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Research article

Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions

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Abstract: In this paper, we introduce a class of the multiplicative tempered fractional integral operators. Then, we investigate two Hermite–Hadamard type inequalities for this class. By using the established identity and the multiplicative convexity, we establish some integral inequalities for the multiplicative tempered fractional integrals involving the λ-incomplete gamma functions. And our results obtained in the present paper generalize some results given by Budak and Tunç (2020) and Ali et al. (2019). Also, we provide three examples to demonstrate the simplicities of the calculations.

Keywords: Hermite–Hadamard inequality; fractional integrals; multiplicative tempered fractional integrals; multiplicatively convex mappings
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1. Introduction and preliminaries

The convexity of functions is a powerful tool to deal with many kinds of issues of pure and applied science. In recent decades, many authors have devoted themselves to studying the properties and inequalities related to convexity in different directions, see [13, 21, 23, 34, 52] and the references cited therein. One of the most important mathematical inequalities concerning convex mapping is Hermite–Hadamard inequality, which is also utilized widely in many other disciplines of applied mathematics. Let's review it as follows:

Let $f : \mathcal{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $\mathcal{K}$ of real numbers and $\tau_1, \tau_2 \in \mathcal{K}$ with $\tau_1 < \tau_2$. The subsequent inequalities are called Hermite–Hadamard inequalities:

$$f\Big(\frac{\tau_1 + \tau_2}{2}\Big) \le \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(t)\mathrm{d}t \le \frac{f(\tau_1) + f(\tau_2)}{2}. \tag{1.1}$$

Many inequalities have been established in terms of inequalities (1.1) via functions of different classes, such as convex functions [28], s-convex functions [33], (α, m)-convex functions [47], harmonically convex functions [16], h-convex functions [18], strongly exponentially generalized preinvex functions [29], h-preinvex functions [37], p-quasiconvex functions [27], N-quasiconvex functions [3], etc. For more recent results about this topic, the readers may refer to [13, 22, 25, 26, 30, 32, 36] and the references cited therein.

The multiplicatively convex function is one of the most significant functions, which can be defined as follows.

Definition 1. *A mapping f: $I \subseteq \mathbb{R} \to [0, \infty)$ is said to be multiplicatively convex or log-convex, if log f is convex or equivalently for all τ_1, $\tau_2 \in I$ and $t \in [0, 1]$, one has the following inequality:*

$$f(t\tau_1 + (1-t)\tau_2) \leq [f(\tau_1)]^t [f(\tau_2)]^{1-t}.$$

From Definition 1, it follows that

$$f(t\tau_1 + (1-t)\tau_2) \leq [f(\tau_1)]^t [f(\tau_2)]^{1-t} \leq tf(\tau_1) + (1-t)f(\tau_2),$$

which reveals that every multiplicatively convex function is a convex mapping, but the converse is not true.

Many properties and inequalities associated with log-convex mappings have been studied by plenty of researchers. For example, Bai and Qi [9] gave several integral inequalities of the Hermite–Hadamard type for log-convex mappings. Dragomir [20] provided some unweighted and weighted inequalities of Hermite–Hadamard type related to log-convex mappings on real intervals. Set and Ardiç [46] established certain Hermite–Hadamard-like type integral inequalities involving log-convex mappings and p-functions. Zhang and Jiang [53] researched some properties for log-convex mapping. For more results on the basis of log-convex mappings, one can see, for example, [10, 39, 40, 49, 50] and the references cited therein.

In 2008, Bashirov [11] proposed a class of the multiplicative operators called *integral, which is denoted by $\int_a^b (f(x))^{\mathrm{d}x}$ and the ordinary integral is denoted by $\int_a^b f(x)\mathrm{d}x$. Recall that the function f is multiplicatively integrable on $[a, b]$, if f is positive and Riemann integrable on $[a, b]$ and

$$\int_a^b (f(x))^{\mathrm{d}x} = e^{\int_a^b \ln(f(x))\mathrm{d}x}.$$

Definition 2. *[11] Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of function f is given by*

$$\frac{\mathrm{d}^* f}{\mathrm{d}t}(t) = f^*(t) = \lim_{h \to 0} \left(\frac{f(t+h)}{f(h)} \right)^{\frac{1}{h}}.$$

If f has positive values and is differentiable at t, then f^* exists and the relation between f^* and ordinary derivative f' is as follows:

$$f^*(t) = e^{[\ln f(t)]'} = e^{\frac{f'(t)}{f(t)}}.$$

The following properties of *differentiable exist:

Theorem 1. *[11] Let f and g be *differentiable functions. If c is an arbitrary constant, then functions cf, fg, $f+g$, f/g and f^g are *differentiable and*

$(i)\,(cf)^*(t) = f^*(t)$,
$(ii)\,(fg)^*(t) = f^*(t)g^*(t)$,
$(iii)\,(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}}g^*(t)^{\frac{g(t)}{f(t)+g(t)}}$,
$(iv)\,\left(\frac{f}{g}\right)^*(t) = \frac{f^*(t)}{g^*(t)}$,
$(v)\,(f^g)^*(t) = f^*(t)^{g(t)}f(t)^{g'(t)}$.

Moreover, Bashirov et al. show that the multiplicative integral has the following properties:

Proposition 1. *[11] If f is positive and Riemann integrable on $[a,b]$, then f is *integrable on $[a,b]$ and*

$(i)\,\int_a^b ((f(x))^p)^{\mathrm{d}x} = \int_a^b \left((f(x))^{\mathrm{d}x}\right)^p$,
$(ii)\,\int_a^b (f(x)g(x))^{\mathrm{d}x} = \int_a^b (f(x))^{\mathrm{d}x}.\int_a^b (g(x))^{\mathrm{d}x}$,
$(iii)\,\int_a^b \left(\frac{f(x)}{g(x)}\right)^{\mathrm{d}x} = \frac{\int_a^b (f(x))^{\mathrm{d}x}}{\int_a^b (g(x))^{\mathrm{d}x}}$,
$(iv)\,\int_a^b (f(x))^{\mathrm{d}x} = \int_a^c (f(x))^{\mathrm{d}x}.\int_c^b (f(x))^{\mathrm{d}x}, \quad a \le c \le b$,
$(v)\,\int_a^a (f(x))^{\mathrm{d}x} = 1 \quad \text{and} \quad \int_a^b (f(x))^{\mathrm{d}x} = \left(\int_b^a (f(x))^{\mathrm{d}x}\right)^{-1}$.

The interesting geometric mean type inequalities, known as the Hermite–Hadamard inequality for the multiplicatively convex functions, are shown by the following theorem in [7].

Theorem 2. *Let f be a positive and multiplicatively convex function on interval $[a,b]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \le \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^{\frac{1}{b-a}} \le \sqrt{f(a)f(b)}. \tag{1.2}$$

Fractional calculus, as an advantageous tool, reveals its significance to implement differentiation and integration of real or complex number orders. Furthermore, it recently emerged rapidly due to its applications in modelling a number of problems especially in dealing with the dynamics of the complex systems, decision making in structural engineering and probabilistic problems, etc., see, for instance, [6,31]. The research of mathematical inequalities including many different types of fractional integral operators, especially the Hermite–Hadamard type inequalities, is a current research focus. For example, refer to [8,19,22] for Riemann–Liouville integrals, to k-Riemann–Liouville integrals [41], to Hadamard fractional integrals [4,48], to conformable fractional integrals [2,14], to Katugampola fractional integrals [17,51], and to exponential kernel integrals [5], etc.

An imperative generalization of Riemann–Liouville fractional integrals was considered by Abdeljawad and Grossman in [1], which is named the multiplicative Riemann–Liouville fractional integrals.

Definition 3. *[1] The multiplicative left-sided Riemann–Liouville fractional integral* ${}_a\mathcal{I}_*^\alpha f(x)$ *of order* $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ *is defined by*

$$ {}_a\mathcal{I}_*^\alpha f(x) = e^{\left(\mathcal{I}_{a+}^\alpha(\ln\circ f)\right)(x)}, $$

and the multiplicative right-sided one ${}_*\mathcal{I}_b^\alpha f(x)$ *is defined by*

$$ {}_*\mathcal{I}_b^\alpha f(x) = e^{\left(\mathcal{I}_{b-}^\alpha(\ln\circ f)\right)(x)}, $$

where the symbols $\mathcal{I}_{a+}^\alpha f(x)$ *and* $\mathcal{I}_{b-}^\alpha f(x)$ *denote respectively the left-sided and right-sided Riemann–Liouville fractional integrals, which are defined by*

$$ \mathcal{I}_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a, $$

and

$$ \mathcal{I}_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b, $$

respectively.

On the other hand, Sarikaya et al. proved the following noteworthy inequalities which are the Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals.

Theorem 3. *[44] Let* $f : [a, b] \to \mathbb{R}$ *be a positive function with* $0 \le a < b$ *and* $f \in L^1([a, b])$. *If* f *is a convex function on* $[a, b]$, *then the following inequalities for fractional integrals hold:*

$$ f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\mathcal{I}_{a+}^\alpha f(b) + \mathcal{I}_{b-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{1.3} $$

with $\alpha > 0$.

Also, Sarikaya and Yildirim built another form relevant to Riemann–Liouville fractional Hermite–Hadamard type inequalities as follows.

Theorem 4. *[45] Under the same assumptions of Theorem 3, we have*

$$ f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{I}_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{1.4} $$

Sabzikar et al. provided the following tempered fractional operators.

Definition 4. *[35] Let* $[a, b]$ *be a real interval and* $\lambda \ge 0$, $\alpha > 0$. *Then for a function* $f \in L^1([a, b])$, *the left-sided and right-sided tempered fractional integrals are, respectively, defined by*

$$ \mathcal{I}_{a+}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t)\mathrm{d}t, \quad x > a, $$

and

$$ \mathcal{I}_{b-}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t)\mathrm{d}t, \quad x < b. $$

For several recent related results involving the tempered fractional integrals, see [24, 38, 42, 43] and the references included there.

Motivated by the results in the papers above, especially these developed in [12, 38], this work aims to investigate some inequalities of Hermite–Hadamard type, which involve the tempered fractional integrals and the notion of the λ-incomplete gamma function for the multiplicatively convex functions. For this purpose, we establish two Hermite–Hadamard type inequalities for the multiplicative tempered fractional integrals, then we present an integral identity for *differentiable mappings, from which we provide certain estimates of the upper bounds for trapezoid inequalities via the multiplicative tempered fractional integral operators.

2. Main results

As one can see, the definitions of the tempered fractional integrals and the multiplicative fractional integrals have similar configurations. This observation leads us to present the following definition of fractional integral operators, to be referred to as the multiplicative tempered fractional integrals.

Definition 5. *The multiplicative left-sided tempered fractional integral* ${}_a\mathcal{I}_*^{\alpha,\lambda} f(x)$ *of order* $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$*, is defined by*

$$ {}_a\mathcal{I}_*^{\alpha,\lambda} f(x) = e^{\left(\mathcal{I}_{a+}^{\alpha,\lambda}(\ln\circ f)\right)(x)}, \lambda \geq 0, $$

and the multiplicative right-sided one ${}_*\mathcal{I}_b^{\alpha,\lambda} f(x)$ *is defined by*

$$ {}_*\mathcal{I}_b^{\alpha,\lambda} f(x) = e^{\left(\mathcal{I}_{b-}^{\alpha,\lambda}(\ln\circ f)\right)(x)}, \lambda \geq 0, $$

where the symbols $\mathcal{I}_{a+}^{\alpha,\lambda} f(x)$ *and* $\mathcal{I}_{b-}^{\alpha,\lambda} f(x)$ *denote the left-sided and right-sided tempered fractional integrals, respectively.*

Observe that, for $\lambda = 0$, the multiplicative tempered fractional integrals become to the multiplicative Riemann–Liouville fractional integrals.

The following facts will be required in establishing our main results.

Remark 1. *For the real numbers* $\alpha > 0$ *and* $x, \lambda \geq 0$*, the following identities hold:*

$$ (i)\ \gamma_{\lambda(b-a)}(\alpha, 1) = \frac{\gamma_\lambda(\alpha, b-a)}{(b-a)^\alpha}, \tag{2.1} $$

$$ (ii)\ \int_0^1 \gamma_{\lambda(b-a)}(\alpha, x)\mathrm{d}x = \frac{\gamma_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\gamma_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}, \tag{2.2} $$

where $\gamma_\lambda(\cdot,\cdot)$ *is the* λ*-incomplete gamma function [38], which is defined as follows:*

$$ \gamma_\lambda(\alpha, x) = \int_0^x t^{\alpha-1} e^{-\lambda t}\mathrm{d}t. $$

If $\lambda = 1$, the λ-incomplete gamma function reduces to the incomplete gamma function [15]:

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} \mathrm{d}t.$$

Proof. (i) By using the changed variable $u = (b-a)t$ in the (2.1), we get

$$\gamma_{\lambda(b-a)}(\alpha, 1) = \int_0^1 t^{\alpha-1} e^{-\lambda(b-a)t} \mathrm{d}t = \int_0^{b-a} \left(\frac{u}{b-a}\right)^{\alpha-1} e^{-\lambda u} \left(\frac{1}{b-a}\right) \mathrm{d}u = \frac{\gamma_\lambda(\alpha, b-a)}{(b-a)^\alpha},$$

which ends the identity (2.1) .

(ii) From the definition of λ-incomplete gamma function, we have

$$\int_0^1 \gamma_{\lambda(b-a)}(\alpha, x) \mathrm{d}x = \int_0^1 \int_0^x y^{\alpha-1} e^{-\lambda(b-a)y} \mathrm{d}y \mathrm{d}x.$$

By changing the order of the integration, we get

$$\begin{aligned}
\int_0^1 \gamma_{\lambda(b-a)}(\alpha, x) \mathrm{d}x &= \int_0^1 \int_y^1 y^{\alpha-1} e^{-\lambda(b-a)y} \mathrm{d}x \mathrm{d}y \\
&= \int_0^1 (1-y) y^{\alpha-1} e^{-\lambda(b-a)y} \mathrm{d}y \\
&= \int_0^1 y^{\alpha-1} e^{-\lambda(b-a)y} \mathrm{d}y - \int_0^1 y^{\alpha} e^{-\lambda(b-a)y} \mathrm{d}y.
\end{aligned}$$

Applying the Remark 1 (i), we get the identity (2.2).

Our first main result is presented by the following theorem.

Theorem 5. *Let f be a positive and multiplicatively convex function on interval $[a, b]$, then we have the following Hermite–Hadamard inequalities for the multiplicative tempered fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \leq \left[{}_a\mathcal{I}_*^{\alpha,\lambda} f(b) \cdot_* \mathcal{I}_b^{\alpha,\lambda} f(a) \right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha, b-a)}} \leq \sqrt{f(a) f(b)}, \tag{2.3}$$

where $\gamma_\lambda(\cdot, \cdot)$ is the λ-incomplete gamma function.

Proof. On account of the multiplicative convexity of f on interval $[a, b]$, we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{at + (1-t)b + (1-t)a + tb}{2}\right) \\
&\leq \left[f(at + (1-t)b)\right]^{\frac{1}{2}} \left[f((1-t)a + tb)\right]^{\frac{1}{2}},
\end{aligned}$$

i.e.

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[\ln f(at + (1-t)b) + \ln f((1-t)a + tb)\right]. \tag{2.4}$$

Multiplying both sides of (2.4) by $t^{\alpha-1}e^{-\lambda(b-a)t}$ then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\begin{aligned}&\ln f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\mathrm{d}t\\&\qquad\leq\frac{1}{2}\left[\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\ln f\left(at+(1-t)b\right)\mathrm{d}t+\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\ln f\left((1-t)a+tb\right)\mathrm{d}t\right].\end{aligned}$$

Utilizing the changed variable, we have

$$\begin{aligned}&\frac{1}{(b-a)^{\alpha}}\ln f\left(\frac{a+b}{2}\right)\int_0^{b-a} x^{\alpha-1}e^{-\lambda x}\mathrm{d}x\\&\qquad\leq\frac{1}{2(b-a)^{\alpha}}\left[\int_a^b (b-x)^{\alpha-1}e^{-\lambda(b-x)}\ln f(x)\mathrm{d}x+\int_a^b (x-a)^{\alpha-1}e^{-\lambda(x-a)}\ln f(x)\mathrm{d}x\right].\end{aligned}$$

That is,

$$\frac{\gamma_{\lambda}(\alpha,b-a)}{(b-a)^{\alpha}}\ln f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)^{\alpha}}\left[\int_a^b (b-x)^{\alpha-1}e^{-\lambda(b-x)}\ln f(x)\mathrm{d}x+\int_a^b (x-a)^{\alpha-1}e^{-\lambda(x-a)}\ln f(x)\mathrm{d}x\right],$$

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right].$$

Thus we get,

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq e^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right]}\\&=\left[e^{\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)}e^{\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)}\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}\\&=\left[{}_{a}\mathcal{I}_{*}^{\alpha,\lambda}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}},\end{aligned}$$

which completes the proof of the first inequality in (2.3).

On the other hand, as f is multiplicatively convex on interval $[a,b]$, we have

$$f\left(at+(1-t)b\right)\leq[f(a)]^{t}[f(b)]^{1-t},$$

and

$$f\left((1-t)a+tb\right)\leq[f(a)]^{1-t}[f(b)]^{t}.$$

Thus,

$$\begin{aligned}&\ln f\left(at+(1-t)b\right)+\ln f\left((1-t)a+tb\right)\\&\leq t\ln f(a)+(1-t)\ln f(b)+(1-t)\ln f(a)+t\ln f(b)\\&=\ln f(a)+\ln f(b).\end{aligned}\tag{2.5}$$

Multiplying both sides of (2.5) by $t^{\alpha-1}e^{-\lambda(b-a)t}$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\ln f\left(at+(1-t)b\right)\mathrm{d}t+\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\ln f\left((1-t)a+tb\right)\mathrm{d}t$$
$$\leq\left[\ln f(a)+\ln f(b)\right]\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\mathrm{d}t.$$

Hence,

$$\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right]\leq\frac{1}{2}\left[\ln f(a)+\ln f(b)\right].$$

Consequently, we have the following inequality

$$e^{\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right]\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}\leq\sqrt{f(a)f(b)},$$

i.e.

$$\left[{}_a\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_*\mathcal{I}_b^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}\leq\sqrt{f(a)f(b)}.$$

This ends the proof.

Remark 2. *Considering Theorem 5, we have the following conclusions:*
(i) The inequalities (2.3) are equivalent to the following inequalities:

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right]\leq\frac{1}{2}\left[\ln f(a)+\ln f(b)\right].$$

(ii) If we choose $\lambda=0$, then we have the following inequalities:

$$f\left(\frac{a+b}{2}\right)\leq\left[{}_a\mathcal{I}_*^{\alpha}f(b)\cdot_*\mathcal{I}_b^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}\leq\sqrt{f(a)f(b)},$$

which is given by Budak in [12].
(iii) If we choose $\lambda=0$ and $\alpha=1$, then we obtain Theorem 2 given by Ali et al. in [7].

Corollary 1. *Suppose that f and g are two positive and multiplicatively convex functions on $[a,b]$, then we have*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\left[{}_a\mathcal{I}_*^{\alpha,\lambda}fg(b)\cdot_*\mathcal{I}_b^{\alpha,\lambda}fg(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}\leq\sqrt{f(a)f(b)}\cdot\sqrt{g(a)g(b)}.\tag{2.6}$$

Proof. As f and g are positive and multiplicatively convex, the function fg is positive and multiplicatively convex. If we apply Theorem 5 to the function fg, then we obtain the required inequalities (2.6).

Remark 3. *If we take $\lambda=0$ in Corollary 1, then we have the following inequalities:*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\left[{}_a\mathcal{I}_*^{\alpha}fg(b)\cdot_*\mathcal{I}_b^{\alpha}fg(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}\leq\sqrt{f(a)f(b)}\cdot\sqrt{g(a)g(b)},$$

which is established by Budak in [12]. Especially if we take $\alpha=1$, we obtain Theorem 7 in [7].

Hermite–Hadamard's inequalities involving midpoint can be represented in the multiplicative tempered fractional integral forms as follows:

Theorem 6. *Under the same assumptions of Theorem 5, we have*

$$f\left(\frac{a+b}{2}\right) \leq \left[{}_{\frac{a+b}{2}}\mathcal{I}_{*}^{\alpha,\lambda} f(b) \cdot_{*} \mathcal{I}_{\frac{a+b}{2}}^{\alpha,\lambda} f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}} \leq \sqrt{f(a) f(b)}, \tag{2.7}$$

where $\gamma_{\lambda}(\cdot,\cdot)$ is the λ-incomplete gamma function.

Proof. On account of the multiplicative convexity of f on interval $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) = f\left[\frac{1}{2}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + \frac{1}{2}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right],$$

i.e.

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\ln f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + \ln f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right]. \tag{2.8}$$

Multiplying both sides of (2.8) by $t^{\alpha-1}e^{-\frac{\lambda(b-a)}{2}t}$ then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\begin{aligned}\ln f\left(\frac{a+b}{2}\right)&\int_{0}^{1} t^{\alpha-1}e^{-\frac{\lambda(b-a)}{2}t}\mathrm{d}t\\ &\leq \frac{1}{2}\left[\int_{0}^{1} t^{\alpha-1}e^{-\frac{\lambda(b-a)}{2}t}\ln f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t + \int_{0}^{1} t^{\alpha-1}e^{-\frac{\lambda(b-a)}{2}t}\ln f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t\right].\end{aligned}$$

That is,

$$\frac{2^{\alpha}}{(b-a)^{\alpha}}\gamma_{\lambda}\left(\alpha,\frac{b-a}{2}\right)\ln f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}}{(b-a)^{\alpha}}\Gamma(\alpha)\left[\mathcal{I}_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha,\lambda}\ln f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha,\lambda}\ln f(a)\right],$$

which yields that,

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq e^{\left[\mathcal{I}_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha,\lambda}\ln f(b) + \mathcal{I}_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha,\lambda}\ln f(a)\right]\frac{\Gamma(\alpha)}{2\gamma_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}}\\ &= \left[{}_{\frac{a+b}{2}}\mathcal{I}_{*}^{\alpha,\lambda} f(b) \cdot_{*} \mathcal{I}_{\frac{a+b}{2}}^{\alpha,\lambda} f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}}.\end{aligned}$$

This completes the proof of the first inequality in inequalities (2.7).

On the other hand, as f is multiplicatively convex, we get

$$f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \leq [f(a)]^{\frac{t}{2}}[f(b)]^{\frac{2-t}{2}},$$

and

$$f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \leq [f(a)]^{\frac{2-t}{2}}[f(b)]^{\frac{t}{2}}.$$

Thus, we have

$$\ln f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+\ln f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq \ln f(a)+\ln f(b). \tag{2.9}$$

Multiplying both sides of (2.9) by $t^{\alpha-1}e^{-\frac{\lambda(b-a)}{2}t}$ then integrating the resulting inequality with respect to t over [0,1], we have

$$\frac{2^{\alpha}}{(b-a)^{\alpha}}\Gamma(\alpha)\left[\mathcal{I}^{\alpha,\lambda}_{(\frac{a+b}{2})^{+}}\ln f(b)+\mathcal{I}^{\alpha,\lambda}_{(\frac{a+b}{2})^{-}}\ln f(a)\right]\leq \frac{2^{\alpha}}{(b-a)^{\alpha}}\gamma_{\lambda}\left(\alpha,\frac{b-a}{2}\right)[\ln f(a)+\ln f(b)],$$

i.e.

$$\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,\frac{b-a}{2})}\left[\mathcal{I}^{\alpha,\lambda}_{(\frac{a+b}{2})^{+}}\ln f(b)+\mathcal{I}^{\alpha,\lambda}_{(\frac{a+b}{2})^{-}}\ln f(a)\right]\leq \frac{1}{2}\left[\ln f(a)+\ln f(b)\right].$$

Consequently, we get the inequality

$$\left[{}_{\frac{a+b}{2}}\mathcal{I}^{\alpha,\lambda}_{*}f(b)\cdot_{*}\mathcal{I}^{\alpha,\lambda}_{\frac{a+b}{2}}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,\frac{b-a}{2})}}\leq \sqrt{f(a)f(b)}.$$

This ends the proof.

Next, we are going to establish several integral inequalities concerning the multiplicative tempered fractional integral operators. To this end, we present the following lemma.

Lemma 1. *Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}^{+}$ be a *differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If f^{*} is integrable on $[a,b]$, then we have*

$$\frac{\sqrt{f(a)f(b)}}{\left[{}_{a}\mathcal{I}^{\alpha,\lambda}_{*}f(b)\cdot_{*}\mathcal{I}^{\alpha,\lambda}_{b}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}=\int_{0}^{1}\left[f^{*}(ta+(1-t)b)^{\eta\left(\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right)}\right]^{\mathrm{d}t}, \tag{2.10}$$

where

$$\eta=\frac{(b-a)^{\alpha}}{2\gamma_{\lambda}(\alpha,b-a)}. \tag{2.11}$$

Proof. Applying the multiplicative integration by parts, we have

$$\begin{aligned}&\int_{0}^{1}\left[f^{*}(ta+(1-t)b)^{\eta\left(\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right)}\right]^{\mathrm{d}t}\\&\quad=\frac{f(a)^{\eta\gamma_{\lambda(b-a)}(\alpha,1)}}{f(b)^{-\eta\gamma_{\lambda(b-a)}(\alpha,1)}}\cdot\frac{1}{\int_{0}^{1}\left(f(ta+(1-t)b)^{\eta\left(t^{\alpha-1}e^{-\lambda(b-a)t}+(1-t)^{\alpha-1}e^{-\lambda(b-a)(1-t)}\right)}\right)^{\mathrm{d}t}}\end{aligned}$$

$$= \frac{[f(a)\cdot f(b)]^{\eta\gamma_{\lambda(b-a)}(\alpha,1)}}{\exp\Big\{\int_0^1 \eta \ln f(ta+(1-t)b)\cdot t^{\alpha-1}e^{-\lambda(b-a)t}\mathrm{d}t + \int_0^1 \eta \ln f(ta+(1-t)b)\cdot(1-t)^{\alpha-1}e^{-\lambda(b-a)(1-t)}\mathrm{d}t\Big\}}$$
$$= \frac{[f(a)\cdot f(b)]^{\eta\frac{\gamma_\lambda(\alpha,b-a)}{(b-a)^\alpha}}}{\exp\{I_1+I_2\}}.$$

Utilizing the changed variable, we obtain

$$\begin{aligned} I_1 &= \eta\int_0^1 \ln f(ta+(1-t)b)t^{\alpha-1}e^{-\lambda(b-a)t}\mathrm{d}t \\ &= \frac{\eta}{(b-a)^\alpha}\int_a^b \ln f(u)(b-u)^{\alpha-1}e^{-\lambda(b-u)}\mathrm{d}u \\ &= \frac{\eta\Gamma(\alpha)}{(b-a)^\alpha}\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \eta\int_0^1 \ln f(ta+(1-t)b)(1-t)^{\alpha-1}e^{-\lambda(b-a)(1-t)}\mathrm{d}t \\ &= \frac{\eta}{(b-a)^\alpha}\int_a^b \ln f(u)(u-a)^{\alpha-1}e^{-\lambda(u-a)}\mathrm{d}u \\ &= \frac{\eta\Gamma(\alpha)}{(b-a)^\alpha}\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a). \end{aligned}$$

Then, we have

$$\begin{aligned} &\int_0^1 \Big[f^*(at+(1-t)b)^{\eta\left(\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right)}\Big]^{\mathrm{d}t} \\ &= \frac{\sqrt{f(a)f(b)}}{\exp\left\{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}\left[\mathcal{I}_{a+}^{\alpha,\lambda}\ln f(b)+\mathcal{I}_{b-}^{\alpha,\lambda}\ln f(a)\right]\right\}} \\ &= \frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_* \mathcal{I}_b^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}}. \end{aligned}$$

This ends the proof.

Remark 4. *Considering Lemma 1, we have the following conclusions:*
(i) If we take $\lambda = 0$, then we have

$$\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha}f(b)\cdot_* \mathcal{I}_b^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}} = \int_0^1 \left(f^*(ta+(1-t)b)^{\frac{1}{2}[t^\alpha-(1-t)^\alpha]}\right)^{\mathrm{d}t}. \tag{2.12}$$

(ii) If we take $\lambda = 0$ and $\alpha = 1$, then we have

$$\frac{\sqrt{f(a)f(b)}}{\int_a^b \left(f(u)^{\frac{1}{b-a}}\right)^{\mathrm{d}u}} = \int_0^1 \left(f^*(ta+(1-t)b)^{\frac{1}{2}(2t-1)}\right)^{\mathrm{d}t}. \tag{2.13}$$

It is worth mentioning that, to the best of our knowledge, the identities (2.12) and (2.13) obtained here are new in the literature.

Theorem 7. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a *differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f^*|$ is multiplicatively convex on $[a, b]$, then we have*

$$\left| \frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda} f(b) \cdot_* \mathcal{I}_b^{\alpha,\lambda} f(a) \right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}} \right| \leq \left[|f^*(a)| \cdot |f^*(b)| \right]^{\eta\delta}, \tag{2.14}$$

where η is defined by (2.11) in Lemma 1 and

$$\delta = \frac{\gamma_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\gamma_\lambda(\alpha, \frac{b-a}{2})}{(b-a)^\alpha} + \frac{2\gamma_\lambda(\alpha+1, \frac{b-a}{2})}{(b-a)^{\alpha+1}} - \frac{\gamma_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}. \tag{2.15}$$

Proof. Making use of Lemma 1, we deduce

$$\begin{aligned} &\left| \frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda} f(b) \cdot_* \mathcal{I}_b^{\alpha,\lambda} f(a) \right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}} \right| \\ &= \left| \int_0^1 \left[f^*(at + (1-t)b)^{\eta\left(\gamma_{\lambda(b-a)}(\alpha,t) - \gamma_{\lambda(b-a)}(\alpha,1-t)\right)} \right]^{\mathrm{d}t} \right| \\ &\leq \exp\left\{ \int_0^1 \left| \ln f^*(at + (1-t)b)^{\eta[\gamma_{\lambda(b-a)}(\alpha,t) - \gamma_{\lambda(b-a)}(\alpha,1-t)]} \right| \mathrm{d}t \right\} \\ &= \exp\left\{ \int_0^1 \left| \eta[\gamma_{\lambda(b-a)}(\alpha,t) - \gamma_{\lambda(b-a)}(\alpha,1-t)] \right| \cdot |\ln f^*(at + (1-t)b)| \, \mathrm{d}t \right\}. \end{aligned} \tag{2.16}$$

As $t \in [0, 1]$, we can know

$$\left| \gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t) \right| = \begin{cases} \displaystyle\int_t^{1-t} u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u, & 0 \leq t \leq \frac{1}{2}, \\ \displaystyle\int_{1-t}^{t} u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u, & \frac{1}{2} < t \leq 1. \end{cases} \tag{2.17}$$

Since $|f^*|$ is multiplicatively convex, we get

$$|\ln f^*(ta + (1-t)b)| \leq t \ln|f^*(a)| + (1-t)\ln|f^*(b)|. \tag{2.18}$$

If we apply (2.17) and (2.18) to the inequality (2.16), we obtain

$$
\begin{aligned}
&\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a}\mathcal{I}_{*}^{\alpha,\lambda}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}\right|\\
&\quad\leq\exp\left\{\eta\int_{0}^{\frac{1}{2}}\int_{t}^{1-t}u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u[t\ln|f^{*}(a)|+(1-t)\ln|f^{*}(b)|]\mathrm{d}t\right.\\
&\qquad\left.+\eta\int_{\frac{1}{2}}^{1}\int_{1-t}^{t}u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u[t\ln|f^{*}(a)|+(1-t)\ln|f^{*}(b)|]\mathrm{d}t\right\}\\
&\quad=\exp\left\{\eta\ln|f^{*}(a)|\int_{0}^{\frac{1}{2}}\int_{t}^{1-t}tu^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\mathrm{d}t+\eta\ln|f^{*}(b)|\int_{0}^{\frac{1}{2}}\int_{t}^{1-t}(1-t)u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\mathrm{d}t\right.\\
&\qquad\left.+\eta\ln|f^{*}(a)|\int_{\frac{1}{2}}^{1}\int_{1-t}^{t}tu^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\mathrm{d}t+\eta\ln|f^{*}(b)|\int_{\frac{1}{2}}^{1}\int_{1-t}^{t}(1-t)u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\mathrm{d}t\right\}\\
&\quad=\exp\Big\{\eta\Big(\ln|f^{*}(a)|\cdot\Delta_{1}+\ln|f^{*}(b)|\cdot\Delta_{2}+\ln|f^{*}(a)|\cdot\Delta_{3}+\ln|f^{*}(b)|\cdot\Delta_{4}\Big)\Big\}.
\end{aligned}
$$

Here, let's evaluate an integral by changing the order of it.

$$
\begin{aligned}
\Delta_{1}&=\int_{0}^{\frac{1}{2}}\int_{t}^{1-t}tu^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\mathrm{d}t\\
&=\int_{0}^{\frac{1}{2}}\int_{0}^{u}tu^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}t\mathrm{d}u+\int_{\frac{1}{2}}^{1}\int_{0}^{1-u}tu^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}t\mathrm{d}u\\
&=\frac{1}{2}\Big[\int_{0}^{\frac{1}{2}}u^{\alpha+1}e^{-\lambda(b-a)u}\mathrm{d}u+\int_{\frac{1}{2}}^{1}(u^{2}-2u+1)u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\Big]\\
&=\frac{1}{2}\Big[\gamma_{\lambda(b-a)}\Big(\alpha+2,\frac{1}{2}\Big)+\int_{\frac{1}{2}}^{1}u^{\alpha+1}e^{-\lambda(b-a)u}\mathrm{d}u-2\int_{\frac{1}{2}}^{1}u^{\alpha}e^{-\lambda(b-a)u}\mathrm{d}u+\int_{\frac{1}{2}}^{1}u^{\alpha-1}e^{-\lambda(b-a)u}\mathrm{d}u\Big]\\
&=\frac{1}{2}\left\{\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)+\left[\gamma_{\lambda(b-a)}(\alpha+2,1)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)\right]\right.\\
&\qquad\left.-2\left[\gamma_{\lambda(b-a)}(\alpha+1,1)-\gamma_{\lambda(b-a)}\left(\alpha+1,\frac{1}{2}\right)\right]+\left[\gamma_{\lambda(b-a)}(\alpha,1)-\gamma_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]\right\}.
\end{aligned}
\tag{2.19}
$$

Analogously, we can get

$$
\begin{aligned}
\Delta_{2}=\frac{1}{2}&\left\{2\gamma_{\lambda(b-a)}\left(\alpha+1,\frac{1}{2}\right)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)+\left[\gamma_{\lambda(b-a)}(\alpha,1)-\gamma_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]\right.\\
&\left.-\left[\gamma_{\lambda(b-a)}(\alpha+2,1)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)\right]\right\},
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\Delta_{3}=\frac{1}{2}&\left\{2\gamma_{\lambda(b-a)}\left(\alpha+1,\frac{1}{2}\right)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)+\left[\gamma_{\lambda(b-a)}(\alpha,1)-\gamma_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]\right.\\
&\left.-\left[\gamma_{\lambda(b-a)}(\alpha+2,1)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)\right]\right\},
\end{aligned}
\tag{2.21}
$$

and

$$\Delta_4 = \frac{1}{2}\left\{\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)+\left[\gamma_{\lambda(b-a)}(\alpha+2,1)-\gamma_{\lambda(b-a)}\left(\alpha+2,\frac{1}{2}\right)\right]\right.$$
$$\left.-2\left[\gamma_{\lambda(b-a)}(\alpha+1,1)-\gamma_{\lambda(b-a)}\left(\alpha+1,\frac{1}{2}\right)\right]+\left[\gamma_{\lambda(b-a)}(\alpha,1)-\gamma_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]\right\}. \tag{2.22}$$

Consequently,

$$\begin{aligned}&\ln|f^*(a)|\cdot\Delta_1+\ln|f^*(b)|\cdot\Delta_2+\ln|f^*(a)|\cdot\Delta_3+\ln|f^*(b)|\cdot\Delta_4\\&=[\ln|f^*(a)|+\ln|f^*(b)|]\left[\frac{\gamma_\lambda(\alpha,b-a)}{(b-a)^\alpha}-\frac{\gamma_\lambda(\alpha,\frac{b-a}{2})}{(b-a)^\alpha}+\frac{2\gamma_\lambda(\alpha+1,\frac{b-a}{2})}{(b-a)^{\alpha+1}}-\frac{\gamma_\lambda(\alpha+1,b-a)}{(b-a)^{\alpha+1}}\right].\end{aligned}$$

Thus, we deduce

$$\begin{aligned}&\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_*\mathcal{I}_b^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}}\right|\\&\leq\exp\left\{\eta[\ln|f^*(a)|+\ln|f^*(b)|]\left[\frac{\gamma_\lambda(\alpha,b-a)}{(b-a)^\alpha}-\frac{\gamma_\lambda(\alpha,\frac{b-a}{2})}{(b-a)^\alpha}\right.\right.\\&\qquad\left.\left.+\frac{2\gamma_\lambda(\alpha+1,\frac{b-a}{2})}{(b-a)^{\alpha+1}}-\frac{\gamma_\lambda(\alpha+1,b-a)}{(b-a)^{\alpha+1}}\right]\right\}\\&=\exp\Big\{\eta\delta[\ln|f^*(a)|+\ln|f^*(b)|]\Big\}\\&=\Big[|f^*(a)|\cdot|f^*(b)|\Big]^{\eta\delta}.\end{aligned}$$

The proof is completed.

Theorem 8. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}^+$ be a *differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. For $q>1$ with $p^{-1}+q^{-1}=1$, if $|f^*|^q$ is multiplicatively convex on $[a,b]$, then we have*

$$\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_*\mathcal{I}_b^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}}\right|\leq\exp\left\{\eta\cdot\tau^{\frac{1}{p}}\left(\frac{\ln|f^*(a)|^q+\ln|f^*(b)|^q}{2}\right)^{\frac{1}{q}}\right\}, \tag{2.23}$$

where η is defined by (2.11) in Lemma 1 and

$$\tau=\int_0^1\left|\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right|^p\mathrm{d}t.$$

Proof. Making use of Lemma 1 and Hölder's inequality, we deduce

$$\begin{aligned}
&\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a}\mathcal{I}_{*}^{\alpha,\lambda}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}\right| \\
&\quad=\left|\int_{0}^{1}\left[f^{*}(at+(1-t)b)^{\eta\left(\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right)}\right]^{\mathrm{d}t}\right| \\
&\quad\leq\exp\left\{\int_{0}^{1}\left|\ln f^{*}(at+(1-t)b)^{\eta\left[\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right]}\right|\mathrm{d}t\right\} \\
&\quad=\exp\left\{\int_{0}^{1}\left|\eta\left[\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right]\cdot\ln f^{*}(at+(1-t)b)\right|\mathrm{d}t\right\} \\
&\quad=\exp\left\{\int_{0}^{1}\left|\eta\left[\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right]\right|\cdot|\ln f^{*}(at+(1-t)b)|\,\mathrm{d}t\right\}.
\end{aligned}\tag{2.24}$$

Due to the Hölder's inequality, we have

$$\begin{aligned}
\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a}\mathcal{I}_{*}^{\alpha,\lambda}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}\right| &\leq\exp\left\{\eta\left(\int_{0}^{1}\left|\gamma_{\lambda(b-a)}(\alpha,t)-\gamma_{\lambda(b-a)}(\alpha,1-t)\right|^{p}\mathrm{d}t\right)^{\frac{1}{p}}\right. \\
&\quad\left.\times\left(\int_{0}^{1}|\ln f^{*}(ta+(1-t)b)|^{q}\,\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{2.25}$$

By virtue of the multiplicative convexity of $|f^{*}|^{q}$, we obtain

$$\begin{aligned}
\int_{0}^{1}|\ln f^{*}(at+(1-t)b)|^{q}\,\mathrm{d}t &\leq\int_{0}^{1}\left[t\ln|f^{*}(a)|^{q}+(1-t)\ln|f^{*}(b)|^{q}\right]\mathrm{d}t \\
&=\frac{\ln|f^{*}(a)|^{q}+\ln|f^{*}(b)|^{q}}{2}.
\end{aligned}\tag{2.26}$$

Combining (2.26) with (2.25), we know that Theorem 8 is true. Thus the proof is completed.

Remark 5. *Considering Theorem 8, we have the following conclusions:*
(i) If we choose $\lambda=0$*, then we have*

$$\begin{aligned}
\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a}\mathcal{I}_{*}^{\alpha}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right| &\leq\exp\left\{\frac{1}{2}\left(\int_{0}^{1}|t^{\alpha}-(1-t)^{\alpha}|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\ln|f^{*}(a)|^{q}+\ln|f^{*}(b)|^{q}}{2}\right)^{\frac{1}{q}}\right\} \\
&\leq\exp\left\{\frac{1}{2}\left(\frac{1}{\alpha p+1}\left(2-\frac{1}{2^{\alpha p-1}}\right)\right)^{\frac{1}{p}}\left(\frac{\ln|f^{*}(a)|^{q}+\ln|f^{*}(b)|^{q}}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

To prove the second inequality above, we use the fact

$$[(1-t)^{\alpha}-t^{\alpha}]^{p}\leq(1-t)^{\alpha p}-t^{\alpha p},$$

for $t \in [0, \frac{1}{2}]$ *and*

$$[t^{\alpha} - (1-t)^{\alpha}]^{p} \le t^{\alpha p} - (1-t)^{\alpha p},$$

for $t \in [\frac{1}{2}, 1]$*, which follows from* $(A-B)^q \le A^q - B^q$ *for any* $A \ge B \ge 0$ *and* $q \ge 1$*.*
(ii) If we choose $\lambda = 0$ *and* $\alpha = 1$*, then we have*

$$\frac{\sqrt{f(a)f(b)}}{\int_a^b \left(f(u)^{\frac{1}{b-a}}\right)^{\mathrm{d}u}} \le \exp\left\{\frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\ln|f^*(a)|^q + \ln|f^*(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Theorem 9. *Let* $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}^{+}$ *be a* **differentiable mapping on* I°*,* $a, b \in I^{\circ}$ *with* $a < b$*. If* $|f^*|^q$*,* $q > 1$*, is multiplicatively convex on* $[a, b]$*, then we have*

$$\begin{aligned}&\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_{*}^{\alpha,\lambda} f(b) \cdot_{*} \mathcal{I}_{b}^{\alpha,\lambda} f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}\right| \\ &\qquad \le \exp\left\{2^{1-\frac{1}{q}} \cdot \eta \cdot \delta\Big(\ln|f^*(a)|^q + \ln|f^*(b)|^q\Big)^{\frac{1}{q}}\right\},\end{aligned} \tag{2.27}$$

where η *is defined by (2.11) in Lemma 1 and* δ *is defined by (2.15) in Theorem 7, respectively.*

Proof. Continuing from the inequality (2.24) in the proof of Theorem 8, using the power-mean inequality, we have

$$\begin{aligned}&\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_{*}^{\alpha,\lambda} f(b) \cdot_{*} \mathcal{I}_{b}^{\alpha,\lambda} f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_{\lambda}(\alpha,b-a)}}}\right| \\ &\le \exp\left\{\eta\left(\int_0^1 \left|\gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t)\right| \mathrm{d}t\right)^{1-\frac{1}{q}}\right. \\ &\qquad \left.\times\left(\int_0^1 \left|\gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t)\right| \cdot |\ln f^*(at + (1-t)b)|^q \,\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

For the convenience of expression, let us define the quantities

$$J_1 = \int_0^1 \left|\gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t)\right| \mathrm{d}t,$$

and

$$J_2 = \int_0^1 \left|\gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t)\right| \cdot |\ln f^*(at + (1-t)b)|^q \,\mathrm{d}t.$$

According to the equalities (2.17), we have

$$\begin{aligned} J_1 &= \int_0^{\frac{1}{2}} \int_t^{1-t} u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t + \int_{\frac{1}{2}}^1 \int_{1-t}^{t} u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t \\ &= 2\left\{2\gamma_{\lambda(b-a)}\left(\alpha+1, \frac{1}{2}\right) + \left[\gamma_{\lambda(b-a)}(\alpha, 1) - \gamma_{\lambda(b-a)}\left(\alpha, \frac{1}{2}\right)\right] - \gamma_{\lambda(b-a)}(\alpha+1, 1)\right\}. \end{aligned} \tag{2.28}$$

Utilizing the multiplicative convexity of $|f^*|^q$, we obtain

$$\begin{aligned} J_2 &\leq \int_0^1 \left|\gamma_{\lambda(b-a)}(\alpha, t) - \gamma_{\lambda(b-a)}(\alpha, 1-t)\right| \cdot \left[t\ln|f^*(a)|^q + (1-t)\ln|f^*(b)|^q\right]\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} \int_t^{1-t} u^{\alpha-1} e^{-\lambda(b-a)u} \cdot \left[t\ln|f^*(a)|^q + (1-t)\ln|f^*(b)|^q\right]\mathrm{d}u\mathrm{d}t \\ &\quad + \int_{\frac{1}{2}}^1 \int_{1-t}^{t} u^{\alpha-1} e^{-\lambda(b-a)u} \cdot \left[t\ln|f^*(a)|^q + (1-t)\ln|f^*(b)|^q\right]\mathrm{d}u\mathrm{d}t \\ &= \ln|f^*(a)|^q \cdot \int_0^{\frac{1}{2}} \int_t^{1-t} t u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t + \ln|f^*(b)|^q \cdot \int_0^{\frac{1}{2}} \int_t^{1-t} (1-t) u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t \\ &\quad + \ln|f^*(a)|^q \cdot \int_{\frac{1}{2}}^1 \int_{1-t}^{t} t u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t + \ln|f^*(b)|^q \cdot \int_{\frac{1}{2}}^1 \int_{1-t}^{t} (1-t) u^{\alpha-1} e^{-\lambda(b-a)u} \mathrm{d}u\mathrm{d}t \\ &= \ln|f^*(a)|^q \cdot \Delta_1 + \ln|f^*(b)|^q \cdot \Delta_2 + \ln|f^*(a)|^q \cdot \Delta_3 + \ln|f^*(b)|^q \cdot \Delta_4, \end{aligned}$$

where $\Delta_i (i = 1, 2, 3, 4)$ are given by (2.19)–(2.22) in the proof of Theorem 7, respectively. Consequently,

$$\begin{aligned} &\ln|f^*(a)|^q \cdot \Delta_1 + \ln|f^*(b)|^q \cdot \Delta_2 + \ln|f^*(a)|^q \cdot \Delta_3 + \ln|f^*(b)|^q \cdot \Delta_4 \\ &= \left[\ln|f^*(a)|^q + \ln|f^*(b)|^q\right]\left[\frac{\gamma_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\gamma_\lambda(\alpha, \frac{b-a}{2})}{(b-a)^\alpha} + \frac{2\gamma_\lambda(\alpha+1, \frac{b-a}{2})}{(b-a)^{\alpha+1}} - \frac{\gamma_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}\right]. \end{aligned} \tag{2.29}$$

Combining (2.28) with (2.29), we have

$$\begin{aligned} &\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda} f(b) \cdot_* \mathcal{I}_b^{\alpha,\lambda} f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha, b-a)}}}\right| \\ &\leq \exp\left\{\eta\,(2\delta)^{1-\frac{1}{q}} \cdot \left(\delta\left(\ln|f^*(a)|^q + \ln|f^*(b)|^q\right)\right)^{\frac{1}{q}}\right\} \\ &= \exp\left\{2^{1-\frac{1}{q}} \cdot \eta \cdot \delta\left(\ln|f^*(a)|^q + \ln|f^*(b)|^q\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

The proof is completed.

3. Examples

The main point of the results established in this paper is that the calculation of the right-hand side is much easier than that of the left-hand side. To show this, three interesting examples are demonstrated below.

Example 1. Let the log-convex function f: $(0,\infty)\to(0,\infty)$ be defined by $f(x)=2^{x^2-3}$. If we take $a=1, b=2$, $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{4}$, then all assumptions in Theorem 5 are satisfied.
The left-hand side term of (2.3) is

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{1+2}{2}\right)=2^{-\frac{3}{4}}\approx 0.5946.$$

The middle term of (2.3) is

$$\begin{aligned}&\left[{}_a\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_* \mathcal{I}_b^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}\\ &=\left[e^{\mathcal{I}_{1+}^{\frac{1}{2},\frac{1}{4}}\ln f(2)}\cdot e^{\mathcal{I}_{2-}^{\frac{1}{2},\frac{1}{4}}\ln f(1)}\right]^{\frac{\Gamma\left(\frac{1}{2}\right)}{2\gamma_{\frac{1}{4}}\left(\frac{1}{2},1\right)}}\\ &=\left[e^{\int_1^2\left(u^2-3\right)\ln 2\cdot(2-u)^{-\frac{1}{2}}e^{-\frac{1}{4}(2-u)}\mathrm{d}u+\int_1^2\left(u^2-3\right)\ln 2\cdot(u-1)^{-\frac{1}{2}}e^{-\frac{1}{4}(u-1)}\mathrm{d}u}\right]^{\frac{1}{2\int_0^1 u^{-\frac{1}{2}}e^{-\frac{1}{4}u}\mathrm{d}u}}\\ &\approx 0.6461.\end{aligned}$$

The right-hand side term of (2.3) is

$$\sqrt{f(a)f(b)}=\sqrt{f(1)f(2)}=2^{-\frac{1}{2}}\approx 0.7071.$$

It is clear that $0.5946<0.6461<0.7071$, which demonstrates the result described in Theorem 5.

Example 2. Let the log-convex function f: $(0,\infty)\to(0,\infty)$ be defined by $f(x)=e^{x^2}$. If we take $a=1, b=2$, $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{2}$, then all assumptions in Theorem 6 are satisfied.
The left-hand side term of (2.7) is

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{1+2}{2}\right)=e^{\frac{9}{4}}\approx 9.4877.$$

The middle term of (2.7) is

$$\begin{aligned}&\left[{}_{\frac{a+b}{2}}\mathcal{I}_*^{\alpha,\lambda}f(b)\cdot_* \mathcal{I}_{\frac{a+b}{2}}^{\alpha,\lambda}f(a)\right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda\left(\alpha,\frac{b-a}{2}\right)}}\\ &=\left[e^{\mathcal{I}_{\frac{3}{2}+}^{\frac{1}{2},\frac{1}{2}}\ln f(2)}\cdot e^{\mathcal{I}_{\frac{3}{2}-}^{\frac{1}{2},\frac{1}{2}}\ln f(1)}\right]^{\frac{\Gamma\left(\frac{1}{2}\right)}{2\gamma_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2}\right)}}\\ &=\left[e^{\int_{\frac{3}{2}}^2 u^2(2-u)^{-\frac{1}{2}}e^{-\frac{1}{2}(2-u)}\mathrm{d}u+\int_1^{\frac{3}{2}}u^2(u-1)^{-\frac{1}{2}}e^{-\frac{1}{2}(u-1)}\mathrm{d}u}\right]^{\frac{1}{2\int_0^{\frac{1}{2}}u^{-\frac{1}{2}}e^{-\frac{1}{2}u}\mathrm{d}u}}\\ &\approx 10.9088.\end{aligned}$$

The right-hand side term of (2.7) is

$$\sqrt{f(a)f(b)}=\sqrt{f(1)f(2)}=e^{\frac{5}{2}}\approx 12.1825.$$

It is clear that $9.4877<10.9088<12.1825$, which demonstrates the result described in Theorem 6.

Example 3. Let the log-convex function $\frac{f'(x)}{f(x)}$: $(0,\infty) \to (0,\infty)$ be defined by $\frac{f'(x)}{f(x)} = \frac{1}{x}$. We can get $f^*(x) = e^{\frac{1}{x}}$, $f(x) = x$. If we take $a = 1, b = 2$, $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{2}$, then all assumptions in Theorem 7 are satisfied.

The left-hand side term of (2.14) is

$$\begin{aligned}
&\left| \frac{\sqrt{f(a)f(b)}}{\left[{}_a\mathcal{I}_*^{\alpha,\lambda} f(b) \cdot_* \mathcal{I}_b^{\alpha,\lambda} f(a) \right]^{\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,b-a)}}} \right| \\
&= \left| \frac{\sqrt{f(1)f(2)}}{\left[e^{\mathcal{I}_{1+}^{\frac{1}{2},\frac{1}{2}} \ln f(2)} \cdot e^{\mathcal{I}_{2-}^{\frac{1}{2},\frac{1}{2}} \ln f(1)} \right]^{\frac{\Gamma\left(\frac{1}{2}\right)}{2\gamma_{\frac{1}{2}}(\frac{1}{2},1)}}} \right| \\
&= \left| \frac{\sqrt{2}}{\left[e^{\int_1^2 \ln u \cdot (2-u)^{-\frac{1}{2}} e^{-\frac{1}{2}(2-u)} \mathrm{d}u + \int_1^2 \ln u \cdot (u-1)^{-\frac{1}{2}} e^{-\frac{1}{2}(u-1)} \mathrm{d}u} \right]^{\frac{1}{2\gamma_{\frac{1}{2}}\left(\frac{1}{2},1\right)}}} \right| \\
&\approx 0.9702.
\end{aligned}$$

The right-hand side term of (2.14) is

$$\left[|f^*(a)| \cdot |f^*(b)| \right]^{\eta\delta} = \left(e^{\frac{3}{2}} \right)^{\frac{1}{2\gamma_{\frac{1}{2}}(\frac{1}{2},1)} \left[\gamma_{\frac{1}{2}}\left(\frac{1}{2},1\right) - \gamma_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2}\right) + 2\gamma_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right) - \gamma_{\frac{1}{2}}\left(\frac{3}{2},1\right) \right]} \approx 1.1480.$$

It is clear that $0.9702 < 1.1480$, which demonstrates the result described in Theorem 7.

4. Conclusions

To the best of our knowledge, this is a first pervasive work on the multiplicative tempered fractional Hermite–Hadamard type inequalities via the multiplicatively convex functions. Two Hermite–Hadamard type inequalities for the multiplicative tempered fractional integrals are hereby established. An integral identity for *differentiable mappings is presented. By using it, some estimates of the upper bounds pertaining to trapezoid type inequalities via the multiplicative tempered fractional integral operators are obtained. Inequalities obtained in this paper generalize some results given by Budak and Tunç (2020) and Ali et al. (2019). Also, three examples show that the calculation of the right-hand side is much easier than that of the left-hand side. The ideas and techniques of this article may inspire further research in this field. This promising field about the multiplicative tempered fractional inequalities is worth further exploration.

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Conflict of interest

The authors declare no conflict of interest.

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