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Research article

Series expansions of powers of arcsine, closed forms for special values of Bell polynomials, and series representations of generalized logsine functions

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Dedicated to Dr. Prof. Aliakbar Montazer Haghighi at Prairie View A&M University in USA

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Abstract: In the paper, the authors

1. establish general expressions of series expansions of $(\arcsin x)^\ell$ for $\ell \in \mathbb{N}$;
2. find closed-form formulas for the sequence

$$\mathrm{B}_{2n,k}\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{2n-k+2}\right),$$

   where $\mathrm{B}_{n,k}$ denotes the second kind Bell polynomials;
3. derive series representations of generalized logsine functions.

The series expansions of the powers $(\arcsin x)^\ell$ were related with series representations for generalized logsine functions by Andrei I. Davydychev, Mikhail Yu. Kalmykov, and Alexey Sheplyakov. The above sequence represented by special values of the second kind Bell polynomials appeared in the study of Grothendieck's inequality and completely correlation-preserving functions by Frank Oertel.

Keywords: general expression; closed-form formula; arcsine; series expansion; power; special value; second kind Bell polynomials; series representation; generalized logsine function
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1. Motivations and outline

In [12, Definition 11.2] and [18, p. 134, Theorem A], the second kind Bell polynomials $\mathrm{B}_{n,k}$ for $n \ge k \ge 0$ are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\ell \in \mathbb{N}_0^{n-k+1}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i},$$

where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the sum is taken over $\ell = (\ell_1, \ell_2, \dots, \ell_{n-k+1})$ with $\ell_i \in \mathbb{N}_0$ satisfying $\sum_{i=1}^{n-k+1} \ell_i = k$ and $\sum_{i=1}^{n-k+1} i\ell_i = n$. This kind of polynomials are very important in combinatorics, analysis, and the like. See the review and survey article [53] and closely related references therein.

In [36, pp. 13–15], when studying Grothendieck's inequality and completely correlation-preserving functions, Oertel obtained the interesting identity

$$\sum_{k=1}^{2n} (-1)^k \frac{(2n+k)!}{k!} \mathrm{B}^{\circ}_{2n,k}\left(0, \frac{1}{6}, 0, \frac{3}{40}, 0, \frac{5}{112}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{(2n-k+2)!}\right) = (-1)^n$$

for $n \in \mathbb{N}$, where

$$\mathrm{B}^{\circ}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \frac{k!}{n!} \mathrm{B}_{n,k}(1!x_1, 2!x_2, \dots, (n-k+1)!x_{n-k+1}). \tag{1.1}$$

In [36, p. 15], Oertel wrote that "However, already in this case we don't know a closed form expression for the numbers

$$\mathrm{B}^{\circ}_{2n,k}\left(0, \frac{1}{6}, 0, \frac{3}{40}, 0, \frac{5}{112}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{(2n-k+2)!}\right). \tag{1.2}$$

An even stronger problem appears in the complex case, since already a closed-form formula for the coefficients of the Taylor series of the inverse of the Haagerup function is still unknown".

By virtue of the relation (1.1), we see that, to find a closed-form formula for the sequence (1.2), it suffices to discover a closed-form formula for

$$\mathrm{B}_{2n,k}\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{2n-k+2}\right). \tag{1.3}$$

In this paper, one of our aims is to derive closed-form formulas for the sequence (1.3). The first main result can be stated as the following theorem.

Theorem 1.1. *For $k, n \ge 0$, $m \in \mathbb{N}$, and $x_m \in \mathbb{C}$, we have*

$$\mathrm{B}_{2n+1,k}\left(0, x_2, 0, x_4, \dots, \frac{1+(-1)^k}{2} x_{2n-k+2}\right) = 0. \tag{1.4}$$

For $k, n \in \mathbb{N}$, we have

$$\mathrm{B}_{2n,2k-1}\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, 0, \frac{[(2n-2k+1)!!]^2}{2n-2k+3}\right)$$
$$= \frac{2^{2n}}{(2k-1)!}\left[\sum_{p=1}^{k} (-4)^{p-1} \frac{\binom{2k-1}{2p-1}}{\binom{2n+2p-1}{2p-1}} \sum_{q=0}^{2p-2} T\left(n+p-1; q, 2p-2; \frac{1}{2}\right)\right.$$

$$-\sum_{p=1}^{k-1}(-1)^{p-1}\frac{\binom{2k-1}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)\Bigg]$$

and

$$\begin{aligned}
&\mathrm{B}_{2n,2k}\left(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{[(2n-2k-1)!!]^2}{2n-2k+1},0\right)\\
&=\frac{2^{2n}}{(2k)!}\left[\sum_{p=1}^{k}(-1)^{p-1}\frac{\binom{2k}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)\right.\\
&\quad\left.-\sum_{p=1}^{k}(-4)^{p-1}\frac{\binom{2k}{2p-1}}{\binom{2n+2p-1}{2p-1}}\sum_{q=0}^{2p-2}T\left(n+p-1;q,2p-2;\frac{1}{2}\right)\right],
\end{aligned}$$

where $s(n,k)$, which can be generated by

$$\langle x\rangle_n=\sum_{m=0}^{n}s(n,m)x^m, \tag{1.5}$$

denote the first kind Stirling numbers and

$$T(r;q,j;\rho)=(-1)^q\left[\sum_{m=q}^{r}(-\rho)^m s(r,m)\binom{m}{q}\right]\left[\sum_{m=j-q}^{r}(-\rho)^m s(r,m)\binom{m}{j-q}\right]. \tag{1.6}$$

In Section 2, for proving Theorem 1.1, we will establish two general expressions for power series expansions of $(\arcsin x)^{2\ell-1}$ and $(\arcsin x)^{2\ell}$ respectively.

In Section 3, with the aid of general expressions for power series expansions of the functions $(\arcsin x)^{2\ell-1}$ and $(\arcsin x)^{2\ell}$ established in Section 2, we will prove Theorem 1.1 in details.

In Section 4, basing on arguments in [20, p. 308] and [28, Section 2.4] and utilizing general expressions for power series expansions of $(\arcsin x)^{2\ell-1}$ and $(\arcsin x)^{2\ell}$ established in Section 2, we will derive series representations of generalized logsine functions which were originally introduced in [34] and have been investigating actively, deeply, and systematically by mathematicians [9, 10, 14–17, 29–31, 37, 38, 57] and physicists [3, 19, 20, 28].

Finally, in Section 5, we will list several remarks on our main results and related stuffs.

2. Power series expansions for the powers of the arcsine function

To prove Theorem 1.1, we need to establish the following general expressions of the power series expansions of $(\arcsin x)^{\ell}$ for $\ell\in\mathbb{N}$.

Theorem 2.1. *For $\ell\in\mathbb{N}$ and $|x|<1$, the functions $(\arcsin x)^{\ell}$ can be expanded into power series*

$$(\arcsin x)^{2\ell-1}=(-4)^{\ell-1}\sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\left[\sum_{q=0}^{2\ell-2}T\left(n+\ell-1;q,2\ell-2;\frac{1}{2}\right)\right]\frac{x^{2n+2\ell-1}}{\binom{2n+2\ell-1}{2\ell-1}} \tag{2.1}$$

or

$$(\arcsin x)^{2\ell} = (-1)^{\ell-1} \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} \Biggl[\sum_{q=0}^{2\ell-2} T(n+\ell-1; q, 2\ell-2; 1)\Biggr] \frac{x^{2n+2\ell}}{\binom{2n+2\ell}{2\ell}}, \tag{2.2}$$

where $s(n,k)$ denotes the first kind Stirling numbers generated in (1.5) *and $T(r;q,j;\rho)$ is defined by* (1.6)*.*

Proof. In [4, pp. 262–263, Proposition 15], [7, p. 3], [20, p. 308], and [28, pp. 49–50], it was stated that the generating expression for the series expansion of $(\arcsin x)^n$ with $n \in \mathbb{N}$ is

$$\exp(t \arcsin x) = \sum_{\ell=0}^{\infty} \frac{b_\ell(t) x^\ell}{\ell!},$$

where $b_0(t) = 1$, $b_1(t) = t$, and

$$b_{2\ell}(t) = \prod_{k=0}^{\ell-1} [t^2 + (2k)^2], \quad b_{2\ell+1}(t) = t \prod_{k=1}^{\ell} [t^2 + (2k-1)^2]$$

for $\ell \in \mathbb{N}$. This means that, when writing

$$b_\ell(t) = \sum_{k=0}^{\ell} \beta_{\ell,k} t^k, \quad \ell \ge 0,$$

where $\beta_{0,0} = 1, \beta_{2\ell,0} = 0, \beta_{2\ell,2k+1} = 0$, and $\beta_{2\ell-1,2k} = 0$ for $k \ge 0$ and $\ell \ge 1$, we have

$$\sum_{\ell=0}^{\infty} (\arcsin x)^\ell \frac{t^\ell}{\ell!} = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell!} \sum_{k=0}^{\ell} \beta_{\ell,k} t^k = \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \frac{x^\ell}{\ell!} \beta_{\ell,k} t^k = \sum_{\ell=0}^{\infty} \Biggl[\sum_{m=\ell}^{\infty} \beta_{m,\ell} \frac{x^m}{m!}\Biggr] t^\ell.$$

Equating coefficients of t^ℓ gives

$$(\arcsin x)^\ell = \ell! \sum_{m=\ell}^{\infty} \beta_{m,\ell} \frac{x^m}{m!} = \ell! \sum_{n=0}^{\infty} \beta_{n+\ell,\ell} \frac{x^{n+\ell}}{(n+\ell)!}, \quad \ell \in \mathbb{N}. \tag{2.3}$$

It is not difficult to see that

$$b_{2\ell}(t) = 4^{\ell-1} t^2 \biggl(1 - \frac{it}{2}\biggr)_{\ell-1} \biggl(1 + \frac{it}{2}\biggr)_{\ell-1} \quad \text{and} \quad b_{2\ell+1}(t) = 4^\ell t \biggl(\frac{1}{2} - \frac{it}{2}\biggr)_\ell \biggl(\frac{1}{2} + \frac{it}{2}\biggr)_\ell,$$

where $i = \sqrt{-1}$ is the imaginary unit and

$$(z)_n = \prod_{\ell=0}^{n-1} (z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is called the rising factorial of $z \in \mathbb{C}$, while

$$\langle z \rangle_n = \prod_{\ell=0}^{n-1} (z-\ell) = \begin{cases} z(z-1)\cdots(z-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases} \tag{2.4}$$

is called the falling factorial of $z \in \mathbb{C}$. Making use of the relation

$$(-z)_n = (-1)^n \langle z \rangle_n \quad \text{or} \quad \langle -z \rangle_n = (-1)^n (z)_n$$

in [52, p. 167], we acquire

$$b_{2\ell}(t) = 4^{\ell-1} t^2 \left\langle \frac{it}{2} - 1 \right\rangle_{\ell-1} \left\langle -\frac{it}{2} - 1 \right\rangle_{\ell-1} \quad \text{and} \quad b_{2\ell+1}(t) = 4^{\ell} t \left\langle \frac{it}{2} - \frac{1}{2} \right\rangle_{\ell} \left\langle -\frac{it}{2} - \frac{1}{2} \right\rangle_{\ell}.$$

Utilizing the relation (1.5) in [59, p. 19, (1.26)], we obtain

$$\begin{aligned}
b_{2\ell}(t) &= 4^{\ell-1} t^2 \sum_{m=0}^{\ell-1} \frac{s(\ell-1,m)}{2^m} (it-2)^m \sum_{m=0}^{\ell-1} (-1)^m \frac{s(\ell-1,m)}{2^m} (it+2)^m \\
&= 4^{\ell-1} t^2 \sum_{m=0}^{\ell-1} \frac{s(\ell-1,m)}{2^m} \sum_{k=0}^{m} \binom{m}{k} i^k t^k (-2)^{m-k} \sum_{m=0}^{\ell-1} (-1)^m \frac{s(\ell-1,m)}{2^m} \sum_{k=0}^{m} \binom{m}{k} i^k t^k 2^{m-k} \\
&= 4^{\ell-1} t^2 \sum_{m=0}^{\ell-1} (-1)^m s(\ell-1,m) \sum_{k=0}^{m} \frac{(-1)^k}{2^k} \binom{m}{k} i^k t^k \sum_{m=0}^{\ell-1} (-1)^m s(\ell-1,m) \sum_{k=0}^{m} \frac{1}{2^k} \binom{m}{k} i^k t^k \\
&= 4^{\ell-1} t^2 \sum_{k=0}^{\ell-1} \left[\sum_{m=k}^{\ell-1} (-1)^{m+k} \frac{s(\ell-1,m)}{2^k} \binom{m}{k} \right] i^k t^k \sum_{k=0}^{\ell-1} \left[\sum_{m=k}^{\ell-1} (-1)^m \frac{s(\ell-1,m)}{2^k} \binom{m}{k} \right] i^k t^k \\
&= 4^{\ell-1} t^2 \sum_{k=0}^{2(\ell-1)} \sum_{q=0}^{k} \left[\sum_{m=q}^{\ell-1} (-1)^{m+q} \frac{s(\ell-1,m)}{2^q} \binom{m}{q} \sum_{m=k-q}^{\ell-1} (-1)^m \frac{s(\ell-1,m)}{2^{k-q}} \binom{m}{k-q} \right] i^k t^k \\
&= 4^{\ell-1} t^2 \sum_{k=0}^{2(\ell-1)} \frac{1}{2^k} \sum_{q=0}^{k} \left[\sum_{m=q}^{\ell-1} (-1)^{m+q} s(\ell-1,m) \binom{m}{q} \sum_{m=k-q}^{\ell-1} (-1)^m s(\ell-1,m) \binom{m}{k-q} \right] i^k t^k \\
&= 4^{\ell-1} \sum_{k=0}^{2(\ell-1)} \frac{i^k}{2^k} \left[\sum_{q=0}^{k} \left(\sum_{m=q}^{\ell-1} (-1)^m s(\ell-1,m) \binom{m}{q} \right) \sum_{m=k-q}^{\ell-1} (-1)^m s(\ell-1,m) \binom{m}{k-q} \right] t^{k+2} \\
&= 4^{\ell-1} \sum_{k=0}^{2(\ell-1)} \frac{i^k}{2^k} \left[\sum_{q=0}^{k} T(\ell-1; q, k; 1) \right] t^{k+2}
\end{aligned}$$

and

$$\begin{aligned}
b_{2\ell+1}(t) &= 4^{\ell} t \sum_{m=0}^{\ell} \frac{s(\ell,m)}{2^m} (it-1)^m \sum_{m=0}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} (it+1)^m \\
&= 4^{\ell} t \sum_{m=0}^{\ell} \frac{s(\ell,m)}{2^m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} i^k t^k \sum_{m=0}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} \sum_{k=0}^{m} \binom{m}{k} i^k t^k \\
&= 4^{\ell} t \sum_{k=0}^{\ell} \left[\sum_{m=k}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} \binom{m}{k} \right] (-i)^k t^k \sum_{k=0}^{\ell} \left[\sum_{m=k}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} \binom{m}{k} \right] i^k t^k \\
&= 4^{\ell} \sum_{k=0}^{2\ell} i^k \left[\sum_{q=0}^{k} (-1)^q \left(\sum_{m=q}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} \binom{m}{q} \right) \sum_{m=k-q}^{\ell} (-1)^m \frac{s(\ell,m)}{2^m} \binom{m}{k-q} \right] t^{k+1}
\end{aligned}$$

$$= 4^{\ell}\sum_{k=0}^{2\ell} i^k\left[\sum_{q=0}^{k} T\left(\ell; q, k; \frac{1}{2}\right)\right]t^{k+1}.$$

This means that

$$\sum_{k=0}^{2\ell}\beta_{2\ell,k}t^k = \sum_{k=-2}^{2(\ell-1)}\beta_{2\ell,k+2}t^{k+2} = \sum_{k=0}^{2(\ell-1)}\beta_{2\ell,k+2}t^{k+2} = 4^{\ell-1}\sum_{k=0}^{2(\ell-1)}\frac{i^k}{2^k}\left[\sum_{q=0}^{k}T(\ell-1; q, k; 1)\right]t^{k+2}$$

and

$$\sum_{k=0}^{2\ell+1}\beta_{2\ell+1,k}t^k = \sum_{k=-1}^{2\ell}\beta_{2\ell+1,k+1}t^{k+1} = \sum_{k=0}^{2\ell}\beta_{2\ell+1,k+1}t^{k+1} = 4^{\ell}\sum_{k=0}^{2\ell} i^k\left[\sum_{q=0}^{k}T\left(\ell; q, k; \frac{1}{2}\right)\right]t^{k+1}.$$

Further equating coefficients of t^{k+2} and t^{k+1} respectively arrives at

$$\beta_{2\ell,k+2} = 4^{\ell-1}\frac{i^k}{2^k}\sum_{q=0}^{k}T(\ell-1; q, k; 1) \quad \text{and} \quad \beta_{2\ell+1,k+1} = 4^{\ell}i^k\sum_{q=0}^{k}T\left(\ell; q, k; \frac{1}{2}\right)$$

for $k \ge 0$.

Replacing ℓ by $2\ell - 1$ for $\ell \in \mathbb{N}$ in (2.3) leads to

$$\begin{aligned}
(\arcsin x)^{2\ell-1} &= (2\ell-1)!\sum_{n=0}^{\infty}\beta_{n+2\ell-1,2\ell-1}\frac{x^{n+2\ell-1}}{(n+2\ell-1)!}\\
&= (2\ell-1)!\sum_{n=0}^{\infty}\beta_{2n+2\ell-1,2\ell-1}\frac{x^{2n+2\ell-1}}{(2n+2\ell-1)!}\\
&= (2\ell-1)!\sum_{n=0}^{\infty}\left[4^{n+\ell-1}i^{2(\ell-1)}\sum_{q=0}^{2(\ell-1)}T\left(n+\ell-1; q, 2\ell-2; \frac{1}{2}\right)\right]\frac{x^{2n+2\ell-1}}{(2n+2\ell-1)!}\\
&= (-1)^{\ell-1}4^{\ell-1}(2\ell-1)!\sum_{n=0}^{\infty}\left[4^{n}\sum_{q=0}^{2(\ell-1)}T\left(n+\ell-1; q, 2\ell-2; \frac{1}{2}\right)\right]\frac{x^{2n+2\ell-1}}{(2n+2\ell-1)!}\\
&= (-4)^{\ell-1}\sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\left[\sum_{q=0}^{2\ell-2}T\left(n+\ell-1; q, 2\ell-2; \frac{1}{2}\right)\right]\frac{x^{2n+2\ell-1}}{\binom{2n+2\ell-1}{2\ell-1}}.
\end{aligned}$$

Replacing ℓ by 2ℓ for $\ell \in \mathbb{N}$ in (2.3) leads to

$$\begin{aligned}
(\arcsin x)^{2\ell} &= (2\ell)!\sum_{n=0}^{\infty}\beta_{n+2\ell,2\ell}\frac{x^{n+2\ell}}{(n+2\ell)!}\\
&= (2\ell)!\sum_{n=0}^{\infty}\beta_{2n+2\ell,2\ell}\frac{x^{2n+2\ell}}{(2n+2\ell)!}\\
&= (-1)^{\ell-1}(2\ell)!\sum_{n=0}^{\infty}\left[4^{n}\sum_{q=0}^{2(\ell-1)}T(n+\ell-1; q, 2\ell-2; 1)\right]\frac{x^{2n+2\ell}}{(2n+2\ell)!}\\
&= (-1)^{\ell-1}\sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1; q, 2\ell-2; 1)\right]\frac{x^{2n+2\ell}}{\binom{2n+2\ell}{2\ell}}.
\end{aligned}$$

The proof of Theorem 2.1 is complete. □

3. Proof of Theorem 1.1

We now start out to prove Theorem 1.1.

In the last line of [18, p. 133], there exists the formula

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!} \tag{3.1}$$

for $k \ge 0$. When taking $x_{2m-1} = 0$ for $m \in \mathbb{N}$, the left hand side of the formula (3.1) is even in $t \in (-\infty, \infty)$ for all $k \ge 0$. Therefore, the formula (1.4) is valid.

Ones know that the power series expansion

$$\arcsin t = \sum_{\ell=0}^{\infty} \frac{[(2\ell-1)!!]^2}{(2\ell+1)!} t^{2\ell+1}, \quad |t| < 1 \tag{3.2}$$

is valid, where $(-1)!! = 1$. This implies that

$$\begin{aligned}
&\mathrm{B}_{2n,k}\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{2n-k+2}\right)\\
&= \mathrm{B}_{2n,k}\left(\frac{(\arcsin t)''|_{t=0}}{2}, \frac{(\arcsin t)'''|_{t=0}}{3}, \frac{(\arcsin t)^{(4)}|_{t=0}}{4}, \dots, \frac{(\arcsin t)^{(2n-k+2)}|_{t=0}}{2n-k+2}\right).
\end{aligned}$$

Employing the formula

$$\mathrm{B}_{n,k}\left(\frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_{n-k+2}}{n-k+2}\right) = \frac{n!}{(n+k)!} \mathrm{B}_{n+k,k}(0, x_2, x_3, \dots, x_{n+1})$$

in [18, p. 136], we derive

$$\begin{aligned}
&\mathrm{B}_{2n,k}\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{2n-k+2}\right)\\
&= \frac{(2n)!}{(2n+k)!} \mathrm{B}_{2n+k,k}(0, (\arcsin t)''|_{t=0}, (\arcsin t)'''|_{t=0}, \dots, (\arcsin t)^{(2n+1)}|_{t=0}).
\end{aligned}$$

Making use of the formula (3.1) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{B}_{n+k,k}(x_1, x_2, \dots, x_{n+1}) \frac{k!n!}{(n+k)!} \frac{t^{n+k}}{n!} &= \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k,\\
\sum_{n=0}^{\infty} \frac{\mathrm{B}_{n+k,k}(x_1, x_2, \dots, x_{n+1})}{\binom{n+k}{k}} \frac{t^{n+k}}{n!} &= \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k,\\
\mathrm{B}_{n+k,k}(x_1, x_2, \dots, x_{n+1}) &= \binom{n+k}{k} \lim_{t\to0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n}\left[\sum_{m=0}^{\infty} x_{m+1} \frac{t^m}{(m+1)!}\right]^k,\\
\mathrm{B}_{2n+k,k}(x_1, x_2, \dots, x_{2n+1}) &= \binom{2n+k}{k} \lim_{t\to0} \frac{\mathrm{d}^{2n}}{\mathrm{d}\, t^{2n}}\left[\sum_{m=0}^{\infty} x_{m+1} \frac{t^m}{(m+1)!}\right]^k.
\end{aligned}$$

Setting $x_1 = 0$ and $x_m = (\arcsin t)^{(m)}|_{t=0}$ for $m \ge 2$ gives

$$\begin{aligned}
\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\sum_{m=0}^{\infty} x_{m+1}\frac{t^m}{(m+1)!}\right]^k &= \frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^k \\
&= \frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t - t}{t}\right)^k \\
&= \frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\sum_{p=0}^{k}(-1)^{k-p}\binom{k}{p}\left(\frac{\arcsin t}{t}\right)^p \\
&= \sum_{p=1}^{k}(-1)^{k-p}\binom{k}{p}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^p.
\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^{2k-1} &= \sum_{p=1}^{2k-1}(-1)^{2k-p-1}\binom{2k-1}{p}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^p \\
= \sum_{p=1}^{k}\binom{2k-1}{2p-1}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p-1} &- \sum_{p=1}^{k-1}\binom{2k-1}{2p}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^{2k} &= \sum_{p=1}^{2k}(-1)^{2k-p}\binom{2k}{p}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^p \\
= \sum_{p=1}^{k}\binom{2k}{2p}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p} &- \sum_{p=1}^{k}\binom{2k}{2p-1}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p-1}.
\end{aligned}$$

From the power series expansions (2.1) and (2.2) in Theorem 2.1, it follows that

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p-1} &= (-1)^{p-1}4^{p-1}(2p-1)! \\
&\quad\times\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\sum_{j=0}^{\infty}\left[4^j\sum_{q=0}^{2p-2}T\left(j+p-1;q,2p-2;\frac{1}{2}\right)\right]\frac{t^{2j}}{(2j+2p-1)!} \\
&= (-1)^{p-1}\frac{4^{n+p-1}}{\binom{2n+2p-1}{2n}}\sum_{q=0}^{2p-2}T\left(n+p-1;q,2p-2;\frac{1}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^{2p} &= (-1)^{p-1}(2p)!\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\sum_{j=0}^{\infty}\left[4^j\sum_{q=0}^{2p-2}T(j+p-1;q,2p-2;1)\right]\frac{t^{2j}}{(2j+2p)!} \\
&= (-1)^{p-1}\frac{4^n}{\binom{2n+2p}{2n}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1).
\end{aligned}$$

Therefore, we arrive at

$$\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^{2k-1}=4^n\sum_{p=1}^{k}(-4)^{p-1}\frac{\binom{2k-1}{2p-1}}{\binom{2n+2p-1}{2p-1}}\sum_{q=0}^{2p-2}T\Bigl(n+p-1;q,2p-2;\frac{1}{2}\Bigr)$$
$$-4^n\sum_{p=1}^{k-1}(-1)^{p-1}\frac{\binom{2k-1}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)$$

and

$$\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^{2k}=4^n\sum_{p=1}^{k}(-1)^{p-1}\frac{\binom{2k}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)$$
$$-4^n\sum_{p=1}^{k}(-4)^{p-1}\frac{\binom{2k}{2p-1}}{\binom{2n+2p-1}{2p-1}}\sum_{q=0}^{2p-2}T\Bigl(n+p-1;q,2p-2;\frac{1}{2}\Bigr).$$

Consequently, we acquire

$$\begin{aligned}
&\mathrm{B}_{2n,2k-1}\Bigl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,0,\frac{[(2n-2k+1)!!]^2}{2n-2k+3}\Bigr)\\
&=\frac{(2n)!}{(2n+2k-1)!}\mathrm{B}_{2n+2k-1,2k-1}(0,(\arcsin t)''|_{t=0},(\arcsin t)'''|_{t=0},\dots,(\arcsin t)^{(2n+1)}|_{t=0})\\
&=\frac{(2n)!}{(2n+2k-1)!}\binom{2n+2k-1}{2k-1}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\Bigl(\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\Bigr)^{2k-1}\\
&=\frac{1}{(2k-1)!}\Biggl[4^n\sum_{p=1}^{k}(-4)^{p-1}\frac{\binom{2k-1}{2p-1}}{\binom{2n+2p-1}{2p-1}}\sum_{q=0}^{2p-2}T\Bigl(n+p-1;q,2p-2;\frac{1}{2}\Bigr)\\
&-4^n\sum_{p=0}^{k-1}(-1)^{p-1}\frac{\binom{2k-1}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)\Biggr]
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{B}_{2n,2k}\Bigl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{[(2n-2k-1)!!]^2}{2n-2k+1},0\Bigr)\\
&=\frac{(2n)!}{(2n+2k)!}\mathrm{B}_{2n+2k,2k}(0,(\arcsin t)''|_{t=0},(\arcsin t)'''|_{t=0},\dots,(\arcsin t)^{(2n+1)}|_{t=0})\\
&=\frac{(2n)!}{(2n+2k)!}\binom{2n+2k}{2k}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\Bigl(\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\Bigr)^{2k}\\
&=\frac{1}{(2k)!}\Biggl[4^n\sum_{p=1}^{k}(-1)^{p-1}\frac{\binom{2k}{2p}}{\binom{2n+2p}{2p}}\sum_{q=0}^{2p-2}T(n+p-1;q,2p-2;1)\\
&-4^n\sum_{p=1}^{k}(-4)^{p-1}\frac{\binom{2k}{2p-1}}{\binom{2n+2p-1}{2p-1}}\sum_{q=0}^{2p-2}T\Bigl(n+p-1;q,2p-2;\frac{1}{2}\Bigr)\Biggr].
\end{aligned}$$

The proof of Theorem 1.1 is complete.

4. Series representation of generalized logsine functions

The logsine function

$$\operatorname{Ls}_j(\theta)=-\int_0^\theta\biggl(\ln\biggl|2\sin\frac{x}{2}\biggr|\biggr)^{j-1}\,\mathrm{d}\,x$$

and generalized logsine function

$$\operatorname{Ls}_j^{(\ell)}(\theta)=-\int_0^\theta x^\ell\biggl(\ln\biggl|2\sin\frac{x}{2}\biggr|\biggr)^{j-\ell-1}\,\mathrm{d}\,x$$

were introduced originally in [34, pp. 191–192], where ℓ, j are integers, $j \ge \ell+1 \ge 1$, and θ is an arbitrary real number. There have been many papers such as [3, 9, 10, 14–17, 19, 20, 28–31, 37, 38, 57] devoted to investigation and applications of the (generalized) logsine functions in mathematics, physics, engineering, and other mathematical sciences.

Theorem 4.1. *Let $\langle z\rangle_n$ for $z\in\mathbb{C}$ and $n\in\{0\}\cup\mathbb{N}$ denote the falling factorial defined by* (2.4) *and let $T(r;q,j;\rho)$ be defined by* (1.6)*. In the region $0<\theta\le\pi$ and for $j,\ell\in\mathbb{N}$, generalized logsine functions $\operatorname{Ls}_j^{(\ell)}(\theta)$ have the following series representations:*

1. for $j\ge 2\ell+1\ge 3$,

$$\begin{aligned}\operatorname{Ls}_j^{(2\ell-1)}(\theta)=&-\frac{\theta^{2\ell}}{2\ell}\biggl[\ln\biggl(2\sin\frac{\theta}{2}\biggr)\biggr]^{j-2\ell}-(-1)^\ell(j-2\ell)(2\ell-1)!(\ln 2)^{j-1}\biggl(\frac{2\sin\frac{\theta}{2}}{\ln 2}\biggr)^{2\ell}\\&\times\sum_{n=0}^{\infty}\frac{(2\sin\frac{\theta}{2})^{2n}}{(2n+2\ell)!}\Biggl[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\Biggr]\\&\times\Biggl[\sum_{\alpha=0}^{j-2\ell-1}\biggl(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\biggr)^{\alpha}\binom{j-2\ell-1}{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell)^{k+1}(\ln\sin\frac{\theta}{2})^k}\Biggr];\end{aligned}\tag{4.1}$$

2. for $j\ge 2\ell+2\ge 4$,

$$\begin{aligned}\operatorname{Ls}_j^{(2\ell)}(\theta)=&-\frac{\theta^{2\ell+1}}{2\ell+1}\biggl[\ln\biggl(2\sin\frac{\theta}{2}\biggr)\biggr]^{j-2\ell-1}+(-1)^\ell\frac{(j-2\ell-1)(2\ell)!(\ln 2)^{j-1}}{2}\biggl(\frac{4\sin\frac{\theta}{2}}{\ln 2}\biggr)^{2\ell+1}\\&\times\sum_{n=0}^{\infty}\Biggl[\frac{(2\sin\frac{\theta}{2})^{2n}}{(2n+2\ell+1)!}\sum_{q=0}^{2\ell}T\biggl(n+\ell;q,2\ell;\frac{1}{2}\biggr)\Biggr]\\&\times\Biggl[\sum_{\alpha=0}^{j-2\ell-2}\binom{j-2\ell-2}{\alpha}\biggl(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\biggr)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell+1)^{k+1}(\ln\sin\frac{\theta}{2})^k}\Biggr];\end{aligned}\tag{4.2}$$

3. for $j\ge 2\ell-1\ge 1$,

$$\begin{aligned}\operatorname{Ls}_j^{(2\ell-2)}(\theta)=&(-1)^\ell 2^{4\ell-3}(2\ell-2)!(\ln 2)^j\biggl(\frac{\sin\frac{\theta}{2}}{\ln 2}\biggr)^{2\ell-1}\\&\times\sum_{n=0}^{\infty}\Biggl[\frac{(2\sin\frac{\theta}{2})^{2n}}{(2n+2\ell-2)!}\sum_{q=0}^{2\ell-2}T\biggl(n+\ell-1;q,2\ell-2;\frac{1}{2}\biggr)\Biggr]\\&\times\sum_{\alpha=0}^{j-2\ell+1}\binom{j-2\ell+1}{\alpha}\biggl(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\biggr)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell-1)^{k+1}(\ln\sin\frac{\theta}{2})^k};\end{aligned}\tag{4.3}$$

4. for $j \ge 2\ell - 1 \ge 1$,

$$\begin{aligned}
\operatorname{Ls}_j^{(2\ell-1)}(\theta) &= (-1)^{\ell}(2\ell-1)!(\ln 2)^j\left(\frac{2\sin\frac{\theta}{2}}{\ln 2}\right)^{2\ell}\\
&\quad\times\sum_{n=0}^{\infty}\left[\frac{(2\sin\frac{\theta}{2})^{2n}}{(2n+2\ell-1)!}\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell}\binom{j-2\ell}{\alpha}\left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell)^{k+1}(\ln\sin\frac{\theta}{2})^k}.
\end{aligned}\tag{4.4}$$

Proof. In [28, p. 49, Section 2.4], it was obtained that

$$\operatorname{Ls}_j^{(k)}(\theta) = -\frac{\theta^{k+1}}{k+1}\left[\ln\left(2\sin\frac{\theta}{2}\right)\right]^{j-k-1} + \frac{2^{k+1}(j-k-1)}{k+1}\int_0^{\sin(\theta/2)}\frac{(\arcsin x)^{k+1}\ln^{j-k-2}(2x)}{x}\,\mathrm{d}\,x \tag{4.5}$$

for $0 < \theta \le \pi$ and $j - k - 2 \ge 0$. Making use of Theorem 2.1 and the formula

$$\int x^n \ln^m x\,\mathrm{d}\,x = x^{n+1}\sum_{k=0}^{m}(-1)^k\langle m\rangle_k\frac{\ln^{m-k}x}{(n+1)^{k+1}}, \quad m,n\ge 0 \tag{4.6}$$

in [22, p. 238, 2.722], we acquire

$$\begin{aligned}
&\int_0^{\sin(\theta/2)}\frac{(\arcsin x)^{2\ell}\ln^{j-2\ell-1}(2x)}{x}\,\mathrm{d}\,x\\
&= (-1)^{\ell-1}(2\ell)!\sum_{n=0}^{\infty}\frac{4^n}{(2n+2\ell)!}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]\int_0^{\sin(\theta/2)}x^{2n+2\ell-1}\ln^{j-2\ell-1}(2x)\,\mathrm{d}\,x\\
&= (-1)^{\ell-1}(2\ell)!\sum_{n=0}^{\infty}\frac{4^n}{(2n+2\ell)!}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]\\
&\quad\times\left[\int_0^{\sin(\theta/2)}x^{2n+2\ell-1}(\ln 2+\ln x)^{j-2\ell-1}\,\mathrm{d}\,x\right]\\
&= (-1)^{\ell-1}(2\ell)!\sum_{n=0}^{\infty}\frac{4^n}{(2n+2\ell)!}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]\\
&\quad\times\left[\sum_{\alpha=0}^{j-2\ell-1}\binom{j-2\ell-1}{\alpha}(\ln 2)^{j-2\ell-\alpha-1}\int_0^{\sin(\theta/2)}x^{2n+2\ell-1}(\ln x)^{\alpha}\,\mathrm{d}\,x\right]\\
&= (-1)^{\ell-1}(2\ell)!\sum_{n=0}^{\infty}\frac{4^n}{(2n+2\ell)!}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]\\
&\quad\times\left[\sum_{\alpha=0}^{j-2\ell-1}\binom{j-2\ell-1}{\alpha}(\ln 2)^{j-2\ell-\alpha-1}\left(\sin\frac{\theta}{2}\right)^{2n+2\ell}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell)^{k+1}}\left(\ln\sin\frac{\theta}{2}\right)^{\alpha-k}\right]\\
&= (-1)^{\ell-1}(2\ell)!(\ln 2)^{j-2\ell-1}\left(\sin\frac{\theta}{2}\right)^{2\ell}\sum_{n=0}^{\infty}\frac{4^n}{(2n+2\ell)!}\left(\sin\frac{\theta}{2}\right)^{2n}\left[\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\right]
\end{aligned}$$

$$\times\left[\sum_{\alpha=0}^{j-2\ell-1}\left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\alpha}\binom{j-2\ell-1}{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell)^{k+1}(\ln\sin\frac{\theta}{2})^k}\right]$$

for $j\ge 2\ell+1\ge 3$. Substituting this result into (4.5) for $k=2\ell-1$ yields (4.1).

Similarly, by virtue of Theorem 2.1 and the formula (4.6), we also have

$$\begin{aligned}
&\int_0^{\sin(\theta/2)}\frac{(\arcsin x)^{2\ell+1}\ln^{j-2\ell-2}(2x)}{x}\,\mathrm{d}\,x\\
&=(-1)^{\ell}4^{\ell}(2\ell+1)!\sum_{n=0}^{\infty}\left[\frac{4^n}{(2n+2\ell+1)!}\sum_{q=0}^{2\ell}T\left(n+\ell;q,2\ell;\frac{1}{2}\right)\right]\int_0^{\sin(\theta/2)}x^{2n+2\ell}\ln^{j-2\ell-2}(2x)\,\mathrm{d}\,x\\
&=(-1)^{\ell}4^{\ell}(2\ell+1)!\sum_{n=0}^{\infty}\left[\frac{4^n}{(2n+2\ell+1)!}\sum_{q=0}^{2\ell}T\left(n+\ell;q,2\ell;\frac{1}{2}\right)\right]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell-2}\binom{j-2\ell-2}{\alpha}(\ln 2)^{j-2\ell-\alpha-2}\int_0^{\sin(\theta/2)}x^{2n+2\ell}(\ln x)^{\alpha}\,\mathrm{d}\,x\\
&=(-1)^{\ell}4^{\ell}(2\ell+1)!\sum_{n=0}^{\infty}\left[\frac{4^n}{(2n+2\ell+1)!}\sum_{q=0}^{2\ell}T\left(n+\ell;q,2\ell;\frac{1}{2}\right)\right]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell-2}\binom{j-2\ell-2}{\alpha}(\ln 2)^{j-2\ell-\alpha-2}\left(\sin\frac{\theta}{2}\right)^{2n+2\ell+1}\sum_{k=0}^{\alpha}(-1)^k\langle\alpha\rangle_k\frac{(\ln\sin\frac{\theta}{2})^{\alpha-k}}{(2n+2\ell+1)^{k+1}}\\
&=(-1)^{\ell}4^{\ell}(2\ell+1)!\left(\sin\frac{\theta}{2}\right)^{2\ell+1}(\ln 2)^{j-2\ell-2}\sum_{n=0}^{\infty}\left[\frac{4^n}{(2n+2\ell+1)!}\left(\sin\frac{\theta}{2}\right)^{2n}\sum_{q=0}^{2\ell}T\left(n+\ell;q,2\ell;\frac{1}{2}\right)\right]\\
&\quad\times\left[\sum_{\alpha=0}^{j-2\ell-2}\binom{j-2\ell-2}{\alpha}\left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell+1)^{k+1}(\ln\sin\frac{\theta}{2})^k}\right]
\end{aligned}$$

for $\ell\in\mathbb{N}$ and $j\ge 2(\ell+1)\ge 4$. Substituting this result into (4.5) for $k=2\ell$ yields (4.2).

In [20, p. 308], it was derived that

$$\operatorname{Ls}_j^{(k)}(\theta)=-2^{k+1}\int_0^{\sin(\theta/2)}\frac{(\arcsin x)^k}{\sqrt{1-x^2}}\ln^{j-k-1}(2x)\,\mathrm{d}\,x \tag{4.7}$$

for $0<\theta\le\pi$ and $j\ge k+1\ge 1$. Differentiating with respect to x on both sides of the formulas (2.1) and (2.2) in Theorem 2.1 results in

$$\frac{(\arcsin x)^{2\ell-2}}{\sqrt{1-x^2}}=(-1)^{\ell-1}4^{\ell-1}(2\ell-2)!\sum_{n=0}^{\infty}\left[4^n\sum_{q=0}^{2\ell-2}T\left(n+\ell-1;q,2\ell-2;\frac{1}{2}\right)\right]\frac{x^{2n+2\ell-2}}{(2n+2\ell-2)!} \tag{4.8}$$

and

$$\frac{(\arcsin x)^{2\ell-1}}{\sqrt{1-x^2}}=(-1)^{\ell-1}(2\ell-1)!\sum_{n=0}^{\infty}\left[4^n\sum_{q=0}^{2\ell-2}T(n+\ell;q,2\ell;1)\right]\frac{x^{2n+2\ell-1}}{(2n+2\ell-1)!} \tag{4.9}$$

for $\ell \in \mathbb{N}$. Substituting the power series expansions (4.8) and (4.9) into (4.7) and employing the indefinite integral (4.6) respectively reveal

$$\begin{aligned}
\operatorname{Ls}_j^{(2\ell-2)}(\theta) &= -2^{2\ell-1}\int_0^{\sin(\theta/2)} \frac{(\arcsin x)^{2\ell-2}}{\sqrt{1-x^2}} \ln^{j-2\ell+1}(2x)\,\mathrm{d}\,x\\
&= (-1)^\ell 2^{4\ell-3}(2\ell-2)!\sum_{n=0}^{\infty}\biggl[\frac{4^n}{(2n+2\ell-2)!}\sum_{q=0}^{2\ell-2}T\Bigl(n+\ell-1;q,2\ell-2;\frac{1}{2}\Bigr)\biggr]\\
&\quad\times\int_0^{\sin(\theta/2)} x^{2n+2\ell-2}(\ln 2+\ln x)^{j-2\ell+1}\,\mathrm{d}\,x\\
&= (-1)^\ell 2^{4\ell-3}(2\ell-2)!\sum_{n=0}^{\infty}\biggl[\frac{4^n}{(2n+2\ell-2)!}\sum_{q=0}^{2\ell-2}T\Bigl(n+\ell-1;q,2\ell-2;\frac{1}{2}\Bigr)\biggr]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell+1}\binom{j-2\ell+1}{\alpha}(\ln 2)^{j-2\ell-\alpha+1}\int_0^{\sin(\theta/2)} x^{2n+2\ell-2}(\ln x)^{\alpha}\,\mathrm{d}\,x\\
&= (-1)^\ell 2^{4\ell-3}(2\ell-2)!(\ln 2)^j\biggl(\frac{\sin\frac{\theta}{2}}{\ln 2}\biggr)^{2\ell-1}\sum_{n=0}^{\infty}\biggl[\frac{4^n}{(2n+2\ell-2)!}\Bigl(\sin\frac{\theta}{2}\Bigr)^{2n}\\
&\quad\times\sum_{q=0}^{2\ell-2}T\Bigl(n+\ell-1;q,2\ell-2;\frac{1}{2}\Bigr)\biggr]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell+1}\binom{j-2\ell+1}{\alpha}\biggl(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\biggr)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell-1)^{k+1}(\ln\sin\frac{\theta}{2})^k}
\end{aligned}$$

for $j \ge 2\ell-1 \ge 1$ and

$$\begin{aligned}
\operatorname{Ls}_j^{(2\ell-1)}(\theta) &= -2^{2\ell}\int_0^{\sin(\theta/2)} \frac{(\arcsin x)^{2\ell-1}}{\sqrt{1-x^2}} \ln^{j-2\ell}(2x)\,\mathrm{d}\,x\\
&= (-1)^\ell 2^{2\ell}(2\ell-1)!\sum_{n=0}^{\infty}\biggl[\frac{4^n}{(2n+2\ell-1)!}\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\biggr]\\
&\quad\times\int_0^{\sin(\theta/2)} x^{2n+2\ell-1}(\ln 2+\ln x)^{j-2\ell}\,\mathrm{d}\,x\\
&= (-1)^\ell 2^{2\ell}(2\ell-1)!\sum_{n=0}^{\infty}\biggl[\frac{4^n}{(2n+2\ell-1)!}\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\biggr]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell}\binom{j-2\ell}{\alpha}(\ln 2)^{j-2\ell-\alpha}\int_0^{\sin(\theta/2)} x^{2n+2\ell-1}(\ln x)^{\alpha}\,\mathrm{d}\,x\\
&= (-1)^\ell(2\ell-1)!(\ln 2)^j\biggl(\frac{2\sin\frac{\theta}{2}}{\ln 2}\biggr)^{2\ell}\sum_{n=0}^{\infty}\biggl[\frac{(2\sin\frac{\theta}{2})^{2n}}{(2n+2\ell-1)!}\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1)\biggr]\\
&\quad\times\sum_{\alpha=0}^{j-2\ell}\binom{j-2\ell}{\alpha}\biggl(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\biggr)^{\alpha}\sum_{k=0}^{\alpha}\frac{(-1)^k\langle\alpha\rangle_k}{(2n+2\ell)^{k+1}(\ln\sin\frac{\theta}{2})^k}
\end{aligned}$$

for $j \geq 2\ell \geq 1$. The series representations (4.3) and (4.4) are thus proved. The proof of Theorem 4.1 is complete. □

5. Remarks

Finally, we list several remarks on our main results and related stuffs.

Remark 5.1. For $n \geq k \geq 1$, the first kind Stirling numbers $s(n,k)$ can be explicitly computed by

$$|s(n+1,k+1)| = n! \sum_{\ell_1=k}^{n} \frac{1}{\ell_1} \sum_{\ell_2=k-1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-1}=2}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}} \sum_{\ell_k=1}^{\ell_{k-1}-1} \frac{1}{\ell_k}. \tag{5.1}$$

The formula (5.1) was derived in [41, Corollary 2.3] and can be reformulated as

$$\frac{|s(n+1,k+1)|}{n!} = \sum_{m=k}^{n} \frac{|s(m,k)|}{m!}$$

for $n \geq k \geq 1$. From the equation (1.5), by convention, we assume $s(n,k) = 0$ for $n < k$ and $k, n < 0$. In recent years, the first kind Stirling numbers $s(n,k)$ have been investigated in [39–42, 45] and closely related references therein.

Remark 5.2. For $|x| < 1$, we have the following series expansions of $\arcsin x$ and its powers.

1. The series expansion (3.2) of $\arcsin x$ can be rewritten as

$$\frac{\arcsin x}{x} = 1! \sum_{n=0}^{\infty} [(2n-1)!!]^2 \frac{x^{2n}}{(2n+1)!}, \tag{5.2}$$

   where $(-1)!! = 1$. Various forms of (5.2) can be found in [1, 4.4.40] and [2, p. 121, 6.41.1].

2. The series expansion of $(\arcsin x)^2$ can be rearranged as

$$\left(\frac{\arcsin x}{x}\right)^2 = 2! \sum_{n=0}^{\infty} [(2n)!!]^2 \frac{x^{2n}}{(2n+2)!}. \tag{5.3}$$

   The variants of (5.3) can be found in [2, p. 122, 6.42.1], [4, pp. 262–263, Proposition 15], [5, pp. 50–51 and p. 287], [6, p. 384], [7, p. 2, (2.1)], [13, Lemma 2], [20, p. 308], [21, pp. 88–90], [22, p. 61, 1.645], [32, p. 1011], [33, p. 453], [47, Section 6.3], [58], [60, p. 59, (2.56)], or [62, p. 676, (2.2)]. It is clear that the series expansion (5.3) and its equivalent forms have been rediscovered repeatedly. For more information on the history, dated back to 1899 or earlier, of the series expansion (5.3) and its equivalent forms, see [7, p. 2] and [32, p. 1011].

3. The series expansion of $(\arcsin x)^3$ can be reformulated as

$$\left(\frac{\arcsin x}{x}\right)^3 = 3! \sum_{n=0}^{\infty} [(2n+1)!!]^2 \left[\sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right] \frac{x^{2n}}{(2n+3)!}. \tag{5.4}$$

   Different variants of (5.4) can be found in [2, p. 122, 6.42.2], [4, pp. 262–263, Proposition 15], [11, p. 188, Example 1], [20, p. 308], [21, pp. 88–90], [22, p. 61, 1.645], or [27, pp. 154–155, (832)].

4. The series expansion of $(\arcsin x)^4$ can be restated as

$$\left(\frac{\arcsin x}{x}\right)^4 = 4!\sum_{n=0}^{\infty}[(2n+2)!!]^2\left[\sum_{k=0}^{n}\frac{1}{(2k+2)^2}\right]\frac{x^{2n}}{(2n+4)!}. \tag{5.5}$$

   There exist three variants of (5.5) in [4, pp. 262–263, Proposition 15], [7, p. 3, (2.2)], and [20, p. 309].
5. Basing on the formula (2.21) in [28, p. 50], we concretely obtain

$$\left(\frac{\arcsin x}{x}\right)^5 = \frac{5!}{2}\sum_{n=0}^{\infty}[(2n+3)!!]^2\left[\left(\sum_{k=0}^{n+1}\frac{1}{(2k+1)^2}\right)^2 - \sum_{k=0}^{n+1}\frac{1}{(2k+1)^4}\right]\frac{x^{2n}}{(2n+5)!}. \tag{5.6}$$

6. In [7], the special series expansions

$$\left(\arcsin\frac{x}{2}\right)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{x^{2n}}{\binom{2n}{n}n^2},$$

$$\left(\arcsin\frac{x}{2}\right)^4 = \frac{3}{2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m^2}\right)\frac{x^{2n}}{\binom{2n}{n}n^2},$$

$$\left(\arcsin\frac{x}{2}\right)^6 = \frac{45}{4}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m^2}\sum_{\ell=1}^{m-1}\frac{1}{\ell^2}\right)\frac{x^{2n}}{\binom{2n}{n}n^2},$$

$$\left(\arcsin\frac{x}{2}\right)^8 = \frac{315}{2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m^2}\sum_{\ell=1}^{m-1}\frac{1}{\ell^2}\sum_{p=1}^{\ell-1}\frac{1}{p^2}\right)\frac{x^{2n}}{\binom{2n}{n}n^2}$$

   were listed. In general, it was obtained in [7, pp. 1–2] that

$$\left(\arcsin\frac{x}{2}\right)^{2\ell} = (2\ell)!\sum_{n=1}^{\infty}H_\ell(n)\frac{x^{2n}}{\binom{2n}{n}n^2}, \quad \ell\in\mathbb{N} \tag{5.7}$$

   and

$$\left(\arcsin\frac{x}{2}\right)^{2\ell+1} = (2\ell+1)!\sum_{n=1}^{\infty}G_\ell(n)\frac{\binom{2n}{n}}{2^{4n+1}}\frac{x^{2n+1}}{2n+1}, \quad \ell\in\{0\}\cup\mathbb{N}, \tag{5.8}$$

   where $H_1(n)=\frac{1}{4}$, $G_0(n)=1$,

$$H_{\ell+1}(n) = \frac{1}{4}\sum_{m_1=1}^{n-1}\frac{1}{(2m_1)^2}\sum_{m_2=1}^{m_1-1}\frac{1}{(2m_2)^2}\cdots\sum_{m_\ell=1}^{m_{\ell-1}-1}\frac{1}{(2m_\ell)^2},$$

   and

$$G_\ell(n) = \sum_{m_1=0}^{n-1}\frac{1}{(2m_1+1)^2}\sum_{m_2=0}^{m_1-1}\frac{1}{(2\ell_2+1)^2}\cdots\sum_{m_\ell=0}^{m_{\ell-1}-1}\frac{1}{(2m_\ell+1)^2}.$$

   The convention is that the sum is zero if the starting index exceeds the finishing index.

7. In [7, (2.9) and (4.3)], [25, p. 480, (88.2.2)], and [56, p. 124], there exist the formulas

$$\begin{aligned}\left(\frac{\arcsin x}{x}\right)^{\ell} = \sum_{n=0}^{\infty}\Bigg[\left(\prod_{k=1}^{\ell-1}\left\{\sum_{n_k=0}^{n_{k-1}}\frac{(2n_{k-1}-2n_k)!}{[(n_{k-1}-n_k)!]^2(2n_{k-1}-2n_k+1)}\frac{1}{2^{2n_{k-1}-2n_k}}\right\}\right)\\ \times\frac{(2n_{\ell-1})!}{(n_{\ell-1}!)^2(2n_{\ell-1}+1)}\frac{1}{2^{2n_{\ell-1}}}\Bigg]x^{2n}\end{aligned} \tag{5.9}$$

and

$$\left(\frac{\arcsin x}{x}\right)^{\ell} = \ell!\sum_{n=0}^{\infty}\left[\sum_{n_1=0}^{n}\frac{\binom{2n_1}{n_1}}{2n_1+1}\sum_{n_2=n_1}^{n}\frac{\binom{2n_2-2n_1}{n_2-n_1}}{2n_2+2}\cdots\sum_{n_\ell=n_{\ell-1}}^{n}\frac{\binom{2n_\ell-2n_{\ell-1}}{n_\ell-n_{\ell-1}}}{2n_\ell+\ell}\frac{1}{4^{n_\ell}}\right]x^n. \tag{5.10}$$

All the power series expansions from (5.2) to (5.6) can also be deduced from Theorem 2.1.

By the way, we notice that the quantity in the pair of bigger brackets, the coefficient of x^{2n}, in the formula (5.9) has no explicit relation with n. This means that there must be some misprints and typos somewhere in the formula (5.9). On 30 January 2021, Christophe Vignat (Tulane University) pointed out that $n_0 = n$ is the missing information in the formula (5.9).

In [28, pp. 49–50, Section 2.4], the power series expansions of $(\arcsin x)^k$ for $2 \le k \le 13$ were concretely and explicitly written down in alternative forms. The main idea in the study of the power series expansions of $(\arcsin x)^k$ for $2 \le k \le 13$ was related with series representations for generalized logsine functions in [28, p. 50, (2.24) and (2.25)]. The special interest is special values of generalized logsine functions defined by [28, p. 50, (2.26) and (2.27)].

In [54, Theorem 1.4] and [55, Theorem 2.1], the nth derivative of $\arcsin x$ was explicitly computed.

In [43, 44], three series expansions (5.2), (5.3), (5.4) and their first derivatives were used to derive known and new combinatorial identities and others.

Because coefficients of $x^{2n+2\ell-1}$ and $x^{2n+2\ell}$ in (2.1) and (2.2) contain three times sums, coefficients of x^{2n} and x^{2n+1} in (5.7) and (5.8) contain ℓ times sums, coefficients of x^{2n} in (5.9) contain $\ell - 1$ times sums, and coefficients of x^n in (5.10) contain ℓ times sums, we conclude that the series expansions (2.1) and (2.2) are more elegant, more operable, more computable, and more applicable.

Remark 5.3. Two expressions (2.1) and (2.2) in Theorem 2.1 for series expansions of $(\arcsin x)^{2\ell-1}$ and $(\arcsin x)^{2\ell}$ are very close and similar to, but different from, each other. Is there a unified expression for series expansions of $(\arcsin x)^{2\ell-1}$ and $(\arcsin x)^{2\ell}$? If yes, two closed-form formulas for $\mathrm{B}_{2n,k}$ in Theorem 1.1 would also be unified. We believe that the formula

$$\exp\left(2a\arcsin\frac{x}{2}\right) = \sum_{n=0}^{\infty}\frac{(ia)_{n/2}}{(ia+1)_{-n/2}}\frac{(-ix)^n}{n!} \tag{5.11}$$

mentioned in [7, p. 3, (2.7)] and collected in [25, p. 210, (10.49.33)] would be useful for unifying two expressions (2.1) and (2.2) in Theorem 2.1, where extended Pochhammer symbols

$$(ia)_{n/2} = \frac{\Gamma(ia+\frac{n}{2})}{\Gamma(ia)} \quad \text{and} \quad (ia+1)_{-n/2} = \frac{\Gamma(ia+1-\frac{n}{2})}{\Gamma(ia+1)} \tag{5.12}$$

were defined in [25, p. 5, Section 2.2.3], and the Euler gamma function $\Gamma(z)$ is defined [59, Chapter 3] by

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

What are closed forms and why do we care closed forms? Please read the paper [8].

Remark 5.4. In [2, p. 122, 6.42], [27, pp. 154–155, (834)], [33, p. 452, Theorem], and [47, Section 6.3, Theorem 21, Sections 8 and 9], it was proved or collected that

$$\frac{\arcsin x}{\sqrt{1-x^2}}=\sum_{n=0}^{\infty}2^{2n}(n!)^2\frac{x^{2n+1}}{(2n+1)!},\quad |x|\le 1. \tag{5.13}$$

In [6, p. 385], [47, Theorem 24], and [61, p. 174, (10)], it was proved that

$$\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{\binom{2n}{n}}=\frac{x^2}{1-x^2}+\frac{x\arcsin x}{(1-x^2)^{3/2}},\quad |x|<1. \tag{5.14}$$

These series expansions (5.13) and (5.14) can be derived directly from the series expansion for $(\arcsin x)^2$ and are a special case of (4.9) for $\ell=1$.

Remark 5.5. The series expansion of the function $\sqrt{1-x^2}\arcsin x$ was listed in [2, p. 122, 6.42.4] which can be corrected and reformulated as

$$\sqrt{1-x^2}\arcsin x=x-1!\sum_{n=1}^{\infty}[(2n-2)!!]^2(2n)\frac{x^{2n+1}}{(2n+1)!},\quad |x|\le 1. \tag{5.15}$$

Basing on the relation

$$(1-x^2)[(\arcsin x)^{\ell}]'=\ell\sqrt{1-x^2}\,(\arcsin x)^{\ell-1}$$

and utilizing series expansions of $(\arcsin x)^3$ and $(\arcsin x)^4$, after simple operations, we can readily derive

$$\sqrt{1-x^2}\,(\arcsin x)^2=x^2-2!\sum_{n=1}^{\infty}[(2n-1)!!]^2\left[(2n+1)\sum_{k=0}^{n-1}\frac{1}{(2k+1)^2}-1\right]\frac{x^{2n+2}}{(2n+2)!} \tag{5.16}$$

and

$$\sqrt{1-x^2}\,(\arcsin x)^3=x^3-3!\sum_{n=1}^{\infty}[(2n)!!]^2\left[(2n+2)\sum_{k=0}^{n-1}\frac{1}{(2k+2)^2}-1\right]\frac{x^{2n+3}}{(2n+3)!}. \tag{5.17}$$

From (4.8) and (4.9), we can generalize the series expansions (5.15), (5.16), and (5.17) as

$$\begin{aligned}\sqrt{1-x^2}\,(\arcsin x)^{2\ell-2}&=x^{2\ell-2}+(-1)^{\ell-1}4^{\ell-1}(2\ell-2)!\\&\quad\times\sum_{n=1}^{\infty}[A(\ell,n)-(2n+2\ell-2)(2n+2\ell-3)A(\ell,n-1)]\frac{x^{2n+2\ell-2}}{(2n+2\ell-2)!}\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}\sqrt{1-x^2}\,(\arcsin x)^{2\ell-1}&=x^{2\ell-1}+(-1)^{\ell-1}(2\ell-1)!\\&\quad\times\sum_{n=1}^{\infty}[B(\ell,n)-(2n+2\ell-1)(2n+2\ell-2)B(\ell,n-1)]\frac{x^{2n+2\ell-1}}{(2n+2\ell-1)!}\end{aligned} \tag{5.19}$$

for $\ell\in\mathbb{N}$, where

$$A(\ell,n)=4^n\sum_{q=0}^{2\ell-2}T\left(n+\ell-1;q,2\ell-2;\frac{1}{2}\right),$$

$$B(\ell,n)=4^n\sum_{q=0}^{2\ell-2}T(n+\ell-1;q,2\ell-2;1),$$

and $T(r;q,j;\rho)$ is defined by (1.6). Considering both coefficients of $x^{2\ell-2}$ and $x^{2\ell-1}$ in the power series expansions (5.18) and (5.19) must be 1, we acquire two combinatorial identities

$$\sum_{q=0}^{2\ell}T\left(\ell;q,2\ell;\frac{1}{2}\right)=\frac{(-1)^\ell}{4^\ell}\quad\text{and}\quad\sum_{q=0}^{2\ell}T(\ell;q,2\ell;1)=(-1)^\ell$$

for $\ell\in\{0\}\cup\mathbb{N}$, where $T(r;q,j;\rho)$ is defined by (1.6).

Remark 5.6. Making use of Theorem 1.1, we readily obtain the first several values of the sequence (1.3) in Tables 1 and 2.

Table 1. The sequence $\mathrm{B}_{2n,2k-1}$ in (1.3) for $1\le n,k\le 8$.

$\mathrm{B}_{2n,2k-1}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=1$	$\frac{1}{3}$	0	0	0	0	0	0	0
$n=2$	$\frac{9}{5}$	0	0	0	0	0	0	0
$n=3$	$\frac{225}{7}$	$\frac{5}{9}$	0	0	0	0	0	0
$n=4$	1225	42	0	0	0	0	0	0
$n=5$	$\frac{893025}{11}$	3951	$\frac{35}{9}$	0	0	0	0	0
$n=6$	$\frac{108056025}{13}$	$\frac{2515524}{5}$	1155	0	0	0	0	0
$n=7$	1217431215	85621185	314314	$\frac{5005}{81}$	0	0	0	0
$n=8$	$\frac{4108830350625}{17}$	18974980350	$\frac{284770486}{3}$	$\frac{140140}{3}$	0	0	0	0

Table 2. The sequence $\mathrm{B}_{2n,2k}$ in (1.3) for $1\le n,k\le 8$.

$\mathrm{B}_{2n,2k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=1$	0	0	0	0	0	0	0	0
$n=2$	$\frac{1}{3}$	0	0	0	0	0	0	0
$n=3$	9	0	0	0	0	0	0	0
$n=4$	$\frac{2067}{5}$	$\frac{35}{27}$	0	0	0	0	0	0
$n=5$	30525	210	0	0	0	0	0	0
$n=6$	$\frac{23483925}{7}$	35211	$\frac{385}{27}$	0	0	0	0	0
$n=7$	516651345	$\frac{106790684}{15}$	7007	0	0	0	0	0
$n=8$	106480673775	$\frac{8891683281}{5}$	2892890	$\frac{25025}{81}$	0	0	0	0

In the papers [46, 48–55] and closely related references therein, the authors and their coauthors discovered and applied closed form expressions for many special values of the second kind Bell polynomials $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$.

Remark 5.7. Taking $\theta=\frac{\pi}{3}$ in (4.3) and (4.4) give

$$\mathrm{Ls}_j^{(2\ell-2)}\left(\frac{\pi}{3}\right)=(-1)^\ell(4\ell-4)!!(\ln 2)^{j-2\ell+1}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+2\ell-2)!}\sum_{q=0}^{2\ell-2}T\left(n+\ell-1;q,2\ell-2;\frac{1}{2}\right)\right]$$

$$\times \sum_{\alpha=0}^{j-2\ell+1} (-1)^{\alpha} \binom{j-2\ell+1}{\alpha} \sum_{k=0}^{\alpha} \frac{\langle \alpha \rangle_k}{(2n+2\ell-1)^{k+1}(\ln 2)^k}$$

and

$$\mathrm{Ls}_j^{(2\ell-1)}\left(\frac{\pi}{3}\right) = (-1)^{\ell}(2\ell-1)!(\ln 2)^{j-2\ell} \sum_{n=0}^{\infty} \left[\frac{1}{(2n+2\ell-1)!} \sum_{q=0}^{2\ell-2} T(n+\ell-1; q, 2\ell-2; 1)\right]$$
$$\times \sum_{\alpha=0}^{j-2\ell} (-1)^{\alpha} \binom{j-2\ell}{\alpha} \sum_{k=0}^{\alpha} \frac{\langle \alpha \rangle_k}{(2n+2\ell)^{k+1}(\ln 2)^k}$$

for $\ell \in \mathbb{N}$, where $\langle z \rangle_n$ for $z \in \mathbb{C}$ and $n \in \{0\} \cup \mathbb{N}$ denotes the falling factorial defined by (2.4) and $T(r; q, j; \rho)$ is defined by (1.6). In [28, p. 50], it was stated that the values $\mathrm{Ls}_j^{(\ell)}(\frac{\pi}{3})$ have been related to special interest in the calculation of the multiloop Feynman diagrams [19, 20].

Similarly, we can also deduce series representations for special values of the logsine function $\mathrm{Ls}_j^{(\ell)}(\theta)$ at $\theta = \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}$ and $\theta = \pi$. These special values were originally derived in [30, 31, 34] and also considered in [3, 9, 10, 14–17, 19, 20, 28, 29, 37, 38, 57] and closely related references therein.

Remark 5.8. This paper is a revised version of electronic arXiv preprints [23, 24].

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6.3. Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

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