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Research article

Modulational instability, multiple Exp-function method, SIVP, solitary and cross-kink solutions for the generalized KP equation

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Abstract: The multiple Exp-function method is employed for seeking the multiple soliton solutions to the generalized (3+1)-dimensional Kadomtsev-Petviashvili (gKP) equation, where contains one-wave, two-wave, and triple-wave solutions. The periodic wave including (exponential, cosh hyperbolic, and cos periodic), cross-kink containing (exponential, sinh hyperbolic, and sin periodic), and solitary containing (exponential, tanh hyperbolic, and tan periodic) wave solutions are obtained. In continuing, the modulation instability is engaged to discuss the stability of obtained solutions. Also, the semi-inverse variational principle is applied for the gKP equation with four major cases. The physical phenomena of these received multiple soliton solutions are analyzed and demonstrated in figures by choosing the specific parameters. By means of symbolic computation these analytical solutions and corresponding rogue waves are obtained with the help of Maple software. Via various three-dimensional, curve, and density charts, dynamical characteristics of these waves are exhibited.

Keywords: multiple Exp-function method; generalized Kadomtsev-Petviashvili equation; modulation instability; semi-inverse variational principle
Mathematics Subject Classification: 35A20, 35A24, 35A25, 35B10, 70K50

1. Introduction

Nonlinear evolution equations (NLEEs) play an important part in the study of nonlinear science, particular in plasma physics, quantum field theory, nonlinear wave propagation and nonlinear optical fibers so that it attracted the attention of a large number of scholars. The extended auxiliary equation technique [1], the Bernoulli's equation approach [2], the Exp-function technique [3], the homotopy analysis technique [4], the homotopy perturbation technique [5], the improved $tan(\phi/2)$-expansion technique ([6, 7]), the Hirota's bilinear technique [8–15], the He's variational principle [16, 17], the binary Darboux transformation [18], the Lie group analysis [19, 20], the Bäcklund transformation method [21], optimal galerkin-homotopy asymptotic method applied [22], and the multiple rogue waves method ([23, 24]) have been proposed to solve NLEEs. By using these approaches, various exact solutions including soliton solution, lump solution, rogue wave solution, periodic solution, interaction solution, rational solution and high-order rational solution were obtained ([25, 26]).

In this paper, we mainly consider the following dynamical model, which can be used to describe some interesting (3+1)-dimensional waves of physics, namely, the generalized Kadomtsev-Petviashvili (gKP) equation [27]. That is

$$(\Psi_t + 6\Psi\Psi_x + \Psi_{xxx})_x + a\Phi_{yy} = 0, \tag{1.1}$$

and also above equation is integrable. Author of [28] introduced the modification of KP (mKP) equation [29] given

$$4\Psi_t - 6\Psi^2\Psi_x + \Psi_{xxx} + 6\Psi_x\partial_x^{-1}\Psi_y + \Psi_x^{-1}\Phi_{yy} = 0. \tag{1.2}$$

The generalized KP (gKP) equation has been researched by some scholars [30–32] in which is given as

$$(\Psi_t + \alpha\Psi_x + \beta\Psi\Psi_x + \gamma\Psi\Psi_{xxt})_x + \Psi_{yy} = 0. \tag{1.3}$$

The another type of gKP equation is given in [33] as below

$$\Psi_{xxxy} + 3(\Psi_x\Psi_y)_x + \Psi_{tx} + \Psi_{ty} + \Psi_{tz} - \Psi_{zz} = 0. \tag{1.4}$$

We first present the bilinear form for Eq (1.4), by taking the following first-order logarithmic transformation

$$\Psi = 2(\ln f)_x, \tag{1.5}$$

then, Eq (1.4) is turned into the bilinear form

$$\left(D_x^3 D_y + D_x D_t + D_y D_t + D_z D_t - D_z^2\right)\mathfrak{f}.\mathfrak{f} = 0, \tag{1.6}$$

in which D_t, D_x, D_y and D_z are Hirota's bilinear frames. Cao [34] investigated the generalized B-type KP equation as follows

$$\Psi_{xxxy} + 3(\Psi_x\Psi_y)_x - 3\Psi_{xz} - \Psi_{ty} = 0. \tag{1.7}$$

Guan et al. [35] derived a (3+1)-dimensional gKP equation in below form

$$\Psi_{xxxy} + 3(\Psi_x\Psi_y)_x + \alpha\Psi_{xxxz} + 3\alpha(\Psi_x\Psi_z)_x + \lambda_1\Psi_{xt} + \lambda_2\Phi_{yt} + \lambda_3\Phi_{zt} + \omega_1\Phi_{xz} + \omega_2\Phi_{yz} + \omega_3\Phi_{zz} = 0, \tag{1.8}$$

and some lump soliton solutions have been constructed using the Hirota bilinear method in [36]. Via transformation $\Psi = 2(\ln f)_x$, the bilinear form of equation (1.8) reads:

$$\left(D_x^3 D_y + \alpha D_x^3 D_z + \lambda_1 D_x D_t + \lambda_2 D_y D_t + \lambda_3 D_z D_t + \omega_1 D_x D_z + \omega_2 D_y D_z + \omega_3 D_z^2\right) \mathfrak{f}.\mathfrak{f} = 0. \tag{1.9}$$

Most classical test functions for solving NLPDEs by using the several particular functions can be constructed via Hirota bilinear technique. In other words, Hirota operator covers most of the classical hypothesis function method. For example, the fractional generalized CBS-BK equation [37], the generalized Bogoyavlensky-Konopelchenko equation [38], the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation [39], and the (2+1)-dimensional generalized variable-coefficient KP-Burgers-type equation [40]. Diverse kinds of studies on solve NLPDEs were perused via mighty authors in which some of them can be stated, for instance, multivariate rogue wave to some PDEs [41], interaction lump solutions the gKP equation [42], the (1+1)-dimensional coupled integrable dispersionless equations [43]. Therefore, we embark on the new research topic of constructing the analytic solutions of nonlinear PDEs by exploring the bilinear method. According to recent studies, we can obtain some of the new exact analytic solutions of nonlinear PDEs by way of constructing their corresponding bilinear differential equations in [44–47]. Here, we will study the multiple Exp-function method (MEFM) for determining the multiple soliton solutions (MSSs). The MEFM employed by some of powerful authors for various nonlinear equations have been surveyed in more studies in [31, 48–50]. Authors of [51] utilized the reduced differential transform method for solving partial differential equations. Also, Yu and Sun [52] studied the dimensionally reduced generalized KP equations by help of Hirita bilinear method and obtained of lump solutions.

The outline of our paper is as follows: the multiple Exp-function scheme has been summarized in section 2. In sections 3, the KP equation, will investigate to finding 1-wave, 2-wave, and three-wave solutions. The Sections 4–6 devote to determined the periodic, cross-kink, and solitary wave solutions. Moreover, in Section 7, the modulation instability analysis is investigated. Finally in Section 8, the SIVP technique is considered with four cases for finding the solitary, bright, dark and singular wave solutions. A few of conclusions and outlook will be given in the final section.

2. Multiple Exp-function method

This method was summarized and improved for achieving the analytic solutions of NLPDEs:

Step 1. Assume a nonlinear PDE is given in general frame as follows

$$\mathcal{N}(x, y, t, \Psi, \Psi_x, \Psi_y, \Psi_z, \Psi_t, \Psi_{xx}, \Psi_{tt}, ...) = 0. \tag{2.1}$$

Take the novel variables $\xi_i = \xi_i(x, y, z, t), 1 \leq i \leq n$, by differentiable frames:

$$\xi_{i,x} = \alpha_i \xi_i, \;\; \xi_{i,y} = \beta_i \xi_i, \;\; \xi_{i,z} = \gamma_i \xi_i, \;\; \xi_{i,t} = -\delta_i \xi_i, \;\; 1 \leq i \leq n, \tag{2.2}$$

where $\alpha_i, \beta_i, \gamma_i, 1 \leq i \leq n$, are unfound amounts. It noted that one can get as the following function

$$\xi_i = \varpi_i e^{\theta_i}, \;\; \theta_i = \alpha_i x + \beta_i y + \gamma_i z - \delta_i t, \;\; 1 \leq i \leq n, \tag{2.3}$$

where $\varpi_i, 1 \leq i \leq n$, unspecified amounts.

Step 2. Assuming the solution of the Eq (2.1) is function of variables $\xi_i, 1 \leq i \leq n$:

$$\Psi = \frac{\Delta(\xi_1, \xi_2, ..., \xi_n)}{\Omega(\xi_1, \xi_2, ..., \xi_n)}, \;\; \Delta = \sum_{r,s=1}^{n} \sum_{i,j=0}^{M} \Delta_{rs,ij} \xi_r^i \xi_s^j, \;\; \Omega = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} \Omega_{rs,ij} \xi_r^i \xi_s^j, \tag{2.4}$$

in which $\Delta_{rs,ij}$ and $\Omega_{rs,ij}$ are amounts to be remained. Replacing Eq (2.4) into Eq (2.1) can be achieved the below form as:

$$\Psi = \frac{\Delta(\varpi_1 e^{\alpha_1 x+\beta_1 y+\gamma_1 z-\delta_1 t}, ..., \varpi_n e^{\alpha_n x+\beta_n y+\gamma_n z-\delta_n t})}{\Omega(\varpi_1 e^{\alpha_1 x+\beta_1 y+\gamma_1 z-\delta_1 t}, ..., \varpi_n e^{\alpha_n x+\beta_n y+\gamma_n z-\delta_n t})}, \tag{2.5}$$

and also we have

$$\begin{aligned} \Delta_t &= \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,t}, \ \ \Omega_t = \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,t}, \ \ \Delta_x = \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,x}, \ \ \Omega_x = \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,x}, \ \ \Delta_y = \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,y}, \\ \Delta_z &= \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,z}, \ \ \Omega_y = \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,y}, \ \ \Omega_z = \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,z}, \end{aligned} \tag{2.6}$$

$$\Psi_t = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,t} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,t}}{\Omega^2}, \ \ \Psi_x = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,x} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,x}}{\Omega^2},$$

$$\Psi_y = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,y} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,y}}{\Omega^2}, \ \ \Psi_z = \frac{\Omega \sum_{i=1}^{n} \Delta_{\xi_i}\xi_{i,z} - \Delta \sum_{i=1}^{n} \Omega_{\xi_i}\xi_{i,z}}{\Omega^2}.$$

3. MSSs for the (3+1) gKP equation

3.1. *Option I: 1-wave solution*

The one-wave function of the solution will be reduced as below form

$$\Psi = \frac{2\Delta_1}{\Omega_1}, \ \ \Omega_1 = 1 + \rho_1 + \rho_2\, e^{\alpha_1 x+\beta_1 y+\gamma_1 z-\delta_1 t}, \ \ \Delta_1 = \sigma_1 + \sigma_2\, e^{\alpha_1 x+\beta_1 y+\gamma_1 z-\delta_1 t}, \tag{3.1}$$

in which ρ_1, ρ_2, σ_1 and σ_2 are unspecified amounts. Substituting (3.1) into Eq (1.8), the below cases will be concluded as:

Case I:

$$\alpha_1 = \alpha_1, \ \beta_1 = \beta_1, \ \rho_1 = \rho_1, \ \rho_2 = \frac{\sigma_2\,(\rho_1+1)}{\sigma_1}, \ \sigma_1 = \sigma_1, \ \sigma_2 = \sigma_2, \ \delta_1 = \delta_1, \ \gamma_1 = \gamma_1, \ \gamma_2 = \gamma_2. \tag{3.2}$$

Case II:

$$\alpha_1 = \alpha_1, \ \beta_1 = -\frac{\alpha\,\alpha_1{}^3\gamma_1 - \alpha_1\delta_1\lambda_1 + \alpha_1\gamma_1\omega_1 - \delta_1\gamma_1\lambda_3 + \gamma_1{}^2\omega_3}{\alpha_1{}^3 - \delta_1\lambda_2 + \gamma_1\omega_2}, \ \rho_1 = -1, \ \rho_2 = \rho_2, \tag{3.3}$$

$$\sigma_1 = \sigma_1, \ \sigma_2 = \sigma_2, \ \delta_1 = \delta_1, \ \gamma_1 = \gamma_1, \ \gamma_2 = \gamma_2.$$

Case III:

$$\alpha_1 = \frac{\gamma_1\,(\alpha\,\delta_1\lambda_2 - \alpha\,\gamma_1\omega_2 - \delta_1\lambda_3 + \gamma_1\omega_3)}{\delta_1\lambda_1 - \gamma_1\omega_1}, \ \beta_1 = -\alpha\gamma_1, \ \rho_1 = \rho_1, \ \rho_2 = \rho_2, \tag{3.4}$$

$$\sigma_1 = \sigma_1, \ \sigma_2 = \sigma_2, \ \delta_1 = \delta_1, \ \ \gamma_1 = \gamma_1, \ \gamma_2 = \gamma_2.$$

Case IV:

$$\alpha_1 = \vartheta, \ \beta_1 = \beta_1, \ or \ \beta_1 = -\alpha\gamma_1, \ \ \rho_1 = -1, \ \rho_2 = \rho_2, \ \sigma_1 = \sigma_1,$$

$$\sigma_2 = \sigma_2,\ \delta_1 = \frac{\vartheta^3 + \gamma_1\omega_2}{\lambda_2},\ \ \gamma_1 = \gamma_1,\ \ \gamma_2 = \gamma_2, \tag{3.5}$$

in which ϑ, solves the equation $\lambda_1\vartheta^4 + (\gamma_1\lambda_3 - \alpha\gamma_1\lambda_2)\vartheta^3 + (\gamma_1\lambda_1\omega_2 - \gamma_1\lambda_2\omega_1)\vartheta - \gamma_1^2\omega_3\lambda_2 + \gamma_1^2\lambda_3\omega_2 = 0$. For example, the 1-wave solution for Case III will be considered as

$$\Psi = 2\frac{\sigma_1 + \sigma_2 e^{\frac{\gamma_1(\alpha\delta_1\lambda_2 - \alpha\gamma_1\omega_2 - \delta_1\lambda_3 + \gamma_1\omega_3)}{\delta_1\lambda_1 - \gamma_1\omega_1}x - \alpha\gamma_1 y + \gamma_1 z - \delta_1 t}}{1 + \rho_1 + \rho_2 e^{\frac{\gamma_1(\alpha\delta_1\lambda_2 - \alpha\gamma_1\omega_2 - \delta_1\lambda_3 + \gamma_1\omega_3)}{\delta_1\lambda_1 - \gamma_1\omega_1}x - \alpha\gamma_1 y + \gamma_1 z - \delta_1 t}}. \tag{3.6}$$

3.2. Option II: 2-wave solutions

The two-wave function of the solution will be reduced as below form

$$\Psi = \frac{2\Delta_2}{\Omega_2}, \tag{3.7}$$

$$\begin{aligned}\Omega_2 &= 1 + \sigma_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \sigma_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t} + \sigma_1\sigma_2\sigma_{12} e^{(\alpha_1+\alpha_2)x + (\beta_1+\beta_2)y + (\gamma_1+\gamma_2)z - (\delta_1+\delta_2)t}, \\ \Delta_2 &= \rho_1 e^{\alpha_1 x + \beta_1 y + \gamma_1 z - \delta_1 t} + \rho_2 e^{\alpha_2 x + \beta_2 y + \gamma_2 z - \delta_2 t} + \rho_1\rho_2\rho_{12} e^{(\alpha_1+\alpha_2)x + (\beta_1+\beta_2)y + (\gamma_1+\gamma_2)z - (\delta_1+\delta_2)t}.\end{aligned} \tag{3.8}$$

Substituting (3.7) in terms of (3.8) into Eq (1.8), the below cases will be resulted as:
Case I:

$$\begin{aligned}\alpha_1 &= 0,\ \alpha_2 = \alpha_2,\ \beta_1 = \beta_1,\ \beta_2 = -\alpha\gamma_2,\ \delta_1 = \frac{\gamma_1(\beta_1\omega_2 + \omega_3\gamma_1)}{\beta_1\lambda_2 + \gamma_1\lambda_3}, \\ \delta_2 &= \frac{\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \gamma_2 = \gamma_2,\ \rho_1 = 0,\ \rho_2 = \rho_2, \\ &\rho_{12} = \rho_{12},\ \sigma_1 = \sigma_1,\ \sigma_2 = \sigma_2,\ \sigma_{12} = 1.\end{aligned} \tag{3.9}$$

Case II:

$$\begin{aligned}\alpha_1 &= \alpha_1,\ \alpha_2 = 0,\ \beta_1 = -\alpha\gamma_1,\ \beta_2 = \beta_2,\ \delta_1 = \frac{\gamma_1(\alpha\gamma_1\omega_2 - \alpha_1\omega_1 - \omega_3\gamma_1)}{\alpha\gamma_1\lambda_2 - \alpha_1\lambda_1 - \gamma_1\lambda_3}, \\ \delta_2 &= \frac{\gamma_2(\beta_2\omega_2 + \gamma_2\omega_3)}{\beta_2\lambda_2 + \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \gamma_2 = \gamma_2,\ \rho_1 = 0, \\ &\rho_2 = \rho_2,\ \rho_{12} = \rho_{12},\ \sigma_1 = \sigma_1,\ \sigma_2 = \sigma_2,\ \sigma_{12} = 1.\end{aligned} \tag{3.10}$$

Case III:

$$\begin{aligned}\alpha_1 &= \alpha_1,\ \alpha_2 = \alpha_2,\ \beta_1 = -\alpha\gamma_1,\ \beta_2 = -\alpha\gamma_2,\ \delta_1 = \frac{\gamma_1(\alpha\gamma_1\omega_2 - \alpha_1\omega_1 - \omega_3\gamma_1)}{\alpha\gamma_1\lambda_2 - \alpha_1\lambda_1 - \gamma_1\lambda_3}, \\ \delta_2 &= \frac{\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \gamma_2 = \gamma_2,\ \rho_1 = 0, \\ &\rho_2 = \rho_2,\ \rho_{12} = \rho_{12},\ \sigma_1 = \sigma_1,\ \sigma_2 = \sigma_2,\ \sigma_{12} = 1.\end{aligned} \tag{3.11}$$

Case IV:

$$\alpha_1 = 0,\ \alpha_2 = \alpha_2,\ \beta_1 = \beta_1,\ \beta_2 = -\alpha\gamma_2,\ \delta_1 = \frac{\gamma_1(\beta_1\omega_2 + \omega_3\gamma_1)}{\beta_1\lambda_2 + \gamma_1\lambda_3},$$

$$\delta_2 = \frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \ \gamma_2 = \gamma_2,\ \ \rho_1 = \rho_1,\ \ \rho_2 = 0, \tag{3.12}$$

$$\rho_{12} = \rho_{12},\ \ \sigma_1 = \sigma_1,\ \ \sigma_2 = \sigma_2,\ \ \sigma_{12} = 1.$$

Case V:

$$\begin{aligned}
\alpha_1 &= \alpha_1,\ \ \alpha_2 = \alpha_2,\ \ \beta_1 = -\alpha\gamma_1,\ \ \beta_2 = -\alpha\gamma_2,\ \ \delta_1 = \frac{\gamma_1\left(\alpha\gamma_1\omega_2 - \alpha_1\omega_1 - \omega_3\gamma_1\right)}{\alpha\gamma_1\lambda_2 - \alpha_1\lambda_1 - \gamma_1\lambda_3}, \\
\delta_2 &= \frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \ \gamma_2 = \gamma_2,\ \ \rho_1 = \rho_1,\ \ \rho_2 = 0,
\end{aligned} \tag{3.13}$$

$$\rho_{12} = \rho_{12},\ \ \sigma_1 = \sigma_1,\ \ \sigma_2 = \sigma_2,\ \ \sigma_{12} = 1.$$

Case VI:

$$\begin{aligned}
\alpha_1 &= \alpha_1,\ \ \alpha_2 = \alpha_2,\ \ \beta_1 = -\frac{\alpha\alpha_1\gamma_2}{\alpha_2},\ \ \beta_2 = -\alpha\gamma_2,\ \ \delta_1 = \frac{\alpha_1\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha_2\left(\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3\right)}, \\
\delta_2 &= \frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \gamma_1,\ \ \gamma_2 = \gamma_2,\ \ \rho_1 = \rho_1,\ \ \rho_2 = 0,
\end{aligned} \tag{3.14}$$

$$\rho_{12} = \rho_{12},\ \ \sigma_1 = \sigma_1,\ \ \sigma_2 = \sigma_2,\ \ \sigma_{12} = \sigma_{12}.$$

Case VII:

$$\begin{aligned}
\alpha_1 &= \alpha_1,\ \ \alpha_2 = \alpha_2,\ \ \beta_1 = -\frac{\alpha\alpha_1\gamma_2}{\alpha_2},\ \ \beta_2 = -\alpha\gamma_2,\ \ \delta_1 = \frac{\alpha_1\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha_2\left(\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3\right)}, \\
\delta_2 &= \frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \frac{\alpha_1\gamma_2}{\alpha_2},\ \ \gamma_2 = \gamma_2,\ \ \rho_1 = \rho_1,\ \ \rho_2 = \rho_2,
\end{aligned} \tag{3.15}$$

$$\rho_{12} = \rho_{12},\ \ \sigma_1 = 0,\ \ \sigma_2 = \sigma_2,\ \ \sigma_{12} = \sigma_{12}.$$

Case VIII:

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}\alpha_2,\ \ \alpha_2 = \alpha_2,\ \ \beta_1 = -\frac{1}{2}\alpha\gamma_2,\ \ \beta_2 = -\alpha\gamma_2,\ \ \delta_1 = \frac{1}{2}\,\frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \\
\delta_2 &= \frac{\gamma_2\left(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3\right)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3}, \gamma_1 = \frac{1}{2}\gamma_2,\ \ \gamma_2 = \gamma_2,\ \ \rho_1 = \rho_1,
\end{aligned} \tag{3.16}$$

$$\rho_2 = \rho_2,\ \ \rho_{12} = \rho_{12},\ \ \sigma_1 = \sigma_1,\ \ \sigma_2 = \sigma_2,\ \ \sigma_{12} = 0.$$

For instance, the 2-wave solution for Case I will be taken as

$$\Psi_1 = 2\rho_2 \mathrm{e}^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3} + x\alpha_2 - y\alpha\gamma_2 + z\gamma_2} \Bigg/ \Bigg(1 + \sigma_1 \mathrm{e}^{-\frac{t\gamma_1(\beta_1\omega_2 + \omega_3\gamma_1)}{\beta_1\lambda_2 + \gamma_1\lambda_3} + y\beta_1 + z\gamma_1} + \tag{3.17}$$

$$\sigma_2 \mathrm{e}^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3} + x\alpha_2 - y\alpha\gamma_2 + z\gamma_2} + \sigma_1\sigma_2 \mathrm{e}^{-\frac{t\gamma_1(\beta_1\omega_2 + \omega_3\gamma_1)}{\beta_1\lambda_2 + \gamma_1\lambda_3} - \frac{t\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3} + x\alpha_2 + y\beta_1 - y\alpha\gamma_2 + z\gamma_1 + z\gamma_2}\Bigg).$$

Also, the 2-wave solution for Case III will be considered as

$$\Psi_2 = 2\rho_2 \mathrm{e}^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2 - \alpha_2\omega_1 - \gamma_2\omega_3)}{\alpha\gamma_2\lambda_2 - \alpha_2\lambda_1 - \gamma_2\lambda_3} + x\alpha_2 - y\alpha\gamma_2 + z\gamma_2} \Bigg/ \Bigg(1 + \sigma_1 \mathrm{e}^{-\frac{t\gamma_1(\alpha\gamma_1\omega_2 - \alpha_1\omega_1 - \omega_3\gamma_1)}{\alpha\gamma_1\lambda_2 - \alpha_1\lambda_1 - \gamma_1\lambda_3} + x\alpha_1 - y\alpha\gamma_1 + z\gamma_1} + \tag{3.18}$$

$$\sigma_2 e^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+x\alpha_2-y\alpha\gamma_2+z\gamma_2}+\sigma_1\sigma_2 e^{-\frac{t\gamma_1(\alpha\gamma_1\omega_2-\alpha_1\omega_1-\omega_3\gamma_1)}{\alpha\gamma_1\lambda_2-\alpha_1\lambda_1-\gamma_1\lambda_3}-\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+x\alpha_1+x\alpha_2-y\alpha\gamma_1-y\alpha\gamma_2+z\gamma_1+z\gamma_2}\Bigg).$$

And finally, the resulting two-wave solution for Case VIII will be read as

$$\Psi_3(x,y,z,t)=2\Bigg(\rho_1 e^{-\frac{1}{2}\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+\frac{1}{2}x\alpha_2-\frac{1}{2}y\alpha\gamma_2+\frac{1}{2}z\gamma_2}+\rho_2 e^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+x\alpha_2-y\alpha\gamma_2+z\gamma_2}+ \tag{3.19}$$

$$\rho_1\rho_2\rho_{12}\, e^{-\frac{3}{2}\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+\frac{3}{2}x\alpha_2-\frac{3}{2}y\alpha\gamma_2+\frac{3}{2}z\gamma_2}\Bigg)\Bigg/\Bigg(1+\sigma_1 e^{-\frac{1}{2}\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+\frac{1}{2}x\alpha_2-\frac{1}{2}y\alpha\gamma_2+\frac{1}{2}z\gamma_2}$$

$$+\sigma_2 e^{-\frac{t\gamma_2(\alpha\gamma_2\omega_2-\alpha_2\omega_1-\gamma_2\omega_3)}{\alpha\gamma_2\lambda_2-\alpha_2\lambda_1-\gamma_2\lambda_3}+x\alpha_2-y\alpha\gamma_2+z\gamma_2}\Bigg).$$

3.3. Option III: Triple-wave solutions

The triple-wave function of the solution will be reduced as below form

$$\Psi=\frac{\Delta_3}{\Omega_3}, \tag{3.20}$$

$$\Omega_3=1+\rho_1e^{\Lambda_1}+\rho_2e^{\Lambda_2}+\rho_3e^{\Lambda_3}+\rho_1\rho_2\rho_{12}e^{\Lambda_1+\Lambda_2}+\rho_1\rho_3\rho_{13}e^{\Lambda_1+\Lambda_3}+\rho_2\rho_3\rho_{23}e^{\Lambda_2+\Lambda_3}+\rho_1\rho_2\rho_3\rho_{12}\rho_{13}\rho_{23}e^{\Lambda_1+\Lambda_2+\Lambda_3}, \tag{3.21}$$

$$\Delta_3=2\sigma_1e^{\Lambda_1}+2\sigma_2e^{\Lambda_2}+2\sigma_1\sigma_2\sigma_{12}e^{\Lambda_1+\Lambda_2}+2\sigma_1\sigma_3\sigma_{13}e^{\Lambda_1+\Lambda_3}+2\sigma_2\sigma_3\sigma_{23}e^{\Lambda_2+\Lambda_3}+$$
$$2\sigma_1\sigma_2\sigma_3\sigma_{12}\sigma_{13}\sigma_{23}e^{\Lambda_1+\Lambda_2+\Lambda_3},$$

in which $\Lambda_i=\alpha_ix+\beta_iy+\gamma_ix-\delta_it, i=1,2,3$. Inserting (3.20) in terms of (3.21) into Eq (1.8), the below case will be reached:

$$\begin{aligned}\alpha_i &= \alpha_i,\ \ \gamma_i=\gamma_i,\ \ \sigma_i=\sigma_i,\ \ \rho_1=0,\ \ \rho_2=\rho_2,\ \ \rho_3=\rho_3,\ \ \beta_i=-\alpha\gamma_i,\\ \delta_i &= \frac{\gamma_i(\alpha\gamma_i\omega_{\ 2}-\alpha_i\omega_1-\omega_3\gamma_i)}{\alpha\gamma_i\lambda_2-\alpha_i\lambda_1-\gamma_i\lambda_3},\ \ i=1,2,3,\ \ \rho_{ij}=\rho_{ij},\ \ \sigma_{ij}=1,\ \ i,j=1,2,3,\ \ i\neq j.\end{aligned} \tag{3.22}$$

Then, the solution is

$$\Psi_1=\frac{2\sigma_1e^{\Lambda_1}+2\sigma_2e^{\Lambda_2}+2\sigma_1\sigma_2e^{\Lambda_1+\Lambda_2}+2\sigma_1\sigma_3e^{\Lambda_1+\Lambda_3}+2\sigma_2\sigma_3e^{\Lambda_2+\Lambda_3}+2\sigma_1\sigma_2\sigma_3e^{\Lambda_1+\Lambda_2+\Lambda_3}}{1+\rho_2e^{\Lambda_2}+\rho_3e^{\Lambda_3}+\rho_2\rho_3\rho_{23}e^{\Lambda_2+\Lambda_3}}, \tag{3.23}$$

in which $\Lambda_i=\alpha_ix-\alpha\gamma_iy+\gamma_ix-\frac{\gamma_i(\alpha\gamma_i\omega_{\ 2}-\alpha_i\omega_1-\omega_3\gamma_i)}{\alpha\gamma_i\lambda_2-\alpha_i\lambda_1-\gamma_i\lambda_3}t, i=1,2,3$ and $\Psi=\Psi(x,y,z,t)$.

4. Novel periodic wave solutions

The triangular periodic waves for Eq (1.8) can be assumed as below:

$$\mathbf{f}=\exp(\tau_1)+a_{16}\exp(-\tau_1)+\cosh(\tau_2)+\cos(\tau_3)+a_{17},\quad \tau_1=\sum_{i=1}^{4}a_ix_i+a_5,\quad \tau_2=\sum_{i=6}^{9}a_ix_{i-5}+a_{10}, \tag{4.1}$$

$$\tau_3=\sum_{i=11}^{14}a_ix_{i-10}+a_{15},\ \ (x_1,x_2,x_3,x_4)=(\mathbf{x},t)=(x,y,z,t),\ \ \Psi(\mathbf{x},t)=v_0+2\ln(\mathbf{f})_x, \tag{4.2}$$

in which $a_i, i = 1, ..., 17$ are unfound values. Substituting (4.1) and (4.2) into Eq (1.8) the below consequences will be gained:

Case I:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{y(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3)}{a_{14}\omega_2} + a_3 z + a_5} + \cosh\left(a_9 t - \frac{y\,(a_8 a_{14}\omega_3 - a_9 a_{12}\omega_2 - a_9 a_{13}\omega_3)}{a_{14}\omega_2} + a_8 z + a_{10}\right) + \cos\left(t a_{14} + y a_{12} + z a_{13} + a_{15}\right). \quad (4.3)$$

Appending (4.3) into (4.1) and (4.2), the soliton-periodic wave solution of Eq (1.8) as below will be achieved:

$$\Psi_1 = v_0 + \frac{2a_1 e^{a_4 t + a_1 x - \frac{y(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3)}{a_{14}\omega_2} + a_3 z + a_5}}{\mathbf{f}}. \quad (4.4)$$

By selecting the suitable values of parameters including

$$a_1 = 1, a_3 = 1.5, a_4 = 2, a_5 = 1.5, a_8 = 2, a_9 = 1.5, a_{10} = 1,$$
$$a_{12} = 2, a_{13} = 2.5, a_{14} = 1, a_{15} = 3.2, \omega_2 = 1.5, \omega_3 = 1.2,$$

the graphical display of soliton-periodic wave solution is offered in Figure 1 such as 3D plot and density plot.

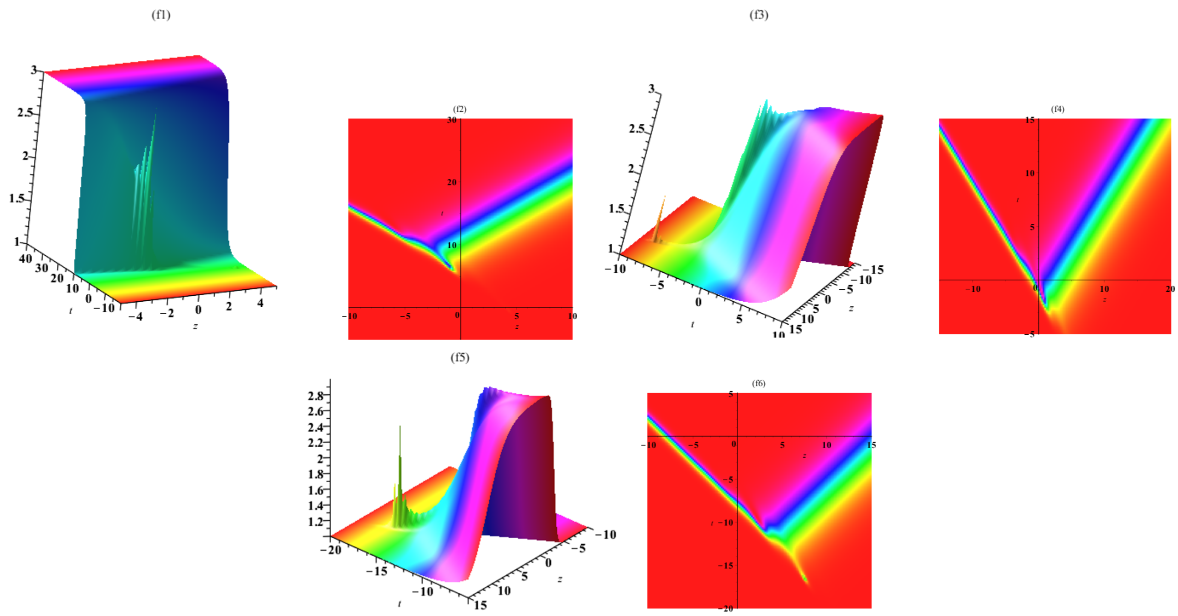

Figure 1. The soliton-periodic solution (4.4) at (f1, f2) $x = -2, y = -2$, (f3, f4) $x = 0, y = 0$, and (f5, f6) $x = 2, y = 2$.

Case II:

$$\mathbf{f} = e^{a_4 t + a_1 x + y a_2 - \frac{(a_1{}^3 a_{14} - a_4 a_{13}\omega_2) z}{a_{14}\omega_2} + a_5} + \cosh\left(a_9 t + y a_7 + \frac{a_9 a_{13} z}{a_{14}} + a_{10}\right) + \cos\left(t a_{14} + y a_{12} + z a_{13} + a_{15}\right). \quad (4.5)$$

Plugging (4.5) into (4.1) and (4.2), obtain a soliton-periodic wave solution of Eq (1.8) as below case:

$$\Psi_2 = v_0 + \frac{2a_1 e^{a_4t+a_1x+ya_2-\frac{\left(a_1{}^3a_{14}-a_4a_{13}\omega_2\right)z}{a_{14}\omega_2}+a_5}}{\mathbf{f}}. \tag{4.6}$$

Case III:

$$\mathbf{f} = e^{a_4t+a_1x-\frac{y(a_3\omega_3+a_4\lambda_3)}{\omega_2}+a_3z+a_5} + \cosh\left(a_9t - \frac{y\,(a_8\omega_3 + a_9\lambda_3)}{\omega_2} + a_8z + a_{10}\right) + \cos\left(-\frac{a_{13}\omega_3 y}{\omega_2} + za_{13} + a_{15}\right). \tag{4.7}$$

Incorporating (4.7) into (4.1) and (4.2), the soliton-periodic wave solution of Eq (1.8) will be gained as below:

$$\Psi_3 = v_0 + \frac{2a_1 e^{a_4t+a_1x-\frac{y(a_3\omega_3+a_4\lambda_3)}{\omega_2}+a_3z+a_5}}{\mathbf{f}}. \tag{4.8}$$

Case IV:

$$\mathbf{f} = e^{a_4t-\frac{y(a_3a_9\omega_3-a_4a_7\omega_2-a_4a_8\omega_3)}{a_9\omega_2}+a_3z+a_5} + \cosh\left(a_9t + ya_7 + a_8z + a_{10}\right) + \cos\left(xa_{11} + a_{15}\right). \tag{4.9}$$

Plugging (4.9) into (4.1) and (4.2), the soliton-periodic wave solution of Eq (1.8) will be achieved as below:

$$\Psi_4 = v_0 - \frac{2\sin\left(xa_{11} + a_{15}\right)a_{11}}{\mathbf{f}}. \tag{4.10}$$

Case V:

$$\mathbf{f} = e^{a_4t+ya_2+\frac{a_4a_8z}{a_9}+a_5} + \cosh\left(a_9t + ya_7 + a_8z + a_{10}\right) + \cos\left(xa_{11} - \frac{\left(a_{11}{}^2\omega_3 + \omega_1\omega_2\right)a_{11}y}{\omega_2{}^2} + \frac{a_{11}{}^3z}{\omega_2} + a_{15}\right). \tag{4.11}$$

Incorporating (4.11) into (4.1) and (4.2), we capture a soliton-periodic wave solution of Eq (1.8) as below:

$$\Psi_5 = v_0 - \frac{2\sin\left(xa_{11} - \frac{\left(a_{11}{}^2\omega_3+\omega_1\omega_2\right)a_{11}y}{\omega_2{}^2} + \frac{a_{11}{}^3z}{\omega_2} + a_{15}\right)a_{11}}{\mathbf{f}}. \tag{4.12}$$

Case VI:

$$\begin{aligned}\mathbf{f} \;=\; & e^{a_4t+\frac{\omega_2(a_3a_9-a_4a_8)x}{a_9a_{11}{}^2}-\frac{1}{3}\frac{y\left(-2\alpha a_9a_{11}{}^6\omega_2+2a_9a_{11}{}^6\omega_3+3\alpha a_3{}^2a_9\omega_2{}^3-3\alpha a_3a_4a_8\omega_2{}^3+2a_9a_{11}{}^4\omega_1\omega_2\right)}{\omega_2{}^3(a_3a_9-a_4a_8)}+a_3z+a_5} + \\ & \cosh\left(a_9t - \frac{1}{3}\frac{y\Omega}{a_4\omega_2{}^3\left(a_3a_9 - a_4a_8\right)\left(a_9{}^2a_{11}{}^6 + \omega_2{}^2\left(a_3a_9 - a_4a_8\right)^2\right)} + a_8z + a_{10}\right) \\ & + \cos\left(xa_{11} - \frac{\alpha a_{11}{}^3y}{\omega_2} + \frac{a_{11}{}^3z}{\omega_2} + a_{15}\right),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Omega \;=\; & 3\,\alpha\,a_4a_8\omega_2{}^5\left(a_3a_9 - a_4a_8\right)^3 + \alpha\,a_9{}^2a_{11}{}^6\omega_2{}^3\left(a_3a_9 + 2\,a_4a_8\right)\left(a_3a_9 - a_4a_8\right) \\ & -a_9{}^2a_{11}{}^4\omega_2{}^2\left(a_3a_9 - a_4a_8\right)^2\left(a_{11}{}^2\omega_3 + \omega_1\omega_2\right) - a_9{}^2a_{11}{}^{12}\left(3\,a_4{}^2 - a_9{}^2\right)\left(\alpha\,\omega_2 - \omega_3\right).\end{aligned}$$

Appending (4.13) into (4.1) and (4.2), the soliton-periodic wave solution of Eq (1.8) will be obtained as below:

$$\Psi_5 = v_0 + \frac{2}{\mathbf{f}}\left[\frac{\omega_2\,(a_3a_9 - a_4a_8)}{a_9{a_{11}}^2}\mathrm{e}^{a_4t+\frac{\omega_2(a_3a_9-a_4a_8)x}{a_9{a_{11}}^2}-\frac{1}{3}\frac{y\left(-2\,\alpha\,a_9{a_{11}}^6\omega_2+2\,a_9{a_{11}}^6\omega_3+3\,\alpha\,{a_3}^2a_9{\omega_2}^3-3\,\alpha\,a_3a_4a_8{\omega_2}^3+2\,a_9{a_{11}}^4\omega_1\omega_2\right)}{{\omega_2}^3(a_3a_9-a_4a_8)}+a_3z+a_5} + \sin\left(-xa_{11} + \frac{\alpha\,{a_{11}}^3y}{\omega_2} - \frac{{a_{11}}^3z}{\omega_2} - a_{15}\right)a_{11}\right]. \tag{4.14}$$

By selecting suitable values of parameters including

$$\alpha = 0.5, a_3 = 1, a_4 = 1.5, a_5 = 2, a_8 = 2, a_9 = 1.5, a_{10} = 1, a_{11} = 2, a_{13} = 2.5, a_{14} = 1,$$
$$a_{15} = 3.2, \omega_1 = 1.5, \omega_2 = 1.2, \omega_3 = 1.5,$$

the graphical display of soliton-periodic wave solution is offered in Figure 2 such as 3D chart and density chart.

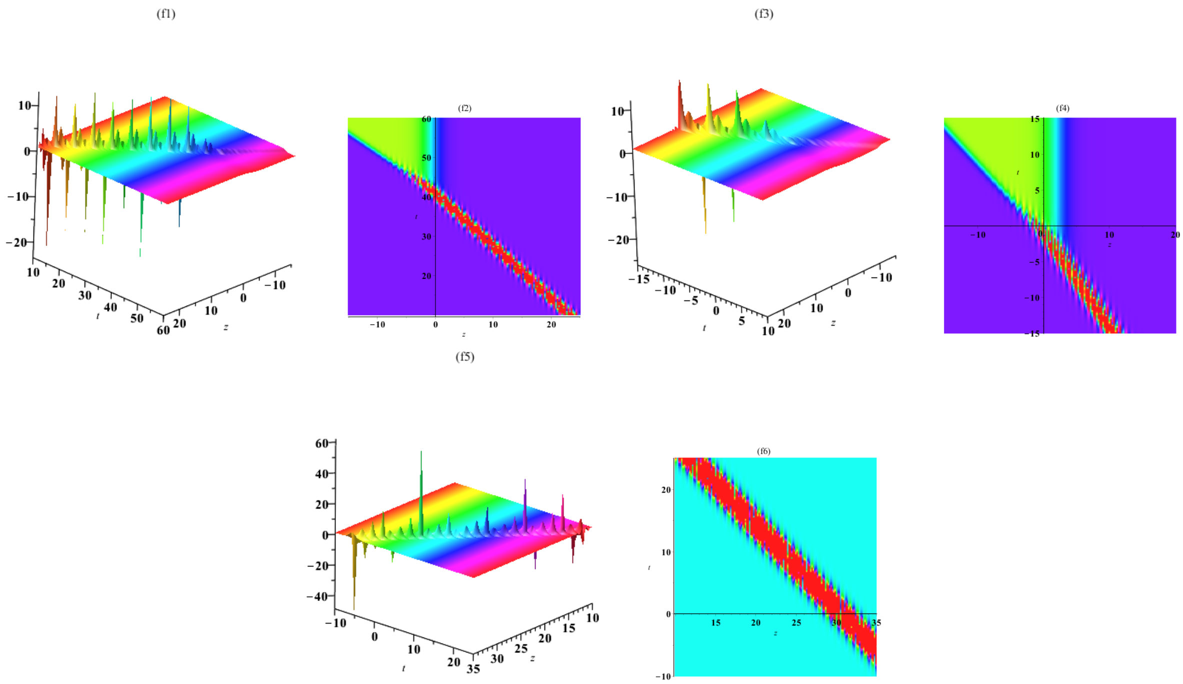

Figure 2. The soliton-periodic solution (4.14) at (f1, f2) $x = -2, y = -2$, (f3, f4) $x = 0, y = 0$, and (f5, f6) $x = 2, y = 2$.

Case VII:

$$\mathbf{f} = \mathrm{e}^{\frac{\Omega a_9t}{{a_{11}}^3}+a_1x-\frac{1}{6}\frac{y\left(-6\,{a_1}^4{a_{11}}^5\omega_3-3\,{a_1}^4{a_{11}}^3\omega_1\omega_2+2\,{a_{11}}^7\omega_1\omega_2+6\,\Omega\,a_1a_8{a_{11}}^2\omega_2\omega_3+3\,\Omega\,a_1a_8\omega_1{\omega_2}^2\right)}{{a_{11}}^5{\omega_2}^2a_1}+\frac{\left(-{a_1}^3{a_{11}}^3+\Omega\,a_8\omega_2\right)z}{{a_{11}}^3\omega_2}+a_5} + \tag{4.15}$$

$$\cosh\left(a_9t + ya_7 + a_8z + a_{10}\right) + \cos\left(xa_{11} - \frac{1}{2}\frac{\left(2\,{a_{11}}^2\omega_3 + \omega_1\omega_2\right)a_{11}y}{{\omega_2}^2} + \frac{{a_{11}}^3z}{\omega_2} + a_{15}\right),$$

$$\Omega = \sqrt{-{a_1}^6 - 2\,{a_1}^4{a_{11}}^2 + {a_{11}}^6}.$$

Incorporating (4.15) into (4.1) and (4.2), the soliton-periodic wave solution of Eq (1.8) will be received as below:

$$\begin{aligned}\Psi_5 &= v_0+\frac{2}{\mathbf{f}}\Bigg[a_1 e^{\frac{\Omega a_9 t}{{a_{11}}^3}+a_1 x-\frac{1}{6}\frac{y\left(-6{a_1}^4{a_{11}}^5\omega_3-3{a_1}^4{a_{11}}^3\omega_1\omega_2+2{a_{11}}^7\omega_1\omega_2+6\Omega a_1 a_8{a_{11}}^2\omega_2\omega_3+3\Omega a_1 a_8\omega_1{\omega_2}^2\right)}{{a_{11}}^5{\omega_2}^2 a_1}+\frac{\left(-{a_1}^3{a_{11}}^3+\Omega a_8\omega_2\right)z}{{a_{11}}^3\omega_2}+a_5} \\ &\quad -\sin\left(xa_{11}-\frac{1}{2}\frac{\left(2{a_{11}}^2\omega_3+\omega_1\omega_2\right)a_{11}y}{{\omega_2}^2}+\frac{{a_{11}}^3 z}{\omega_2}+a_{15}\right)a_{11}\Bigg]. \qquad (4.16)\end{aligned}$$

By selecting the specific amounts of parameters including

$$\begin{aligned}\alpha &= 0.5, a_1=1, a_4=a_9=\omega_1=\omega_3=1.5,, a_5=2, a_7=1.5, a_8=2, \\ a_{10} &= 1, a_{11}=2, a_{13}=2.5, a_{14}=1, a_{15}=3.2, \omega_2=1.2,\end{aligned}$$

the graphical display of soliton-periodic wave solution is offered in Figure 3 such as 3D chart, density chart, and 2D chart and below cases:

$$(f3)\ y=1,2,3,\quad (f6)\ y=1,2,3,\ \ and\ \ (f9)\ y=-1,-2,-3.$$

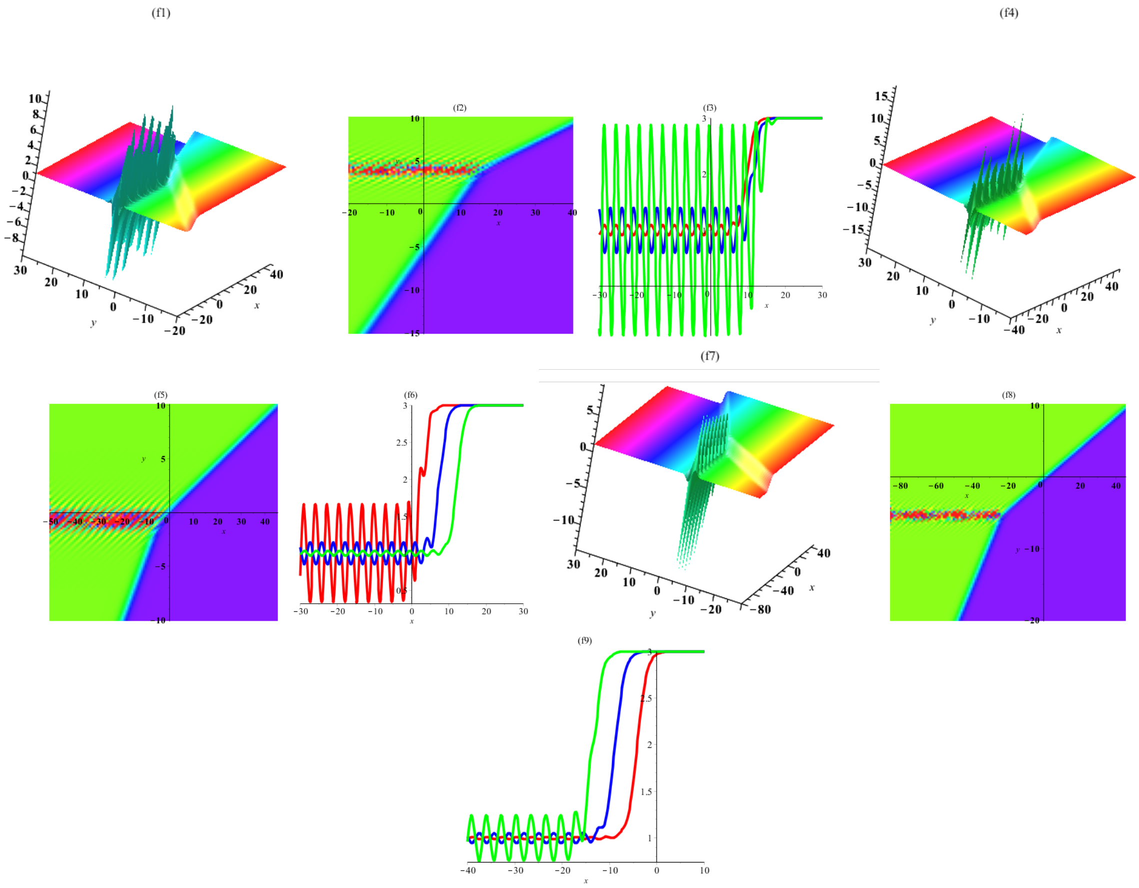

Figure 3. The soliton-periodic solution (4.16) at (f1, f2) $z=-2, t=-2$, (f4, f5) $z=0, t=0$, and (f7, f8) $z=2, t=2$.

5. Novel cross-kink wave solutions

Three function containing exponential, hyperbolic, and triangular periodic waves for Eq (1.8) can be assumed as the following:

$$\mathbf{f} = \exp(\tau_1) + a_{16}\exp(-\tau_1) + \sinh(\tau_2) + \sin(\tau_3) + a_{17}, \quad \tau_1 = \sum_{i=1}^{4} a_i x_i + a_5, \quad \tau_2 = \sum_{i=6}^{9} a_i x_{i-5} + a_{10}, \tag{5.1}$$

$$\tau_3 = \sum_{i=11}^{14} a_i x_{i-10} + a_{15}, \quad (x_1, x_2, x_3, x_4) = (\mathbf{x}, t) = (x, y, z, t), \quad \Psi(\mathbf{x}, t) = v_0 + 2\ln(\mathbf{f})_x, \tag{5.2}$$

in which $a_i, i = 1, ..., 17$ are unfound values. Substituting (5.2) into Eq (1.8) the below consequences will be gained:

Case I:

$$\mathbf{f} = \mathrm{e}^{a_4 t + a_1 x - \frac{(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3)y}{a_{14}\omega_2} + a_3 z + a_5} + \sinh\left(a_9 t - \frac{(a_8 a_{14}\omega_3 - a_9 a_{12}\omega_2 - a_9 a_{13}\omega_3)\, y}{a_{14}\omega_2} + a_8 z + a_{10}\right) \tag{5.3}$$

$$+ \sin\left(t a_{14} + y a_{12} + z a_{13} + a_{15}\right).$$

Substituting (5.3) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be gained as the following:

$$\Psi_1 = v_0 + \frac{2a_1 \mathrm{e}^{a_4 t + a_1 x - \frac{(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3)y}{a_{14}\omega_2} + a_3 z + a_5}}{\mathbf{f}}. \tag{5.4}$$

By selecting the suitable values of parameters including

$$\begin{aligned} a_1 &= 1, a_3 = 3, a_4 = 2, a_5 = 1.5, a_8 = 1.7, a_9 = 1.5, a_{10} = 1.5, \\ a_{12} &= 2.5, a_{13} = 1.1, a_{14} = 2.1, a_{15} = 3.2, \omega_2 = 1, \omega_3 = 1.5, \end{aligned}$$

the graphical representation of cross-kink wave solution is offered in Figure 4 such 3D plot and density plot.

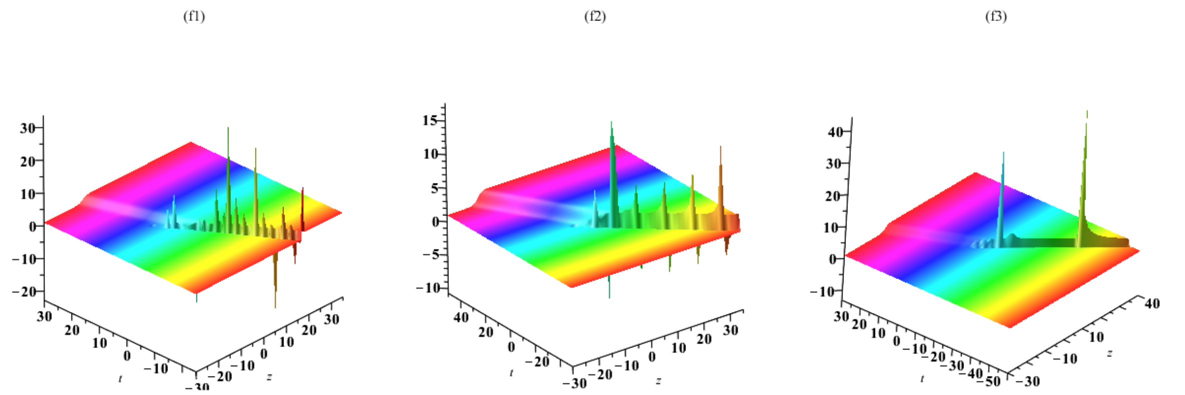

Figure 4. The cross-kink wave solution (5.4) at (f1, f2) $x = 3, y = 2$, (f3, f4) $x = 0, y = 2$, and (f5, f6) $x = -3, y = 2$.

Case II:

$$\mathbf{f} = e^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 a_{14} - a_4 a_{13}\omega_2\right) z}{a_{14}\omega_2} + a_5} + \sinh\left(a_9 t + a_7 y + \frac{a_9 a_{13} z}{a_{14}} + a_{10}\right) + \sin\left(t a_{14} + y a_{12} + z a_{13} + a_{15}\right). \quad (5.5)$$

Putting (5.5) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be received as the following:

$$\Psi_2 = v_0 + \frac{2a_1 e^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 a_{14} - a_4 a_{13}\omega_2\right) z}{a_{14}\omega_2} + a_5}}{\mathbf{f}}. \quad (5.6)$$

By selecting the suitable values of parameters including

$$\begin{aligned} a_1 &= 1, a_3 = 3, a_4 = 2, a_5 = 1.5, a_8 = 1.7, a_9 = 1.5, a_{10} = 1.5, \\ a_{12} &= 2.5, a_{13} = 1.1, a_{14} = 2.1, a_{15} = 3.2, \omega_2 = 1, \omega_3 = 1.5, \end{aligned}$$

the graphical exhibition of cross-kink solution is offered in Figure 5 such as 3D chart and density chart.

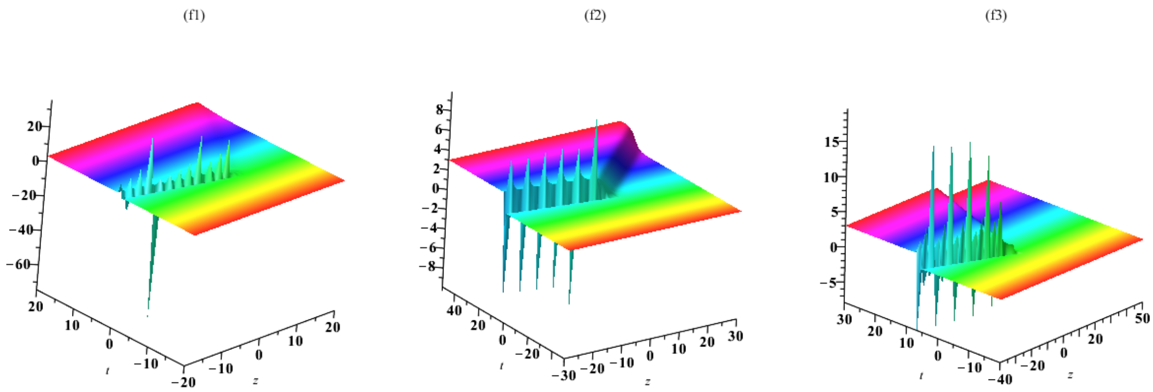

Figure 5. The cross-kink wave solution (5.6) at (f1, f2) $x = 3, y = 2$, (f3, f4) $x = 0, y = 2$, and (f5, f6) $x = -3, y = 2$.

Case III:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{(a_3\omega_3 + a_4\lambda_3) y}{\omega_2} + a_3 z + a_5} + \sinh\left(a_9 t - \frac{(a_8\omega_3 + a_9\lambda_3) y}{\omega_2} + a_8 z + a_{10}\right) + \sin\left(-\frac{a_{13}\omega_3 y}{\omega_2} + a_{13} z + a_{15}\right). \quad (5.7)$$

Plugging (5.7) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be obtained as the following:

$$\Psi_3 = v_0 + \frac{2a_1 e^{a_4 t + a_1 x - \frac{(a_3\omega_3 + a_4\lambda_3) y}{\omega_2} + a_3 z + a_5}}{\mathbf{f}}. \quad (5.8)$$

Case IV:

$$\begin{aligned} \mathbf{f} &= e^{-\frac{a_9\left(a_{11}{}^2\left(\omega_3 a_1{}^2 - \omega_1\omega_2\right)\left(a_1{}^3 + a_3\omega_2\right) - a_1{}^5\omega_1\omega_2\right) t}{a_1{}^2 a_7 a_{11}{}^2 \omega_2{}^2} + a_1 x - \frac{a_3\left(\omega_3 a_1{}^2 - \omega_1\omega_2\right) y}{a_1{}^2\omega_2} + a_3 z + a_5} + \sin\left(x a_{11} + a_{15}\right) \\ &\quad + \sinh\left(a_9 t + a_7 y - \frac{\left(a_1{}^3 + a_3\omega_2\right) a_1{}^2 a_7 a_{11}{}^2 \omega_2 z}{a_{11}{}^2\left(\omega_3 a_1{}^2 - \omega_1\omega_2\right)\left(a_1{}^3 + a_3\omega_2\right) - a_1{}^5\omega_1\omega_2} + a_{10}\right). \end{aligned} \quad (5.9)$$

Inserting (5.9) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be gained as the following:

$$\Psi_4 = v_0 + 2\frac{a_1 \mathrm{e}^{-\frac{a_9\left(a_{11}^2\left(\omega_3 a_1^2-\omega_1\omega_2\right)\left(a_1^3+a_3\omega_2\right)-a_1^5\omega_1\omega_2\right)t}{a_1^2 a_7 a_{11}^2 \omega_2^2}+a_1 x-\frac{a_3\left(\omega_3 a_1^2-\omega_1\omega_2\right)y}{a_1^2\omega_2}+a_3 z+a_5} + \cos\left(x a_{11} + a_{15}\right) a_{11}}{\mathbf{f}}. \quad (5.10)$$

Case V:

$$\mathbf{f} = \mathrm{e}^{a_4 t + a_2 y + \frac{a_4 a_8 z}{a_9} + a_5} + \sinh\left(a_9 t + a_7 y + a_8 z + a_{10}\right) + \sin\left(x a_{11} - \frac{a_{11}\left(a_{11}^2\omega_3 + \omega_1\omega_2\right)y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right). \quad (5.11)$$

Substituting (5.11) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be received as the following:

$$\Psi_5 = v_0 + \frac{2\cos\left(x a_{11} - \frac{a_{11}\left(a_{11}^2\omega_3+\omega_1\omega_2\right)y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right)a_{11}}{\mathbf{f}}. \quad (5.12)$$

By selecting the suitable values of parameters including

$$\begin{aligned} a_1 &= 1, a_3 = 3, a_4 = 2, a_5 = 1.5, a_8 = 1.7, a_9 = 1.5, a_{10} = 1.5, \\ a_{12} &= 2.5, a_{13} = 1.1, a_{14} = 2.1, a_{15} = 3.2, \omega_2 = 1, \omega_3 = 1.5, \end{aligned}$$

the graphical exhibition of cross-kink solution is offered in Figure 6 such as 3D chart and density chart.

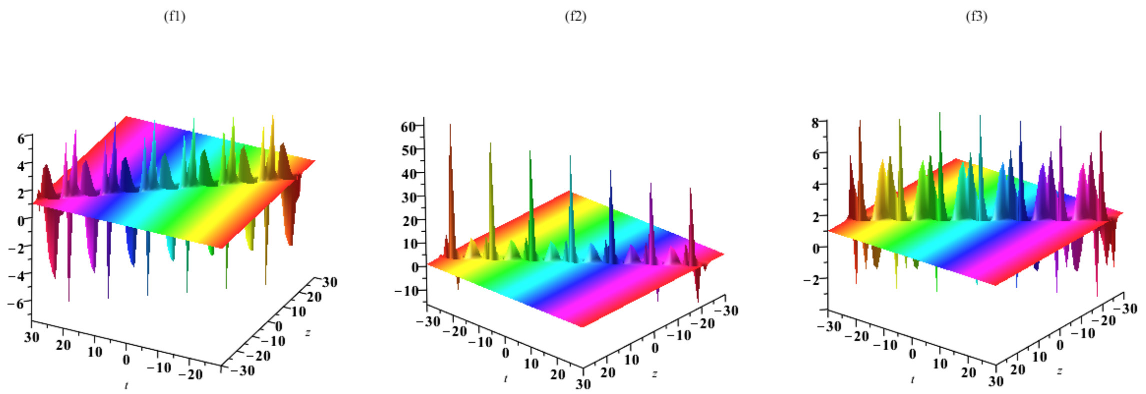

Figure 6. The cross-kink wave solution (5.6) at (f1, f2) $x = 3, y = 2$, (f3, f4) $x = 0, y = 2$, and (f5, f6) $x = -3, y = 2$.

Case VI:

$$\begin{aligned} \mathbf{f} &= \mathrm{e}^{\frac{\Omega a_9 t}{a_{11}^3} + a_1 x - \frac{1}{12}\frac{\left(3\Omega a_1 a_8 \omega_2\left(4 a_{11}^2\omega_3 + 3\omega_1\omega_2\right) - 12 a_1^4 a_{11}^5 \omega_3 - a_{11}^3\omega_1\omega_2\left(9 a_1^4 - 2 a_{11}^2 a_1^2 - 2 a_{11}^4\right)\right)y}{a_{11}^5\omega_2^2 a_1} + \frac{\left(-a_1^3 a_{11}^3 + \Omega a_8 \omega_2\right)z}{\omega_2 a_{11}^3} + a_5} \\ & \quad \sinh\left(a_9 t + a_7 y + a_8 z + a_{10}\right) + \sin\left(x a_{11} - \frac{1}{12}\frac{a_{11}\left(12 a_{11}^2\omega_3 + 7\,\omega_1\omega_2\right)y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right), \quad (5.13) \end{aligned}$$

$$\Omega = \sqrt{-3\,a_1^6 - 4\,a_1^4 a_{11}^2 + a_{11}^6}.$$

Putting (5.13) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be concluded as the following:

$$\begin{aligned}\Psi_5 &= v_0 + \frac{2}{\mathbf{f}}\left[a_1 e^{\frac{\Omega a_9 t}{a_{11}^3} + a_1 x - \frac{1}{12}\frac{\left(3\Omega a_1 a_8 \omega_2\left(4a_{11}^2\omega_3 + 3\omega_1\omega_2\right) - 12a_1^4 a_{11}^5\omega_3 - a_{11}^3\omega_1\omega_2\left(9a_1^4 - 2a_{11}^2 a_1^2 - 2a_{11}^4\right)\right)y}{a_{11}^5\omega_2^2 a_1} + \frac{\left(-a_1^3 a_{11}^3 + \Omega a_8\omega_2\right)z}{\omega_2 a_{11}^3} + a_5}\right. \\ &\quad \left. + \cos\left(xa_{11} - \frac{1}{12}\frac{a_{11}\left(12a_{11}^2\omega_3 + 7\omega_1\omega_2\right)y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right)a_{11}\right]. \qquad (5.14)\end{aligned}$$

By selecting suitable values of parameters including

$$\begin{aligned}\alpha &= 0.5, a_3 = 1, a_4 = a_9 = \omega_1 = \omega_3 = 1.5, a_5 = 2, a_8 = 2, \\ a_{10} &= 1, a_{11} = 2, a_{13} = 2.5, a_{14} = 1, a_{15} = 3.2, \omega_2 = 1.2,\end{aligned}$$

the graphical representation of cross-kink solution is offered in Figure 7 such as 3D chart and density chart.

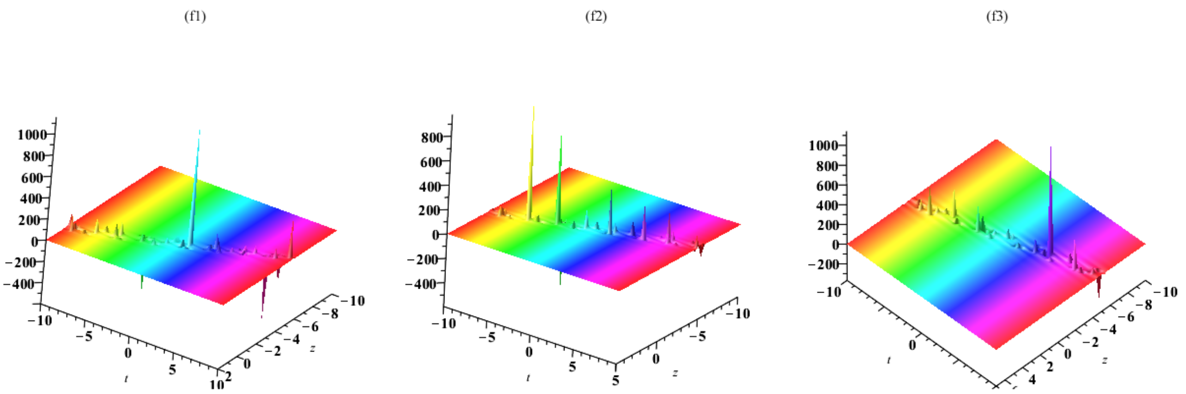

Figure 7. The cross-kink wave solution (5.14) at (f1, f2) $x = -2, y = -2$, (f3, f4) $x = 0, y = 0$, and (f5, f6) $x = 2, y = 2$.

Case VII:

$$\begin{aligned}\mathbf{f} &= e^{\frac{\Omega a_9 t}{a_{11}^3} + a_1 x - \frac{1}{6}\frac{y\left(-6a_1^4 a_{11}^5\omega_3 - 3a_1^4 a_{11}^3\omega_1\omega_2 + 2a_{11}^7\omega_1\omega_2 + 6\Omega a_1 a_8 a_{11}^2\omega_2\omega_3 + 3\Omega a_1 a_8\omega_1\omega_2^2\right)}{a_{11}^5\omega_2^2 a_1} + \frac{\left(-a_1^3 a_{11}^3 + \Omega a_8\omega_2\right)z}{a_{11}^3\omega_2} + a_5} + \\ &\quad \cosh\left(a_9 t + ya_7 + a_8 z + a_{10}\right) + \cos\left(xa_{11} - \frac{1}{2}\frac{\left(2a_{11}^2\omega_3 + \omega_1\omega_2\right)a_{11}y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right), \qquad (5.15)\end{aligned}$$

$$\Omega = \sqrt{-a_1^6 - 2a_1^4 a_{11}^2 + a_{11}^6}.$$

Inserting (5.15) into (5.1) and (5.2), the cross-kink solution of Eq (1.8) will be gained as the following:

$$\begin{aligned}\Psi_5 &= v_0 + \frac{2}{\mathbf{f}}\left[a_1 e^{\frac{\Omega a_9 t}{a_{11}^3} + a_1 x - \frac{1}{6}\frac{y\left(-6a_1^4 a_{11}^5\omega_3 - 3a_1^4 a_{11}^3\omega_1\omega_2 + 2a_{11}^7\omega_1\omega_2 + 6\Omega a_1 a_8 a_{11}^2\omega_2\omega_3 + 3\Omega a_1 a_8\omega_1\omega_2^2\right)}{a_{11}^5\omega_2^2 a_1} + \frac{\left(-a_1^3 a_{11}^3 + \Omega a_8\omega_2\right)z}{a_{11}^3\omega_2} + a_5}\right. \\ &\quad \left. - \sin\left(xa_{11} - \frac{1}{2}\frac{\left(2a_{11}^2\omega_3 + \omega_1\omega_2\right)a_{11}y}{\omega_2^2} + \frac{a_{11}^3 z}{\omega_2} + a_{15}\right)a_{11}\right]. \qquad (5.16)\end{aligned}$$

By selecting suitable values of parameters including

$$\alpha = 0.5, a_1 = 1, a_5 = 1.5, a_8 = 2, a_9 = 0.5, a_{10} = 1.5, a_{15} = 3.2, \omega_1 = 1.5, \omega_2 = 1.2, \omega_3 = 1.5,$$

the graphical representation of cross-kink solution is offered in Figure 8 such as 3D chart and density chart.

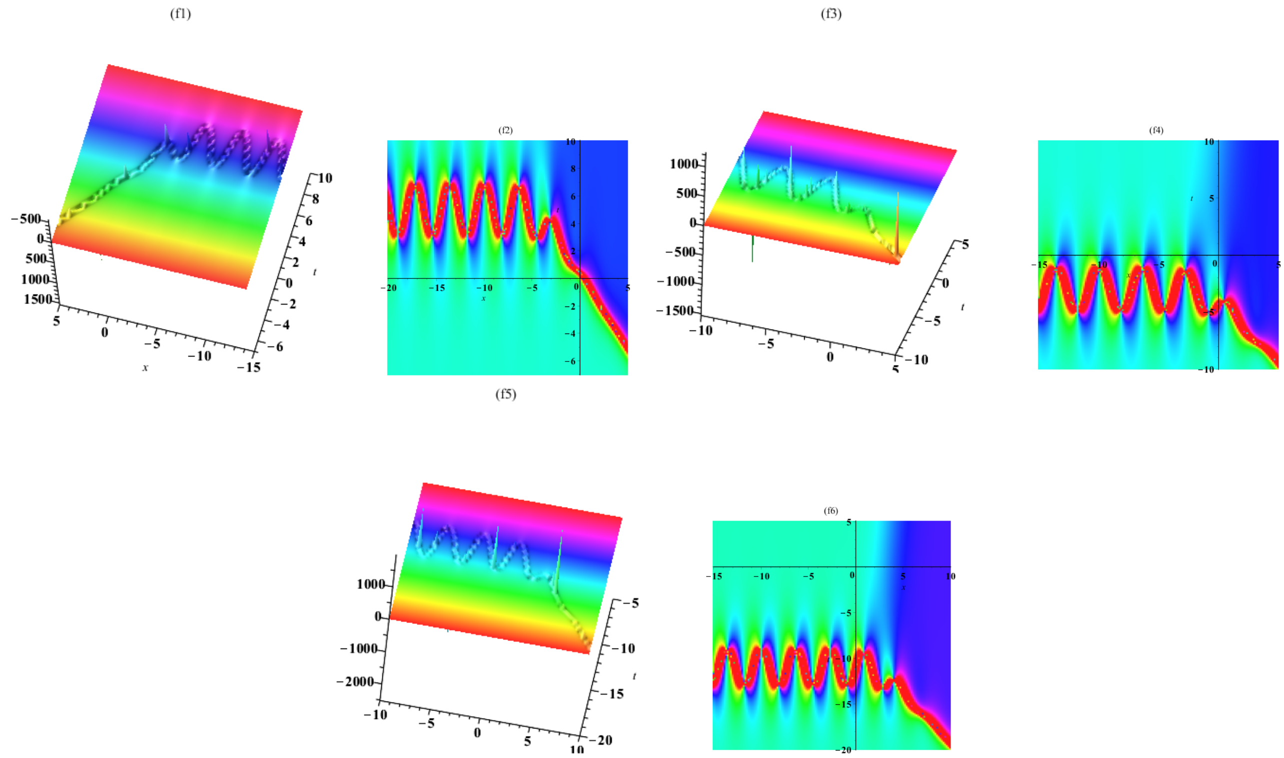

Figure 8. The lump-periodic solution (5.16) at (f1, f2) $z = -2$, (f4, f5) $z = 0$, and (f7, f8) $z = 2$.

6. Novel solitary wave solutions

Three function containing exponential, hyperbolic, and triangular periodic waves for Eq (1.8) can be assumed as the following:

$$\mathbf{f} = \exp(\tau_1) + a_{16}\exp(-\tau_1) + \tanh(\tau_2) + \tan(\tau_3) + a_{17}, \quad \tau_1 = \sum_{i=1}^{4} a_i x_i + a_5, \quad \tau_2 = \sum_{i=6}^{9} a_i x_{i-5} + a_{10}, \tag{6.1}$$

$$\tau_3 = \sum_{i=11}^{14} a_i x_{i-10} + a_{15}, \quad (x_1, x_2, x_3, x_4) = (\mathbf{x}, t) = (x, y, z, t), \quad \Psi(\mathbf{x}, t) = v_0 + 2\ln(\mathbf{f})_x, \tag{6.2}$$

in which $a_i, i = 1, ..., 17$ are unfound values. Substituting (6.2) into Eq (1.8) the below consequences will be gained:

Case I:

$$\mathbf{f} = e^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 + a_4 \lambda_2\right) z}{\omega_2} + a_5} + \tanh\left(y a_7 + a_{10}\right) + \tan\left(y a_{12} + a_{15}\right) + a_{17}. \tag{6.3}$$

Substituting (6.3) into (6.1) and (6.2), we can capture a solitary wave solution of Eq (1.8) as the following:

$$\Psi_1 = v_0 + \frac{2\,a_1 \mathrm{e}^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 + a_4 \lambda_2\right) z}{\omega_2} + a_5}}{\mathbf{f}}. \tag{6.4}$$

Case II:

$$\mathbf{f} = \mathrm{e}^{a_1 x + a_2 y - \frac{a_1{}^3 z}{\omega_2} + a_5} + \tanh\left(y a_7 + a_{10}\right) + \tan\left(y a_{12} + a_{15}\right) + a_{17}. \tag{6.5}$$

Inserting (6.5) into (6.1) and (6.2), we can capture a solitary wave solution of Eq (1.8) as below:

$$\Psi_2 = v_0 + \frac{2\,a_1 \mathrm{e}^{a_1 x + a_2 y - \frac{a_1{}^3 z}{\omega_2} + a_5}}{\mathbf{f}}. \tag{6.6}$$

Case III:

$$\mathbf{f} = \mathrm{e}^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 \omega_3 + a_4 \lambda_3 \omega_2\right) z}{\omega_2 \omega_3} + a_5} + \tanh\left(t a_9 + y a_7 - \frac{z a_9 \lambda_3}{\omega_3} + a_{10}\right) + \tan\left(y a_{12} + a_{15}\right) + a_{17}. \tag{6.7}$$

Putting (6.7) into (6.1) and (6.2), the solitary wave solution of Eq (1.8) can be indicated as below:

$$\Psi_3 = v_0 + \frac{2\,a_1 \mathrm{e}^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1{}^3 \omega_3 + a_4 \lambda_3 \omega_2\right) z}{\omega_2 \omega_3} + a_5}}{\mathbf{f}}. \tag{6.8}$$

Case IV:

$$\mathbf{f} = \mathrm{e}^{a_4 t + a_1 x + a_2 y + z a_3 + a_5} - \tanh\left(\frac{y a_8 \omega_3}{\omega_2} - z a_8 - a_{10}\right) - \tan\left(\frac{y a_{13} \omega_3}{\omega_2} - a_{13} z - a_{15}\right) + a_{17}. \tag{6.9}$$

Putting (6.9) into (6.1) and (6.2), the solitary wave of Eq (1.8) can be stated as the following:

$$\Psi_4 = v_0 + \frac{2\,a_1 \mathrm{e}^{a_4 t + a_1 x + a_2 y + z a_3 + a_5}}{\mathbf{f}}. \tag{6.10}$$

By selecting the suitable values of parameters including

$$\begin{aligned} a_1 &= 1, a_2 = 1.5, a_3 = 2, a_4 = 2, a_5 = 1.5, a_8 = 2, a_9 = 1.5, a_{10} = 1.5, \\ a_{13} &= 2, a_{13} = 1.1, a_{15} = 3.2, a_{17} = 2, \omega_2 = -1.5, \end{aligned}$$

$$\omega_3 = 1.2, \lambda_2 = 1, \lambda_3 = 1.5, x = -2, t = 2,$$

with the following components

$$\begin{aligned} \alpha &= \frac{a_1{}^3 \omega_3 - a_1 \omega_1 \omega_2 - a_2 \omega_2{}^2 - a_3 \omega_2 \omega_3 + a_4 \lambda_2 \omega_3 - a_4 \lambda_3 \omega_2}{\omega_2 a_1{}^3}, \\ \lambda_1 &= -\frac{a_1{}^3 a_2 \omega_2 + a_1{}^3 a_3 \omega_3 + a_2 a_4 \lambda_2 \omega_2 + a_3 a_4 \lambda_2 \omega_3}{a_1 a_4 \omega_2}, \end{aligned}$$

the graphical representation of rational solitary solution is offered in Figure 9 such as 3D chart and density chart.

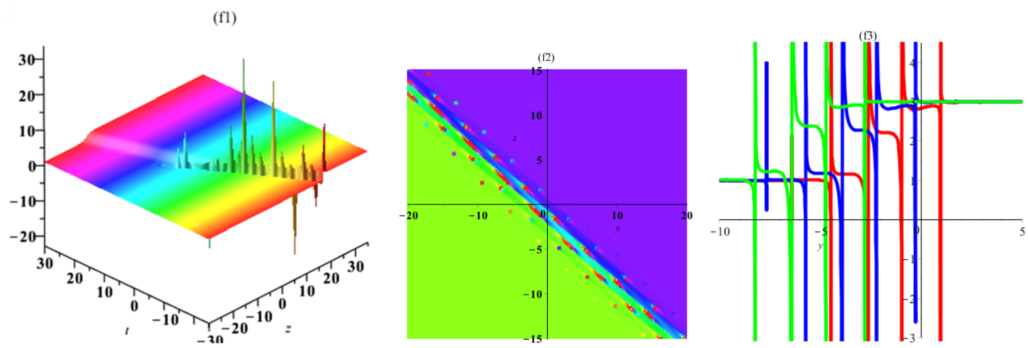

Figure 9. The rational solitary wave solution (6.10) at (f1) $z = 0$, (f2) $z = 1$, and (f3) $z = 3$.

Case V:

$$f = e^{a_1 x - \frac{a_3\omega_3 y}{\omega_2} + z a_3 + a_5} - \tanh\left(\frac{y a_8 \omega_3}{\omega_2} - z a_8 - a_{10}\right) - \tan\left(\frac{y a_{13}\omega_3}{\omega_2} - a_{13} z - a_{15}\right) + a_{17}. \tag{6.11}$$

Incorporating (6.11) into (6.1) and (6.2), the solitary solution of Eq (1.8) will be gained as below:

$$\Psi_5 = v_0 + \frac{2\, a_1 e^{a_1 x - \frac{a_3\omega_3 y}{\omega_2} + z a_3 + a_5}}{\mathbf{f}}, \quad \alpha = \frac{{a_1}^2\omega_3 - \omega_1\omega_2}{{a_1}^2\omega_2}. \tag{6.12}$$

Case VI:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{(a_3\omega_3 + a_4\lambda_3) y}{\omega_2} + z a_3 + a_5} + \tanh\left(t a_9 - \frac{y\,(a_8\omega_3 + a_9\lambda_3)}{\omega_2} + z a_8 + a_{10}\right) - \tan\left(\frac{y a_{13}\omega_3}{\omega_2} - a_{13} z - a_{15}\right) + a_{17}. \tag{6.13}$$

Appending (6.13) into (6.1) and (6.2), the solitary solution of Eq (1.8) can be written as the following:

$$\Psi_6 = v_0 + \frac{2\, a_1 e^{a_4 t + a_1 x - \frac{(a_3\omega_3 + a_4\lambda_3) y}{\omega_2} + z a_3 + a_5}}{\mathbf{f}}, \quad \alpha = \frac{{a_1}^2\omega_3 - \omega_1\omega_2}{{a_1}^2\omega_2}. \tag{6.14}$$

Case VII:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{a_3\omega_3 y}{\omega_2} + z a_3 + a_5} + \tanh\left(t a_9 - \frac{y a_8\omega_3}{\omega_2} + z a_8 + a_{10}\right) - \tan\left(\frac{y a_{13}\omega_3}{\omega_2} - a_{13} z - a_{15}\right) + a_{17}. \tag{6.15}$$

Appending (6.15) into (6.1) and (6.2), the solitary solution of Eq (1.8) will be obtained as below:

$$\Psi_7 = v_0 + \frac{2\, a_1 e^{a_4 t + a_1 x - \frac{a_3\omega_3 y}{\omega_2} + z a_3 + a_5}}{\mathbf{f}}, \quad \alpha = \frac{{a_1}^2\omega_3 - \omega_1\omega_2}{{a_1}^2\omega_2}, \quad \lambda_2 = \frac{\lambda_3\omega_2}{\omega_3}. \tag{6.16}$$

By choosing the specific amounts of parameters including

$$\begin{aligned} a_1 &= 1, a_2 = 1.5, a_3 = 2, a_4 = 2, a_5 = 1.5, a_8 = 2, a_9 = 1.3, a_{10} = 1.5, \\ a_{13} &= 2, a_{13} = 2, a_{15} = 3.2, a_{17} = 2, \omega_2 = -1.5, \\ \omega_3 &= 1.2, \lambda_2 = 1, \lambda_3 = 1.5, x = -2, t = 2, \end{aligned}$$

the graphical representation of rational solitary solution is offered in Figure 10 such as 3D chart and density chart.

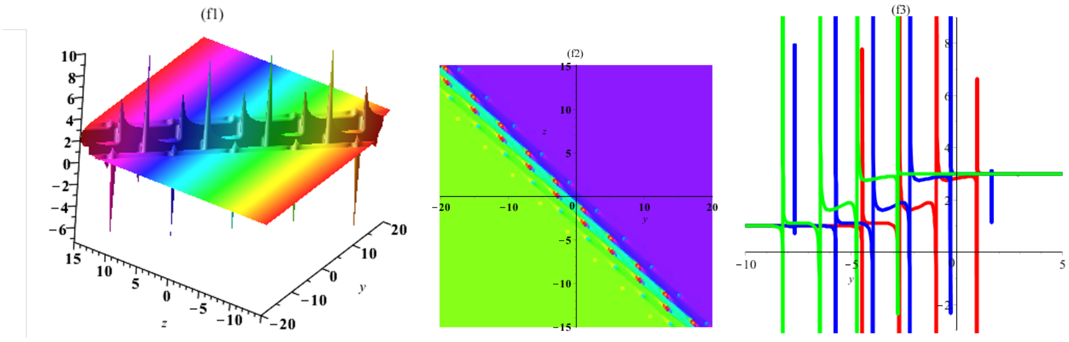

Figure 10. The rational solitary wave solution (6.16) at (f1) $z = 0$, (f2) $z = 1$, and (f3) $z = 3$.

Case VIII:

$$\mathbf{f} = e^{a_4 t + a_1 x + a_2 y - \frac{(a_1{}^3 a_{14} - a_4 a_{13}\omega_2) z}{a_{14}\omega_2} + a_5} + \tanh\left(t a_9 + y a_7 + \frac{z a_9 a_{13}}{a_{14}} + a_{10}\right) + \tan\left(t a_{14} + y a_{12} + a_{13} z + a_{15}\right) + a_{17}. \tag{6.17}$$

Incorporating (6.17) into (6.1) and (6.2), the solitary solution of Eq (1.8) will be received as below:

$$\Psi_8 = v_0 + \frac{2\, a_1 e^{a_4 t + a_1 x + a_2 y - \frac{(a_1{}^3 a_{14} - a_4 a_{13}\omega_2) z}{a_{14}\omega_2} + a_5}}{\mathbf{f}}, \quad \alpha = \frac{a_1{}^2 \omega_3 - \omega_1 \omega_2}{a_1{}^2 \omega_2}, \quad \lambda_2 = -\frac{a_{13}\omega_2}{a_{14}}, \quad \lambda_3 = -\frac{a_{13}\omega_3}{a_{14}}. \tag{6.18}$$

Case IX:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3) y}{a_{14}\omega_2} + a_3 z + a_5} \tag{6.19}$$

$$+ \tanh\left(t a_9 - \frac{y\left(a_8 a_{14}\omega_3 - a_9 a_{12}\omega_2 - a_9 a_{13}\omega_3\right)}{a_{14}\omega_2} + z a_8 + a_{10}\right) \tag{6.20}$$

$$+ \tan\left(t a_{14} + y a_{12} + z a_{13} + a_{15}\right) + a_{17}. \tag{6.21}$$

Appending (6.21) into (6.1) and (6.2), the solitary solution of Eq (1.8) can be reached as below:

$$\Psi_9 = v_0 + \frac{2 a_1 e^{a_4 t + a_1 x - \frac{(a_3 a_{14}\omega_3 - a_4 a_{12}\omega_2 - a_4 a_{13}\omega_3) y}{a_{14}\omega_2} + a_3 z + a_5}}{\mathbf{f}}, \quad \alpha = \frac{a_1{}^2 \omega_3 - \omega_1 \omega_2}{a_1{}^2 \omega_2}, \tag{6.22}$$

$$\lambda_1 = -\frac{a_1{}^2\left(a_{12}\omega_2 + a_{13}\omega_3\right)}{a_{14}\omega_2}, \quad \lambda_3 = -\frac{a_{12}\omega_2 + a_{13}\omega_3}{a_{14}}.$$

Case X:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{a_3 \omega_3 y}{\omega_2} + a_3 z + a_5} + \tanh\left(-\frac{y a_8 \omega_3}{\omega_2} + z a_8 + a_{10}\right) + \tan\left(t a_{14} - \frac{y a_{13}\omega_3}{\omega_2} + z a_{13} + a_{15}\right) + a_{17}. \tag{6.23}$$

Inserting (6.21) into (6.1) and (6.2), the solitary solution of Eq (1.8) will be gained as below form:

$$\Psi_{10} = v_0 + \frac{2\, a_1 e^{a_4 t + a_1 x - \frac{a_3 \omega_3 y}{\omega_2} + a_3 z + a_5}}{\mathbf{f}}, \quad \alpha = \frac{a_1{}^2 \omega_3 - \omega_1 \omega_2}{a_1{}^2 \omega_2}, \quad \lambda_2 = \frac{\lambda_3 \omega_2}{\omega_3}. \tag{6.24}$$

By choosing the specific amounts of parameters including

$$\begin{aligned} a_1 &= 1, a_3 = 1.5, a_4 = 2, a_5 = 1.5, a_7 = 2, a_8 = 2, a_9 = 2.1, \\ a_{10} &= 1.5, a_{13} = 2, a_{13} = 2, a_{14} = 2.5, a_{15} = 3.2, \\ a_{17} &= 2, \omega_2 = -1.5, \omega_3 = 1.2, \lambda_3 = 1.5, x = -2, t = 2, \end{aligned}$$

the graphical representation of rational solitary solution is offered in Figure 11 such as 3D chart and density chart.

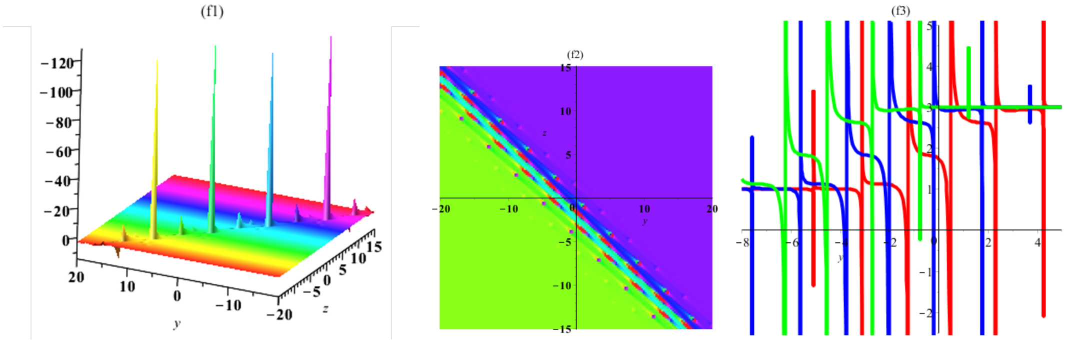

Figure 11. The rational solitary wave solution (6.24) at (f1) $z = -2$, (f2) $z = 0$, and (f3) $z = 2$.

Case XI:

$$\mathbf{f} = e^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1^3 a_{14} - a_4 a_{13} \omega_2\right) z}{a_{14} \omega_2} + a_5} + \tanh\left(\frac{t a_{14} a_8}{a_{13}} + y a_7 + z a_8 + a_{10}\right) + \tan\left(t a_{14} + z a_{13} + a_{15}\right) + a_{17}. \quad (6.25)$$

Inserting (6.25) into (6.1) and (6.2), the solitary solution of Eq (1.8) can be stated as below case:

$$\Psi_{11} = v_0 + \frac{2 a_1 e^{a_4 t + a_1 x + a_2 y - \frac{\left(a_1^3 a_{14} - a_4 a_{13} \omega_2\right) z}{a_{14} \omega_2} + a_5}}{\mathbf{f}}, \ \alpha = -\frac{a_1^3 a_2 + a_1 a_3 \omega_1 + a_2 a_3 \omega_2 + a_3^2 \omega_3}{a_1^3 a_3}, \quad (6.26)$$

$$\begin{aligned} \lambda_1 &= \frac{a_1^3 a_2 a_{12} a_{13} - a_1^3 a_3 a_{12}^2 + a_2 a_3 a_{12} a_{13} \omega_2 + a_2 a_3 a_{13}^2 \omega_3 + a_2 a_3 a_{13} a_{14} \lambda_3}{a_3 a_{12} a_1 a_{14}} \\ &\quad + \frac{-a_3^2 a_{12}^2 \omega_2 - a_3^2 a_{12} a_{13} \omega_3 - a_3^2 a_{12} a_{14} \lambda_3}{a_3 a_{12} a_1 a_{14}}, \end{aligned}$$

$$\lambda_2 = -\frac{a_{13}\left(a_{12} \omega_2 + a_{13} \omega_3 + a_{14} \lambda_3\right)}{a_{12} a_{14}}.$$

Case XII:

$$\mathbf{f} = e^{a_4 t + a_1 x - \frac{a_3 \omega_3 y}{\omega_2} + a_3 z + a_5} + \tanh\left(t a_9 + y a_7 - \frac{z a_7 \omega_2}{\omega_3} + a_{10}\right) + \tan\left(t a_{14} - \frac{y a_{13} \omega_3}{\omega_2} + z a_{13} + a_{15}\right) + a_{17}. \quad (6.27)$$

Inserting (6.27) into (6.1) and (6.2), the solitary solution of Eq (1.8) will be gained as below form:

$$\Psi_{12} = v_0 + \frac{2 a_1 e^{a_4 t + a_1 x - \frac{a_3 \omega_3 y}{\omega_2} + a_3 z + a_5}}{\mathbf{f}}, \ \alpha = -\frac{a_1^3 a_2 + a_1 a_3 \omega_1 + a_2 a_3 \omega_2 + a_3^2 \omega_3}{a_1^3 a_3}, \quad (6.28)$$

$$\lambda_1 = \frac{{a_1}^3 a_2 a_{12} a_{13} - {a_1}^3 a_3 {a_{12}}^2 + a_2 a_3 a_{12} a_{13} \omega_2 + a_2 a_3 {a_{13}}^2 \omega_3 + a_2 a_3 a_{13} a_{14} \lambda_3}{a_3 a_{12} a_1 a_{14}} + \frac{-{a_3}^2 {a_{12}}^2 \omega_2 - {a_3}^2 a_{12} a_{13} \omega_3 - {a_3}^2 a_{12} a_{14} \lambda_3}{a_3 a_{12} a_1 a_{14}},$$

$$\lambda_2 = -\frac{a_{13}\left(a_{12}\omega_2 + a_{13}\omega_3 + a_{14}\lambda_3\right)}{a_{12} a_{14}}.$$

7. Stability analysis of Eq (1.8)

In the current section, we will analyze the continuous modulational instability of the nonlinear generalized KP equation. In addition, the feasibility of the localized waves in the present system is certified by linear stability analysis. First, we search the perturbed solution for the giving Eq (1.8) of the form

$$\Psi(x,y,z,t) = \zeta + \delta\,\Theta, \tag{7.1}$$

where $\Theta = \Theta(x,y,z,t)$ and ζ is a steady state solution. Inserting (7.1) into Eq (1.8), become

$$\delta\,\frac{\partial^4}{\partial x^3 \partial y}\Theta + \alpha\,\delta\,\frac{\partial^4}{\partial x^3 \partial z}\Theta + 3\,\delta^2\left(\frac{\partial^2}{\partial x^2}\Theta\right)\frac{\partial}{\partial y}\Theta + 3\,\delta^2\left(\frac{\partial}{\partial x}\Theta\right)\frac{\partial^2}{\partial x \partial y}\Theta + 3\,\alpha\,\delta^2\left(\frac{\partial^2}{\partial x^2}\Theta\right)\frac{\partial}{\partial z}\Theta + \tag{7.2}$$

$$3\,\alpha\,\delta^2\left(\frac{\partial}{\partial x}\Theta\right)\frac{\partial^2}{\partial x \partial z}\Theta + \lambda_1\delta\,\frac{\partial^2}{\partial t \partial x}\Theta + \lambda_2\delta\,\frac{\partial^2}{\partial t \partial y}\Theta + \lambda_3\delta\,\frac{\partial^2}{\partial t \partial z}\Theta + \omega_1\delta\,\frac{\partial^2}{\partial x \partial z}\Theta + \omega_2\delta\,\frac{\partial^2}{\partial y \partial z}\Theta + \omega_3\delta\,\frac{\partial^2}{\partial z^2}\Theta = 0,$$

by linerization Eq (7.2), one gets

$$\delta\frac{\partial^4}{\partial x^3 \partial y}\Theta + \alpha\delta\frac{\partial^4}{\partial x^3 \partial z}\Theta + \lambda_1\delta\frac{\partial^2}{\partial t \partial x}\Theta + \lambda_2\delta\frac{\partial^2}{\partial t \partial y}\Theta + \lambda_3\delta\frac{\partial^2}{\partial t \partial z}\Theta + \omega_1\delta\frac{\partial^2}{\partial x \partial z}\Theta + \omega_2\delta\frac{\partial^2}{\partial y \partial z}\Theta + \omega_3\delta\frac{\partial^2}{\partial z^2}\Theta = 0. \tag{7.3}$$

Theorem 7.1. *Assume that the solution of Eq (7.3) has the following case as*

$$\Theta(x,y,z,t) = \rho_1\, e^{i(Mx+Ny+Pz+Bt)}, \tag{7.4}$$

in which M, N, P are the normalized wave numbers, by plugging (7.4) into Eq (7.3), separation the coefficients of $e^{i(Mx+Ny+Pz+Wt)}$ one gets

$$B(M,N,P) = \frac{M^3 P\alpha + M^3 N - MP\omega_1 - NP\omega_2 - P^2\omega_3}{M\lambda_1 + N\lambda_2 + P\lambda_3}. \tag{7.5}$$

Proof. By putting (7.4) into (7.3), becomes

$$\delta\,\frac{\partial^4}{\partial x^3 \partial y}\overline{\Theta} + \alpha\,\delta\,\frac{\partial^4}{\partial x^3 \partial z}\overline{\Theta} + \lambda_1\delta\,\frac{\partial^2}{\partial t \partial x}\overline{\Theta} + \lambda_2\delta\,\frac{\partial^2}{\partial t \partial y}\overline{\Phi} + \lambda_3\delta\,\frac{\partial^2}{\partial t \partial z}\overline{\Theta} + \omega_1\delta\,\frac{\partial^2}{\partial x \partial z}\overline{\Theta} + \omega_2\delta\,\frac{\partial^2}{\partial y \partial z}\overline{\Theta} + \omega_3\delta\,\frac{\partial^2}{\partial z^2}\overline{\Theta}$$
$$= \mathrm{e}^{i(Mx+Ny+Pz+Bt)}\delta\,\rho_1\left(M^3 P\alpha + M^3 N - MP\omega_1 - MB\lambda_1 - NP\omega_2 - NB\lambda_2 - P^2\omega_3 - PB\lambda_3\right), \tag{7.6}$$

in which $\overline{\Theta} = \Theta(x,y,z,t)$. By solving and simplifying we can determine the function of $B(M,N,P)$ as the following

$$B(M,N,P) = \frac{M^3 P\alpha + M^3 N - MP\omega_1 - NP\omega_2 - P^2\omega_3}{M\lambda_1 + N\lambda_2 + P\lambda_3}. \tag{7.7}$$

Accordingly, the considered solution was obtained. Thereupon the proof is perfect.

It is easy to notice that modulation stability occurs when $M\lambda_1 + N\lambda_2 + P\lambda_3 \neq 0$. So, the modulation stability achieve spectrum $\Upsilon(B)$ will be as below form:

$$\Upsilon(B) = \frac{M^3P\alpha + M^3N - MP\omega_1 - NP\omega_2 - P^2\omega_3}{M\lambda_1 + N\lambda_2 + P\lambda_3}. \tag{7.8}$$

In Figures 12–14 can be discovered that while the sign of $B(M,N,P)$ is positive for all quantity of M. Furthermore, in Figure 12 can be observed if the $B(M,N,P)$ is positive or negative for some quantities of M. Finally, in Figure 13 and 14 can be perceived that while the sign of $B(M,N,P)$ is positive for all quantity of M.

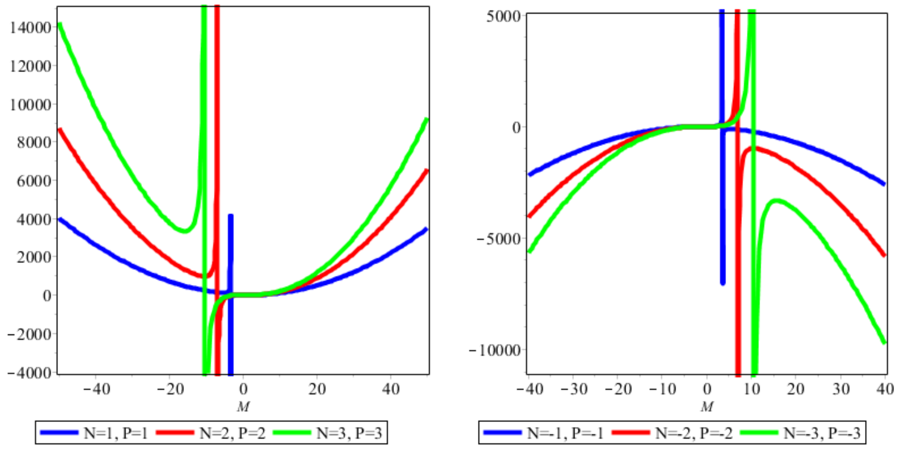

Figure 12. The graphic representation $\Upsilon(B)$ for the wave number M via considering the diverse quantities $\lambda_1 = 1, \lambda_2 = 1.5, \lambda_3 = 2, \omega_1 = 2.2, \omega_2 = -2, \omega_3 = 3, \alpha = 0.5$.

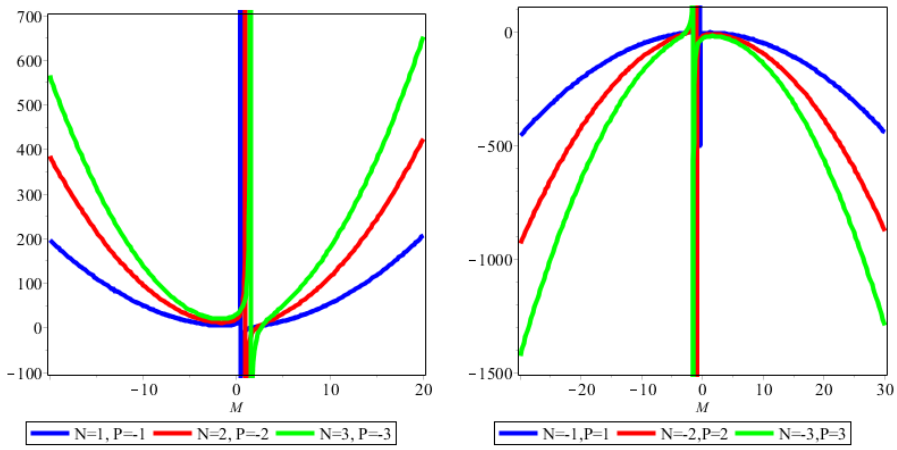

Figure 13. The graphic representation $\Upsilon(B)$ for the wave number M via considering the diverse quantities $\lambda_1 = 1, \lambda_2 = 1.5, \lambda_3 = 2, \omega_1 = 2.2, \omega_2 = -2, \omega_3 = 3, \alpha = 0.5$.

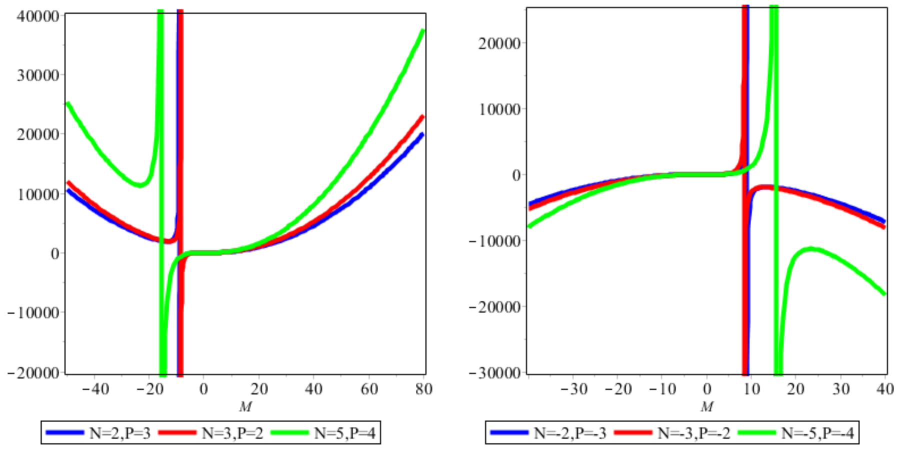

Figure 14. The graphic representation $\Upsilon(B)$ for the wave number M via considering the diverse quantities $\lambda_1 = 1, \lambda_2 = 1.5, \lambda_3 = 2, \omega_1 = 2.2, \omega_2 = -2, \omega_3 = 3, \alpha = 0.5$.

8. Usage of SIVP for Eq (1.8)

Via employing the wave alteration $\xi = k(x + ay + bz - ct)$ in Eq (1.8) once can gain to the below ODE as

$$k^2(\alpha b + a)\Psi'''' + 6k(\alpha b + a)\Psi'\Psi'' + (ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1)\Psi'' = 0, \tag{8.1}$$

in which $\Psi = \Psi(\xi)$ and $\Psi' = \frac{d\Psi}{d\xi}$. According to the SIVP [16, 17] and by multiplying Eq (8.1) with Ψ' and integrating once respect to ξ, the following stationary integral will be arises

$$J = \int_0^\infty \left[A_1\left(\Psi'\Psi''' - \frac{1}{2}(\Psi''^2\right) + \frac{1}{3}A_2(\Psi'^3 + \frac{1}{2}A_3(\Psi'^2\right] d\xi, \tag{8.2}$$

in which

$$A_1 = k^2(\alpha b + a), \quad A_2 = 6k(\alpha b + a), \quad A_3 = ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1.$$

8.1. Case I

We utilize the solitary wave function as the following

$$u(\xi) = \delta\, sech(\mu\xi). \tag{8.3}$$

Hence, the stationary integral transforms to

$$\begin{aligned} J \;\; = \;\; & \frac{1}{30}k^2\delta^2\mu\; - 21\,\alpha\, bk^2\mu^2 - 21\,ak^2\mu^2 - 8\,\alpha\, b\delta\, k\mu - 8\,a\delta\, k\mu + 5\,ab\omega_2 \\ & -5\,ac\lambda_2 + 5\,b^2\omega_3 - 5\,bc\lambda_{\,3} + 5\,b\omega_1 - 5\,c\lambda_1). \end{aligned} \tag{8.4}$$

Based on the SIVP and using derivative J respect to A and B, one get

$$\frac{\partial J}{\partial A} \;\; = \;\; \frac{1}{15}k^2\delta\,\mu\; - 21\,\alpha\, bk^2\mu^2 - 21\,ak^2\mu^2 - 8\,\alpha\, b\delta\, k\mu - 8\,a\delta\, k\mu + 5\,ab\omega_2 - 5\,ac\lambda_2$$

$$+5\,b^2\omega_3 - 5\,bc\lambda_3 + 5\,b\omega_1 - 5\,c\lambda_1) + \frac{1}{30}k^2\delta^2\mu\,(-8\,\alpha\,bk\mu - 8\,ak\mu) = 0, \tag{8.5}$$

and

$$\begin{aligned}\frac{\partial J}{\partial B} &= \frac{1}{30}k^2\delta^2 - 21\,\alpha\,bk^2\mu^2 - 21\,a\,k^2\mu^2 - 8\,\alpha\,b\delta\,k\mu - 8\,a\delta\,k\mu + 5\,ab\omega_2 - 5\,ac\lambda_2 + 5\,b^2\omega_3\\ &-5\,bc\lambda_3 + 5\,b\omega_1 - 5\,c\lambda_1) + \frac{1}{30}k^2\delta^2\mu\left(-42\,\alpha\,bk^2\mu - 42\,ak^2\mu - 8\,\alpha\,b\delta\,k - 8\,a\delta\,k\right) = 0. \end{aligned}\tag{8.6}$$

Solve the Eqs (8.5) and (8.6), become

$$\delta = \pm\frac{21}{2}\frac{ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1}{\sqrt{-(21\,\alpha\,b + 21\,a)(ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1)}}, \tag{8.7}$$

$$\mu = \pm\frac{1}{21}\frac{\sqrt{-(21\,\alpha\,b + 21\,a)(ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1)}}{(\alpha\,b + a)\,k}.$$

The condition can be obtained as below

$$(\alpha\,b + a)\left(ab\omega_2 - ac\lambda_2 + b^2\omega_3 - bc\lambda_3 + b\omega_1 - c\lambda_1\right) < 0. \tag{8.8}$$

Finally, the solitary solution received by utilizing of SIVP can be reached as

$$\Psi(x, y, z, t) = \delta\ \mathrm{sech}\left[k\mu(x + ay + bz - ct)\right]. \tag{8.9}$$

8.2. Case II

We utilize the bright wave function as the following

$$u(\xi) = \delta\ \mathrm{sech}^2(\mu\xi). \tag{8.10}$$

Hence, the stationary integral transforms to

$$\begin{aligned} J &= -\frac{2\,\delta^2\mu(120\,\alpha\,bk^2\mu^2 + 120\,ak^2\mu^2 + 35\,\alpha\,b\delta\,k\mu + 35\,a\delta\,k\mu - 14\,ab\omega_2}{105}\\ &+\frac{14\,ac\lambda_2 - 14\,b^2\omega_3 + 14\,bc\lambda_3 - 14\,b\omega_1 + 14\,c\lambda_1)}{105}. \end{aligned}\tag{8.11}$$

Based on the SIVP and using derivative J respect to A and B, one get

$$\begin{aligned}\frac{\partial J}{\partial A} &= -\frac{4\,\delta\,\mu(120\,\alpha\,bk^2\mu^2 + 120\,ak^2\mu^2 + 35\,\alpha\,b\delta\,k\mu + 35\,a\delta\,k\mu - 14\,ab\omega_2 + 14\,ac\lambda_2}{105}\\ &+\frac{-14\,b^2\omega_3 + 14\,bc\lambda_3 - 14\,b\omega_1 + 14\,c\lambda_1) - 2\,\delta^2\mu\,(35\,\alpha\,bk\mu + 35\,ak\mu)}{105} = 0. \end{aligned}\tag{8.12}$$

and

$$\frac{\partial J}{\partial B} = -\frac{2\,\delta^2(120\,\alpha\,bk^2\mu^2 + 120\,a\,k^2\mu^2 + 35\,\alpha\,b\delta\,k\mu + 35\,a\delta\,k\mu - 14\,ab\omega_2 + 14\,ac\lambda_2 - 14\,b^2\omega_3}{105}$$

$$+\frac{+14\,bc\lambda_3-14\,b\omega_1+14\,c\lambda_1)-2\,\delta^2\mu\left(240\,\alpha\,bk^2\mu+240\,ak^2\mu+35\,\alpha\,b\delta\,k+35\,a\delta\,k\right)}{105}=0. \quad (8.13)$$

Solve the Eqs (8.12) and (8.13), become

$$\delta=\pm\frac{48}{5}\frac{ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1}{\sqrt{-(21\,\alpha\,b+21\,a)(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1)}}, \quad (8.14)$$

$$\mu=\pm\frac{1}{30}\frac{\sqrt{-(21\,\alpha\,b+21\,a)(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1)}}{(\alpha\,b+a)\,k}.$$

The condition of definition of the above relations can be expressed as

$$(\alpha\,b+a)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)<0. \quad (8.15)$$

Finally, the solitary solution gained by utilizing of SIVP will be as

$$\Psi(x,y,z,t)=\delta\ \mathrm{sech}^2\left[k\mu(x+ay+bz-ct)\right]. \quad (8.16)$$

8.3. Case III

Assume the dark soliton wave solution be as the below case as

$$u(\xi)=\delta\ \tanh^2(\mu\xi). \quad (8.17)$$

Hence, the stationary integral transforms to

$$J=\frac{2\,\delta^2\mu(-120\,\alpha\,bk^2\mu^2-120\,ak^2\mu^2+35\,\alpha\,b\delta\,k\mu+35\,a\delta\,k\mu+14\,ab\omega_2}{105}+\frac{-14\,ac\lambda_2+14\,b^2\omega_3-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)}{105}. \quad (8.18)$$

Based on the SIVP and using derivative J respect to A and B, one get

$$\frac{\partial J}{\partial\delta}=\frac{4\,\delta\,\mu(-120\,\alpha\,bk^2\mu^2-120\,ak^2\mu^2+35\,\alpha\,b\delta\,k\mu+35\,a\delta\,k\mu+14\,ab\omega_2-14\,ac\lambda_2}{105}+\frac{+14\,b^2\omega_3-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)+2\,\delta^2\mu\,(35\,\alpha\,bk\mu+35\,ak\mu)}{105}=0, \quad (8.19)$$

and

$$\frac{\partial J}{\partial\mu}=\frac{2\,\delta^2(-120\,\alpha\,bk^2\mu^2-120\,a\,k^2\mu^2+35\,\alpha\,b\delta\,k\mu+35\,a\delta\,k\mu+14\,ab\omega_2-14\,ac\lambda_2+}{105}+\frac{14\,b^2\omega_3-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)+2\,\delta^2\mu\left(-240\,\alpha\,bk^2\mu-240\,ak^2\mu+35\,\alpha\,b\delta\,k+35\,a\delta\,k\right)}{105}=0. \quad (8.20)$$

Solve the Eqs (8.19) and (8.20), one get

$$\delta=\mp\frac{48}{5}\frac{ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1}{\sqrt{-(21\,\alpha\,b+21\,a)(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1)}}, \quad (8.21)$$

$$\mu = \pm\frac{1}{30}\frac{\sqrt{-\left(21\,\alpha\,b+21\,a\right)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)}}{\left(\alpha\,b+a\right)k}.$$

The condition of definition of the above relations can be presented the following form

$$(\alpha\,b+a)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)<0. \tag{8.22}$$

Finally, the dark solution acquired by utilizing of SIVP will be as

$$\Psi(x,y,z,t)=\delta\tanh^2\left[k\mu(x+ay+bz-ct)\right]. \tag{8.23}$$

8.4. Case IV

Let the singular soliton wave solution be as the below case as

$$u(\xi)=\delta\ csch^2(\mu\xi). \tag{8.24}$$

Then, the stationary integral transforms to

$$\begin{aligned} J &= -\frac{2\,\delta^2\mu(-120\,\alpha\,bk^2\mu^2-120\,ak^2\mu^2-70\,\alpha\,b\delta\,k\mu-70\,a\delta\,k\mu+14\,ab\omega_2}{105} \\ &\quad+\frac{-14\,ac\lambda_2+14\,b^2\omega_3-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)}{105}. \end{aligned} \tag{8.25}$$

Based on the SIVP and using derivative J respect to A and B, become

$$\begin{aligned} \frac{\partial J}{\partial\delta} &= -\frac{4\,\delta\,\mu(-120\,\alpha\,bk^2\mu^2-120\,ak^2\mu^2-70\,\alpha\,b\delta\,k\mu-70\,a\delta\,k\mu+14\,ab\omega_2-14\,ac\lambda_2}{105} \\ &\quad+\frac{+14\,b^2\omega_3-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)-2\,\delta^2\mu\,(-70\,\alpha\,bk\mu-70\,ak\mu)}{105}=0 \end{aligned} \tag{8.26}$$

and

$$\begin{aligned} \frac{\partial J}{\partial\mu} &= -\frac{2\,\delta^2(-120\,\alpha\,bk^2\mu^2-120\,ak^2\mu^2-70\,\alpha\,b\delta\,k\mu-70\,a\delta\,k\mu+14\,ab\omega_2-14\,ac\lambda_2+14\,b^2\omega_3}{105} \\ &\quad+\frac{-14\,bc\lambda_3+14\,b\omega_1-14\,c\lambda_1)-2\,\delta^2\mu\left(-240\,\alpha\,bk^2\mu-240\,ak^2\mu-70\,\alpha\,b\delta\,k-70\,a\delta\,k\right)}{105}=0. \end{aligned} \tag{8.27}$$

Solve the Eqs (8.26) and (8.27), one get

$$\delta=\pm\frac{24}{5}\frac{ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1}{\sqrt{-\left(21\,\alpha\,b+21\,a\right)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)}}, \tag{8.28}$$

$$\mu=\pm\frac{1}{30}\frac{\sqrt{-\left(21\,\alpha\,b+21\,a\right)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)}}{\left(\alpha\,b+a\right)k}.$$

The condition of definition of the above relations can be presented as the below form as

$$(\alpha\,b+a)\left(ab\omega_2-ac\lambda_2+b^2\omega_3-bc\lambda_3+b\omega_1-c\lambda_1\right)<0. \tag{8.29}$$

Finally, the singular solution acquired by utilizing of SIVP will be as

$$\Psi(x,y,z,t)=\delta\,\mathrm{csch}^2\left[k\mu(x+ay+bz-ct)\right]. \tag{8.30}$$

9. Conclusion

In this article, the MEFM employed for searching the MSSs for the gKP equation, which contains 1-wave, 2-wave, and 3-wave solutions. The periodic wave, cross-kink, and solitary wave solutions have been obtained. In continuing, the modulation instability applied to discuss the stability of earned solutions. It is quite visible that these novel schemes have plenty of family solutions containing rational exponential, hyperbolic, and periodic functions with selecting particular parameters. Also, the semi-inverse variational principle will be used for the gKP equation. Four major cases containing the solitary, bright, dark and singular wave solutions were studied from four different ansatzes.

By means of symbolic computation, these analytical solutions and corresponding rogue waves are obtained. Via various curve plots, density plot and three-dimensional plots, dynamical characteristics of these rouge waves are exhibited. Because of the strong nonlinear characteristic of Hirota bilinear method, the test function constructed by the Hirota operator, which can be regarded as the test function constructed by considered model. The results are beneficial to the study of the plasma, optics, acoustics, fluid dynamics and fluid mechanics. All computations in this paper have been employed quickly with the help of the Maple 18. Moreover, the method applied in this paper provides an effective tool to obtain exact solutions of nonlinear system and can in common use for other NLEEs.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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