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Research article

Estimates of trapezium-type inequalities for h-convex functions with applications to quadrature formulae

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Abstract: In this article, we develop a new class of trapezium-type inequalities up to twice differentiable h-convex mappings for fractional integrals of Riemann-type. We conclude numerous existing results in literature from our general inequalities. Based on our consequences, we will obtain some quadrature formulas as applications.

Keywords: mid-point type inequalities; generalized Riemann-Liouville fractional integral operator; h-convex; s-convex
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1. Introduction

It is familiar that the convexity has an important and key part in many areas due to its broad applications. This idea has been broadened and summed up in different directions (see, e.g., [1–8]). Sanja Varošanec present a class of convex functions [9] which is very helpful for the mathematicians working in the field of mathematical inequalities. There is a famous inequality for convex functions known as Hermite-Hadamard inequality (see, e.g., [10, p. 137]). This inequality provides bounds of the mean value of a continuous convex function $\Psi : (a, b) \to \mathbb{R}$.

If $\Psi : \triangle \to \mathbb{R}$ on an interval $\triangle$ of real numbers, such that $\rho, \varrho \in U$ with $\rho < \varrho$, we can write

$$\Psi\left(\frac{\rho+\varrho}{2}\right) \leq \frac{1}{\varrho-\rho}\int_{\rho}^{\varrho}\Psi(\varpi)d\varpi \leq \frac{\Psi(\rho)+\Psi(\varrho)}{2}. \quad (1.1)$$

Fractional calculus has vast applications in inequalities as well as in other physical sciences [11, 12]. Recently Hermite-Hadamard inequalities have been reinvestigated and developed by many researchers by using several fractional calculus operators [13–22]. In [23] Sarikaya et al. studied the Hemite-Hadamard inequalities for Riemann-Liouville fractional integral (RLFI) using convex functions. Lio et al. in [24] investigated the said inequalities by involving RLFI operator for twice differentiable geometric-arithmetically s-convex functions. Such inequalities were examined by Wu et al. in [25] and Iqbal et al. in [26] for k-fractional operators via different convexities. Mevlut Tunc in [27] explored the inequality given by (1.1) for h-convex functions via RLFI operator. To enhance the flow of the work, we present some mathematical preliminaries of the theory of fractional calculus that are required to set up our results.

Definition 1.1. *([24]) A function $\Psi : D \subset \mathbb{R}^+ \to \mathbb{R}^+$ is known as geometric-arithmetically s-convex on D if for $s \in (0, 1]$ and $\forall$, $\rho, \varrho \in D$ with $\nu \in [0, 1]$, we have*

$$\Psi(\rho^{\nu}\varrho^{1-\nu}) \leq \nu^{s}\Psi(\rho) + (1-\nu)^{s}\Psi(\varrho).$$

We utilize the class of h-convex $SX(h, D)$, h-concave $SV(h, D)$, s-convex in second sense K_s^2 and quasi convex $P(I)$ functions.

Sanja Varošanec present the following concept of convex and h-convex functions in [9] which are explained as follows:

Definition 1.2. *A function $\Psi : [\rho, \varrho] \to \mathbb{R}$ is said to be convex if the inequality*

$$\Psi(\omega l + (1-\omega)r) \leq \omega\Psi(l) + (1-\omega)\Psi(r)$$

holds for all $l, r \in [\rho, \varrho]$ with $\omega \in [0, 1]$.

Definition 1.3. *Let $h > 0$ be a function such that $h : D \subset \mathbb{R} \to \mathbb{R}$. Then the function $\Psi > 0$ is h-convex, if for $\Psi : D \subset \mathbb{R} \to \mathbb{R}$, the inequality*

$$\Psi(\omega\rho + (1-\omega)\varrho) \leq h(\omega)\Psi(\rho) + h(1-\omega)\Psi(\varrho), \quad \forall \ \rho, \varrho \in D \quad (1.2)$$

is true. The function $\Psi \in SV(h, D)$ is h-concave, if the inequality (1.2) is reversed.

For more detail about the classes $P(D)$ and K_s^2, we refer the reader to visit the article [28].

Definition 1.4. *([29]) The left and right sided RLFI $I^{\alpha}_{\rho^+}\Psi$ and $I^{\alpha}_{\varrho^-}\Psi$ of order $\alpha > 0$ on an interval $[\rho, \varrho]$ are defined by*

$$I^{\alpha}_{\rho^+}\Psi(\omega) = \frac{1}{\Gamma(\alpha)}\int_{\rho}^{\omega}(\omega-t)^{\alpha-1}\Psi(t)dt, \quad \omega > \rho$$

and

$$I_{\varrho^-}^{\alpha}\Psi(\omega) = \frac{1}{\Gamma(\alpha)}\int_{\omega}^{\varrho}(t-\omega)^{\alpha-1}\Psi(t)dt, \quad \omega < \varrho$$

respectively. Here $\Gamma(.)$ *represents the Euler-Gamma function defined by*

$$\Gamma(t) = \int_{0}^{\infty}\omega^{t-1}e^{-\omega}d\omega \ , \mathbb{R}(t) > 0.$$

The following results presented by Sarikaya et al. in [23] must be recalled in order to achieve our objectives.

Theorem 1.5. *If* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *is a positive convex function on* $[\rho, \varrho]$ *with* $0 \leq \rho < \varrho$ *and* $\Psi \in L_1[\rho, \varrho]$*, then the inequality*

$$\Psi\left(\frac{\rho+\varrho}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\varrho}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho) + I_{\varrho^-}^{\alpha}\Psi(\rho)\right] \leq \frac{\Psi(\rho)+\Psi(\varrho)}{2}$$

holds.

Lemma 1.6. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a differentiable mapping on* (ρ, ϱ) *with* $\rho < \varrho$*. If* $\Psi' \in L_1[\rho, \varrho]$*, then the identity*

$$\frac{\Psi(\rho)+\Psi(\varrho)}{2} - \frac{\Gamma(\alpha+1)}{2(\rho-a)^{\varrho}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho) + I_{\varrho^-}^{\alpha}\Psi(\rho)\right]$$
$$= \frac{\varrho-\rho}{2}\int_{0}^{1}\left[(1-\omega)^{\alpha} - \omega^{\alpha}\right]\Psi'(\omega\rho + (1-\omega)\varrho)d\omega$$

holds.

The identity proved by Wang et al. in [30] stated as follows:

Lemma 1.7. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a twice differentiable mapping on* (ρ, ϱ) *with* $\rho < \varrho$*. If* $\Psi'' \in L_1[\rho, \varrho]$*, then the equality*

$$\frac{\Psi(\rho)+\Psi(\varrho)}{2} - \frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho) + I_{\varrho^-}^{\alpha}\Psi(\rho)\right]$$
$$= \frac{(\varrho-\rho)^2}{2}\int_{0}^{1}\frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}\Psi''(\omega\rho + (1-\omega)\varrho)d\omega$$

is true.

Lemma 1.8. *([31]) For* $\omega \in [0, 1]$*, we have*

$$(1-\omega)^{\alpha} \leq 2^{1-\alpha} - \omega^{\alpha} \qquad \alpha \in [0, 1]$$

and

$$(1-\omega)^{\alpha} \geq 2^{1-\alpha} - \omega^{\alpha} \qquad \alpha \in [1, \infty].$$

Lemma 1.9. *([19]) Assume that* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *is twice differentiable on* (ρ, ϱ) *with* $\rho < \varrho$. *If* $\Psi'' \in L_1[\rho, \varrho]$, *then the identity*

$$\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^+}\Psi(\varrho) + I^{\alpha}_{\varrho^-}\Psi(\rho)\Big] - \Psi\Big(\frac{\rho+\varrho}{2}\Big) = \frac{(\varrho-\rho)^2}{2}\int_0^1 m(\omega)\Psi''(\omega\rho + (1-\omega)\varrho)d\omega$$

is true, where

$$m(\omega) = \begin{cases} \omega - \frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}, & \omega \in [0, \frac{1}{2}); \\ (1-\omega) - \frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}, & \omega \in [\frac{1}{2}, 1). \end{cases}$$

The k-RLFI introduced by Mubeen et al. in [32] is given in the following definition.

Definition 1.10. *The k-RLFI* $I^{\alpha}_{\rho^+,k}\Psi$ *and* $I^{\alpha}_{\varrho^-,k}\Psi$ *of order* $\alpha > 0$, $k > 0$ *and* $\rho \geq 0$ *for* $\Psi \in L_1[\rho, \varrho]$ *are defined by*

$$I^{\alpha}_{\rho^+,k}\Psi(\omega) = \frac{1}{k\Gamma_k(\alpha)}\int_{\rho}^{\omega}(\omega-\mu)^{\frac{\alpha}{k}-1}\Psi(\mu)d\mu, \qquad \omega > \rho$$

and

$$I^{\alpha}_{\varrho^-,k}\Psi(\omega) = \frac{1}{k\Gamma_k(\alpha)}\int_{\omega}^{\varrho}(\mu-\omega)^{\frac{\alpha}{k}-1}\Psi(\mu)d\mu, \qquad \omega < \varrho$$

respectively. Here $\Gamma_k(.)$ *is the k-Gamma function and* $I^{0}_{\rho^+,k}\Psi = I^{0}_{\varrho^-,k}\Psi = \Psi$.

The following results are given in [25].

Theorem 1.11. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a positive convex mapping for* $0 \leq \rho < \varrho$ *and* $\Psi \in L_1[\rho, \varrho]$, *then the below inequalities*

$$\Psi\Big(\frac{\rho+\varrho}{2}\Big) \leq \frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho) + I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big] \leq \frac{\Psi(\rho)+\Psi(\varrho)}{2}$$

holds.

Lemma 1.12. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a differentiable mapping on* (ρ, ϱ) *with* $\rho < \varrho$. *If* $\Psi' \in L_1[\rho, \varrho]$, *then the identity for generalized RLFI*

$$\frac{\Psi(\rho)+\Psi(\varrho)}{2} - \frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho) + I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big] = \frac{\varrho-\rho}{2}\int_0^1[(1-\omega)^{\frac{\alpha}{k}} - \omega^{\frac{\alpha}{k}}]\Psi'(\omega\rho + (1-\omega)\varrho)d\omega$$

is true.

Recently, Iqbal et al. prove the following lemma's in [26].

Lemma 1.13. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a twice differentiable mapping on* (ρ, ϱ) *with* $\rho < \varrho$. *If* $\Psi'' \in L_1[\rho, \varrho]$, *then the upcoming identity for generalized RLFI*

$$\begin{aligned}&\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]\\&=\frac{(\varrho-\rho)^2}{2}\int_0^1\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{(\frac{\alpha}{k}+1)}\Psi''(\omega\rho+(1-\omega)\varrho)d\omega\end{aligned}$$

holds.

Lemma 1.14. *Assume that* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *is twice differentiable mapping on* (ρ, ϱ) *with* $\rho < \varrho$. *Let* $k > 0$ *and* $\Psi'' \in L_1[\rho, \varrho]$, *then*

$$\begin{aligned}&\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]-\Psi\Big(\frac{\rho+\varrho}{2}\Big)\\&=\frac{(\varrho-\rho)^2}{2}\int_0^1 m(\omega)\Psi''(\omega\rho+(1-\omega)\varrho)d\omega,\end{aligned}$$

where

$$m(\omega)=\begin{cases}\omega-\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}, & \omega\in[0,\frac{1}{2});\\(1-\omega)-\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}, & \omega\in[\frac{1}{2},1).\end{cases}$$

Motivated by a certain class of h-convex functions presented by Sanja Varošanec in [9] and the fractional Hermite-Hadamard inequalities [23–27], we will study such inequalities for h-convex functions via Riemann-type integrals. Our main results are stated in sections below.

2. Main results

This section focuses on trapezoid-type inequalities for twice-differentiable h-convex functions. In approximating the Riemann integral by a trapezoidal formula, such a type of inequality offers a priori error bounds. In general, they also show that the mid-point rule provides the best approximation of all Riemann sums sampled at the inside points of a given partition in the class.

Theorem 2.1. *Let* $I^{\alpha}_{\rho^+}\Psi$ *and* $I^{\alpha}_{\varrho^-}\Psi$ *be the left and right sided RLFI of order* $\alpha > 0$. *Let* $\Psi : [\rho, \varrho] \to \mathbb{R}$ *be a positive mapping with* $\Psi \in L_1[\rho, \varrho]$ *and* $0 \le \rho < \varrho$. *If* Ψ *is h-convex on* $[\rho, \varrho]$, *then we have the inequality*

$$\begin{aligned}\Psi\Big(\frac{\rho+\varrho}{2}\Big)&\le\frac{h(\frac{1}{2})\Gamma(\alpha+1)}{(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^+}\Psi(\varrho)+I^{\alpha}_{\varrho^-}\Psi(\rho)\Big]\\&\le\alpha h\Big(\frac{1}{2}\Big)\Big[\Psi(\rho)+\Psi(\varrho)\Big]\int_0^1\omega^{\alpha-1}\Big[h(\omega)+h(1-\omega)\Big]d\omega.\end{aligned}\tag{2.1}$$

Proof. Since Ψ is an h-convex function, so we can write

$$\Psi(\omega x + (1-\omega)y) \leq h(\omega)\Psi(x) + h(1-\omega)\Psi(y), \tag{2.2}$$

$$\Psi((1-\omega)x + \omega y) \leq h(1-\omega)\Psi(x) + h(\omega)\Psi(y). \tag{2.3}$$

For $\omega = \frac{1}{2}$, we get

$$\Psi(\frac{x+y}{2}) \leq h(\frac{1}{2})\Psi(x) + h(\frac{1}{2})\Psi(y) = h(\frac{1}{2})[\Psi(x) + \Psi(y)].$$

If we choose $x = \omega\rho + (1-\omega)\varrho$ and $y = \rho(1-\omega) + \varrho\omega$, then we have

$$\Psi(\frac{\rho+\varrho}{2}) \leq h(\frac{1}{2})[\Psi(\omega\rho + (1-\omega)\varrho) + \Psi(\rho(1-\omega) + \omega\varrho)].$$

Multiplying above inequality by $\omega^{\alpha-1}$ and integrating with respect to ω over $[0,1]$, we get

$$\Psi\Big(\frac{\rho+\varrho}{2}\Big)\int_0^1 \omega^{\alpha-1}d\omega$$
$$\leq h(\frac{1}{2})\left[\int_0^1 \omega^{\alpha-1}\Psi(\omega\rho + (1-\omega)\varrho)d\omega + \int_0^1 \omega^{\alpha-1}\Psi((1-\omega)\rho + \omega\varrho)d\omega\right],$$

which can also be written as

$$\frac{1}{\alpha}\Psi(\frac{\rho+\varrho}{2}) \leq h(\frac{1}{2})\left[\int_0^1 \omega^{\alpha-1}\Psi(\omega\rho + (1-\omega)\varrho)d\omega + \int_0^1 \omega^{\alpha-1}\Psi((1-\omega)\rho + \omega\varrho)d\omega\right]. \tag{2.4}$$

Now, replacing $\omega\rho + (1-\omega)\varrho = u$ and $(1-\omega)\rho + \omega\varrho = v$, the inequality (2.4) becomes

$$\frac{1}{\alpha}\Psi(\frac{\rho+\varrho}{2}) \leq \frac{h(\frac{1}{2})\Gamma(\alpha)}{(\varrho-\rho)^{\alpha}}\Big[\frac{1}{\Gamma(\alpha)}\int_\rho^\varrho (\varrho-u)^{\alpha-1}\Psi(u)du + \frac{1}{\Gamma(\alpha)}\int_\rho^\varrho (v-\rho)^{\alpha-1}\Psi(v)dv\Big].$$

By managing the terms, we get

$$\Psi\Big(\frac{\rho+\varrho}{2}\Big) \leq \frac{h(\frac{1}{2})\Gamma(\alpha+1)}{(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^+}\Psi(\varrho) + I^{\alpha}_{\varrho^-}\Psi(\rho)\Big]. \tag{2.5}$$

By replacing $x = \rho$, $y = \varrho$ in (2.2) and (2.3) respectively, then adding, we have

$$\Psi(\omega\rho + (1-\omega)\varrho) + \Psi((1-\omega)\rho + \omega\varrho) \leq [h(\omega) + h(1-\omega)][\Psi(\rho) + \Psi(\varrho)]. \tag{2.6}$$

Multiplying (2.6) by $\omega^{\alpha-1}$, then integrating over $[0,1]$, we get

$$\frac{h(\frac{1}{2})\Gamma(\alpha+1)}{(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^+}\Psi(\varrho) + I^{\alpha}_{\varrho^-}\Psi(\rho)\Big] \leq \alpha h(\frac{1}{2})[\Psi(\rho) + \Psi(\varrho)]\int_0^1 \omega^{\alpha-1}[h(\omega) + h(1-\omega)]d\omega. \tag{2.7}$$

Combining (2.5) and (2.7), we get the desired result.

Remark 2.2. *If we choose $h(\omega) = \omega$ in Theorem 2.1, then we get the result of Sarikaya et al. [23, Theorem 2].*

Example 2.3. *By plotting graphs of double inequality (2.1) corresponding to the choice $\Psi(\wp) = e^{\wp}$ and $h(\omega) = \omega$, we prove that both inequalities are correct. Obviously*

$$I^{\alpha}_{\rho^+} e^{\varrho} = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{\varrho} (\varrho - \wp)^{\alpha-1} e^{\wp} d\wp \tag{2.8}$$

and

$$I^{\alpha}_{\varrho^-} e^{\rho} = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{\varrho} (\wp - \rho)^{\alpha-1} e^{\wp} d\wp. \tag{2.9}$$

By utilizing these expressions into (2.1), we get

$$2e^{\frac{\rho+\varrho}{2}} \leq \frac{\alpha}{(\varrho - \rho)^{\alpha}} \int_{0}^{1} [(\varrho - \wp)^{\alpha-1} + (\wp - \rho)^{\alpha-1}] e^{\wp} d\wp \leq e^{\rho} + e^{\varrho}. \tag{2.10}$$

The three functions given by the double inequality on the left, middle and right sides (2.10) are plotted in Figure 1 against $\alpha \in (0, 1]$. The functions curves indicate that dual inequality is correct.

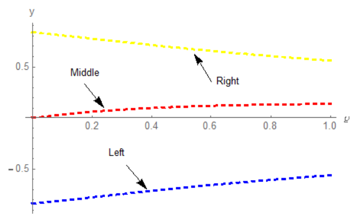

Figure 1. For the case $\rho = 0$, $\varrho = 1$ the graphs illustrate that the double inequality (2.10) is correct.

Theorem 2.4. *Let $I^{\alpha}_{\rho^+}\Psi$ and $I^{\alpha}_{\varrho^-}\Psi$ be the left and right sided RLFI of order $\alpha > 0$. Let $\Psi \in SX(h, D)$, $\rho, \varrho \in D$ with $\rho < \varrho$ and $\Psi \in L_1[\rho, \varrho]$. Let $|\Psi''|^b$ be h-convex function, then one has the following inequality*

$$\left| \frac{\Psi(\rho) + \Psi(\varrho)}{2} - \frac{\Gamma(\alpha+1)}{2(\varrho - \rho)^{\alpha}} \left[I^{\alpha}_{\rho^+}\Psi(\varrho) + I^{\alpha}_{\varrho^-}\Psi(\rho) \right] \right|$$
$$\leq \frac{(\varrho - \rho)^2}{2(\alpha+1)} \left(1 - \frac{2}{(\alpha+1)g + 1} \right)^{\frac{1}{g}} \left(|\Psi''(\rho)|^b \int_{0}^{1} h(\omega) d\omega + |\Psi''(\varrho)|^b \int_{0}^{1} h(1-\omega) d\omega \right)^{\frac{1}{b}}, \tag{2.11}$$

where $\frac{1}{g} + \frac{1}{b} = 1$.

Proof. By using Lemma 1.7, Definition 1.3 and Hölder's inequality respectively, we have

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho)+I_{\varrho^-}^{\alpha}\Psi(\rho)\right]\right|$$

$$\leq \frac{(\varrho-\rho)^2}{2}\int_0^1\left|\frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}\right|\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega$$

$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1})^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|^b d\omega\right)^{\frac{1}{b}}.$$

By utilizing the fact $(C-D)^g \leq C^g - D^g$ for any $C > D \geq 0$ and $g \geq 1$, we can write

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho)+I_{\varrho^-}^{\alpha}\Psi(\rho)\right]\right|$$

$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-\omega)^{(\alpha+1)g}-\omega^{(\alpha+1)g})d\omega\right)^{\frac{1}{g}}\left(\int_0^1\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|^b d\omega\right)^{\frac{1}{b}}$$

$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(1-\frac{1}{g(\alpha+1)+1}-\frac{1}{g(\alpha+1)+1}\right)^{\frac{1}{g}}$$

$$\left(\int_0^1(h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b)d\omega\right)^{\frac{1}{b}}$$

$$=\frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(1-\frac{2}{g(\alpha+1)+1}\right)^{\frac{1}{g}}\left(|\Psi''(\rho)|^b\int_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1 h(1-\omega)d\omega\right)^{\frac{1}{b}}.$$

This completes the proof of inequality (2.11).

Corollary 2.5. *If we choose $h(\omega)=\omega$, then we get the result for convex function i.e.,*

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho)+I_{\varrho^-}^{\alpha}\Psi(\rho)\right]\right|$$
$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(1-\frac{2}{(\alpha+1)g+1}\right)^{\frac{1}{g}}\left(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{2}\right)^{\frac{1}{b}}.$$

Corollary 2.6. *Corresponding to the choice $h(\omega)=\omega^s$, where $s\in(0,1)$, we get the result for geometric-arithmetically s-convex function.*

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^+}^{\alpha}\Psi(\varrho)+I_{\varrho^-}^{\alpha}\Psi(\rho)\right]\right|$$
$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(1-\frac{2}{(\alpha+1)g+1}\right)^{\frac{1}{g}}\left(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{s+1}\right)^{\frac{1}{b}}.$$

Example 2.7. *By plotting graphs of dual inequality in Theorem 2.4 for a convex function $\Psi(\wp) = e^{\wp}$, corresponding to $b = 2$, $g = 2$ and $h(\wp) = \wp$, we prove the validity of the results.*

$$\begin{aligned} &\frac{-(\varrho-\rho)^2}{(\alpha+1)}\left(\frac{2\alpha+1}{2\alpha+3}\right)^{\frac{1}{2}}\left(\frac{e^{2\rho}+e^{2\varrho}}{2}\right)^{\frac{1}{2}} \\ &\leq e^{\rho}+e^{\varrho}-\frac{\alpha}{(\varrho-\rho)^{\alpha}}\int\limits_{\rho}^{\varrho}[(\varrho-\wp)^{\alpha-1}+(\wp-\rho)^{\alpha-1}]e^{\wp}d\wp \\ &\leq \frac{(\varrho-\rho)^2}{(\alpha+1)}\left(\frac{2\alpha+1}{2\alpha+3}\right)^{\frac{1}{2}}\left(\frac{e^{2\rho}+e^{2\varrho}}{2}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

The three functions given by the double inequality on the left, middle and right sides (2.12) are plotted in Figure 2 against $\alpha \in (0, 1]$. The graphs of the functions prove the validity of dual inequality.

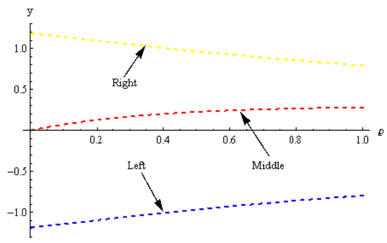

Figure 2. For the case $\rho = 0$, $\varrho = 1$ the graphs illustrate that the double inequality (2.12) is correct.

Theorem 2.8. *Let $I^{\alpha}_{\rho^+}\Psi$ and $I^{\alpha}_{\varrho^-}\Psi$ be the left and right sided RLFI of order $\varrho > 0$. Let $\Psi \in SX(h, D)$, $\rho, \varrho \in D$ with $\rho < \varrho$ and $\Psi \in L_1[\rho, \varrho]$. Let $|\Psi''|^b$ be h-convex, then one has inequality via fractional integrals*

$$\begin{aligned} &\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I^{\alpha}_{\rho^+}\Psi(\varrho)+I^{\alpha}_{\varrho^-}\Psi(\rho)\right]\right| \\ &\leq \frac{(\varrho-\rho)^2\max\{2^{1-\alpha}-1, 1-2^{1-\alpha}\}}{2(\alpha+1)}\left(|\Psi''(\rho)|^b\int\limits_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int\limits_0^1 h(1-\omega)d\omega\right)^{\frac{1}{b}}, \end{aligned}$$

where $\frac{1}{g}+\frac{1}{b}=1$.

Proof. Using Lemma 1.7, Definition 1.3, Lemma 1.8 and Hölder's inequality respectively, we have

$$\begin{aligned} &\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I^{\alpha}_{\rho^+}\Psi(\varrho)+I^{\alpha}_{\varrho^-}\Psi(\rho)\right]\right| \\ &\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\int\limits_0^1\left|1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}\right|\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega \\ &\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int\limits_0^1\left|1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}\right|^g d\omega\right)^{\frac{1}{g}}\left(\int\limits_0^1\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|^b d\omega\right)^{\frac{1}{b}}. \end{aligned} \tag{2.13}$$

Now, 2-cases arise.
Case 1 : For $\varrho \in [0,1]$, the inequality (2.13) becomes

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I^{\alpha}_{\rho^{+}}\Psi(\varrho)+I^{\alpha}_{\varrho^{-}}\Psi(\rho)\right]\right| \\
&\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int_0^1 (2^{1-\alpha}-1)^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1 \left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|^b d\omega\right)^{\frac{1}{b}} \\
&\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}(2^{1-\alpha}-1)\left(\int_0^1 \left[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b\right]d\omega\right)^{\frac{1}{b}} \\
&= \frac{(\varrho-\rho)^2(2^{1-\alpha}-1)}{2(\alpha+1)}\left(|\Psi''(\rho)|^b\int_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1 h(1-\omega)d\omega\right)^{\frac{1}{b}}. \qquad (2.14)
\end{aligned}$$

Case-2: For $\alpha \in [1,\infty)$, the inequality (2.13) can be written as:

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I^{\alpha}_{\rho^{+}}\Psi(\varrho)+I^{\alpha}_{\varrho^{-}}\Psi(\rho)\right]\right| \\
&\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int_0^1 (1-2^{1-\alpha})^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1 \left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|^b d\omega\right)^{\frac{1}{b}} \\
&\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}(1-2^{1-\alpha})\left(\int_0^1 [h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b]d\omega\right)^{\frac{1}{b}} \\
&= \frac{(\varrho-\rho)^2(1-2^{1-\alpha})}{2(\alpha+1)}\left(|\Psi''(\rho)|^b\int_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1 h(1-\omega)d\omega\right)^{\frac{1}{b}}. \qquad (2.15)
\end{aligned}$$

By combining (2.14) and (2.15), we get the required result.

Corollary 2.9. *If we choose $h(\omega)=\omega$ in Theorem 2.8, then we get the result for the classical convex function.*

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I^{\alpha}_{\rho^{+}}\Psi(\varrho)+I^{\alpha}_{\varrho^{-}}\Psi(\rho)\right]\right| \\
&\leq \frac{(\varrho-\rho)^2\max\{2^{1-\alpha}-1,1-2^{1-\alpha}\}}{2(\alpha+1)}\left(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{2}\right)^{\frac{1}{b}}.
\end{aligned}$$

Remark 2.10. *If we choos $h(\omega)=\omega^s$ in Theorem 2.8, where $s\in(0,1)$, then we get the result of Liao et al. [24, Theorem 3.2].*

Theorem 2.11. *Let $I^{\alpha}_{\rho^{+}}\Psi$ and $I^{\alpha}_{\varrho^{-}}\Psi$ be the left and right sided RLFI of order $\alpha>0$. Let $\Psi\in SX(h,D)$, $\rho,\varrho\in D$ with $\rho<\varrho$ and $\Psi\in L_1[\rho,\varrho]$. Let $|\Psi''|$ be h-convex, then one has the following inequality*

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^{+}}^{\alpha}\Psi(\varrho)+I_{\varrho^{-}}^{\alpha}\Psi(\rho)\right]\right|$$
$$\leq\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\Big[\int_{0}^{1}[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|]d\omega$$
$$-\int_{0}^{1}[h(\omega)(1-\omega)^{\alpha+1}|\Psi''(\rho)|+h(1-\omega)(1-\omega)^{\alpha+1}|\Psi''(\varrho)|]d\omega$$
$$-\int_{0}^{1}[h(\omega)\omega^{\alpha+1}|\Psi''(\rho)|+h(1-\omega)\omega^{\alpha+1}|\Psi''(\varrho)|]d\omega\Big]. \tag{2.16}$$

Proof. By using Lemma 1.7 and Definition 1.3 respectively, we have

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\left[I_{\rho^{+}}^{\alpha}\Psi(\varrho)+I_{\varrho^{-}}^{\alpha}\Psi(\rho)\right]\right|$$
$$\leq\frac{(\varrho-\rho)^{2}}{2}\int_{0}^{1}\left|\frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}\right|\left|\Psi''(\omega a+(1-\omega)\varrho)\right|d\omega$$
$$=\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\int_{0}^{1}\left(1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}\right)\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega$$
$$\leq\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\int_{0}^{1}\left(1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}\right)\Big[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|\Big]d\omega$$
$$=\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\Big[\int_{0}^{1}[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|]d\omega$$
$$-\int_{0}^{1}[h(\omega)(1-\omega)^{\alpha+1}|\Psi''(\rho)|+h(1-\omega)(1-\omega)^{\alpha+1}|\Psi''(\varrho)|]d\omega$$
$$-\int_{0}^{1}[h(\omega)\omega^{\alpha+1}|\Psi''(\rho)|+h(1-\omega)\omega^{\alpha+1}|\Psi''(\varrho)|]d\omega\Big].$$

Hence the inequality (2.16) holds.

Remark 2.12. *By choosing $h(\omega)=\omega^{s}$, where $s\in(0,1)$, we get the result of Liao et al. [24, Theorem 3.1].*

Example 2.13. *By plotting the graphs of (2.16) for a convex function $\Psi(\wp)=e^{\wp}$ corresponding to $h(\wp)=\wp$, we prove the validity of the results.*

$$\frac{-\alpha(\varrho-\rho)^2(e^\rho+e^\varrho)}{(\alpha+1)(\alpha+2)}$$
$$\leq e^\rho+e^\varrho-\frac{\alpha}{(\varrho-\rho)^\alpha}\int_\rho^\varrho[(\varrho-\wp)^{\alpha-1}+(\wp-\rho)^{\alpha-1}]e^\wp d\wp$$
$$\leq \frac{\alpha(\varrho-\rho)^2(e^\rho+e^\varrho)}{(\alpha+1)(\alpha+2)}. \tag{2.17}$$

The graphs of the functions given by the left, middle and right sides of the double inequality (2.17) are plotted in Figure 3 against $\alpha \in (0, 1]$. *The functions curves indicate that dual inequality is correct.*

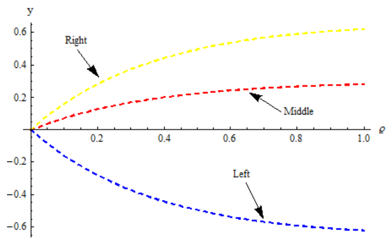

Figure 3. The graphs illustrate the validity of dual inequality (2.17) for the case $\rho = 0$ and $\varrho = 1$.

Theorem 2.14. *Let* $I^\alpha_{\rho^+}\Psi$ *and* $I^\alpha_{\varrho^-}\Psi$ *be the left and right sided RLFI of order* $\alpha > 0$. *Let* $\Psi \in SX(h, D)$, $\rho, \varrho \in D$ *with* $\rho < \varrho$ *and* $\Psi \in L_1[\rho, \varrho]$. *Let* $|\Psi''|^b$ *be h-convex on* $[\rho, \varrho]$ *for fixed* $\alpha \in (0, \infty)$, *then one has the inequality*

$$\left|\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^\alpha}\left[I^\alpha_{\rho^+}\Psi(\varrho)+I^\alpha_{\varrho^-}\Psi(\rho)\right]-\Psi\left(\frac{\rho+\varrho}{2}\right)\right|$$
$$\leq \frac{(\varrho-\rho)^2}{2(\alpha+1)}\left(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\rho)|^b]d\omega\right)^{\frac{1}{b}}$$
$$\times\left(\frac{(\alpha+1)2^{-g-1}+(\varrho+0.5)^{g+1}-\varrho^{g+1}}{g+1}\right)^{\frac{1}{g}}, \tag{2.18}$$

where $\frac{1}{g}+\frac{1}{b}=1$.

Proof. By using Lemma 1.9, Hölder's inequality and Definition 1.3 respectively, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^{+}}\Psi(\varrho)+I^{\alpha}_{\varrho^{-}}\Psi(\rho)\Big]-\Psi\Big(\frac{\rho+\varrho}{2}\Big)\right|\\
&\leq\frac{(\varrho-\rho)^{2}}{2}\int_{0}^{1}|m(\omega)||\Psi''(\omega\rho+(1-\omega)\varrho)|d\omega\\
&\leq\frac{(\varrho-\rho)^{2}}{2}\Big(\int_{0}^{1}|m(\omega)|^{g}d\omega\Big)^{\frac{1}{g}}\Big(\int_{0}^{1}|\Psi''(\omega\rho+(1-\omega)\varrho)|^{b}d\omega\Big)^{\frac{1}{b}}\\
&\leq\frac{(\varrho-\rho)^{2}}{2}\Big(\int_{0}^{1}|m(\omega)|^{g}d\omega\Big)^{\frac{1}{g}}\Big(\int_{0}^{1}[h(\omega)|\Psi''(\rho)|^{b}+h(1-\omega)|\Psi''(\varrho)|^{b}]d\omega\Big)^{\frac{1}{b}}\\
&=\frac{(\varrho-\rho)^{2}}{2}\Big(\int_{0}^{1}[h(\omega)|\Psi''(\rho)|^{b}+h(1-\omega)|\Psi''(\varrho)|^{b}]d\omega\Big)^{\frac{1}{b}}\\
&\times\Big[\Big(\int_{0}^{\frac{1}{2}}\Big|\Big(\omega-\frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}\Big)\Big|^{g}d\omega\Big)\\
&+\Big(\int_{\frac{1}{2}}^{1}\Big|\Big(1-\omega-\frac{1-(1-\omega)^{\alpha+1}-\omega^{\alpha+1}}{\alpha+1}\Big)\Big|^{g}d\omega\Big)\Big]^{\frac{1}{g}}.
\end{aligned}$$

By simple calculation and using the fact $(1-\omega)^{\alpha+1}+\omega^{\alpha+1}\leq 1$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\varrho-\rho)^{\alpha}}\Big[I^{\alpha}_{\rho^{+}}\Psi(\varrho)+I^{\alpha}_{\varrho^{-}}\Psi(\rho)\Big]-\Psi\Big(\frac{\rho+\varrho}{2}\Big)\right|\\
&\leq\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\Big(\int_{0}^{1}[h(\omega)|\Psi''(\rho)|^{b}+h(1-\omega)|\Psi''(\varrho)|^{b}]d\omega\Big)^{\frac{1}{b}}\\
&\Big[(\alpha+1)\int_{0}^{\frac{1}{2}}\omega^{g}d\omega+\int_{\frac{1}{2}}^{1}(\alpha+1-\omega)^{g}d\omega\Big]^{\frac{1}{g}}\\
&=\frac{(\varrho-\rho)^{2}}{2(\alpha+1)}\Big(\int_{0}^{1}[h(\omega)|\Psi''(\rho)|^{b}+h(1-\omega)|\Psi''(\varrho)|^{b}]d\omega\Big)^{\frac{1}{b}}\\
&\Big(\frac{(\alpha+1)2^{-g-1}+(\varrho+0.5)^{g+1}-\varrho^{g+1}}{g+1}\Big)^{\frac{1}{g}}.
\end{aligned}$$

This completes the proof of Theorem 2.14.

Remark 2.15. *If we choose $h(\omega)=\omega^{s}$ in Theorem 2.14, where $s\in(0,1)$, then we get the result of Liao et al. [24, Theorem 4.2].*

3. Mid-point type inequalities for h-convex functions via k-fractional integrals

In this section, we present the estimates of mid-point type inequalities for k-RLFI via h-convex functions.

Theorem 3.1. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha > 0$. Let $\Psi : [\rho, \varrho] \to \mathbb{R}$ be a positive function with $0 \le \rho < \varrho$ and $\Psi \in L_1[\rho, \varrho]$. If Ψ is h-convex function on $[\rho, \varrho]$, then*

$$\begin{aligned}\Psi(\frac{\rho+\varrho}{2}) &\le \frac{h(\frac{1}{2})\Gamma_k(\alpha+k)}{(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho) + I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big] \\ &\le \frac{\alpha h(\frac{1}{2})}{k}[\Psi(\rho)+\Psi(\varrho)]\int_0^1 \omega^{\frac{\alpha}{k}-1}[h(\omega)+h(1-\omega)]d\omega.\end{aligned}$$

Proof. Since Ψ is h-convex, then we can write

$$\Psi(\omega x + (1-\omega)y) \le h(\omega)\Psi(x) + h(1-\omega)\Psi(y), \tag{3.1}$$

$$\Psi((1-\omega)x + \omega y) \le h(1-\omega)\Psi(x) + h(\omega)\Psi(y). \tag{3.2}$$

Corresponding to the choice $\omega = \frac{1}{2}$, we can write

$$\Psi(\frac{x+y}{2}) \le h(\frac{1}{2})\Psi(x) + h(\frac{1}{2})\Psi(y) = h(\frac{1}{2})[\Psi(x)+\Psi(y)].$$

By replacing $x = \omega a + (1-\omega)\varrho$ and $y = a(1-\omega) + \varrho\omega$, we get

$$\Psi(\frac{\rho+\varrho}{2}) \le h(\frac{1}{2})[\Psi(\omega\rho + (1-\omega)\varrho) + \Psi(a(1-\omega)+\omega\varrho)].$$

Multiplying by $\omega^{\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, we get

$$\Psi(\frac{\rho+\varrho}{2})\int_0^1 \omega^{\frac{\alpha}{k}-1}d\omega \le h(\frac{1}{2})\left[\int_0^1 \omega^{\frac{\alpha}{k}-1}\Psi(\omega\rho+(1-\omega)\varrho)d\omega + \int_0^1 \omega^{\frac{\alpha}{k}-1}\Psi((1-\omega)\rho+\omega\varrho)d\omega\right].$$

This can also be written as:

$$\frac{k}{\alpha}\Psi(\frac{\rho+\varrho}{2}) \le h(\frac{1}{2})\left[\int_0^1 \omega^{\frac{\alpha}{k}-1}\Psi(\omega\rho+(1-\omega)\varrho)d\omega + \int_0^1 t^{\frac{\alpha}{k}-1}\Psi((1-\omega)\rho+\omega\varrho)d\omega\right]. \tag{3.3}$$

Substituting $\omega\rho + (1-\omega)\varrho = u$ and $(1-\omega)\rho + \omega\varrho = v$, the inequality (3.3) can be written as:

$$\begin{aligned}
&\frac{1}{\alpha}\Psi(\frac{\rho+\varrho}{2})\\
&\leq \frac{h(\frac{1}{2})\Gamma_k(\alpha)}{(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[\frac{1}{k\Gamma_k(\alpha)}\int\limits_{\rho}^{\varrho}(\varrho-u)^{\frac{\alpha}{k}-1}\Psi(u)du\\
&+\frac{1}{k\Gamma_k(\alpha)}\int\limits_{\rho}^{\varrho}(v-\rho)^{\frac{\alpha}{k}-1}\Psi(v)dv\Big]
\end{aligned}$$

which implies

$$\Psi(\frac{\rho+\varrho}{2})\leq \frac{h(\frac{1}{2})\Gamma_k(\alpha+k)}{(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]. \tag{3.4}$$

Choose $x=\rho$ and $y=\varrho$ in (3.1) and (3.2), then adding, we get

$$\Psi(\omega\rho+(1-\omega)\varrho)+\Psi((1-\omega)\rho+\omega\varrho)\leq [h(\omega)+h(1-\omega)][\Psi(\rho)+\Psi(\varrho)]. \tag{3.5}$$

Multiplying (3.5) by $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{h(\frac{1}{2})\Gamma_k(\alpha+k)}{(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]\\
&\leq \frac{\alpha h(\frac{1}{2})}{k}[\Psi(\rho)+\Psi(\varrho)]\int\limits_0^1\omega^{\frac{\alpha}{k}-1}[h(\omega)+h(1-\omega)]d\omega.
\end{aligned}\tag{3.6}$$

Combining (3.4) and (3.6), we get the required result.

Remark 3.2. *If we choose $h(\omega)=\omega$ in Theorem 3.1, then we get the result of Iqbal et al. [25, Theorem 2.2].*

Theorem 3.3. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha>0$. Let $\Psi\in SX(h,D)$, $\rho,\varrho\in D$ with $\rho<\varrho$ and $\Psi\in L_1[\rho,\varrho]$. Let $|\Psi''|^b$ be h-convex, then the inequality*

$$\begin{aligned}
&\Big|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]\Big|\\
&\leq \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}(1-\frac{2}{(\frac{\alpha}{k}+1)g+1})^{\frac{1}{g}}(|\Psi''(\rho)|^b\int\limits_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int\limits_0^1 h(1-\omega)d\omega)^{\frac{1}{b}}
\end{aligned}$$

holds, where $\frac{1}{g}+\frac{1}{b}=1$.

Proof. By using Lemma 1.13, Definition 1.3 and Hölder's inequality respectively, we can have

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\le \frac{(\varrho-\rho)^2}{2}\int_0^1\left|\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{(\frac{\alpha}{k}+1)}\right|\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega\\
&\le \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(\int_0^1(1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1})^g d\omega\Big)^{\frac{1}{g}}\Big(\int_0^1|\Psi''(\omega\rho+(1-\omega)\varrho)\Big|^b d\omega\Big)^{\frac{1}{b}}\\
&\le \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(1-\frac{1}{g(\frac{\alpha}{k}+1)+1}-\frac{1}{g(\frac{\alpha}{k}+1)+1}\Big)^{\frac{1}{g}}\Big(\int_0^1(h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b)d\omega\Big)^{\frac{1}{b}}\\
&= \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(1-\frac{2}{g(\frac{\alpha}{k}+1)+1}\Big)^{\frac{1}{g}}(|\Psi''(\rho)|^b\int_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1 h(1-\omega)d\omega)^{\frac{1}{b}}.
\end{aligned}$$

Using the fact $(1-(1-\omega)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})^g \le 1-(1-\omega)^{g(\frac{\alpha}{k}+1)}-\omega^{g(\frac{\alpha}{k}+1)}$ for any $\omega\in[0,1]$, which follows from $(C-D)^g\le C^g-D^g$, for any $C>D\ge 0$ and $g\ge 1$.

Corollary 3.4. *If we choose $h(\omega)=\omega$ in Theorem 3.3, then the result for a simple convex function is given by*

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\le \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(1-\frac{2}{(\frac{\alpha}{k}+1)g+1}\Big)^{\frac{1}{g}}\Big(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{2}\Big)^{\frac{1}{b}}.
\end{aligned}$$

Corollary 3.5. *If we choose $h(\omega)=\omega^s$ corresponding to $s\in(0,1)$ in Theorem 3.3, then we get the result for geometrically arithmetically s-convex function i.e.,*

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\le \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(1-\frac{2}{(\frac{\alpha}{k}+1)g+1}\Big)^{\frac{1}{g}}\Big(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{s+1}\Big)^{\frac{1}{b}}.
\end{aligned}$$

Theorem 3.6. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha>0$. Let $\Psi\in SX(h,D)$, $\rho,\varrho\in I$ with $\rho<\varrho$ and $\Psi\in L_1[\rho,\varrho]$. Let $|\Psi''|^b$ be h-convex, then one has the inequality via fractional integrals.*

$$\begin{aligned}
&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\le \frac{(\varrho-\rho)^2\max\{2^{1-\frac{\alpha}{k}}-1,1-2^{1-\frac{\alpha}{k}}\}}{2(\frac{\alpha}{k}+1)}\Big(|\Psi''(\rho)|^b\int_0^1 h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1 h(1-\omega)d\omega\Big)^{\frac{1}{b}},
\end{aligned}$$

where $\frac{1}{g}+\frac{1}{b}=1$.

Proof. By using Lemma 1.13, Hölder's inequality, Lemma 1.8 and Definition 1.3 respectively, we can write

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(a)\right]\right|$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\int_0^1\left|1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}\right|\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\left(\int_0^1\left|1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}\right|^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\omega\rho+(1-\omega)\varrho)\Big|^b d\omega\right)^{\frac{1}{b}}. \tag{3.7}$$

Now, we have the following two cases:
Case 1: For $\alpha\in[0,1]$, the inequality (3.7) can be written as:

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\left(\int_0^1(2^{1-\frac{\alpha}{k}}-1)^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\omega\rho+(1-\omega)\varrho)|^b d\omega\right)^{\frac{1}{b}}$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}(2^{1-\frac{\alpha}{k}}-1)(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b]d\omega)^{\frac{1}{b}}$$
$$=\frac{(\varrho-\rho)^2(2^{1-\frac{\alpha}{k}}-1)}{2(\frac{\alpha}{k}+1)}\left(|\Psi''(\rho)|^b\int_0^1h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1h(1-\omega)d\omega\right)^{\frac{1}{b}}. \tag{3.8}$$

Case 2: For the choice of $\alpha\in[1,\infty)$, the inequality (3.7) can be written as:

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\left(\int_0^1(1-2^{1-\frac{\alpha}{k}})^g d\omega\right)^{\frac{1}{g}}\left(\int_0^1|\Psi''(\omega\rho+(1-\omega)\varrho)|^b d\omega\right)^{\frac{1}{b}}$$
$$\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}(1-2^{1-\frac{\alpha}{k}})\left(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b]d\omega\right)^{\frac{1}{b}}$$
$$=\frac{(\varrho-\rho)^2(1-2^{1-\frac{\alpha}{k}})}{2(\frac{\alpha}{k}+1)}\left(|\Psi''(\rho)|^b\int_0^1h(\omega)d\omega+|\Psi''(\varrho)|^b\int_0^1h(1-\omega)d\omega\right)^{\frac{1}{b}}. \tag{3.9}$$

Combining inequalities (3.8) and (3.9), we get the inequality (3.7).

Remark 3.7. *If we choose $h(\omega)=\omega^s$ corresponding to $s\in(0,1)$ in Theorem 3.6, then we get the result of Iqbal et al. [26, Theorem 2.4].*

Theorem 3.8. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha > 0$. Let $\Psi \in SX(h, D)$, $\rho, \varrho \in D$ with $\rho < \varrho$ and $\Psi \in L_1[\rho, \varrho]$. Let $|\Psi''|$ be h-convex, then the inequality*

$$\begin{aligned}&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\leq \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big[\int_0^1[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|]d\omega\\
&-\int_0^1\Big[h(\omega)(1-\omega)^{\frac{\alpha}{k}+1}|\Psi''(\rho)|+h(1-\omega)(1-\omega)^{\frac{\alpha}{k}+1}|\Psi''(\varrho)|\Big]d\omega\\
&-\int_0^1\Big[h(\omega)\omega^{\frac{\alpha}{k}+1}|\Psi''(\rho)|+h(1-\omega)\omega^{\frac{\alpha}{k}+1}|\Psi''(\varrho)|\Big]d\omega\Big]\end{aligned}$$

holds.

Proof. Using Lemma 1.13 and Definition 1.3 respectively, we get

$$\begin{aligned}&\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\right]\right|\\
&\leq \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\int_0^1\left(1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}\right)\left|\Psi''(\omega\rho+(1-\omega)\varrho)\right|d\omega\\
&\leq \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\int_0^1\left(1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}\right)\Big[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|\Big]d\omega\\
&= \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big[\int_0^1[h(\omega)|\Psi''(\rho)|+h(1-\omega)|\Psi''(\varrho)|]d\omega\\
&-\int_0^1[h(\omega)(1-\omega)^{\frac{\alpha}{k}+1}|\Psi''(\rho)|+h(1-\omega)(1-\omega)^{\frac{\alpha}{k}+1}|\Psi''(\varrho)|]d\omega\\
&-\int_0^1[h(\omega)\omega^{\frac{\alpha}{k}+1}|\Psi''(\rho)|+h(1-\omega)\omega^{\frac{\alpha}{k}+1}|\Psi''(\varrho)|]d\omega\Big].\end{aligned}$$

This completes the proof of Theorem 3.8.

Remark 3.9. *By choosing $h(\omega) = \omega^s$, where $s \in (0, 1)$, we get the result of Iqbal et al. [26, Theorem 2.3].*

Theorem 3.10. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha > 0$. Let $\Psi \in SX(h, D)$, $\rho, \varrho \in D$ with $\rho < \varrho$ and $\Psi \in L_1[\rho, \varrho]$. If $|\Psi''|^b$ is h-convex on $[\rho, \varrho]$ for $\alpha \in (0, \infty)$ and*

$\varrho \in (0, \infty)$, *we get the inequality*

$$\begin{aligned}&\left|\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]-\Psi\Big(\frac{\rho+\varrho}{2}\Big)\right|\\&\leq\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\rho)|^b]d\omega\Big)^{\frac{1}{b}}\\&\times\Big(\frac{(\frac{\alpha}{k}+1)2^{-g-1}+(\frac{\alpha}{k}+0.5)^{g+1}-(\frac{\alpha}{k})^{g+1}}{g+1}\Big)^{\frac{1}{g}},\end{aligned}$$

where $\frac{1}{g}+\frac{1}{b}=1$.

Proof. Using Lemma 1.14, Hölder's inequality and Definition 1.3 respectively, we have

$$\begin{aligned}&\left|\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]-\Psi\Big(\frac{a+\varrho}{2}\Big)\right|\\&\leq\frac{(\varrho-\rho)^2}{2}\Big(\int_0^1|m(\omega)|^gd\omega\Big)^{\frac{1}{g}}\Big(\int_0^1|\Psi''(\omega\rho+(1-\omega)\varrho)|^bd\omega\Big)^{\frac{1}{b}}\\&\leq\frac{(\varrho-\rho)^2}{2}\Big(\int_0^1|m(\omega)|^gd\omega\Big)^{\frac{1}{g}}\Big(\int_0^1\Big[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b\Big]d\omega\Big)^{\frac{1}{b}}\\&=\frac{(\varrho-\rho)^2}{2}\Big(\int_0^1\Big[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b\Big]d\omega\Big)^{\frac{1}{b}}\\&\times\Big[\Big(\int_0^{\frac{1}{2}}\Big|\Big(\omega-\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\Big)\Big|^gd\omega\Big)\\&+\Big(\int_{\frac{1}{2}}^1\Big|\Big(1-\omega-\frac{1-(1-\omega)^{\frac{\alpha}{k}+1}-\omega^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\Big)\Big|^gd\omega\Big)\Big]^{\frac{1}{g}}\\&=\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(\int_0^1\Big[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b\Big]d\omega\Big)^{\frac{1}{b}}\\&\times\Big[\Big(\int_0^{\frac{1}{2}}\Big|\Big(\Big(\frac{\alpha}{k}+1\Big)\omega-1+\Big(1-\omega\Big)^{\frac{\alpha}{k}+1}+\omega^{\frac{\alpha}{k}+1}\Big)\Big|^gd\omega\Big)\\&+\Big(\int_{\frac{1}{2}}^1\Big|\Big(\frac{\alpha}{k}+1-(\frac{\alpha}{k}+1)\omega-1+(1-\omega)^{\frac{\alpha}{k}+1}+\omega^{\frac{\alpha}{k}+1}\Big)\Big|^gd\omega\Big)\Big]^{\frac{1}{g}}\end{aligned}$$

Since $(1-\omega)^{\frac{\alpha}{k}+1}+\omega^{\frac{\alpha}{k}+1}\leq 1$, therefore we can write

$$\left|\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]-\Psi\Big(\frac{\rho+\varrho}{2}\Big)\right|$$
$$\leq \frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b]d\omega\Big)^{\frac{1}{b}}$$
$$\times\Big[(\frac{\alpha}{k}+1)\int_0^{\frac{1}{2}}\omega^g d\omega+\int_{\frac{1}{2}}^1(\frac{\alpha}{k}+1-\omega)^g d\omega\Big]^{\frac{1}{g}}$$
$$=\frac{(\varrho-\rho)^2}{2(\frac{\alpha}{k}+1)}\Big(\int_0^1[h(\omega)|\Psi''(\rho)|^b+h(1-\omega)|\Psi''(\varrho)|^b]d\omega\Big)^{\frac{1}{b}}$$
$$\Big[\frac{(\frac{\alpha}{k}+1)2^{-g-1}+(\varrho+0.5)^{g+1}-\varrho^{g+1}}{g+1}\Big]^{\frac{1}{g}},$$

The proof of Theorem 3.10 is done.

Remark 3.11. *If we choose $h(\omega)=\omega^s$, $s\in(0,1]$ in Theorem 3.10, then we get the result of Iqbal et al. [26, Theorem 2.6].*

Theorem 3.12. *Let $I^{\alpha}_{\rho^+,k}\Psi$ and $I^{\alpha}_{\varrho^-,k}\Psi$ be the left and right sided k-RLFI of order $\alpha>0$. Let $h: J\subset\mathbb{R}\to\mathbb{R}$ and $\Psi:[\rho,\varrho]\to\mathbb{R}$ be positive functions with $0\leq\rho<\varrho$ and $h^b(\omega)\in L_1[0,1]$, $\Psi\in L_1[\rho,\varrho]$. If $|\Psi'|$ is h-convex mapping on $[\rho,\varrho]$, then the inequality*

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]\right|$$
$$\leq\frac{(\varrho-\rho)[|\Psi'(\rho)|+|\Psi'(\varrho)|]}{2}\Big[\Big(\frac{2^{\frac{g\alpha}{k}+1}-1}{2^{\frac{p\alpha}{k}+1}(\frac{g\alpha}{k}+1)}\Big)^{\frac{1}{g}}-\Big(\frac{1}{2^{\frac{g\alpha}{k}+1}(\frac{g\alpha}{k}+1)}\Big)^{\frac{1}{g}}\Big]$$
$$\times\Big[\Big(\int_0^{\frac{1}{2}}\Big(h(\omega)\Big)^b d\omega\Big)^{\frac{1}{b}}+\Big(\int_{\frac{1}{2}}^1\Big(h(\omega)\Big)^b d\omega\Big)^{\frac{1}{b}}\Big]$$

holds, where $\alpha>0$, $g>0$ and $\frac{1}{g}+\frac{1}{b}=1$.

Proof. Since $|\Psi'|$ is h-convex, then by Lemma 1.12, we can write

$$\left|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}\Big[I^{\alpha}_{\rho^+,k}\Psi(\varrho)+I^{\alpha}_{\varrho^-,k}\Psi(\rho)\Big]\right|$$
$$\leq\frac{\varrho-\rho}{2}\int_0^1\Big|(1-\omega)^{\frac{\alpha}{k}}-\omega^{\frac{\alpha}{k}}\Big|\Big|\Psi'(\omega\rho+(1-\omega)\varrho)\Big|d\omega$$
$$\leq\frac{\varrho-\rho}{2}\Big[\int_0^{\frac{1}{2}}[(1-\omega)^{\frac{\alpha}{k}}-\omega^{\frac{\alpha}{k}}][h(\omega)|\Psi'(\rho)|+h(1-\omega)|\Psi'(\varrho)|]d\omega$$

$$+\int_{\frac{1}{2}}^{1}[\omega^{\frac{\alpha}{k}}-(1-\omega)^{\frac{\alpha}{k}}][h(\omega)|\Psi'(\rho)|+h(1-\omega)|\Psi'(\varrho)|]d\omega\Big]$$

$$=\frac{\varrho-\rho}{2}\Big[|\Psi'(\rho)|\int_{0}^{\frac{1}{2}}(1-\omega)^{\frac{\alpha}{k}}h(\omega)d\omega-|\Psi'(\rho)|\int_{0}^{\frac{1}{2}}\omega^{\frac{\alpha}{k}}h(\omega)d\omega$$

$$+|\Psi'(\varrho)|\int_{0}^{\frac{1}{2}}(1-\omega)^{\frac{\alpha}{k}}h(1-\omega)d\omega-|\Psi'(\varrho)|\int_{0}^{\frac{1}{2}}\omega^{\frac{\alpha}{k}}h(1-\omega)d\omega$$

$$+|\Psi'(\rho)|\int_{\frac{1}{2}}^{1}\omega^{\frac{\alpha}{k}}h(\omega)d\omega-|\Psi'(\rho)|\int_{\frac{1}{2}}^{1}(1-\omega)^{\frac{\alpha}{k}}h(\omega)d\omega$$

$$+|\Psi'(\varrho)|\int_{\frac{1}{2}}^{1}\omega^{\frac{\alpha}{k}}h(1-\omega)d\omega-|\Psi'(\varrho)|\int_{\frac{1}{2}}^{1}(1-\omega)^{\frac{\alpha}{k}}h(1-\omega)d\omega\Big]. \tag{3.10}$$

Using Hölder's inequality on the right side of (3.10), we get

$$\Big|\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\frac{\Gamma_k(\alpha+k)}{2(\varrho-\rho)^{\frac{\alpha}{k}}}[I^{\alpha}_{\rho^+}\Psi(\varrho)+I^{\varrho}_{b^-}\Psi(\rho)]\Big|$$
$$=\frac{(\varrho-\rho)[|\Psi'(\rho)|+|\Psi'(\varrho)|]}{2}\Big[\Big(\frac{2^{\frac{g\alpha}{k}+1}-1}{2^{\frac{g\alpha}{k}+1}(\frac{g\alpha}{k}+1)}\Big)^{\frac{1}{g}}-\Big(\frac{1}{2^{\frac{g\alpha}{k}+1}(\frac{g\alpha}{k}+1)}\Big)^{\frac{1}{g}}\Big]$$
$$\times\Big[\Big(\int_{0}^{\frac{1}{2}}(h(\omega))^{b}d\omega\Big)^{\frac{1}{b}}+\Big(\int_{\frac{1}{2}}^{1}(h(\omega))^{b}d\omega\Big)^{\frac{1}{b}}\Big].$$

This completes the proof of the result.

Remark 3.13. *If we choose $k=1$ in Theorem 3.12, we get the result of Tunc. [27, Theorem 5].*

4. Applications to quadrature formulas

This section is devoted to some particular inequalities which generalize some classical results such as trapezoid-type, mid-point-type and Hadamard's inequality.

Proposition 4.1. *(Hadamard's inequality). Under the assumptions of Theorem 2.1 with $\alpha=1$ and $h(\omega)=\omega$, we get the "Hadamard's inequality"*

$$\Psi(\frac{\rho+\varrho}{2})\leq\frac{1}{\varrho-\rho}\int_{\rho}^{\varrho}\Psi(\omega)d\omega\leq\frac{\Psi(\rho)+\Psi(\varrho)}{2}.$$

Proposition 4.2. *Let the assumptions of Theorem 2.4 be fulfilled with $\alpha = 1$ and $h(\omega) = \omega$, then we obtain the "Trapezoid inequality"*

$$\left|(\varrho-\rho)\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\int_{\rho}^{\varrho}\Psi(\omega)d\omega\right| \leq \frac{(\varrho-\rho)^3}{4}\Big(\frac{2g-1}{2g+1}\Big)^{\frac{1}{g}}\Big(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{2}\Big)^{\frac{1}{b}}.$$

Proposition 4.3. *Let the assumptions of Theorem 2.11 be fulfilled with $\alpha = 1$ and $h(\omega) = \omega$, then we obtain the "Trapezoid inequality"*

$$\left|(\varrho-\rho)\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\int_{\rho}^{\varrho}\Psi(\omega)d\omega\right| \leq \frac{(\varrho-\rho)^3}{24}\Big[|\Psi''(\rho)|+|\Psi''(\varrho)|\Big].$$

Proposition 4.4. *Under the assumptions of Theorem 2.14 with $\alpha = 1$ and $h(\omega) = \omega$, we get the following "Mid-point inequality"*

$$\left|(\varrho-\rho)\Psi(\frac{\rho+\varrho}{2})-\int_{\rho}^{\varrho}\Psi(\omega)d\omega\right| \leq \frac{(\varrho-\rho)^3}{4}\Big(\frac{1+3^{g+1}-2^g}{2^g(g+1)}\Big)^{\frac{1}{g}}\Big(\frac{|\Psi''(\rho)|^b+|\Psi''(\varrho)|^b}{2}\Big)^{\frac{1}{b}}.$$

Proposition 4.5. *Let the assumptions of Theorem 3.12 be fulfilled with $\alpha = 1$ and $h(\omega) = \omega$, then we get the following "Trapezoid inequality"*

$$\left|(\varrho-\rho)\frac{\Psi(\rho)+\Psi(\varrho)}{2}-\int_{\rho}^{\varrho}\Psi(\omega)d\omega\right| \leq \frac{(\varrho-\rho)^2[|\Psi'(\rho)|+|\Psi'(\varrho)|]}{2}$$
$$\times\Big[\Big(\frac{2^{g+1}-1}{2^{g+1}(g+1)}\Big)^{\frac{1}{g}}-\Big(\frac{1}{2^{g+1}(g+1)}\Big)^{\frac{1}{g}}\Big]\Big[\Big(\frac{2^{b+1}-1}{2^{b+1}(b+1)}\Big)^{\frac{1}{b}}+\Big(\frac{1}{2^{b+1}(b+1)}\Big)^{\frac{1}{b}}\Big].$$

5. Conclusions

We aim at designing the generalizations of some Trapezoid-type inequalities for Riemann-type fractional integrals in the present article. We use h-convex function to this ends and develops numerous inequalities. We prove the validity of our results by using the examples and their graphs. The work presented consists of the formulas of quadrature as a boundary of the new inequalities. The results of these studies complement those of earlier studies. The earlier findings are supported by simple, recent studies and play an additional role in generalizations.

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Conflict of interest

The authors declares that there is no conflict of interests regarding the publication of this paper.

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