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Research article

Sharp bounds for Gauss Lemniscate functions and Lemniscatic means

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Abstract: For $a, b > 0$ with $a \neq b$, the Gauss lemniscate mean $\mathcal{LM}(a, b)$ is defined by

$$\mathcal{LM}(a,b)=\begin{cases}\dfrac{\sqrt{a^2-b^2}}{\left[\operatorname{arcsl}\left(\sqrt[4]{1-b^2/a^2}\right)\right]^2}, & a>b,\\ \dfrac{\sqrt{b^2-a^2}}{\left[\operatorname{arcslh}\left(\sqrt[4]{b^2/a^2-1}\right)\right]^2}, & a<b,\end{cases}$$

where $\operatorname{arcsl}(x)=\int_0^x \frac{dt}{\sqrt{1-t^4}}$ ($|x|<1$) and $\operatorname{arcslh}(x)=\int_0^x \frac{dt}{\sqrt{1+t^4}}$ ($x\in\mathbb{R}$) is the arc lemniscate sine and hyperbolic arc lemniscate sine functions respectively. In this paper, we mainly establish sharp two-parameter bounds for four symmetric and homogeneous means derived from $\mathcal{LM}(a, b)$, $\mathcal{LM}_{\mathcal{GA}}(a,b)=\mathcal{LM}(\mathcal{G}(a,b),\mathcal{A}(a,b))$, $\mathcal{LM}_{\mathcal{AG}}(a,b)=\mathcal{LM}(\mathcal{A}(a,b),\mathcal{G}(a,b))$, $\mathcal{LM}_{\mathcal{AQ}}(a,b)=\mathcal{LM}(\mathcal{A}(a,b),\mathcal{G}(a,b))$ and $\mathcal{LM}_{\mathcal{QA}}(a,b)=\mathcal{LM}(\mathcal{A}(a,b),\mathcal{G}(a,b))$. The obtained results lead to several asymptotical inequalities for Lemniscate functions. Here $\mathcal{A}(a,b)=(a+b)/2$, $\mathcal{G}(a,b)=\sqrt{ab}$ and $\mathcal{Q}(a,b)=\sqrt{(a^2+b^2)/2}$ are the classical arithmetic, geometric, and quadratic means.

Keywords: Lemniscate function; Lemniscatic mean; arithmetic mean; geometric mean; sharp bounds
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1. Introduction

Gauss's arc lemniscate sine and the hyperbolic arc lemniscate sine functions are defined by as follows:

$$\operatorname{arcsl}(x)=\int_0^x \frac{dt}{\sqrt{1-t^4}}, \quad |x|<1,$$

$$\operatorname{arcslh}(x) = \int_0^x \frac{dt}{\sqrt{1+t^4}}, \quad x \in \mathbb{R},$$

respectively (cf. [1, p.259] or [2, (2.5) and (2.6)]).

Another pair of arc lemniscate functions, Gauss's arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are defined in terms of the arc lemniscate sine and the hyperbolic arc lemniscate sine functions, respectively (cf. [3, (3.5) and (3.6)]):

$$\operatorname{arctl}(x) = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \quad x \in \mathbb{R},$$

$$\operatorname{arctlh}(x) = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \quad |x| < 1.$$

It is not difficult to verify that

$$\omega = \operatorname{arcsl}(1) = \frac{1}{\sqrt{2}}\mathcal{K}\left(1/\sqrt{2}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{2\pi}} = 1.31103\cdots,$$

$$\tau = \operatorname{arctl}(1) = \operatorname{arcsl}\left(1/\sqrt[4]{2}\right) = 0.89558\cdots$$

and

$$\operatorname{arctlh}(1) = \operatorname{arcslh}(+\infty) = \sqrt{2}\omega = 1.85407\cdots,$$

where

$$\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta = \int_0^{\pi/2}\frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \quad 0 < r < 1$$

is the complete elliptic integral of the first kind [4–10], and

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \operatorname{Re}(x) > 0$$

is the classical Euler gamma function [11–14].

For $a, b > 0$ with $a \neq b$, the arithmetic mean $\mathcal{A}(a,b)$, geometric mean $\mathcal{G}(a,b)$, harmonic mean $\mathcal{H}(a,b)$, quadratic mean $\mathcal{Q}(a,b)$, contra-harmonic mean $\mathcal{C}(a,b)$ and the lemniscatic mean $\mathcal{LM}(a,b)$ [3, 15] of a and b are respectively defined by

$$\mathcal{A}(a,b) = \frac{a+b}{2},\ \mathcal{G}(a,b) = \sqrt{ab},\ \mathcal{H}(a,b) = \frac{2ab}{a+b},$$

$$\mathcal{Q}(a,b) = \sqrt{\frac{a^2+b^2}{2}},\ \mathcal{C}(a,b) = \frac{a^2+b^2}{a+b}$$

and

$$\mathcal{LM}(a,b) = \begin{cases} \dfrac{\sqrt{a^2-b^2}}{\left[\operatorname{arcsl}\left(\sqrt[4]{1-b^2/a^2}\right)\right]^2}, & a > b, \\ \dfrac{\sqrt{b^2-a^2}}{\left[\operatorname{arcslh}\left(\sqrt[4]{b^2/a^2-1}\right)\right]^2}, & a < b. \end{cases}$$

Recall that the lemniscatic mean $\mathcal{LM}(a,b)$ of two positive real a and b is an iterative mean, i.e.,

$$\mathcal{LM}(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where

$$a_0 = a,\ b_0 = b,$$
$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_{n+1}a_n}$$

for $n = 0, 1, 2, \cdots$. And the lemniscatic mean is non-symmetric and homogeneous of degree one with respect to its variables a and b. Let

$$\mathcal{LM}_{\mathcal{GA}}(a,b) = \mathcal{LM}[\mathcal{G}(a,b), \mathcal{A}(a,b)],\ \mathcal{LM}_{\mathcal{AG}}(a,b) = \mathcal{LM}[\mathcal{A}(a,b), \mathcal{G}(a,b)],$$

$$\mathcal{LM}_{\mathcal{AQ}}(a,b) = \mathcal{LM}[\mathcal{A}(a,b), \mathcal{Q}(a,b)],\ \mathcal{LM}_{\mathcal{QA}}(a,b) = \mathcal{LM}[\mathcal{Q}(a,b), \mathcal{A}(a,b)].$$

Then one has four symmetric and homogeneous means given by explicitly as follows: (c.f. [3, 15])

$$\mathcal{LM}_{\mathcal{GA}}(a,b) = \mathcal{A}(a,b)\left(\frac{z}{\operatorname{arctlh} z}\right)^2,\ \mathcal{LM}_{\mathcal{AG}}(a,b) = \mathcal{A}(a,b)\left(\frac{z}{\operatorname{arcsl} z}\right)^2, \tag{1.1}$$

$$\mathcal{LM}_{\mathcal{AQ}}(a,b) = \mathcal{A}(a,b)\left(\frac{z}{\operatorname{arcslh} z}\right)^2,\ \mathcal{LM}_{\mathcal{QA}}(a,b) = \mathcal{A}(a,b)\left(\frac{z}{\operatorname{arctl} z}\right)^2 \tag{1.2}$$

with $z = \sqrt{|a-b|/(a+b)}$. Moreover, [3, (6.10)] showed that the derived means satisfy the following inequality chain:

$$\begin{aligned} \mathcal{H}(a,b) &< \mathcal{G}(a,b) < \mathcal{LM}_{\mathcal{GA}}(a,b) < \mathcal{LM}_{\mathcal{AG}}(a,b) < \mathcal{A}(a,b) \\ &< \mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{Q}(a,b) < \mathcal{C}(a,b) \end{aligned} \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$.

In the recent years, the arc lemniscate functions and lemniscatic means have attracted the attention of many researchers because they are closely related to special functions. In particular, many remarkable properties and inequalities involving them can be found in the literature [16–22]. For example, Neuman [3] obtained

Proposition 1.1. ([3, Proposition 6.1]) Inequalities

$$\begin{gathered} \mathcal{A}^2(a,b)\mathcal{G}(a,b)\mathcal{LM}_{\mathcal{GA}}(a,b) < \mathcal{LM}_{\mathcal{AG}}(a,b)^4, \quad \mathcal{G}^2(a,b)\mathcal{A}(a,b)\mathcal{LM}_{\mathcal{AG}}(a,b) < \mathcal{LM}_{\mathcal{GA}}(a,b)^4, \\ \mathcal{A}^2(a,b)\mathcal{Q}(a,b)\mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{LM}_{\mathcal{AQ}}(a,b)^4, \quad \mathcal{Q}^2(a,b)\mathcal{A}(a,b)\mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{LM}_{\mathcal{QA}}(a,b)^4, \\ \mathcal{LM}_{\mathcal{GA}}(a,b)\mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{A}^2(a,b), \quad \mathcal{LM}_{\mathcal{AG}}(a,b)\mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{A}^2(a,b) \end{gathered}$$

hold true.

Let $a, b > 0$, $p \in [1, \infty)$, $q \in [1/2, +\infty)$, $t \in [0, 1/2]$ and

$$\mathcal{GA}_{t,p}(a,b) = \mathcal{G}^p[ta + (1-t)b, tb + (1-t)a]\mathcal{A}^{1-p}(a,b) \tag{1.4}$$

and

$$\mathcal{CA}_{t,q}(a,b) = \mathcal{C}^q[ta + (1-t)b, tb + (1-t)a]\mathcal{A}^{1-q}(a,b). \tag{1.5}$$

Then it is easy to check that

$$\mathcal{GA}_{t,1}(a,b)=\mathcal{G}[ta+(1-t)b,tb+(1-t)a],$$

$$\mathcal{GA}_{t,2}(a,b)=\mathcal{H}[ta+(1-t)b,tb+(1-t)a],$$

$$\mathcal{CA}_{t,\frac{1}{2}}(a,b)=\mathcal{Q}[ta+(1-t)b,tb+(1-t)a],$$

$$\mathcal{CA}_{t,1}(a,b)=\mathcal{C}[ta+(1-t)b,tb+(1-t)a]$$

and the function $t\to\mathcal{GA}_{t,p}(a,b)$ is strictly increasing on the interval $[0,1/2]$ for fixed $a,b>0$ with $a\neq b$ and $p\in[1,+\infty)$, while $t\to\mathcal{CA}_{t,q}(a,b)$ is strictly decreasing on $[0,1/2]$ for fixed $a,b>0$ with $a\neq b$ and $q\in[1/2,+\infty)$. For more inequalities involving $\mathcal{GA}_{t,p}(a,b)$, $\mathcal{CA}_{t,q}(a,b)$ and thier applications in special functions, the readers can refer to the literature [8, 23–35].

Since

$$\begin{aligned}\mathcal{GA}_{0,p}(a,b)&=\mathcal{A}(a,b)\left[\frac{\mathcal{G}(a,b)}{\mathcal{A}(a,b)}\right]^{p}\leq\mathcal{G}(a,b)<\mathcal{LM}_{\mathcal{GA}}(a,b)<\mathcal{LM}_{\mathcal{AG}}(a,b)\\&<\mathcal{A}(a,b)=\mathcal{GA}_{\frac{1}{2},p}(a,b)=\mathcal{CA}_{\frac{1}{2},q}(a,b)<\mathcal{LM}_{\mathcal{AQ}}(a,b)<\mathcal{LM}_{\mathcal{QA}}(a,b)\\&<\mathcal{Q}(a,b)\leq\mathcal{A}(a,b)\left[\frac{\mathcal{C}(a,b)}{\mathcal{A}(a,b)}\right]^{q}=\mathcal{CA}_{0,q}(a,b)\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$, then it is nature to ask that: what are the best possible parameters λ_1, λ_2, λ_3, λ_4, μ_1, μ_2, μ_3, $\mu_4\in[0,1/2]$ such that the double inequalities

$$\mathcal{GA}_{\lambda_1,p}(a,b)<\mathcal{LM}_{\mathcal{GA}}(a,b)<\mathcal{GA}_{\mu_1,p}(a,b),\quad \mathcal{GA}_{\lambda_2,p}(a,b)<\mathcal{LM}_{\mathcal{AG}}(a,b)<\mathcal{GA}_{\mu_2,p}(a,b)$$

$$\mathcal{CA}_{\lambda_3,q}(a,b)<\mathcal{LM}_{\mathcal{AQ}}(a,b)<\mathcal{CA}_{\mu_3,q}(a,b),\quad \mathcal{CA}_{\lambda_4,q}(a,b)<\mathcal{LM}_{\mathcal{QA}}(a,b)<\mathcal{CA}_{\mu_4,q}(a,b)$$

hold for all $p\in[1,\infty)$, $q\in[1/2,+\infty)$ and $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results, we need the derivative formulas of the arc lemniscate functions and several lemmas, which we present in this section.

$$\frac{d\operatorname{arcsl}(x)}{dx}=\left(1-x^4\right)^{-1/2},\quad \frac{d\operatorname{arctlh}(x)}{dx}=\left(1-x^4\right)^{-3/4},\ |x|<1,$$
$$\frac{d\operatorname{arcslh}(x)}{dx}=\left(1+x^4\right)^{-1/2},\quad \frac{d\operatorname{arctl}(x)}{dx}=\left(1+x^4\right)^{-3/4},\ x\in\mathbb{R}.$$

Lemma 2.1 ([36, Theorem 1.25]). Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and differentiable on (a,b). Let $g'(x)\neq0$ on (a,b). Then, if $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), so are

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad\text{and}\quad\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 ([22, Lemma 3.3]). (1) The function $x \to \frac{\sqrt{1-x^4}\text{arcsl}(x)}{x}$ is strictly decreasing from $(0,1)$ onto $(0,1)$;

(2) The function $x \to \frac{\sqrt{1+x^4}\text{arcslh}(x)}{x}$ is strictly increasing from $(0,1)$ onto $(1,\omega)$;

(3) The function $x \to \frac{\sqrt[4]{1+x^4}\text{arctl}(x)}{x}$ is strictly increasing from $(0,1)$ onto $(1,2^{1/4}\tau)$;

(4) The function $x \to \frac{\sqrt[4]{1-x^4}\text{arctlh}(x)}{x}$ is strictly decreasing from $(0,1)$ onto $(0,1)$.

Lemma 2.3. *For* $u \in [0,1]$, $p \in [1,\infty)$. *Let*

$$f(u,p;x) = \frac{1}{2}p\log\left(1-ux^4\right) - 2\log x + 2\log\left[\text{arctlh}(x)\right], \ x \in (0,1). \tag{2.1}$$

Then the following statements are true:

(1) $f(u,p;x) > 0$ *for* $x \in (0,1)$ *if and only if* $u \le 3/(5p)$;

(2) $f(u,p;x) < 0$ *for* $x \in (0,1)$ *if and only if* $u \ge 1 - \left[1/(4\omega^4)\right]^{1/p}$.

Proof. It follows from (2.1) that

$$f(u,p;0^+) = 0, \tag{2.2}$$

$$f(u,p;1^-) = \frac{1}{2}p\log(1-u) + 2\log\left(\sqrt{2}\omega\right), \tag{2.3}$$

$$\frac{df(u,p;x)}{dx} = \frac{2x^3\left[x+(p-1)(1-x^4)^{3/4}\text{arctlh}(x)\right]}{(1-x^4)^{3/4}(1-ux^4)\text{arctlh}(x)}\left[f_p(x)-u\right], \tag{2.4}$$

where

$$f_p(x) = \frac{x-(1-x^4)^{3/4}\text{arctlh}(x)}{x^4\left[x+(p-1)(1-x^4)^{3/4}\text{arctlh}(x)\right]}.$$

Let $\zeta_1(x) = x/(1-x^4)^{3/4} - \text{arctlh}(x)$ and $\zeta_2(x) = x^5/(1-x^4)^{3/4} + (p-1)x^4\text{arctlh}(x)$. Then simple computations lead to

$$\zeta_1(0^+) = \zeta_2(0^+) = 0, \ f_p(x) = \zeta_1(x)/\zeta_2(x), \tag{2.5}$$

$$\frac{\zeta_1'(x)}{\zeta_2'(x)} = \frac{1}{1+\frac{1}{3}(p+1)(1-x^4)+\frac{4}{3}(p-1)(1-x^4)^{3/2}\frac{(1-x^4)^{1/4}\text{arctlh}(x)}{x}}. \tag{2.6}$$

Eq (2.6) and Lemma 2.2(4) show that $\zeta_1'(x)/\zeta_2'(x)$ is strictly increasing on $(0,1)$, so is $f_p(x)$ by (2.5) and Lemma 2.1. Moreover,

$$f_p(0^+) = \lim_{x\to 0^+}\frac{\zeta_1'(x)}{\zeta_2'(x)} = \frac{3}{5p}, \quad f_p(1^-) = 1. \tag{2.7}$$

Following we divide the proof into three cases.

Case 1 $0 \le u \le 3/(5p)$. Then from (2.4) and (2.7) together with the monotonicity of $f_p(x)$ lead to the conclusion that the function $x \mapsto f(u,p;x)$ is strictly increasing on $(0,1)$. Therefore, $f(u,p;x) > 0$ for all $x \in (0,1)$ follows from (2.2).

Case 2 $1 - \left[1/(4\omega^4)\right]^{1/p} \le u \le 1$(Since the function $u \to f(u,p;1^-)$ is strictly decreasing on $(0,1)$, and $f(3/(5p),p;1^-) > 0$, $f(1,p;1^-) = -\infty$, then the function $u \to f(u,p;1^-)$ has a unique zero-point

$1-\left[1/(4\omega^4)\right]^{1/p} \in (3/(5p),1)$ for any fixed $p \in [1,\infty)$). Then it follows from (2.3), (2.4) and (2.7) together with the monotonicity of $f_p(x)$ that

$$f(u,p;1^-) \le 0,$$

and there exists $x_0 \in (0,1)$ such that the function $x \mapsto f(u,p;x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$. Therefore, together with (2.2), $f(u,p;x)<0$ for all $x\in(0,1)$.

Case 3 $3/(5p) < u < 1-\left[1/(4\omega^4)\right]^{1/p}$. Then it is easy to check that the function $x \mapsto f(u,p;x)$ has the same piecewise monotone property in Case 2, and while

$$f(u,p;1^-) > 0.$$

Combining with (2.2), we conclude that there exists $x_0^* \in (0,1)$ such that $f(u,p;x)<0$ for $x \in \left(0,x_0^*\right)$ and $f(u,p;x)>0$ for $x\in\left(x_0^*,1\right)$. □

Lemma 2.4. *For $v\in[0,1]$, $p\in[1,\infty)$. Let*

$$g(v,p;x) = \frac{1}{2}p\log\left(1-vx^4\right) - 2\log x + 2\log\left[\mathrm{arcsl}(x)\right] \tag{2.8}$$

Then the following statements are true:

(1) $g(v,p;x)>0$ *for* $x\in(0,1)$ *if and only if* $v \le 2/(5p)$;

(2) $g(v,p;x)<0$ *for* $x\in(0,1)$ *if and only if* $v \ge 1-\left(1/\omega^4\right)^{1/p}$.

Proof. It follows from (2.8) that

$$g(v,p;0^+) = 0, \tag{2.9}$$

$$g(v,p;1^-) = \frac{1}{2}p\log(1-v) + 2\log\omega, \tag{2.10}$$

$$\frac{dg(v,p;x)}{dx} = \frac{2x^3\left[x+(p-1)\sqrt{1-x^4}\,\mathrm{arcsl}(x)\right]}{\sqrt{1-x^4}(1-vx^4)\mathrm{arcsl}(x)}\left[g_p(x)-v\right], \tag{2.11}$$

where

$$g_p(x) = \frac{x-\sqrt{1-x^4}\,\mathrm{arcsl}(x)}{x^4\left[x+(p-1)\sqrt{1-x^4}\,\mathrm{arcsl}(x)\right]}.$$

Let $\xi_1(x) = x/\sqrt{1-x^4} - \mathrm{arcsl}(x)$ and $\xi_2(x) = x^5/\sqrt{1-x^4} + (p-1)x^4\mathrm{arcsl}(x)$. Then simple computations lead to

$$\xi_1(0^+) = \xi_2(0^+) = 0,\ g_p(x) = \xi_1(x)/\xi_2(x), \tag{2.12}$$

$$\frac{\xi_1'(x)}{\xi_2'(x)} = \frac{1}{1+\frac{1}{2}(p+2)(1-x^4)+2(p-1)(1-x^4)\frac{\sqrt{1-x^4}\mathrm{arcsl}(x)}{x}}. \tag{2.13}$$

Eq (2.13) and Lemma 2.2(1) imply that $\xi_1'(x)/\xi_2'(x)$ is strictly increasing on $(0,1)$. Therefore, the conclusion that $g_p(x)$ is strictly increasing on $(0,1)$ follows from Lemma 2.1 and (2.12). Moreover,

$$g_p(0^+) = \lim_{x\to0^+}\frac{\xi_1'(x)}{\xi_2'(x)} = \frac{2}{5p},\ g_p(1^-) = 1. \tag{2.14}$$

Following we divide the proof into three cases.

Case 1 $0 < v \leq 2/(5p)$. Then (2.11), (2.14) and the monotonicity of the function $g_p(x)$ lead to the conclusion that the function $x \mapsto g(v, p; x)$ is strictly increasing on $(0, 1)$. Therefore, $g(v, p; x) > 0$ for all $x \in (0, 1)$ follows from (2.9).

Case 2 $1 - \left(1/\omega^4\right)^{1/p} \leq v < 1$ (With the similar argument in the proof of Lemma 2.3, we claim that $1 - \left(1/\omega^4\right)^{1/p} \in (2/(5p), 1)$). Then it follows from (2.10), (2.11) and (2.14) together with the monotonicity of $g_p(x)$ that

$$g\left(v, p; 1^-\right) \leq 0,$$

and there exists $x_1 \in (0, 1)$ such that the function $x \mapsto g(v, p; x)$ is strictly decreasing on $(0, x_1)$ and strictly increasing on $(x_1, 1)$. Therefore, together with (2.9), $g(v, p; x) < 0$ for all $x \in (0, 1)$.

Case 3 $2/(5p) < v < 1 - \left(1/\omega^4\right)^{1/p}$. Then the function $x \mapsto g(v, p; x)$ also first decreases and then increases on $(0, 1)$, and from (2.10) we get

$$g(v, p; 1^-) > 0.$$

Combining with (2.9), we conclude that there exists $x_1^* \in (0, 1)$ such that $g(v, p; x) < 0$ for $x \in \left(0, x_1^*\right)$ and $g(v, p; x) > 0$ for $x \in \left(x_1^*, 1\right)$. □

Lemma 2.5. *For $u \in [0, 1]$, $q \in [1/2, \infty)$. Let*

$$F\left(u, q; x\right) = q \log\left(1 + ux^4\right) - 2 \log x + 2 \log\left[\operatorname{arcslh}(x)\right], \ x \in (0, 1). \tag{2.15}$$

Then the following statements are true:

(1) $F\left(u, q; x\right) > 0$ *for* $x \in (0, 1)$ *if and only if* $u \geq 1/\left(5q\right)$*;*

(2) $F\left(u, q; x\right) < 0$ *for* $x \in (0, 1)$ *if and only if* $u \leq \left(\sqrt{2}/\omega\right)^{2/q} - 1$.

Proof. It follows from (2.15) that

$$F(u, q; 0^+) = 0, \tag{2.16}$$

$$F(u, q; 1^-) = q \log\left(1 + u\right) + 2 \log \omega - \log 2, \tag{2.17}$$

$$\frac{dF(u, q; x)}{dx} = \frac{2x^3\left[x + (2q - 1)\sqrt{1 + x^4}\operatorname{arcslh}(x)\right]}{\sqrt{1 + x^4}(1 + ux^4)\operatorname{arcslh}(x)}\left[u - F_q(x)\right], \tag{2.18}$$

where

$$F_q(x) = \frac{\sqrt{1 + x^4}\operatorname{arcslh}(x) - x}{x^4\left[x + (2q - 1)\sqrt{1 + x^4}\operatorname{arcslh}(x)\right]}.$$

Let $\zeta_1^*(x) = \operatorname{arcslh}(x) - x/\sqrt{1 + x^4}$ and $\zeta_2^*(x) = x^5/\sqrt{1 + x^4} + (2q - 1)x^4\operatorname{arcslh}(x)$. Then simple computations lead to

$$\zeta_1^*(0^+) = \zeta_2^*(0^+) = 0, \ F_q(x) = \zeta_1^*(x)/\zeta_2^*(x), \tag{2.19}$$

$$\frac{\zeta_1^{*\prime}(x)}{\zeta_2^{*\prime}(x)} = \frac{1}{1 + (q + 1)(1 + x^4) + 2(2q - 1)(1 + x^4)\frac{\sqrt{1+x^4}\operatorname{arcslh}(x)}{x}}. \tag{2.20}$$

Eq (2.20) and Lemma 2.2(2) show that $\zeta_1^{*\prime}(x)/\zeta_2^{*\prime}(x)$ is strictly decreasing on $(0,1)$, so is $F_q(x)$ by (2.19) and Lemma 2.1. Moreover,

$$F_q(0^+) = \lim_{x\to 0^+} \frac{\zeta_1^{*\prime}(x)}{\zeta_2^{*\prime}(x)} = \frac{1}{5q}, \quad F_q(1^-) = \frac{\omega - 1}{1 + (2q-1)\omega}. \tag{2.21}$$

Following we divide the proof into three cases.

Case 1 $1/(5q) \le u \le 1$. Then (2.18) and (2.21) together with the monotonicity of $F_q(x)$ imply that the function $x \mapsto F(u,q;x)$ is strictly increasing on $(0,1)$. Therefore, $F(u,q;x) > 0$ for all $x \in (0,1)$ follows from (2.16).

Case 2 $0 \le u \le (\omega-1)/[1+(2q-1)\omega]$. Then the function $x \mapsto F(u,q;x)$ is strictly decreasing on $(0,1)$, and consequently $F(u,q;x) < 0$ for all $x \in (0,1)$.

Case 3 $(\omega-1)/[1+(2q-1)\omega] < u < 1/(5q)$. Then it follows from (2.18) and (2.21) together with the monotonicity of $F_q(x)$ that and there exists $x_2 \in (0,1)$ such that the function $x \mapsto F(u,q;x)$ is strictly decreasing on $(0,x_2)$ and strictly increasing on $(x_2,1)$.

Since the function $u \to F(u,q;1^-)$ is strictly increasing on $[0,1]$, and $F(1/(5q),q;1^-) > 0$, $F((\omega-1)/[1+(2q-1)\omega],q;1^-) < 0$, then the function $u \to F(u,q;1^-)$ has a unique zero-point $\left(\sqrt{2}/\omega\right)^{2/q} - 1 \in ((\omega-1)/[1+(2q-1)\omega], 1/(5q))$ for any fixed $q \in [1/2,+\infty)$. Finally we investigate two subcases $\left(\sqrt{2}/\omega\right)^{2/q} - 1 < u < 1/(5q)$ and $(\omega-1)/[1+(2q-1)\omega] < u \le \left(\sqrt{2}/\omega\right)^{2/q} - 1$.

Subcase 3.1 $\left(\sqrt{2}/\omega\right)^{2/q} - 1 < u < 1/(5q)$. Then

$$F(u,q;1^-) > 0$$

and thereby there exists $x_2^* \in (0,1)$ such that $F(u,q;x) < 0$ for $x \in (0,x_2^*)$ and $F(u,q;x) > 0$ for $x \in \left(x_2^*,1\right)$.

Subcase 3.2 $(\omega-1)/[1+(2q-1)\omega] < u \le \left(\sqrt{2}/\omega\right)^{2/q} - 1$. Then

$$F(u,q;1^-) \le 0$$

and therefore $F(u,q;x) < 0$ for $x \in (0,1)$. □

Lemma 2.6. *For $v \in [0,1]$, $q \in [1/2,\infty)$. Let*

$$G(v,q;x) = q\log\left(1+vx^4\right) - 2\log x + 2\log\left[\operatorname{arctl}(x)\right], \ x \in (0,1). \tag{2.22}$$

Then the following statements are true:

(1) $G(v,q;x) > 0$ *for $x \in (0,1)$ if and only if $v \ge 3/(10q)$;*

(2)$G(v,q;x) < 0$ *for $x \in (0,1)$ if and only if $u \le (1/\tau)^{2/q} - 1$.*

Proof. It follows from (2.22) that

$$G(v,q;0^+) = 0, \tag{2.23}$$

$$G(v,q;1^-) = q\log(1+v) + 2\log\tau, \tag{2.24}$$

$$\frac{dG(v,q;x)}{dx} = \frac{2x^3\left[x + (2q-1)(1+x^4)^{3/4}\operatorname{arctl}(x)\right]}{(1+x^4)^{3/4}(1+vx^4)\operatorname{arctl}(x)}\left[v - G_q(x)\right], \tag{2.25}$$

where

$$G_q(x) = \frac{(1+x^4)^{3/4}\mathrm{arctl}(x) - x}{x^4\left[x + (2q-1)(1+x^4)^{3/4}\mathrm{arctl}(x)\right]}.$$

Let $\xi_1^*(x) = \mathrm{arctl}(x) - x/(1+x^4)^{3/4}$ and $\xi_2^*(x) = x^5/(1+x^4)^{3/4} + (2q-1)x^4\mathrm{arctl}(x)$. Then simple computations lead to

$$\xi_1^*(0^+) = \xi_2^*(0^+) = 0,\ G_q(x) = \xi_1^*(x)/\xi_2^*(x), \tag{2.26}$$

$$\frac{\xi_1^{*\prime}(x)}{\xi_2^{*\prime}(x)} = \frac{1}{1 + \frac{1}{3}(2q+1)(1+x^4) + \frac{4}{3}(2q-1)(1+x^4)^{3/2}\frac{\sqrt[4]{1+x^4}\mathrm{arctl}(x)}{x}}. \tag{2.27}$$

Eq (2.27) and Lemma 2.2 (3) show that $\xi_1^{*\prime}(x)/\xi_2^{*\prime}(x)$ is strictly decreasing on $(0,1)$, so is $G_q(x)$ by (2.26) and Lemma 2.1. Moreover,

$$G_q(0^+) = \lim_{x\to 0^+}\frac{\xi_1^{*\prime}(x)}{\xi_2^{*\prime}(x)} = \frac{3}{10q}, \quad G_q(1^-) = \frac{2^{3/4}\tau - 1}{1 + 2^{3/4}(2q-1)\tau}. \tag{2.28}$$

Following we divide the proof into three cases.

Case 1 $3/(10q) \le v \le 1$. Then (2.25) and (2.28) together with the monotonicity of $G_q(x)$ imply that the function $x \mapsto G(v,q;x)$ is strictly increasing on $(0,1)$. Therefore, $G(v,q;x) > 0$ for all $x \in (0,1)$ follows from (2.23).

Case 2 $0 \le v \le (2^{3/4}\tau - 1)/[1 + 2^{3/4}(2q-1)\tau]$. Then the function $x \mapsto G(v,q;x)$ is strictly decreasing on $(0,1)$, and consequently $G(v,q;x) < 0$ for all $x \in (0,1)$.

Case 3 $(2^{3/4}\tau - 1)/[1 + 2^{3/4}(2q-1)\tau] < v < 3/(10q)$. Then it follows from (2.25) and (2.28) together with the monotonicity of $G_q(x)$ that and there exists $x_3 \in (0,1)$ such that the function $x \mapsto G(v,q;x)$ is strictly decreasing on $(0,x_3)$ and strictly increasing on $(x_3,1)$.

With the similar argument in the proof of Lemma 2.5, we also obtain that $(1/\tau)^{2/p} - 1 \in ((2^{3/4}\tau - 1)/[1 + 2^{3/4}(2q-1)\tau], 3/(10q))$. We divide the proof into two Subcases.

Subcase 3.1 $(1/\tau)^{2/p} - 1 < v < 3/(10q)$. Then

$$G(v,q;1^-) > 0$$

and thereby there exists $x_3^* \in (0,1)$ such that $G(v,q;x) < 0$ for $x \in (0,x_3^*)$ and $G(v,q;x) > 0$ for $x \in \left(x_3^*, 1\right)$.

Subcase 3.2 $(2^{3/4}\tau - 1)/[1 + 2^{3/4}(2q-1)\tau] < v \le (1/\tau)^{2/p} - 1$. Then

$$G(v,q;1^-) \le 0$$

and therefore $G(v,q;x) < 0$ for $x \in (0,1)$. □

3. Main results

Theorem 3.1. *Let $p \in [1,\infty)$ and $\lambda_1, \mu_1 \in [0,1/2]$. Then the double inequalities*

$$\mathcal{GA}_{\lambda_1,p}(a,b) < \mathcal{LM}_{\mathcal{GA}}(a,b) < \mathcal{GA}_{\mu_1,p}(a,b)$$

hold for all $a,b > 0$ with $a \ne b$ if and only if $\lambda_1 \le 1/2 - \sqrt{1 - \left[1/\left(4\omega^4\right)\right]^{1/p}}/2$ and $\mu_1 \ge 1/2 - \sqrt{15p}/(10p)$.

Proof. Let $t \in [0, 1/2]$, since $\mathcal{GA}_{t,p}(a,b)$ and $\mathcal{LM}_{\mathcal{GA}}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b > 0$. Let $x = \sqrt{(a-b)/(a+b)} \in (0,1)$. Then from (1.1) and (1.4) we get

$$\begin{aligned}
&\log\left[\frac{\mathcal{GA}_{t,p}(a,b)}{\mathcal{LM}_{\mathcal{GA}}(a,b)}\right] = \log\left[\frac{\mathcal{GA}_{t,p}(a,b)}{\mathcal{A}(a,b)}\right] - \log\left[\frac{\mathcal{LM}_{\mathcal{GA}}(a,b)}{\mathcal{A}(a,b)}\right] \\
=&\frac{1}{2}p\log\left[1-(1-2t)^2x^4\right] - 2\log x + 2\log\left[\operatorname{arctlh}(x)\right] \\
=&f\left((1-2t)^2, p; x\right),
\end{aligned} \tag{3.1}$$

where $f(\cdot, p; x)$ is defined in Lemma 2.3.

Therefore, Theorem 3.1 follows easily from Lemma 2.3 and (3.1). □

Theorem 3.2. *Let $p \in [1, \infty)$ and $\lambda_2, \mu_2 \in [0, 1/2]$. Then the double inequalities*

$$\mathcal{GA}_{\lambda_2,p}(a,b) < \mathcal{LM}_{\mathcal{AG}}(a,b) < \mathcal{GA}_{\mu_2,p}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq 1/2 - \sqrt{1-(1/\omega^4)^{1/p}}/2$ and $\mu_2 \geq 1/2 - \sqrt{10p}/(10p)$.

Proof. Let $t \in [0, 1/2]$. Without loss of generality, we suppose that $x = \sqrt{(a-b)/(a+b)} \in (0,1)$. Then from (1.1) and (1.4) we get

$$\begin{aligned}
&\log\left[\frac{\mathcal{GA}_{t,p}(a,b)}{\mathcal{LM}_{\mathcal{AG}}(a,b)}\right] = \log\left[\frac{\mathcal{GA}_{t,p}(a,b)}{\mathcal{A}(a,b)}\right] - \log\left[\frac{\mathcal{LM}_{\mathcal{AG}}(a,b)}{\mathcal{A}(a,b)}\right] \\
=&\frac{1}{2}p\log\left[1-(1-2t)^2x^4\right] - 2\log(x) + 2\log\left[\operatorname{arcsl}(x)\right] \\
=&g\left((1-2t)^2, p; x\right),
\end{aligned} \tag{3.2}$$

where $g(\cdot, p; x)$ is defined in Lemma 2.4.

Therefore, Theorem 3.2 follows easily from Lemma 2.4 and (3.2). □

Theorem 3.3. *Let $q \in [1/2, \infty)$ and $\lambda_3, \mu_3 \in [0, 1/2]$. Then the double inequalities*

$$\mathcal{CA}_{\lambda_3,q}(a,b) < \mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{CA}_{\mu_3,q}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_3 \geq 1/2 - \sqrt{\left(\sqrt{2}/\omega\right)^{2/q} - 1}/2$ and $\mu_3 \leq 1/2 - \sqrt{5q}/(10q)$.

Proof. Let $t \in [0, 1/2]$, since $\mathcal{CA}_{t,q}(a,b)$ and $\mathcal{LM}_{\mathcal{GA}}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b > 0$. Let $x = \sqrt{(a-b)/(a+b)} \in (0,1)$. Then from (1.2) and (1.5) we get

$$\begin{aligned}
&\log\left[\frac{\mathcal{CA}_{t,q}(a,b)}{\mathcal{LM}_{\mathcal{AQ}}(a,b)}\right] = \log\left[\frac{\mathcal{CA}_{t,q}(a,b)}{\mathcal{A}(a,b)}\right] - \log\left[\frac{\mathcal{LM}_{\mathcal{AQ}}(a,b)}{\mathcal{A}(a,b)}\right] \\
=&q\log\left[1+(1-2t)^2x^4\right] - 2\log x + 2\log\left[\operatorname{arcslh}(x)\right] \\
=&F\left((1-2t)^2, q; x\right),
\end{aligned} \tag{3.3}$$

where $F(\cdot, q; x)$ is defined in Lemma 2.5.

Therefore, Theorem 3.3 follows easily from Lemma 2.5 and (3.3). □

Theorem 3.4. *Let $q \in [1/2, \infty)$ and $\lambda_4, \mu_4 \in [0, 1/2]$. Then the double inequalities*

$$\mathcal{CA}_{\lambda_4,q}(a,b) < \mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{CA}_{\mu_4,q}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_4 \geq 1/2 - \sqrt{(1/\tau)^{2/q} - 1}/2$ and $\mu_4 \leq 1/2 - \sqrt{30q}/(20q)$.

Proof. Let $t \in [0, 1/2]$. Without loss of generality, we suppose that $x = \sqrt{(a-b)/(a+b)} \in (0, 1)$. Then from (1.2) and (1.5) we get

$$\begin{aligned}
&\log\left[\frac{\mathcal{CA}_{t,q}(a,b)}{\mathcal{LM}_{\mathcal{QA}}(a,b)}\right] = \log\left[\frac{\mathcal{CA}_{t,q}(a,b)}{\mathcal{A}(a,b)}\right] - \log\left[\frac{\mathcal{LM}_{\mathcal{QA}}(a,b)}{\mathcal{A}(a,b)}\right] \\
=&q\log\left[1 + (1-2t)^2 x^4\right] - 2\log(x) + 2\log\left[\operatorname{arctl}(x)\right] \\
=&G\left((1-2t)^2, q; x\right),
\end{aligned} \tag{3.4}$$

where $G(\cdot, q; x)$ is defined in Lemma 2.6.

Therefore, Theorem 3.4 follows easily from Lemma 2.6 and (3.4). □

Corollary 3.5. *Let α_1, α_2, α_3, α_4, β_1, β_2, β_3, $\beta_4 \in [0, 1/2]$. Then the double inequalities*

$$\mathcal{H}\left[\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a\right] < \mathcal{LM}_{\mathcal{GA}}(a,b) < \mathcal{H}\left[\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a\right],$$

$$\mathcal{G}\left[\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a\right] < \mathcal{LM}_{\mathcal{GA}}(a,b) < \mathcal{G}\left[\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a\right],$$

$$\mathcal{H}\left[\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\right] < \mathcal{LM}_{\mathcal{AG}}(a,b) < \mathcal{H}\left[\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\right],$$

$$\mathcal{G}\left[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\right] < \mathcal{LM}_{\mathcal{AG}}(a,b) < \mathcal{G}\left[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\right]$$

hold for all $a, b > 0$ with $a \neq b$ with the best possible parameters

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2} - \tfrac{\sqrt{1-1/(2\omega^2)}}{2}, & \beta_1 &= \tfrac{1}{2} - \tfrac{\sqrt{30}}{20}, \\
\alpha_2 &= \tfrac{1}{2} - \tfrac{\sqrt{1-1/(4\omega^4)}}{2}, & \beta_2 &= \tfrac{1}{2} - \tfrac{\sqrt{15}}{10}, \\
\alpha_3 &= \tfrac{1}{2} - \tfrac{\sqrt{1-1/\omega^2}}{2}, & \beta_3 &= \tfrac{1}{2} - \tfrac{\sqrt{5}}{10}, \\
\alpha_4 &= \tfrac{1}{2} - \tfrac{\sqrt{1-1/\omega^4}}{2}, & \beta_4 &= \tfrac{1}{2} - \tfrac{\sqrt{10}}{10}.
\end{aligned}$$

Corollary 3.6. *Let α_1^*, α_2^*, α_3^*, α_4^*, β_1^*, β_2^*, β_3^*, $\beta_4^* \in [0, 1/2]$. Then the double inequalities*

$$\mathcal{Q}[\alpha_1^* a + (1-\alpha_1^*)b, \alpha_1^* b + (1-\alpha_1^*)a] < \mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{Q}[\beta_1^* a + (1-\beta_1^*)b, \beta_1^* b + (1-\beta_1^*)a],$$

$$\mathcal{C}[\alpha_2^* a + (1-\alpha_2^*)b, \alpha_2^* b + (1-\alpha_2^*)a] < \mathcal{LM}_{\mathcal{AQ}}(a,b) < \mathcal{C}[\beta_2^* a + (1-\beta_2^*)b, \beta_2^* b + (1-\beta_2^*)a],$$

$$\mathcal{Q}[\alpha_3^* a + (1-\alpha_3^*)b, \alpha_3^* b + (1-\alpha_3^*)a] < \mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{Q}[\beta_3^* a + (1-\beta_3^*)b, \beta_3^* b + (1-\beta_3^*)a],$$

$$\mathcal{C}[\alpha_4^* a + (1-\alpha_4^*)b, \alpha_4^* b + (1-\alpha_4^*)a] < \mathcal{LM}_{\mathcal{QA}}(a,b) < \mathcal{C}[\beta_4^* a + (1-\beta_4^*)b, \beta_4^* b + (1-\beta_4^*)a]$$

hold for all $a, b > 0$ with $a \neq b$ with the best possible parameters

$$\begin{aligned}
\alpha_1^* &= \tfrac{1}{2} - \tfrac{\sqrt{4/\omega^4-1}}{2}, & \beta_1^* &= \tfrac{1}{2} - \tfrac{\sqrt{10}}{10}, \\
\alpha_2^* &= \tfrac{1}{2} - \tfrac{\sqrt{2/\omega^2-1}}{2}, & \beta_2^* &= \tfrac{1}{2} - \tfrac{\sqrt{5}}{10}, \\
\alpha_3^* &= \tfrac{1}{2} - \tfrac{\sqrt{1/\tau^4-1}}{2}, & \beta_3^* &= \tfrac{1}{2} - \tfrac{\sqrt{15}}{10}, \\
\alpha_4^* &= \tfrac{1}{2} - \tfrac{\sqrt{1/\tau^2-1}}{2}, & \beta_4^* &= \tfrac{1}{2} - \tfrac{\sqrt{30}}{20}.
\end{aligned}$$

4. Inequalities for Lemniscate functions

Theorem 4.1. *The double inequalities*

$$\left(1-\frac{3}{5p}x^4\right)^{-p/4} < \frac{\operatorname{arctlh}(x)}{x} < \left[1-\left(1-2^{-2/p}\omega^{-4/p}\right)x^4\right]^{-p/4}, \tag{4.1}$$

$$\left(1-\frac{2}{5p}x^4\right)^{-p/4} < \frac{\operatorname{arcsl}(x)}{x} < \left[1-\left(1-\omega^{-4/p}\right)x^4\right]^{-p/4} \tag{4.2}$$

hold for all $p \in [1, \infty)$ *and* $x \in (0, 1)$.

Proof. Substituting $\lambda_1 = 1/2 - \sqrt{1-[1/(4\omega^4)]^{1/p}}/2$, $\mu_1 = 1/2 - \sqrt{15p}/(10p)$, and $\lambda_2 = 1/2 - \sqrt{1-(1/\omega^4)^{1/p}}/2$, $\mu_2 = 1/2 - \sqrt{10p}/(10p)$ into Theorems 3.1 and 3.2, we obtain (4.1) and (4.2) immediately. □

Theorem 4.2. *The double inequalities*

$$\left(1+\frac{1}{5q}x^4\right)^{-q/2} < \frac{\operatorname{arcslh}(x)}{x} < \left[1-\left(1-2^{1/q}\omega^{-2/q}\right)x^4\right]^{-q/2}, \tag{4.3}$$

$$\left(1+\frac{3}{10q}x^4\right)^{-q/2} < \frac{\operatorname{arctl}(x)}{x} < \left[1-\left(1-\tau^{-2/q}\right)x^4\right]^{-q/2} \tag{4.4}$$

hold for all $q \in [1/2, \infty)$ *and* $x \in (0, 1)$.

Proof. Substituting $\lambda_3 = 1/2 - \sqrt{\left(\sqrt{2}/\omega\right)^{2/q}-1}/2$, $\mu_3 = 1/2 - \sqrt{5q}/(10q)$ and $\lambda_4 = 1/2 - \sqrt{(1/\tau)^{2/q}-1}/2$, $\mu_4 = 1/2 - \sqrt{30q}/(20q)$ into Theorems 3.3 and 3.4, we obtain (4.3) and (4.4) immediately. □

Lemma 4.3. (1) *The function* $t \to \frac{\log(1-rx^4t)}{t}$ *is strictly decreasing on* $(0, 1)$ *for any fixed* $x \in (0, 1)$ *and* $r \in (-1, 0) \cup (0, 1)$*;*

(2) *The function* $t \to \frac{\log[1-(1-r^t)x^4]}{t}$ *is strictly increasing on* $(0, 1)$ *for any fixed* $x \in (0, 1)$ *and* $r \in (0, +\infty)$.

Proof. For part (1), let $\phi_1(t) = \log(1 - rx^4t)$, $\phi_2(t) = t$ and $\phi(t) = \phi_1(t)/\phi_2(t)$. Then $\phi_1(0) = \phi_2(0) = 0$,

$$\frac{\phi_1'(t)}{\phi_2'(t)} = -\frac{rx^4}{1-(rx^4)t}. \tag{4.5}$$

It is apparent from (4.5) to see that $\phi_1'(t)/\phi_2'(t)$ is strictly decreasing on $(0, 1)$. By application of Lemma 2.1, part (1) is clear.

For part (2), let $\varphi_1(t) = \log\left[1-(1-r^t)x^4\right]$, $\varphi_2(t) = t$ and $\varphi(t) = \varphi_1(t)/\varphi_2(t)$. Then by simple computation we get $\varphi_1(0) = \varphi_2(0) = 0$, and

$$\frac{\varphi_1'(t)}{\varphi_2'(t)} = \frac{x^4r^t\log r}{1-(1-r^t)x^4},\ \frac{d}{dt}\left[\frac{\varphi_1'(t)}{\varphi_2'(t)}\right] = -\frac{(\log r)^2\,x^4(1-x^4)r^t}{[1-(1-r^t)x^4]^2} > 0$$

for all $t \in (0, 1)$. So that $\varphi_1'(t)/\varphi_2'(t)$ is strictly increasing on $(0, 1)$, applying Lemma 2.1, the assertion of part (2) follows. □

Employing Lemma 4.3, it is easy to see that the best estimates in (4.1) and (4.2) are arrived at for $p = 1$, and while these in (4.3) and (4.4) for $q = 1/2$.

Theorem 4.4. *The double sharp inequalities*

$$\left(1-\frac{2}{5}x^4\right)^{-1/4} < \frac{\mathrm{arcsl}(x)}{x} < \left[1-\left(1-\frac{1}{\omega^4}\right)x^4\right]^{-1/4},$$
$$\left(1+\frac{2}{5}x^4\right)^{-1/4} < \frac{\mathrm{arcslh}(x)}{x} < \left[1-\left(1-\frac{4}{\omega^4}\right)x^4\right]^{-1/4},$$
$$\left(1+\frac{3}{5}x^4\right)^{-1/4} < \frac{\mathrm{arctl}(x)}{x} < \left[1-\left(1-\frac{1}{\tau^4}\right)x^4\right]^{-1/4},$$
$$\left(1-\frac{3}{5}x^4\right)^{-1/4} < \frac{\mathrm{arctlh}(x)}{x} < \left[1-\left(1-\frac{1}{4\omega^4}\right)x^4\right]^{-1/4}$$

hold for all $x \in (0, 1)$.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. J. M. Borwein, P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, New York: John Wiley & Sons, 1987.
2. B. C. Carlson, Algorithms involving arithmetic and geometric means, *Am. Math. Mon.*, **78** (1971), 496–505.
3. E. Neuman, On Gauss lemniscate functions and lemniscatic mean, *Math. Panno.*, **18** (2007), 77–94.
4. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, New York: Dover, 1965.
5. H. Alzer, Sharp inequalities for the complete elliptic integral of the first kind, *Math. Proc. Cambridge*, **124** (1998), 309–314.
6. H. Alzer, S. L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, *J. Comput. Appl. Math.*, **172** (2004), 289–312.
7. S. L. Qiu, M. Vuorinen, Special functions in geometric function theory, In: *Handbook of complex analysis: Geometric function theory*, **2** (2005), 621–659.

8. M. K. Wang, Y. M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, *J. Math. Anal. Appl.*, **402** (2013), 119–126.

9. M. K. Wang, H. H. Chu, Y. M. Li, Y. M. Chu, Answers to thress conjectues on convexity of three functions involving complete elliptic integrals of the first kind, *Appl. Anal. Discr. Math.*, **14** (2020), 255–271.

10. Z. H. Yang, W. M. Qian, W. Zhang, Y. M. Chu. Notes on the complete elliptic integral of the first kind, *Math. Inequal. Appl.*, **23** (2020), 77–93.

11. G. E. Andrews, R. Askey, R. Roy, Special functions, In: *Encyclopedia of mathematics and its applications*, Cambridge: Cambridge University Press, **70** (1999).

12. Y. X. Li, A. Rauf, M. Naeem, M. A. Binyamin, A. Aslam, Valency-based topological properties of linear hexagonal chain and hammer-like benzenoid, *Complexity*, **2021** (2021), 9939469.

13. Z. H. Yang, W. M. Qian, Y. M. Chu, W. Zhang, On rational bounds for the gamma function, *J. Inequal. Appl.*, **2017** (2017), 210.

14. T. H. Zhao, Y. M. Chu, H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, *Abstr. Appl. Anal.*, **2011** (2011), 896483.

15. E. Neuman, On Gauss lemniscate functions and lemniscatic mean II, *Math. Pannon.*, **23** (2012), 65–73.

16. Y. X. Li, T. Muhammad, M. Bilal, M. A. Khan, A. Ahmadian, B. A. Pansera, Fractional simulation for Darcy-Forchheimer hybrid nanoliquid flow with partial slip over a spinning disk, *Alex. Eng. J.*, **60** (2021), 4787–4796.

17. M. I. Khan, S. Kadry, Y. M. Chu, W. A. Khane, A. Kumar, Exploration of Lorentz force on a paraboloid stretched surface in flow of Ree-Eyring nanomaterial, *J. Mater. Res. Technol.*, **9** (2020), 10265–10275.

18. J. E. Deng, C. P. Chen, Sharp Shafer-Fink type inequalities for Gauss lemniscate functions, *J. Inequal. Appl.*, **2014** (2014), 35.

19. J. Liu, C. P. Chen, Padé approximant related to inequalities for Gauss lemniscate functions, *J. Inequal. Appl.*, **2016** (2016), 1–16.

20. E. Neuman, Two-sided inequalities for the lemniscate functions, *J. Inequalities Special Funct.*, **1** (2010), 1–7.

21. E. Neuman, On lemniscate functions, *Integr. Transf. Spec. F.*, **24** (2013), 164–171.

22. M. J. Wei, Y. He, G. D. Wang, Shafer-Fink type inequalities for arc lemniscate functions, *RACSAM*, **114** (2020), 53.

23. Y. M. Chu, M. K. Wang, X. Y. Ma, Sharp bounds for Toader mean in terms of contraharmonic mean with applications, *J. Math. Inequal.*, **7** (2013), 161–166.

24. Y. M. Chu, M. K. Wang, Y. F. Qiu, X. Y. Ma, Sharp two parameter bounds for the logarithmic mean and the arithmetic-geometric mean of Gauss, *J. Math. Inequal.*, **7** (2013), 349–355.

25. M. Nowicka, A. Witkowski, Optimal bounds for the tangent and hyperbolic sine means II, *J. Math. Inequal.*, **14** (2020), 23–33.

26. W. M. Qian, Z. Y. He, Y. M. Chu, Approximation for the complete elliptic integral of the first kind, *RACSAM*, **114** (2020), 57.
27. W. M. Qian, X. H. Zhang, Y. M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, *J. Math. Inequal.*, **11** (2017), 121–127.
28. Y. X. Li, M. H. Alshbool, Y. P. Lv, I. Khan, M. R. Khan, A. Issakhov, Heat and mass transfere in MHD Williamson nanofluid flow over an exponentially porous stretching surface, *Case Stud. Therm. Eng.*, **26** (2021), 100975.
29. W. F. Xiang, S. Y. Xu, J. W. Lu, Y. M. Li, Y. M. Chu, Z. Q. Zhang, Event-triggered filtering for discrete-time Markovian jump systems with additive time-varying delays, *Appl. Math. Comput.*, **391** (2021), 125630.
30. M. K. Wang, Y. M. Chu, Y. M. Li, W. Zhang, Asymptotic expansion and bounds for complete elliptic integrals, *Math. Inequal. Appl.*, **23** (2020), 821–841.
31. M. K. Wang, Y. M. Chu, Y. Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, *Appl. Math. Comput.*, **276** (2016), 44–60.
32. M. K. Wang, H. H. Chu, Y. M. Chu, Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals, *J. Math. Anal. Appl.*, **480** (2019), 123388.
33. H. H. Chu, T. H. Zhao, Y. M. Chu, Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means, *Math. Slovaca*, **70** (2020), 1097–1112.
34. W. M. Qian, Y. Y. Yang, H. W. Zhang, Y. M. Chu, Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean, *J. Inequal. Appl.*, **2019** (2019), 287.
35. W. M. Qian, Z. Y. He, H. W. Zhang, Y. M. Chu, Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean, *J. Inequal. Appl.*, **2019** (2019), 168.
36. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Conformal invariants, inequalities, and quasiconformal maps*, New York: John Wiley & Sons, 1997.