If $u \in H^\sigma(I_t)$ and $\sigma \ge 1$, then*

$$|P_M^1 v - v|_{l,I_t} \le CM^{l-\sigma}|v|_{\sigma,I_t}, \quad l = 0, 1,$$

where C is a positive constant independent of M.

The problem (1.1) is expressed in a weak form: Find $E_z \in H_0^1(I_x) \otimes H^1(I_t)$ and $H_y \in L^2(I_x) \otimes H^1(I_t)$ such that

$$\begin{cases} (\epsilon\partial_t E_z, v)_\Omega + (H_y, \partial_x v)_\Omega = 0, & \forall\, v \in H_0^1(I_x) \otimes L^2(I_t), \\ (\mu\partial_t H_y, w)_\Omega - (\partial_x E_z, w)_\Omega = 0, & \forall\, w \in L^2(I_x) \otimes L^2(I_t), \\ E_z(x,0) = E_{z0}(x),\ H_y(x,0) = H_{y0}(x), & \forall x \in I_x. \end{cases} \tag{2.9}$$

The LT-ST scheme to the problem (1.1) is: Find $E_{zL} \in V_N^0 \otimes V_M$ and $H_{yL} \in V_{N-1} \otimes V_M$ such that

$$\begin{cases} (\epsilon\partial_t E_{zL}, v)_\Omega + (H_{yL}, \partial_x v)_\Omega = 0, & \forall\, v \in V_N^0 \otimes V_{M-1}, \\ (\mu\partial_t H_{yL}, w)_\Omega - (\partial_x E_{zL}, w)_\Omega = 0, & \forall\, w \in V_{N-1} \otimes V_{M-1}, \\ E_{zL}(x,0) = I_N^L E_{z0}(x),\ \ H_{yL}(x,0) = P_{N-1} I_N^L H_{y0}(x), & \forall x \in I_x. \end{cases} \tag{2.10}$$

2.2. Stability analysis

In the following section, the stability analysis of (2.10) is considered. Suppose that there are perturbations $\tilde{f}_i (i = 1, 2)$ on the right-hand side. For simplicity, the original notations E_{zL} and H_{yL} are used to represent the solutions to the perturbation problem, which satisfies the following perturbation equation:

$$\begin{cases} (\epsilon\partial_t E_{zL}, v)_\Omega + (H_{yL}, \partial_x v)_\Omega = (\tilde{f}_1, v)_\Omega, & \forall\, v \in V_N^0 \otimes V_{M-1}, \\ (\mu\partial_t H_{yL}, w)_\Omega - (\partial_x E_{zL}, w)_\Omega = (\tilde{f}_2, w)_\Omega, & \forall\, w \in V_{N-1} \otimes V_{M-1}, \\ E_{zL}(x,0) = 0,\ \ H_{yL}(x,0) = 0, & \forall x \in I_x. \end{cases} \tag{2.11}$$

Theorem 2.1. Let E_{zL} and H_{yL} are the solutions to (2.11). Suppose that $\tilde{f}_i (i = 1, 2)$ are perturbations on the right-hand side, such that

$$\|\sqrt{\epsilon}E_{zL}\|_\Omega^2 + \|\sqrt{\mu}H_{yL}\|_\Omega^2 + T\left(\|\sqrt{\epsilon}E_{zL}(T)\|_{I_x}^2 + \|\sqrt{\mu}H_{yL}(T)\|_{I_x}^2\right) \le CT^2\left(\|\tilde{f}_1\|_\Omega^2 + \|\tilde{f}_2\|_\Omega^2\right). \tag{2.12}$$

Proof. Taking $v = \tilde{E}_{zL} := t^{-1}E_{zL} \in V_N^0 \otimes V_{M-1}$ and $w = \tilde{H}_{yL} := t^{-1}H_{yL} \in V_{N-1} \otimes V_{M-1}$ in (2.11), we get

$$\begin{cases} (\epsilon\partial_t(t\tilde{E}_{zL}), \tilde{E}_{zL})_\Omega + (t\tilde{H}_{yL}, \partial_x\tilde{E}_{zL})_\Omega = (\tilde{f}_1, \tilde{E}_{zL})_\Omega, \\ (\mu\partial_t(t\tilde{H}_{yL}), \tilde{H}_{yL})_\Omega - (t\partial_x\tilde{E}_{zL}, \tilde{H}_{yL})_\Omega = (\tilde{f}_2, \tilde{H}_{yL})_\Omega, \end{cases} \tag{2.13}$$

which leads to

$$(\epsilon\partial_t(t\tilde{E}_{zL}), \tilde{E}_{zL})_\Omega + (\mu\partial_t(t\tilde{H}_{yL}), \tilde{H}_{yL})_\Omega = (\tilde{f}_1, \tilde{E}_{zL})_\Omega + (\tilde{f}_2, \tilde{H}_{yL})_\Omega. \tag{2.14}$$

By integration by parts,

$$\begin{aligned} &(\epsilon\partial_t(t\tilde{E}_{zL}), \tilde{E}_{zL})_\Omega = (\epsilon\tilde{E}_{zL}, \tilde{E}_{zL})_\Omega + (\epsilon t\partial_t\tilde{E}_{zL}, \tilde{E}_{zL})_\Omega \\ &= \|\sqrt{\epsilon}\tilde{E}_{zL}\|_\Omega^2 + \frac{1}{2}T\|\sqrt{\epsilon}\tilde{E}_{zL}(T)\|_{I_x}^2 - \frac{1}{2}\|\sqrt{\epsilon}\tilde{E}_{zL}\|_\Omega^2 = \frac{1}{2}\left(\|\sqrt{\epsilon}\tilde{E}_{zL}\|_\Omega^2 + T\|\sqrt{\epsilon}\tilde{E}_{zL}(T)\|_{I_x}^2\right), \\ &(\mu\partial_t(t\tilde{H}_{zL}), \tilde{H}_{zL})_\Omega = (\mu\tilde{H}_{zL}, \tilde{H}_{zL})_\Omega + (\mu t\partial_t\tilde{H}_{zL}, \tilde{H}_{zL})_\Omega \\ &= \frac{1}{2}\left(\|\sqrt{\mu}\tilde{H}_{zL}\|_\Omega^2 + T\|\sqrt{\mu}\tilde{H}_{zL}(T)\|_{I_x}^2\right), \end{aligned} \tag{2.15}$$

and using the Cauchy-Schwarz inequality

$$\begin{aligned} |(\tilde{f}_1, \tilde{E}_{zL})_\Omega + (\tilde{f}_2, \tilde{H}_{yL})_\Omega| &\le \|\tilde{f}_1\|_\Omega\|\tilde{E}_{zL}\|_\Omega + \|\tilde{f}_2\|_\Omega\|\tilde{H}_{yL}\|_\Omega \\ &\le \frac{1}{4}\left(\|\sqrt{\epsilon}\tilde{E}_{zL}\|_\Omega^2 + \|\sqrt{\mu}\tilde{H}_{zL}\|_\Omega^2\right) + \frac{1}{\epsilon}\|\tilde{f}_1\|_\Omega^2 + \frac{1}{\mu}\|\tilde{f}_2\|_\Omega^2, \end{aligned} \tag{2.16}$$

where $\tilde{E}_{zL}(T)=\tilde{E}_{zL}(x,T)$ and $\tilde{H}_{yL}(T)=\tilde{H}_{yL}(x,T)$. Substituting (2.15)–(2.16) into (2.14),

$$\begin{aligned} &\frac{1}{4}\left(\|\sqrt{\epsilon}\tilde{E}_{zL}\|_\Omega^2 + \|\sqrt{\mu}\tilde{H}_{zL}\|_\Omega^2\right) + \frac{T}{2}\left(\|\sqrt{\epsilon}\tilde{E}_{zL}(T)\|_{I_x}^2 + \|\sqrt{\mu}\tilde{H}_{zL}(T)\|_{I_x}^2\right) \\ &\qquad \le \frac{1}{\epsilon}\|\tilde{f}_1\|_\Omega^2 + \frac{1}{\mu}\|\tilde{f}_2\|_\Omega^2. \end{aligned} \tag{2.17}$$

and noting that $\|E_{zL}\|_\Omega \le T\|\tilde{E}_{zL}\|_\Omega$, $\|H_{yL}\|_\Omega \le T\|\tilde{H}_{yL}\|_\Omega$, we get the result of (2.12). □

2.3. Error estimate

In the following section, the error estimate of (2.10) is given. In order to deal with the error of the initial value, the following auxiliary problem is considered [25]

$$\begin{cases} \epsilon\partial_t E = \partial_x H, & (x,t) \in \Omega, \\ \mu\partial_t H = \partial_x E, & (x,t) \in \Omega, \\ E(-1,t) = E(1,t) = 0, & t \in I_t, \\ E(x,0) = I_N^L E_{z0}(x),\ H(x,0) = P_{N-1}I_N^L H_{y0}(x), & x \in I_x. \end{cases} \tag{2.18}$$

Firstly, the estimate between the two solutions to (2.10) and (2.18) is considered. We define

$$E_a = P_N^1 P_M^1 E, \quad H_a = P_{N-1}P_M^1 H. \tag{2.19}$$

By (2.5) and (2.8), we have

$$\begin{aligned} (\partial_t P_M^1 E, v)_{I_t} &= (\partial_t E, v)_{I_t}, \qquad \forall\, v \in V_{M-1}, \\ (\partial_x P_N^1 E, w)_{I_x} &= (\partial_x E, w)_{I_x}, \qquad \forall\, w \in V_{N-1}, \end{aligned}$$

and

$$\begin{cases} (\epsilon\partial_t E_a, v)_\Omega + (H_a, \partial_x v)_\Omega = (\epsilon P_N^1 \partial_t E, v)_\Omega - (P_M^1 \partial_x H, v)_\Omega, & \forall\, v \in V_N^0 \otimes V_{M-1}, \\ (\mu\partial_t H_a, w)_\Omega - (\partial_x E_a, w)_\Omega = (\mu\partial_t H, w)_\Omega - (P_M^1 \partial_x E, w)_\Omega, & \forall\, w \in V_{N-1} \otimes V_{M-1}. \end{cases} \tag{2.20}$$

Let $e_z = E_{zL} - E_a$ and $e_y = H_{yL} - H_a$. By (2.10) and (2.20), the following error equation is obtained

$$\begin{cases} (\epsilon\partial_t e_z, v)_\Omega + (e_y, \partial_x v)_\Omega = (f_1, v)_\Omega, & \forall\, v \in V_N^0 \otimes V_{M-1}, \\ (\mu\partial_t e_y, w)_\Omega - (\partial_x e_z, w)_\Omega = (f_2, w)_\Omega, & \forall\, w \in V_{N-1} \otimes V_{M-1}, \\ e_z(x,0) = 0, \;\; e_y(x,0) = 0, & \forall x \in I_x. \end{cases} \tag{2.21}$$

Due to (2.18), we have $\epsilon\partial_t E = \partial_x H$, $\mu\partial_t H = \partial_x E$, and

$$f_1 = \epsilon(P_M^1 - I)\partial_t E + (I - P_N^1)\partial_x H, \quad f_2 = \mu(P_M^1 - I)\partial_t H.$$

Similar to the proof of Theorem 2.1, we obtain the following error estimate.

Theorem 2.2. Let E_a and H_a be the projections (2.19) of E and H (2.18), respectively. Let E_{zL} and H_{yL} be the solutions to (2.10), respectively. Assuming that $\sigma \ge 1$, $r \ge 2$, $E, H \in C([0,T]; H^r(I_x)) \cap L^2(I_x; H^\sigma(I_t))$, and then there exists a positive constant C such that

$$\begin{aligned} &\|\sqrt{\epsilon}(E_{zL} - E_a)\|_\Omega^2 + \|\sqrt{\mu}(H_{yL} - H_a)\|_\Omega^2 + T\left(\|\sqrt{\epsilon}(E_{zL} - E_a)(T)\|_{I_x}^2 + \|\sqrt{\mu}(H_{yL} - H_a)(T)\|_{I_x}^2\right) \\ &\quad \le CT^2\left[M^{2(1-\sigma)}\left(\|\partial_t^\sigma E\|_\Omega^2 + \|\partial_t^\sigma H\|_\Omega^2\right) + N^{2(1-r)}\|\partial_x^r H\|_\Omega^2\right]. \end{aligned} \tag{2.22}$$

Proof. By (2.12) and (2.21), we have

$$\|\sqrt{\epsilon}e_z\|_\Omega^2 + \|\sqrt{\mu}e_y\|_\Omega^2 + T\left(\|\sqrt{\epsilon}e_z(T)\|_{I_x}^2 + \|\sqrt{\mu}e_y(T)\|_{I_x}^2\right) \le CT^2\left(\|f_1\|_\Omega^2 + \|f_2\|_\Omega^2\right). \tag{2.23}$$

According to Lemma 2.1 and 2.2, it follows that

$$\|f_1\|_\Omega^2 \le C\left(M^{2(1-\sigma)}\|\partial_t^\sigma E\|_\Omega^2 + N^{2(1-r)}\|\partial_x^r H\|_\Omega^2\right), \tag{2.24}$$

$$\|f_2\|_\Omega^2 \le CM^{2(1-\sigma)}\|\partial_t^\sigma H\|_\Omega^2. \tag{2.25}$$

Substituting (2.24)–(2.25) into (2.23), the error estimate (2.22) is obtained. □

Next, the error estimate between the solutions to (2.10) and (1.1) is considered.

Theorem 2.3. Let E_z, H_y, E_{zL}, and H_{yL} be the solutions to (1.1) and (2.10), respectively. Assume that $\sigma \ge 1$, $r \ge 2$, $E, H \in C([0,T]; H^r(I_x)) \cap L^2(I_x; H^\sigma(I_t))$, and then there exists a positive constant C such that

$$\begin{aligned} &\|\sqrt{\epsilon}(E_{zL} - E_z)\|_\Omega^2 + \|\sqrt{\mu}(H_{yL} - H_y)\|_\Omega^2 + T\left(\|\sqrt{\epsilon}(E_{zL} - E_z)(T)\|_{I_x}^2 + \|\sqrt{\mu}(H_{yL} - H_y)(T)\|_{I_x}^2\right) \\ &\quad \le CT^2\left[M^{2(1-\sigma)}\left(\|\partial_t^\sigma E\|_\Omega^2 + \|\partial_t^\sigma H\|_\Omega^2\right) + N^{2(1-r)}\|\partial_x^r H\|_\Omega^2\right] \\ &\qquad + CTN^{-2r}\left(\|E_z\|_{L^\infty(0,T;H^r(I_x))}^2 + \|H_y\|_{L^\infty(0,T;H^r(I_x))}^2\right). \end{aligned} \tag{2.26}$$

Proof. Firstly, the error between the solutions to (2.18) and (1.1) is estimated. Let $e_z = E - E_z$ and $e_y = H - H_y$. By (1.1) and (2.18), we get the following error equation

$$\begin{cases} \epsilon\partial_t e_z = \partial_x e_y, & (x,t) \in \Omega, \\ \mu\partial_t e_y = \partial_x e_z, & (x,t) \in \Omega. \end{cases} \tag{2.27}$$

Then, we consider the inner product on I_x

$$\begin{cases} (\epsilon\partial_t e_z, e_z)_{I_x} = -(e_y, \partial_x e_z)_{I_x}, \\ (\mu\partial_t e_y, e_y)_{I_x} = (\partial_x e_z, e_y)_{I_x}, \end{cases} \tag{2.28}$$

which leads to

$$\|\sqrt{\epsilon}e_z(t)\|_{I_x}^2 + \|\sqrt{\mu}e_y(t)\|_{I_x}^2 = \|\sqrt{\epsilon}e_z(0)\|_{I_x}^2 + \|\sqrt{\mu}e_y(0)\|_{I_x}^2, \quad \forall t > 0. \tag{2.29}$$

Next, integrating over t

$$\|\sqrt{\epsilon}e_z\|_\Omega^2 + \|\sqrt{\mu}e_y\|_\Omega^2 = T(\|\sqrt{\epsilon}e_z(0)\|_{I_x}^2 + \|\sqrt{\mu}e_y(0)\|_{I_x}^2), \tag{2.30}$$

and taking $t = T$, we have

$$\|\sqrt{\epsilon}e_z(T)\|_{I_x}^2 + \|\sqrt{\mu}e_y(T)\|_{I_x}^2 = \|\sqrt{\epsilon}e_z(0)\|_{I_x}^2 + \|\sqrt{\mu}e_y(0)\|_{I_x}^2. \tag{2.31}$$

According to Lemma 2.1, it follows that

$$\begin{aligned} \|e_z(0)\|_{I_x} &= \|E_{z0} - I_N^L E_{z0}\|_{I_x} \le CN^{-r}|E_{z0}|_{r,I_x}, \\ \|e_y(0)\|_{I_x} &= \|H_{y0} - P_{N-1}I_N^L H_{y0}\|_{I_x} \\ &\le \|H_{y0} - P_{N-1}H_{y0}\|_{I_x} + \|P_{N-1}(H_{y0} - I_N^L H_{y0})\|_{I_x} \le CN^{-r}|H_{y0}|_{r,I_x}. \end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.30)–(2.31), we obtain

$$\begin{aligned} &\|\sqrt{\epsilon}(E - E_z)\|_\Omega^2 + \|\sqrt{\mu}(H - H_y)\|_\Omega^2 + T\left(\|\sqrt{\epsilon}(E - E_z)(T)\|_{I_x}^2 + \|\sqrt{\mu}(H - H_y)(T)\|_{I_x}^2\right) \\ &\le CTN^{-2r}\left(|E_{z0}|_{r,I_x}^2 + |H_{y0}|_{r,I_x}^2\right). \end{aligned} \tag{2.33}$$

On the other hand, by Lemmas 2.1–2.2, we have

$$\begin{aligned} &\|\sqrt{\epsilon}(E_a - E)\|_\Omega^2 + \|\sqrt{\mu}(H_a - H)\|_\Omega^2 + T\left(\|\sqrt{\epsilon}(E_a - E)(T)\|_{I_x}^2 + \|\sqrt{\mu}(H_a - H)(T)\|_{I_x}^2\right) \\ &\le CTN^{-2r}\left(|E_{z0}|_{r,I_x}^2 + |H_{y0}|_{r,I_x}^2\right). \end{aligned} \tag{2.34}$$

From (2.22) and (2.33)–(2.34), the error estimate (2.26) is obtained. □

3. Time multi-interval Legendre-tau spectral method

In this section, a time multi-interval Legendre-tau spectral scheme is developed and its error estimate is obtained.

3.1. Preliminaries

Let K be a positive integer and a partition of the computational interval I_t is given as

$$I_t = \bigcup_{k=1}^{K} I_k, \quad I_k = (a_{k-1}, a_k), \quad \tau_k = a_k - a_{k-1}, \quad 1 \le k \le K, \tag{3.1}$$

where

$$0 = a_0 < a_1 < \cdots < a_k < \cdots < a_K = T.$$

Let $\mathcal{M} = (M_1, \cdots, M_K)$ and $L = (N, \mathcal{M})$. We define the space of approximate functions in time as

$$X_{\mathcal{M}} = W_{\mathcal{M}} \cap H^1(I_t), \quad W_{\mathcal{M}} = \{v : v|_{I_k} \in \mathbb{P}_{M_k}(I_k),\ 1 \le k \le K\}, \tag{3.2}$$

where $\mathbb{P}_{M_k}(I_k)$ denotes the space of polynomials of degree at most M_k on I_k. We define the space of the test functions in time as

$$W_{\mathcal{M}-1} = \{v : v|_{I_k} \in \mathbb{P}_{M_k-1}(I_k),\ 1 \le k \le K\}, \tag{3.3}$$

where $\mathcal{M} - 1 = (M_1 - 1, \cdots, M_K - 1)$.

Let $\hat{I} = (-1, 1)$ be a reference interval, $\hat{t}_j^k$ and $\hat{\omega}_j^k (0 \le j \le M_k)$ be the LGL points and the corresponding weights on $\hat{I}$. We denote by $\{\hat{t}_j^k\}$ and $\{\hat{\omega}_j^k\}$ be the LGL points and the corresponding weights on I_k. Next, we define

$$I_{\mathcal{M}}^k = \{t_j^k : t_j^k = \frac{\tau_k \hat{t}_j^k + a_{k-1} + a_k}{2},\ 0 \le j \le M_k, 1 \le k \le K\},$$

where $\tau_k = a_k - a_{k-1}$.

Letting $v^k \equiv v|_{I_k}$, for any $u, v \in C(\bar{I})$ and $\omega_j^k = \frac{1}{2}\tau_k \hat{\omega}_j^k$, we define

$$(u, v)_{\mathcal{M}, I_k} = \sum_{j=0}^{M_k} u^k(t_j^k) v^k(t_j^k) \omega_j^k, \quad (u, v)_{\mathcal{M}} = \sum_{k=1}^{K} (u, v)_{\mathcal{M}, I_k}.$$

Similarly, we denote $\hat{t}_j^{k,C}$ and $\hat{\omega}_j^{k,C}$ be the CGL points and the corresponding weights on $\hat{I}$. Let $\{\hat{t}_j^{k,C}\}$ and $\{\hat{\omega}_j^{k,C}\}$ be the CGL points and the corresponding weights on I_k.

We define LGL interpolation operator $I_{\mathcal{M}}^L : C(\bar{I}) \to W_{\mathcal{M}}$ by

$$I_{\mathcal{M}}^L u(t_j^k) = u(t_j^k), \ \ 0 \le M_k, \ \ 1 \le k \le K.$$

Similarly, for the CGL interpolation operator $I_{\mathcal{M}}^L : C(\bar{I}) \to W_{\mathcal{M}}$, which satisfies

$$I_{\mathcal{M}}^C u(t_j^{k,C}) = u(t_j^{k,C}), \ \ 0 \le M_k, \ \ 1 \le k \le K.$$

Define the following relation

$$v(t) = \hat{v}(\hat{t}), \quad t = \frac{1}{2}(\tau_k \hat{t} + a_{k-1} + a_k), \quad a_{k-1} \le t \le a_k.$$

Let $\hat{P}_{M_k-1} : L^2(\hat{I}) \to \mathbb{P}_{M_k-1}$ the L^2-Legendre projection operator by $P_{\mathcal{M}-1} : L^2(I_t) \to W_{\mathcal{M}-1}$ such that

$$(P_{\mathcal{M}-1}v)|_{I_k}(t) = \hat{P}_{M_k-1}\widehat{(v|_{I_k})}(\hat{t}).$$

Let $\hat{P}_{1,M_k} : H^1(\hat{I}) \to \mathbb{P}_{M_k}$ be the Legendre projection operator, which satisfies

$$\hat{P}_{1,M_k}\hat{v}(\hat{t}) = \hat{v}(-1) + \int_{-1}^{\hat{t}} \hat{P}_{M_k-1}\partial_{\hat{t}}\hat{v}(s)\mathrm{d}s,$$

and $P_{1,\mathcal{M}}$ be generated by $P_{1,M_k} : H^1(I_k) \to \mathbb{P}_{M_k}(I_k)$ such that

$$(P_{1,\mathcal{M}}v)|_{I_k}(t) \equiv P_{1,M_k}v|_{I_k}(t) = \hat{P}_{1,M_k}\widehat{(v|_{I_k})}(\hat{t}). \tag{3.4}$$

The following approximation results can be found in [10].

Lemma 3.1. *If $v \in H^{\sigma}(I_t)$ and $\sigma \geq 1$, then*

$$|P^1_{\mathcal{M}}v - v|_{l,I_t} \leq C\left(\sum_{k=1}^{K}(\tau_k^{-1}M_k)^{2(l-\sigma)}|v|^2_{\sigma,I_k}\right)^{\frac{1}{2}}, \quad l = 0, 1,$$

where C is a generic positive constant independent of τ_k, M_k.

The time multi-interval Legendre-tau spectral method for the problem (1.1) is : Find $E^k_{zN} \in V^0_N \otimes W_M$ and $H^k_{yN} \in V_{N-1} \otimes W_M$ such that

$$\begin{cases} (\epsilon\partial_t E^K_{zL}, v)_\Omega + (H^K_{yL}, \partial_x v)_\Omega = 0, & \forall\ v \in V^0_N \otimes W_{\mathcal{M}-1}, \\ (\mu\partial_t H^K_{yL}, w)_\Omega - (\partial_x E^K_{zL}, w)_\Omega = 0, & \forall\ w \in V_{N-1} \otimes W_{\mathcal{M}-1}, \\ E^K_{zL}(x,0) = I^L_N E_{z0}(x), \quad H^K_{yL}(x,0) = P_{N-1}I^L_N H_{y0}(x), & \forall x \in I_x. \end{cases} \tag{3.5}$$

We set

$$v^k(x,t) = v(x, t + a_{k-1}), \quad t \in \hat{I}_k = (0, \tau_k), \quad 1 \leq k \leq K.$$

Let $\Omega_k = I_x \times \hat{I}_k$, and (3.5) can be written as: For $1 \leq k \leq K$, find $E^k_{zL} \in V^0_N \otimes \mathbb{P}_{M_k}(\hat{I}_k)$ and $H^k_{yL} \in V_{N-1} \otimes \mathbb{P}_{M_k}(\hat{I}_k)$ such that

$$\begin{cases} (\epsilon\partial_t E^k_{zL}, v^k)_{\Omega_k} + (H^k_{yL}, \partial_x v^k)_{\Omega_k} = 0, & \forall\ v^k \in V^0_N \otimes \mathbb{P}_{M_k-1}(\hat{I}_k), \\ (\mu\partial_t H^k_{yL}, w^k)_{\Omega_k} - (\partial_x E^k_{zL}, w^k)_{\Omega_k} = 0, & \forall\ w^k \in V_{N-1} \otimes \mathbb{P}_{M_k-1}(\hat{I}_k), \\ E^k_{zL}(x,0) = E^{k-1}_{zL}(x, \tau_{k-1}), \quad H^k_{yL}(x,0) = H^{k-1}_{yN}(x, \tau_{k-1}), & x \in I_x, \end{cases} \tag{3.6}$$

where $E^0_{zL}(x, \tau_0) = I^L_N E_{z0}(x)$, $H^0_{yL}(x, \tau_0) = P_{N-1}I^L_N H_{y0}(x)$ when $k = 1$.

3.2. Error estimate

In the following, we present the error estimate. In order to deal with the error of the initial value, we consider the following auxiliary problems on Ω_k, $1 \leq k \leq K$,

$$\begin{cases} \epsilon\partial_t E^k = \partial_x H^k, & (x,t) \in \Omega_k, \\ \mu\partial_t H^k = \partial_x E^k, & (x,t) \in \Omega_k, \\ E^k(x,0) = E^k_{zL}(x,0), \quad H^k(x,0) = H^k_{yL}(x,0), & x \in I_x. \end{cases} \tag{3.7}$$

Similar to the process of the single-interval, we define $E_a^k = P_N^1 P_{M_k}^1 E^k$, $H_a^k = P_{N-1} P_{M_k}^1 H^k$, and denote

$$f_1^k = \epsilon(P_{M_k}^1 - I)\partial_t E^k + (I - P_N^1)\partial_x H^k, \quad f_2^k = \mu(P_{M_k}^1 - I)\partial_t H^k.$$

Let $e_z^k = E_{zL}^k - E_a^k$ and $e_y^k = H_{yL}^k - H_a^k$, the following error equation is obtained

$$\begin{cases} (\epsilon\partial_t e_z^k, v^k)_{\Omega_k} + (e_y^k, \partial_x v^k)_{\Omega_k} = (f_1^k, v^k)_{\Omega_k}, & \forall\, v^k \in V_N^0 \otimes \mathbb{P}_{M_k-1}, \\ (\mu\partial_t e_y^k, w)_{\Omega_k} - (\partial_x e_z^k, w^k)_{\Omega_k} = (f_2^k, w^k)_{\Omega_k}, & \forall\, w^k \in V_{N-1} \otimes \mathbb{P}_{M_k-1}, \\ e_z^k(x,0) = 0, \;\; e_y^k(x,0) = 0, & \forall x \in I_x. \end{cases} \tag{3.8}$$

For each subinterval in the multi-interval, using Theorem 2.2 and Lemma 3.1, the error estimate between the solution to (3.6) and the projection of the solution to (3.7) is obtained

$$\begin{aligned} &\| \sqrt{\epsilon}(E_{zL}^k - E_a^k)\|_{\Omega_k}^2 + \| \sqrt{\mu}(H_{yL}^k - H_a^k)\|_{\Omega_k}^2 + \tau_k \left(\| \sqrt{\epsilon}(E_{zL}^k - E_a^k)(\tau_k)\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^k - H_a^k)(\tau_k)\|_{I_x}^2\right) \\ &\le C\tau_k^2 \left[(\tau_k^{-1} M_k)^{2(1-\sigma)} \left(\|\partial_t^\sigma E^k\|_{\Omega_k}^2 + \|\partial_t^\sigma H^k\|_{\Omega_k}^2\right) + N^{2(1-r)}\|\partial_x^r H^k\|_{\Omega_k}^2\right]. \end{aligned} \tag{3.9}$$

Let $\mathbf{e}_z^k = E^k - E_z^k$ and $\mathbf{e}_y^k = H^k - H_y^k$, the results are similar to (2.30)–(2.31) for the multi-interval case,

$$\| \sqrt{\epsilon}\mathbf{e}_z^k\|_{\Omega_k}^2 + \| \sqrt{\mu}\mathbf{e}_y^k\|_{\Omega_k}^2 = \tau_k \left(\| \sqrt{\epsilon}\mathbf{e}_z^k(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^k(0)\|_{I_x}^2\right), \tag{3.10}$$

$$\| \sqrt{\epsilon}\mathbf{e}_z^k(\tau_k)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^k(\tau_k)\|_{I_x}^2 = \| \sqrt{\epsilon}\mathbf{e}_z^k(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^k(0)\|_{I_x}^2. \tag{3.11}$$

Using the triangle inequality, we get

$$\begin{aligned} &\sqrt{\| \sqrt{\epsilon}\mathbf{e}_z^k(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^k(0)\|_{I_x}^2} = \sqrt{\| \sqrt{\epsilon}(E_{zL}^{k-1} - E_z^{k-1})(\tau_{k-1})\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^{k-1} - H_y^{k-1})(\tau_{k-1})\|_{I_x}^2} \\ &\le \sqrt{\| \sqrt{\epsilon}(E_{zL}^{k-1} - E^{k-1})(\tau_{k-1})\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^{k-1} - H^{k-1})(\tau_{k-1})\|_{I_x}^2} \\ &\quad + \sqrt{\| \sqrt{\epsilon}\mathbf{e}_z^{k-1}(\tau_{k-1})\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^{k-1}(\tau_{k-1})\|_{I_x}^2} \\ &= \sqrt{\| \sqrt{\epsilon}(E_{zL}^{k-1} - E^{k-1})(\tau_{k-1})\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^{k-1} - H^{k-1})(\tau_{k-1})\|_{I_x}^2} \\ &\quad + \sqrt{\| \sqrt{\epsilon}\mathbf{e}_z^{k-1}(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^{k-1}(0)\|_{I_x}^2}, \end{aligned}$$

which leads to

$$\begin{aligned} &\sqrt{\| \sqrt{\epsilon}\mathbf{e}_z^k(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^k(0)\|_{I_x}^2} \le \sum_{m=1}^{k-1} \sqrt{\| \sqrt{\epsilon}(E_{zL}^m - E^m)(\tau_m)\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^m - H^m)(\tau_m)\|_{I_x}^2} \\ &\quad + \sqrt{\| \sqrt{\epsilon}\mathbf{e}_z^1(0)\|_{I_x}^2 + \| \sqrt{\mu}\mathbf{e}_y^1(0)\|_{I_x}^2}, \qquad \forall k \ge 2. \end{aligned} \tag{3.12}$$

By the Cauchy-Schwarz inequality, $\sum_{m=1}^{k-1} \tau_m = a_{k-1}$, and (3.9), we derive

$$\begin{aligned} &\left(\sum_{m=1}^{k-1} \sqrt{\| \sqrt{\epsilon}(E_{zL}^m - E_a^m)(\tau_m)\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^m - H_a^m)(\tau_m)\|_{I_x}^2}\right)^2 \\ &\le a_{k-1} \sum_{m=1}^{k-1} \tau_m^{-1} \left(\| \sqrt{\epsilon}(E_{zL}^m - E_a^m)(\tau_m)\|_{I_x}^2 + \| \sqrt{\mu}(H_{yL}^m - H_a^m)(\tau_m)\|_{I_x}^2\right) \\ &\le Ca_{k-1} \sum_{m=1}^{k-1} \left[(\tau_m^{-1} M_m)^{2(1-\sigma)} \left(\|\partial_t^\sigma E^m\|_{\Omega_m}^2 + \|\partial_t^\sigma H^m\|_{\Omega_m}^2\right) + N^{2(1-r)}\|\partial_x^r H^m\|_{\Omega_m}^2\right]. \end{aligned} \tag{3.13}$$

According to (2.7) and Lemma 2.1, it follows that

$$\begin{aligned}
&\left(\sum_{m=1}^{k-1}\sqrt{\|\sqrt{\epsilon}(E_a^m-E^m)(\tau_m)\|_{I_x}^2+\|\sqrt{\mu}(H_a^m-H^m)(\tau_m)\|_{I_x}^2}\right)^2\\
&=\left(\sum_{m=1}^{k-1}\sqrt{\|\sqrt{\epsilon}(P_N^1-I)E^m(\tau_m)\|_{I_x}^2+\|\sqrt{\mu}(P_{N-1}-I)H^m(\tau_m)\|_{I_x}^2}\right)^2 \qquad (3.14)\\
&\leq Ca_{k-1}\sum_{m=1}^{k-1}\tau_m^{-1}N^{-2r}\left(|E^m(\tau_m)|_{r,I_x}^2+|H^m(\tau_m)|_{r,I_x}^2\right).
\end{aligned}$$

As (2.32), we have

$$\|\sqrt{\epsilon}\mathbf{e}_z^1(0)\|_{I_x}^2+\|\sqrt{\mu}\mathbf{e}_y^1(0)\|_{I_x}^2\leq CN^{-2r}\left(|E_{z0}|_{r,I_x}^2+|H_{y0}|_{r,I_x}^2\right).$$

Substituting the above estimation results into (3.10)–(3.11), we obtain

$$\begin{aligned}
&\|\sqrt{\epsilon}(E^k-E_z^k)\|_{\Omega_k}^2+\|\sqrt{\mu}(H^k-H_y^k)\|_{\Omega_k}^2+\tau_k\left(\|\sqrt{\epsilon}(E^k-E_z^k)(\tau_k)\|_{I_x}^2+\|\sqrt{\mu}(H^k-H_y^k)(\tau_k)\|_{I_x}^2\right)\\
&\leq Ca_{k-1}\tau_k\sum_{m=1}^{k-1}\left[(\tau_m^{-1}M_m)^{2(1-\sigma)}\left(\|\partial_t^\sigma E^m\|_{\Omega_m}^2+\|\partial_t^\sigma H^m\|_{\Omega_m}^2\right)+N^{2(1-r)}\|\partial_x^r H^m\|_{\Omega_m}^2\right] \qquad (3.15)\\
&\quad+Ca_{k-1}\tau_k\sum_{m=1}^{k-1}\tau_m^{-1}N^{-2r}\left(|E^m(\tau_m)|_{r,I_x}^2+|H^m(\tau_m)|_{r,I_x}^2\right)+C\tau_kN^{-2r}\left(|E_{z0}|_{r,I_x}^2+|H_{y0}|_{r,I_x}^2\right).
\end{aligned}$$

By (2.7), Lemma 2.1 and 2.2, we get

$$\begin{aligned}
&\|\sqrt{\epsilon}(E_a^k-E^k)\|_{\Omega_k}^2+\|\sqrt{\mu}(H_a^k-H^k)\|_{\Omega_k}^2+\tau_k\left(\|\sqrt{\epsilon}(E_a^k-E^k)(\tau_k)\|_{I_x}^2+\|\sqrt{\mu}(H_a^k-H^k)(\tau_k)\|_{I_x}^2\right)\\
&\leq C\left[(\tau_k^{-1}M_k)^{-2\sigma}\left(\|\partial_t^\sigma E^k\|_{\Omega_k}^2+\|\partial_t^\sigma H^k\|_{\Omega_k}^2\right)+N^{-2r}\left(\|\partial_x^r E^k\|_{\Omega_k}^2+\partial_x^r H^k\|_{\Omega_k}^2\right)\right] \qquad (3.16)\\
&\quad+C\tau_kN^{-2r}\left(|E^k(\tau_k)|_{r,I_x}^2+|H^k(\tau_k)|_{r,I_x}^2\right).
\end{aligned}$$

If $\tau_k\equiv\tau$, $M_k\equiv M$ for simplicity, and combining (3.9) and (3.15)–(3.16), we get the following error estimate.

Theorem 3.1. Let E_z and H_y be solutions to (1.1), respectively. Let E_{zL}^K and H_{yL}^K be solutions to (3.5), respectively. Let E^k and H^k be solutions to (3.7), respectively. Assuming that $\sigma\geq 1$, $r\geq 2$, $E_z,H_y\in C([0,T];H^r(I_x))\cap L^2(I_x;H^\sigma(I_t))$, $E^k,H^k\in C([0,\tau_k];H^r(I_x))\cap L^2(I_x;H^\sigma(\hat{I}^k))$, and then there exists a positive constant C such that

$$\begin{aligned}
&\|\sqrt{\epsilon}(E_{zL}^K-E_z)\|_\Omega^2+\|\sqrt{\mu}(H_{yL}^K-H_y)\|_\Omega^2+\sum_{k=1}^K\tau_k\left(\|\sqrt{\epsilon}(E_{zL}^k-E_z^k)(\tau_k)\|_{I_x}^2+\|\sqrt{\mu}(H_{yL}^k-H_y^k)(\tau_k)\|_{I_x}^2\right) \qquad (3.17)\\
&\leq C\left[(\tau^{-1}M)^{2(1-\sigma)}+N^{2(1-r)}+\tau^{-2}N^{-2r}\right].
\end{aligned}$$

4. Numerical examples

In this section, some numerical results are presented. We define

$$\mathbb{E}_\infty(E_z) = \max_{0\le j\le N} |E_{zL}(x_j^C, t) - E_z(x_j^C, t)|,$$
$$\mathbb{E}_\infty(H_y) = \max_{0\le j\le N} |H_{yL}(x_j^C, t) - H_y(x_j^C, t)|.$$

Example 4.1. The LT-ST spectral method for the 1-D Maxwell equation

Consider the problem (1.1) with $I_x = (0, 1)$, $I_t = (0, 1)$, $\Omega = I_x \times I_t$, $\epsilon = 1$, and $\mu = 1$. The solution is as

$$\begin{cases} \mathrm{E_z(x, t) = cos(3\pi t)sin(3\pi x)}, & (x, t) \in \Omega, \\ \mathrm{H_y(x, t) = sin(3\pi t)cos(3\pi x)}, & (x, t) \in \Omega. \end{cases} \tag{4.1}$$

In Figure 1, the values of $\log_{10} \mathbb{E}_\infty(E_z)$ and $\log_{10} \mathbb{E}_\infty(H_y)$ is obtained when $t = 1$. It can be seen from Figure 1 that the LT-ST method has spectral accuracy both in the time and space, which is consistent with the results of theoretical analysis.

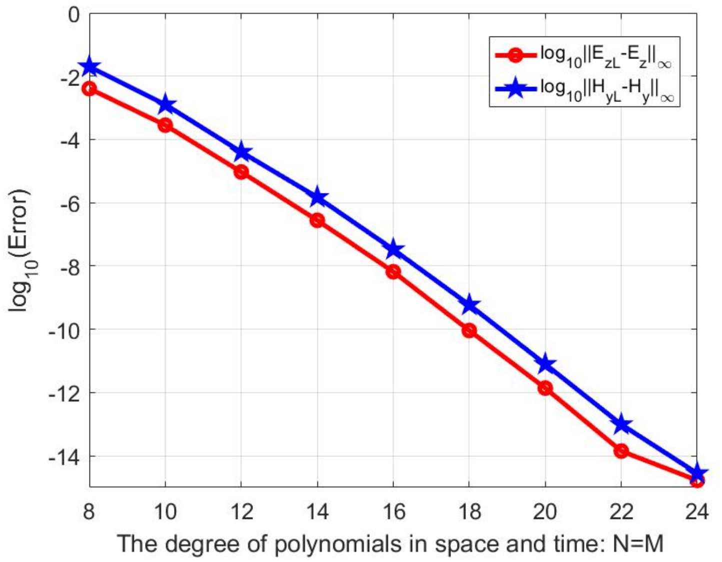

Figure 1. L^∞-error at $t = 1$ of the LT-ST method (2.10).

To check the high accuracy, we compare the numerical errors of our scheme (2.10) with the Legendre-tau spectral method in space and the leapfrog-Crank-Nicolson method in time (LT-LFCN) [19]. For convenience of notation, let (N, τ) be the degree of the polynomial in the space approximation and the time step for the LT-LFCN method.

The L^∞-error of the LT-LFCN scheme and our method (2.10) at $t = 1$ are listed in Table 1. It can be seen from Table 1 that on the same PC machine, the proposed method takes shorter time than the LT-LFCN method.

Table 1. L^∞-error of the LT-LFCN method and the LT-ST method (2.10).

LT-LFCN				LT-ST			
(N,τ)	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time	(N,M)	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time
(8,1e-02)	5.14e-04	6.85e-03	0.19s	(8,8)	4.04e-03	1.99e-02	0.08s
(12,1e-03)	4.35e-06	6.97e-06	0.49s	(12,12)	9.38e-06	3.99e-05	0.10s
(16,1e-04)	1.29e-09	6.96e-07	4.94s	(16,16)	6.57e-09	3.34e-08	0.10s
(20,1e-05)	1.03e-13	6.98e-09	52.81s	(20,20)	1.38e-12	7.86e-12	0.10s
(24,1e-06)	2.83e-14	6.97e-11	598.48s	(24,24)	1.69e-15	2.99e-15	0.11s

Example 4.2. The time multi-interval Legendre-tau spectral method for the 1-D Maxwell equation

Further, the method (3.6) is used to solve Example 4.1 of $N = M_k = 24$ and $0 \le t \le 5$, and the numerical results are shown in Table 2.

Table 2. L^∞-error of the time five-interval Legendre-tau spectral method (3.6) ($N = M_k = 24$).

t	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time
1.00	1.69e-15	2.99e-15	0.11s
2.00	3.10e-15	3.44e-15	0.12s
3.00	3.38e-15	3.44e-15	0.13s
4.00	5.82e-15	7.10e-15	0.16s
5.00	9.49e-15	7.71e-15	0.19s

5. Application to the 1-D nonlinear Maxwell equation

In this section, the proposed method is applied to the numerical solution of the 1-D nonlinear Maxwell equation. The approximating of the nonlinear term is calculated by interpolation at the CGL point, and implemented with the help of Fast Legendre transformation.

5.1. Scheme

Now, we apply the LT-ST method to solve the 1-D nonlinear Maxwell equation as [26]

$$\begin{cases} \epsilon\partial_t E_z + J(E_z) - \partial_x H_y = 0, & (x,t) \in \Omega, \\ \mu\partial_t H_y - \partial_x E_z = 0, & (x,t) \in \Omega, \\ E_z(-1,t) = E_z(1,t) = 0, & t \in I_t, \\ E_z(x,0) = E_{z0}(x),\ H_y(x,0) = H_{y0}(x), & x \in I_x, \end{cases} \tag{5.1}$$

where the nonlinear function $J(E_z) = \sigma(|E_z|)E_z$ with $\sigma(s)$ is a real valued function representing the electric conductivity.

The problem (5.1) can be written in a weak form: Find $E_z \in H_0^1(I_x)\otimes H^1(I_t)$ and $H_y \in L^2(I_x)\otimes H^1(I_t)$ such that

$$\begin{cases} (\epsilon\partial_t E_z, v)_\Omega + (J(E_z), v)_\Omega + (H_y, \partial_x v)_\Omega = 0, & \forall\, v \in H_0^1(I_x)\otimes L^2(I_t), \\ (\mu\partial_t H_y, w)_\Omega - (\partial_x E_z, w)_\Omega = 0, & \forall\, w \in L^2(I_x)\otimes L^2(I_t), \\ E_z(x,0) = E_{z0}(x),\ H_y(x,0) = H_{y0}(x), & \forall x \in I_x. \end{cases} \tag{5.2}$$

Combining the interpolation operator both in space and time, a 2-D interpolation is defined as $I^L_{(N,M)}$. The LT-ST method to the problem (5.1) is: Find $E_{zL} \in V_N^0 \otimes V_M$ and $H_{yL} \in V_{N-1} \otimes V_M$ such that

$$\begin{cases} (\epsilon\partial_t E_{zL}, v)_\Omega + (I^C_{(N,M)} J(E_{zL}), v)_\Omega + (H_{yL}, \partial_x v)_\Omega = 0, & \forall\ v \in V_N^0 \otimes V_{M-1}, \\ (\mu\partial_t H_{yL}, w)_\Omega - (\partial_x E_{zL}, w)_\Omega = 0, & \forall\ w \in V_{N-1} \otimes V_{M-1}, \\ E_{zL}(x,0) = I_N^L E_{z0}(x), \quad H_{yL}(x,0) = P_{N-1} I_N^L H_{y0}(x), & \forall x \in I_x, \end{cases} \tag{5.3}$$

We briefly describe the implementation of scheme (5.3). For simplicity, taking $\Omega = [-1,1]\times[-1,1]$. Let L_k be the Legendre polynomial of degree k, and the basis functions in space are

$$\Phi(x) = \left(\frac{1-x}{2}, \frac{1+x}{2}, \phi_2(x), ..., \phi_N(x)\right),$$

$$\Phi_0(x) = (\phi_2(x), ..., \phi_N(x)), \quad L(x) = (L_0(x), L_1(x), ..., L_{N-1}(x)),$$

where $\phi_k(x) = L_k(x) - L_{k-2}(x)$.

The basis functions in time are

$$\Psi(t) = (1, 1+t, \phi_2(t), ..., \phi_M(t)), \quad L(t) = (L_0(t), L_1(t), ..., L_{M-1}(t)),$$

where $\phi_k(t) = L_k(t) - L_{k-2}(t)$.

The approximate solutions and the test functions are expressed as

$$\begin{aligned} E_{zL}(x,t) &= \Psi(t)\hat{E}\Phi^T(x), & H_{yL}(x,t) &= \Psi(t)\hat{H}L^T(x), \\ v(x,t) &= L(t)\hat{v}\Phi_0^T(x), & w(x,t) &= L(t)\hat{w}L^T(x). \end{aligned}$$

The interpolation polynomial of the nonlinear term can be expressed as $I^C_{(N,M)} J(E_{zL}) = \Psi(t)\hat{J}\Phi^T(x)$. The following algebraic equation is obtained from (5.3)

$$\begin{cases} \epsilon(\partial_t\Psi, L)_{I_t}\hat{E}(\Phi_0, \Phi)_{I_x} + (\Psi, L)_{I_t}\hat{J}(\Phi_0, \Phi)_{I_x} + (\Psi, L)_{I_t}\hat{H}(\partial_x\Phi_0, L)_{I_x} = 0, \\ \mu(\partial_t\Psi, L)_{I_t}\hat{H}(L, L)_{I_x} - (\Psi, L)_{I_t}\hat{E}(L, \partial_x\Phi)_{I_x} = 0, \end{cases} \tag{5.4}$$

where $\hat{E}$ and $\hat{H}$ are matrices composed of coefficients of approximate solutions E_{zL} and H_{yL}, respectively. For simplicity, (5.4) can be rewritten in matrix form as

$$\begin{cases} \epsilon K^t\hat{E}M^x + M^t\hat{J}M^x + M^t\hat{H}K_0^x = 0, \\ \mu K^t\hat{H}D - M^t\hat{E}K^{xT} = 0. \end{cases} \tag{5.5}$$

A simple implicit-explicit iteration method is used to solve (5.5). In order to separate the initial conditions from the coefficient matrix, $\hat{E}$, $\hat{H}$, M_t is divided into the following forms as

$$\hat{E} = \begin{bmatrix} \hat{E}_i \\ \hat{E}_0 \end{bmatrix}, \qquad \hat{H} = \begin{bmatrix} \hat{H}_i \\ \hat{H}_0 \end{bmatrix}, \qquad M^t = \begin{bmatrix} M_i^t & M_0^t \end{bmatrix}, \tag{5.6}$$

where $\hat{E}_i$ and $\hat{H}_i$ are the first rows of the coefficient matrix $\hat{E}$ and $\hat{H}$ respectively, corresponding to the initial value, M_i^t is the first column of M^t. By the properties of the basis function and the orthogonality of Legendre polynomials show that both K^t and D are diagonal matrices, and the elements on the diagonal of K^t are 2 except that the first element is zero. Thus, (5.5) can be expressed as

$$4\epsilon\mu\hat{E}_0 M^x + (M_0^t)^2 \hat{E}_0 K^{xx} = -2\mu M^t \hat{J} M^x - 2\mu M_i^t \hat{H}_i K_0^x - M_0^t M_i^t \hat{E}_i K^{xx}, \tag{5.7}$$

$$2\mu\hat{H}_0 = M_0^t \hat{E}_0 K^{xT} D^{-1} + M_i^t \hat{E}_i K^{xT} D^{-1}. \tag{5.8}$$

Let

$$G = -2\mu M_i^t \hat{H}_i K_0^x - M_0^t M_i^t \hat{E}_i K^{xx},$$

In computations. We use the following simple explicit-implicit iteration scheme for (5.7),

$$4\epsilon\mu\hat{E}_0^{[k+1]} M^x + (M_0^t)^2 \hat{E}_0^{[k+1]} K^{xx} = -2\mu M^t \hat{J}^{[k]} M^x + G, \quad k = 0, 1, \cdots, \tag{5.9}$$

when $k = 0$, using the initial information of E_{zL} in (5.3), and taking $E_{zL}^{[0]}(t) \equiv E_{zL}(0)$ as the initial guess of the iteration. The iterative scheme (5.9) is a linear equation of $\hat{E}_0^{[k+1]}$, which can be solved by the method in [10].

Combining the interpolation operator in space and the multi-interval interpolation operator in time in Section 3, a 2-D interpolation is defined as $I_{(N,\mathcal{M})}^L$. The time multi-interval Legendre-tau spectral method for (5.1) is: Find $E_{zN}^k \in V_N^0 \otimes W_{\mathcal{M}}$ and $H_{yN}^k \in V_{N-1} \otimes W_{\mathcal{M}}$ such that

$$\begin{cases} (\epsilon\partial_t E_{zL}^K, v)_\Omega + (I_{(N,\mathcal{M})}^L J(E_{zL}^K), v)_\Omega + (H_{yL}^K, \partial_x v)_\Omega = 0, & \forall\, v \in V_N^0 \otimes W_{\mathcal{M}-1}, \\ (\mu\partial_t H_{yL}^K, w)_\Omega - (\partial_x E_{zL}^K, w)_\Omega = 0, & \forall\, w \in V_{N-1} \otimes W_{\mathcal{M}-1}, \\ E_{zL}^K(x,0) = I_N^L E_{z0}(x), \quad H_{yL}^K(x,0) = P_{N-1} I_N^L H_{y0}(x), & \forall x \in I_x, \end{cases} \tag{5.10}$$

In computation, the interval is shifted to $\hat{I}_k = (0, \tau_k)$. Let $\Omega_k = I_x \times \hat{I}_k$, and then (5.10) can be written as: Find $E_{zL}^k \in V_N^0 \otimes \mathbb{P}_{M_k}(\hat{I}_k)$ and $H_{yL}^k \in V_{N-1} \otimes \mathbb{P}_{M_k}(\hat{I}_k)$, $1 \le k \le K$, such that

$$\begin{cases} (\epsilon\partial_t E_{zL}^k, v^k)_{\Omega_k} + (I_{(N,M_k)}^L J(E_{zL}^k), v^k)_{\Omega_k} + (H_{yL}^k, \partial_x v^k)_{\Omega_k} = 0, & \forall\, v^k \in V_N^0 \otimes \mathbb{P}_{M_k-1}(\hat{I}_k), \\ (\mu\partial_t H_{yL}^k, w^k)_{\Omega_k} - (\partial_x E_{zL}^k, w^k)_{\Omega_k} = 0, & \forall\, w^k \in V_{N-1} \otimes \mathbb{P}_{M_k-1}(\hat{I}_k), \\ E_{zL}^k(x,0) = E_{zL}^{k-1}(x, \tau_{k-1}), \quad H_{yL}^k(x,0) = H_{yN}^{k-1}(x, \tau_{k-1}), & x \in I_x, \end{cases} \tag{5.11}$$

where $E_{zL}^0(x, \tau_0) = I_N^L E_{z0}(x)$ and $H_{yL}^0(x, \tau_0) = P_{N-1} I_N^L H_{y0}(x)$ when $k = 1$.

5.2. Numerical examples

Example 5.1. The LT-ST method for the 1-D nonlinear Maxwell equation

Consider the problem (5.1), and set the right-hand function of the first equation to $f(x,t)$. According to [26], the nonlinear term is given as

$$J(E_z) = (|E_z|^2 - |E_z|^4) E_z,$$

where $I_x = (0,1)$, $I_t = (0,1)$, $\Omega = I_x \times I_t$, and $\epsilon = \mu = 1$. The solution is

$$\begin{cases} \mathrm{E_z(x,t)} = \cos(3\pi \mathrm{t})\sin(3\pi \mathrm{x}), & (x,t) \in \Omega, \\ \mathrm{H_y(x,t)} = \sin(3\pi \mathrm{t})\cos(3\pi \mathrm{x}), & (x,t) \in \Omega, \end{cases} \tag{5.12}$$

and the right-hand side of the first equation is

$$f(x,t) = \cos(3\pi t)^3 \sin(3\pi x)^3 - \cos(3\pi t)^5 \sin(3\pi x)^5, \quad (x,t) \in \Omega. \tag{5.13}$$

The scheme (5.3) is used to solve Example 5.1, and the values of $\log_{10} \mathbb{E}_\infty(E_z)$ and $\log_{10} \mathbb{E}_\infty(H_y)$ are obtained when $t = 1$. It can be seen from Figure 2 that the method has high accuracy both in time and space.

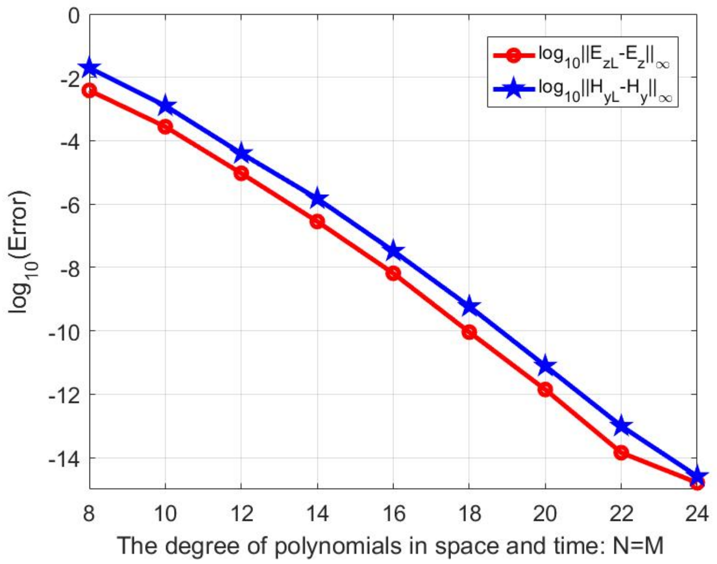

Figure 2. L^∞-error at $t = 1$ of the LT-ST method (5.3).

ItNum represents the number of iterations. Further, the method (5.11) is used to solve Example 5.1 in the case of $N = M_k = 24$ and $0 \le t \le 5$, the numerical results are shown in Table 3.

Table 3. L^∞-error of the time five-interval Legendre-tau spectral method (5.11) ($N = M_k = 24$).

t	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time	ItNum
1.00	1.72e-15	2.83e-15	0.17s	10
2.00	3.72e-15	3.44e-15	0.32s	10
3.00	4.11e-15	4.10e-15	0.49s	10
4.00	6.30e-15	6.55e-15	0.65s	10
5.00	1.04e-14	8.93e-15	0.81s	10

Example 5.2. Comparison of the LT-ST method of 1-D nonlinear Maxwell equation and related computation results

Consider the same problem as in Example 5.1, but the nonlinear is given as [26]

$$J(E_z) = |E_z|^{\frac{1}{2}} E_z.$$

Taking the same solution (5.12), the right-hand function of the first equation is

$$f(x,t) = \cos(3\pi t)\sin(3\pi x)\sqrt{|\cos(3\pi t)\sin(3\pi x)|}, \quad (x,t) \in \Omega. \tag{5.14}$$

The Scheme (5.3) is applied to Example 5.1, and the values of $\log_{10} \mathbb{E}_\infty(E_z)$ and $\log_{10} \mathbb{E}_\infty(H_y)$ is obtained when $t = 1$. Computational results are given in Figure 3 to show that the LT-ST method has high accuracy both in time and space.

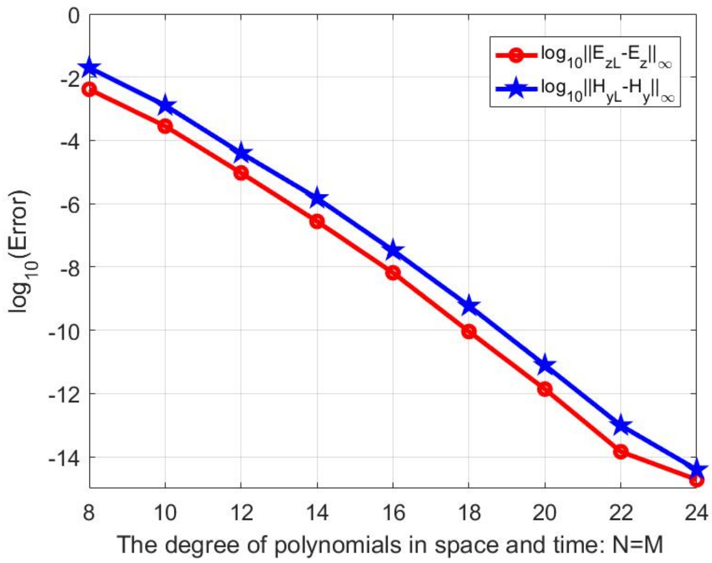

Figure 3. L^∞-error at $t = 1$ of the LT-ST method (5.3).

In order to compare the accuracy with the LT-LFCN method, we use it and the LT-ST method to computate Example 5.2, respectively. The LT-LFCN method is as follows:

Let τ be the time step, $t_k = k\tau$ $(k = 0, 1, \cdots, n_T; T = n_T\tau)$. Denote $u^k(x) := u(x, k\tau)$, and we define

$$u^k_{\hat{t}} = \frac{u^{k+1} - u^{k-1}}{2\tau}, \quad u^{\bar{k}} = \frac{u^{k+1} + u^{k-1}}{2}.$$

The LT-LFCN scheme to the problem (5.1) is: For $1 \le k \le n_T - 1$, find $E^k_{zN} \in V^0_N$ and $H^k_{yN} \in V_{N-1}$ such that

$$\begin{cases} (\epsilon E^k_{zN\hat{t}}, v) + (H^{\bar{k}}_{yN}, \partial_x v) + (I_N J(E^k_{zN}), v) = 0, & \forall\, v \in V^0_N, \\ (\mu H^k_{yN\hat{t}}, w) - (\partial_x E^{\bar{k}}_{zN}, w) = 0, & \forall\, w \in V_{N-1} \\ E^0_{zN} = I^L_N E_{z0},\ E^1_{zN} = I^L_N[E_{z0} + \tau\partial_t E_z(0)], & \\ H^0_{yN} = P^L_{N-1} I_N H_{y0},\ H^1_{yN} = P^L_{N-1} I_N[H_{y0} + \tau\partial_t H_y(0)]. & \end{cases} \tag{5.15}$$

The L^∞-error of the LT-LFCN method (5.15) and the proposed method (5.3) at $t = 1$ are shown in Table 4. The results in Table 4 demonstrate that on the same PC machine, the proposed method provides more accurate results using less time than the LT-LFCN method.

Table 4. L^∞-error of the LT-LFCN method (5.15) and the LT-ST method (5.3).

LT-LFCN				LT-ST			
(N, τ)	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time	(N, M)	$\mathbb{E}_\infty(E_z)$	$\mathbb{E}_\infty(H_y)$	time
(8,1e-02)	1.86e-03	1.98e-02	0.21s	(8,8)	4.13e-03	1.99e-02	0.16s
(12,1e-03)	2.22e-05	1.46e-04	0.71s	(12,12)	9.41e-06	3.98e-05	0.17s
(16,1e-04)	1.65e-07	1.56e-06	7.86s	(16,16)	6.55e-09	3.33e-08	0.18s
(20,1e-05)	1.50e-09	1.56e-08	82.99s	(20,20)	1.37e-12	7.86e-12	0.18s
(24,1e-06)	1.59e-11	1.58e-10	863.42s	(24,24)	1.77e-15	2.77e-15	0.19s

The scheme (5.11) is also used to solve Example 5.2 for long-time computation. Numerical results are given in Table 5 with $N = M_k = 24$ and $0 \le t \le 5$ to show the effectiveness of the LT-ST method.

Table 5. L^∞-error of the time five-interval Legendre-tau spectral method (5.11) ($N = M_k = 24$).

t	$L^\infty(E_z)$	$L^\infty(H_y)$	time	ItNum
1.00	1.77e-15	2.77e-15	0.19s	12
2.00	3.33e-15	4.88e-15	0.36s	11
3.00	3.77e-15	4.21e-15	0.55s	12
4.00	6.77e-15	7.21e-15	0.73s	11
5.00	9.85e-15	9.35e-15	0.91s	11

6. Conclusions

In this paper, the LT-ST method is investigated for the 1-D Maxwell equation and the time multi-interval Legendre-tau spectral method is considered. Error estimates for the method of single and multidomain are given, respectively. Numerical results are consistent with the theoretical analysis. Compared with the LT-LFCN method, the proposed method has advantages in accuracy and computation time. Moreover, the space-time spectral method is developed for the numerical solutions of the 1-D nonlinear Maxwell equation. In the future, the multidomain spectral method in space will be developed to solve the case of inhomogeneous media.

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Conflict of interest

The authors declare no conflict of interest.

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