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Research article

Pullback attractor of Hopfield neural networks with multiple time-varying delays

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Abstract: This paper deals with the attractor problem of Hopfield neural networks with multiple time-varying delays. The mathematical expression of the networks cannot be expressed in the vector-matrix form due to the existence of the multiple delays, which leads to the existence condition of the attractor cannot be easily established by linear matrix inequality approach. We try to derive the existence conditions of the linear matrix inequality form of pullback attractor by employing Lyapunov-Krasovskii functional and inequality techniques. Two examples are given to demonstrate the effectiveness of our theoretical results and illustrate the conditions of the linear matrix inequality form are better than those of the algebraic form.

Keywords: Hopfield neural network; multiple time-varying delays; pullback attractor; linear matrix inequality
Mathematics Subject Classification: 34D20

1. Introduction

Since people found that Hopfield neural network has potential applications in some engineering fields such as classification, associative memory and optimization, dynamic behaviors of the network have received considerable attention. Some interesting and useful results for bifurcations, chaos, periodic solutions, synchronization and stability of the network have come into our view, for example, see [1–35] and the references therein. It is noted that for the neural network, the attractor as a classical dynamical behavior has not been given much attention. It is obvious that the system for which the existence of an attractor can be ensured is always an interesting subject.

The theory of global attractors for autonomous systems has been developed to solve some problems arising in the study of delayed functional differential equations [36]. The classical semigroup property of autonomous systems can not be acquired because the initial time is just as important as the final time in non-autonomous differential equations. The theory of pullback attractors has been developed for stochastic and non-autonomous systems in which the trajectories can be unbounded when time increases to infinity [37–43]. In this case, the global attractor is defined as a parameterized family of sets $\{\mathcal{A}(t)\}_{t\in R}$ depending on the final time, such that attracts solutions of the system 'from $-\infty$', i.e. initial time goes to $-\infty$ while the final time remains fixed.

In this paper, we consider the following Hopfield neural networks with multiple time-varying delays:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + u_i, t \geq 0, \tag{1.1}$$

where c_i, a_{ij}, b_{ij} and u_i are some constants and $c_i > 0$, a_{ij} and b_{ij} present the connection weight coefficients, u_i denotes the external bias, $f_i(\cdot)$ and $g_i(\cdot)$ are continuous nonlinear activation functions.

System (1.1) is a more general mathematical expression. When $\tau_{ij}(t) = \tau_j(t)$, the equation of system (1.1) is the following vector-matrix form studied in [44]:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bg(x(t - \tau(t))) + u, t \geq 0, \tag{1.2}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T, A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, C = diag(c_1, \cdots, c_n)$, $u = (u_1, \cdots, u_n)^T, f(x(t)) = (f_1(x_1(t)), \cdots, f_n(x_n(t)))^T, g(x(t - \tau(t))) = (g_1(x_1(t - \tau_1(t))), \cdots, g_n(x_n(t - \tau_n(t))))^T$.

It is clear that system (1.1) cannot be described in the vector form because it contains multiple delays $\tau_{ij}(t)$, which leads to the existence condition of the attractor of system (1.1) can not be easily established by linear matrix inequality approach. In this case, we need to develop new mathematical techniques and employ suitable Lyapunov functionals for the attractor analysis of system (1.1). In addition to this, based on our careful review of recently published almost all the pullback attractor results for system (1.1), we have realized that for system (1.1), the research on pullback attractor has not received enough attention. These facts have been the main motivations of the current paper to focus on the pullback attractor of system (1.1). We try to derive the existence condition of the linear matrix inequality form for pullback attractor by employing Lyapunov-Krasovskii functional and inequality techniques. At the same time, we also give the existence condition of algebraic form for pullback attractor.

2. Preliminaries

Let $\tau > 0$ be a given positive number and denote by $\mathfrak{L}$ the Banach space $C([-\tau, 0]; R^n)$ endowed with the norm $\|\xi\| = \sup_{s\in[-\tau,0]} |\xi(s)|$, $|\cdot|$ is the Euclidean norm and $C([-\tau, 0]; R^n)$ is the space of all continuous R^n-valued functions defined on $[-\tau, 0]$. Denote by x_t the element in $\mathfrak{L}$ given by $x_t(s) = x(t + s)$ for all $s \in [-\tau, 0]$. $A > 0$ means that matrix A is symmetric positive definite. A^T denotes the transpose of the matrix A and I denotes identity matrix. Let X be a complete metric space and denote by $dist(A, B)$ the Hausdorff semidistance between A and B given by $dist(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b),\ A, B \subseteq X$. $*$ means the symmetric terms of a symmetric matrix.

Functions $\tau_{ij}(t)$, $f_i(\cdot)$ and $g_i(\cdot)$ are required to satisfy that there exist some constants μ, l_i^-, l_i^+, m_i^- and m_i^+ such that for every $z, y \in R(z \neq y)$ and $i, j = 1, \cdots, n$,

$$0 \leq \tau_{ij}(t) \leq \tau, \dot{\tau}_{ij}(t) \leq \mu < 1, t \geq 0, \tag{2.1}$$

$$l_i^- \leq \frac{f_i(z) - f_i(y)}{z - y} \leq l_i^+, m_i^- \leq \frac{g_i(z) - g_i(y)}{z - y} \leq m_i^+. \tag{2.2}$$

It is obvious that (2.2) is less conservative than that in [30] because the constants in (2.2) may be positive, negative numbers or zeros. Meanwhile, (2.2) implies that

$$|f_i(z) - f_i(y)| \leq l_i|z - y|, |g_i(z) - g_i(y)| \leq m_i|z - y|, z, y \in R, \tag{2.3}$$

where $l_i = \max\{|l_i^-|, |l_i^+|\}, m_i = \max\{|m_i^-|, |m_i^+|\}$.

System (1.1) can be written as

$$\frac{dx(t)}{dt} = F(t, \cdot), \tag{2.4}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T$, continuous map $F(t, \cdot)$ is defined as

$$\begin{aligned} F(t,\xi) \quad = \quad & \Big(- c_1\xi_1(0) + \sum_{j=1}^{n} a_{1j}f_j(\xi_j(0)) + \sum_{j=1}^{n} b_{1j}g_j(\xi_j(-\tau_{1j}(0))) + u_1, \cdots, \\ & -c_n\xi_n(0) + \sum_{j=1}^{n} a_{nj}f_j(\xi_j(0)) + \sum_{j=1}^{n} b_{nj}g_j(\xi_j(-\tau_{nj}(0))) + u_n\Big)^T, \xi \in \mathcal{L}. \end{aligned}$$

It follows from [41,43] that for every $(s, \xi) \in R \times \mathfrak{L}$, system (2.4) has a solution $x(t; s, \xi)$. We define a solution operator $\phi(t, s)$ which gives the solution (in $\mathfrak{L}$) at time t when $x_s = \xi$, via $\phi(t, s)\xi = x_t(\cdot; s, \xi)$.

Definition 1. [41] Let ϕ be a process on X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in R}$ is said to be a (global) pullback attractor for ϕ if, for all $s \in R$, it satisfies

$$\phi(t, s)\mathcal{A}(s) = \mathcal{A}(t), for\ all\ t \geq s,$$

$$lim_{s\to\infty} dist(\phi(t, t - s)D, \mathcal{A}(t)) = 0,\ for\ all\ bounded\ subsets\ D\ of\ X.$$

Definition 2. [41] $\{B(t)\}_{t\in R}$ is said to be absorbing with respect to the process ϕ if, for all $t \in R$ and all $D \subset X$ bounded, there exists $T_D(t) > 0$ such that for all $h > T_D(t), \phi(t, t - h)D \subset B(t)$.

Lemma 1. *F* maps bounded sets into bounded sets.

Proof. From (2.2) and (2.3), it follows that for every $\xi \in D = \{\xi : ||\xi|| \leq r, r > 0\} \subset \mathcal{L}$,

$$\begin{aligned} |F(t,\xi)|^2 \quad \leq \quad & \sum_{i=1}^{n} \Big(- c_i\xi_i(0) + \sum_{j=1}^{n} a_{ij}f_j(\xi_j(0)) + \sum_{j=1}^{n} b_{ij}g_j(\xi_j(-\tau_{ij}(0))) + u_i\Big)^2 \\ \leq \quad & \sum_{i=1}^{n} \Big(c_i|\xi_i(0)| + \sum_{j=1}^{n} |a_{ij}|(l_j|\xi_j(0)| + |f_j(0)|) + \sum_{j=1}^{n} |b_{ij}|(m_j|\xi_j(-\tau_{ij}(0))| + |g_j(0)|) + |u_i|\Big)^2 \end{aligned}$$

$$\leq \sum_{i=1}^{n}\Big((c_i+\sum_{j=1}^{n}|a_{ij}|l_j+\sum_{j=1}^{n}|b_{ij}|m_j)r+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|+|u_i|\Big)^2.$$

Lemma 2. [41] Suppose that F and $\phi(t,s)$ map bounded sets into bounded sets, and that there exists a family of bounded absorbing sets $\{B(t)\}_{t\in R}$ for ϕ. Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for problem (2.4).

3. Main results

Set

$$P = diag\{p_1,\cdots,p_n\}, U_i = diag\{u_{i1},\cdots,u_{in}\}(i=1,2), M = diag\{m_1,\cdots,m_n\},$$
$$L_1 = diag\{l_1^-l_1^+,\cdots,l_n^-l_n^+\}, L_2 = diag\{l_1^-+l_1^+,\cdots,l_n^-+l_n^+\},$$
$$M_1 = diag\{m_1^-m_1^+,\cdots,m_n^-m_n^+\}, M_2 = diag\{m_1^-+m_1^+,\cdots,m_n^-+m_n^+\},$$
$$B_1 = diag\{\sum_{j=1}^{n}|b_{1j}|,\cdots,\sum_{j=1}^{n}|b_{nj}|\}, B_2 = diag\{\sum_{j=1}^{n}p_j|b_{j1}|,\cdots,\sum_{j=1}^{n}p_j|b_{jn}|\},$$
$$B_3 = diag\{\sum_{j=1}^{n}|b_{1j}|m_j,\cdots,\sum_{j=1}^{n}|b_{nj}|m_j\}, B_4 = p\times diag\{\sum_{j=1}^{n}|b_{j1}|,\cdots,\sum_{j=1}^{n}|b_{jn}|\}.$$

We first give two sets of sufficient conditions of the linear matrix inequality form.

Theorem 1. Suppose that there exist three symmetric positive definite matrices P, U_1 and U_2 such that

$$\Sigma = \begin{pmatrix} \Sigma_{11} & PA+U_1L_2 & U_2M_2 \\ * & -2U_1 & 0 \\ * & * & -2U_2+\frac{1}{1-\mu}B_2 \end{pmatrix} < 0,$$

where

$$\Sigma_{11} = PB_1 - 2PC - 2U_1L_1 - 2U_2M_1.$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Proof. $\Sigma < 0$ implies there must exist a sufficient small positive constant λ such that

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma_{11}+2\lambda P+2\lambda I & PA+U_1L_2 & U_2M_2 \\ * & 2\lambda I-2U_1 & 0 \\ * & * & 2\lambda I-2U_2+\frac{e^{\lambda\tau}}{1-\mu}B_2 \end{pmatrix} < 0. \tag{3.1}$$

For every solution $x(t)$ satisfying $\|x_{t_0}\| \leq r$, we construct the following Lyapunov-Krasovskii functional

$$V(t) = e^{\lambda t}\sum_{i=1}^{n}p_ix_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}\int_{t-\tau_{ij}(t)}^{t}e^{\lambda(s+\tau)}g_j^2(x_j(s))ds \tag{3.2}$$

and obtain

$$
\begin{aligned}
V(t_0) &= e^{\lambda t_0}\sum_{i=1}^{n} p_i x_i^2(t_0) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}\int_{t_0-\tau_{ij}(t_0)}^{t_0} e^{\lambda(s+\tau)} g_j^2(x_j(s))ds \\
&\le \max_{1\le i\le n}\{p_i\}e^{\lambda t_0}|x(t_0)|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}\int_{t_0-\tau}^{t_0} e^{\lambda(s+\tau)}(m_j|x_j(s)| + |g_j(0)|)^2 ds \\
&\le \max_{1\le i\le n}\{p_i\}e^{\lambda t_0}r^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}\int_{t_0-\tau}^{t_0} e^{\lambda(s+\tau)}(m_j r + |g_j(0)|)^2 ds \\
&\le \max_{1\le i\le n}\{p_i\}e^{\lambda t_0}r^2 + e^{\lambda(t_0+\tau)}\tau\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}(m_j r + |g_j(0)|)^2 \le \alpha, \qquad (3.3)
\end{aligned}
$$

where α is a positive constant.

Computing $\dot{V}(t)$ along the trajectories of system (1.1) and using (2.1), we derive

$$
\begin{aligned}
\dot{V}(t) &= \lambda e^{\lambda t}\sum_{i=1}^{n} p_i x_i^2(t) + 2e^{\lambda t}\sum_{i=1}^{n} p_i x_i(t)\Big(-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) + u_i\Big) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_i|b_{ij}|}{1-\mu}\Big(e^{\lambda(t+\tau)} g_j^2(x_j(t)) - (1-\dot{\tau}_{ij}(t))e^{\lambda(t-\tau_{ij}(t)+\tau)} g_j^2(x_j(t-\tau_{ij}(t)))\Big) \\
&\le e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n} a_{ij} f_j(x_j(t))x_i(t) + 2\sum_{j=1}^{n}|b_{ij}||g_j(x_j(t-\tau_{ij}(t)))||x_i(t)| \\
&\quad + 2|u_i||x_i(t)|\Big) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|\Big(\frac{e^{\lambda(t+\tau)} g_j^2(x_j(t))}{1-\mu} - e^{\lambda t} g_j^2(x_j(t-\tau_{ij}(t)))\Big) \\
&\le e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n} a_{ij} f_j(x_j(t))x_i(t) + \sum_{j=1}^{n}|b_{ij}|(g_j^2(x_j(t-\tau_{ij}(t))) + x_i^2(t)) \\
&\quad + \lambda^{-1}u_i^2 + \lambda x_i^2(t)\Big) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|\Big(\frac{e^{\lambda(t+\tau)} g_j^2(x_j(t))}{1-\mu} - e^{\lambda t} g_j^2(x_j(t-\tau_{ij}(t)))\Big) \\
&= e^{\lambda t}\sum_{i=1}^{n} p_i\Big((2\lambda - 2c_i + \sum_{j=1}^{n}|b_{ij}|)x_i^2(t) + 2\sum_{j=1}^{n} a_{ij} f_j(x_j(t))x_i(t) + \lambda^{-1}u_i^2\Big) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} p_j|b_{ji}|\frac{e^{\lambda(t+\tau)} g_i^2(x_i(t))}{1-\mu} \\
&= e^{\lambda t}\Big(x^T(t)(2\lambda P - 2PC + PB_1)x(t) + 2x^T(t)PAf(x(t)) + \frac{e^{\lambda\tau}}{1-\mu} g^T(x(t))B_2 g(x(t))\Big) \\
&\quad + e^{\lambda t}\lambda^{-1}u^T P u, \qquad (3.4)
\end{aligned}
$$

where $g(x(t)) = (g_1(x_1(t)), \cdots, g_n(x_n(t)))^T$.

Assumption (2.2) implies the following inequalities hold:

$$\begin{aligned}
0 &\le -2\sum_{i=1}^{n} u_{1i}[f_i(x_i(t)) - f_i(0) - l_i^+ x_i(t)][f_i(x_i(t)) - f_i(0) - l_i^- x_i(t)] \\
&= -2\sum_{i=1}^{n} u_{1i}\{f_i^2(x_i(t)) - (l_i^+ + l_i^-)x_i(t)f_i(x_i(t)) + l_i^+ l_i^- x_i^2(t)\} \\
&\quad + \sum_{i=1}^{n} u_{1i}[-2f_i^2(0) + 4f_i(0)f_i(x_i(t)) - 2(l_i^+ + l_i^-)x_i(t)f_i(0)] \\
&\le -2\sum_{i=1}^{n} u_{1i}\{f_i^2(x_i(t)) - (l_i^+ + l_i^-)x_i(t)f_i(x_i(t)) + l_i^+ l_i^- x_i^2(t)\} \\
&\quad + \sum_{i=1}^{n}\{2[\lambda f_i^2(x_i(t)) + \lambda^{-1}u_{1i}^2 f_i^2(0)] + [\lambda x_i^2(t) + \lambda^{-1}(l_i^+ + l_i^-)^2 u_{1i}^2 f_i^2(0)]\} \\
&= f^T(x(t))(2\lambda I - 2U_1)f(x(t)) + 2f^T(x(t))U_1L_2x(t) + x^T(t)(\lambda I - 2U_1L_1)x(t) \\
&\quad +\lambda^{-1}f^T(0)[2U_1^2 + L_2^2U_1^2]f(0)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
0 &\le -2\sum_{i=1}^{n} u_{2i}[g_i(x_i(t)) - g_i(0) - m_i^+ x_i(t)][g_i(x_i(t)) - g_i(0) - m_i^- x_i(t)] \\
&\le g^T(x(t))(2\lambda I - 2U_2)g(x(t)) + 2g^T(x(t))U_2M_2x(t) + x^T(t)(\lambda I - 2U_2M_1)x(t) \\
&\quad +\lambda^{-1}g^T(0)[2U_2^2 + M_2^2U_2^2]g(0).
\end{aligned} \tag{3.6}$$

From (3.1), (3.3)-(3.6), we derive

$$\dot{V}(t) \le e^{\lambda t}y^T(t)\tilde{\Sigma}y(t) + e^{\lambda t}\lambda^{-1}\beta \le e^{\lambda t}\lambda^{-1}\beta$$

and

$$V(t) \le V(t_0) + \int_{t_0}^{t} e^{\lambda s}\lambda^{-1}\beta ds \le \alpha + e^{\lambda t}\lambda^{-2}\beta, \tag{3.7}$$

where $y(t) = (x^T(t), f^T(x(t)), g^T(x(t)))^T$,

$$\beta = u^T P u + f^T(0)[2U_1^2 + L_2^2U_1^2]f(0) + g^T(0)[2U_2^2 + M_2^2U_2^2]g(0).$$

From (3.2) and (3.7), we have

$$|x(t)|^2 \le \frac{e^{-\lambda t}\alpha + \lambda^{-2}\beta}{\min_{1\le i\le n}\{p_i\}} \tag{3.8}$$

and

$$|x(t+\theta)|^2 \le \frac{e^{-\lambda(t+\theta)}\alpha + \lambda^{-2}\beta}{\min_{1\le i\le n}\{p_i\}} \le \frac{e^{-\lambda(t-\tau)}\alpha + \lambda^{-2}\beta}{\min_{1\le i\le n}\{p_i\}},\ \theta \in [-\tau, 0]. \tag{3.9}$$

Inequality (3.9) derives

$$\|x_t\|^2 \le \frac{e^{-\lambda(t-\tau)}\alpha + \lambda^{-2}\beta}{\min_{1\le i\le n}\{p_i\}}. \tag{3.10}$$

Inequality (3.10) and Corollary 6 in [43] show that all solutions exist globally in time and $\phi(t, t_0)$ is bounded. Then,

$$B(t) = \left\{ z \in \mathcal{L} : \|z\|^2 \le \frac{e^{-\lambda(t-\tau)}\alpha + \lambda^{-2}\beta}{\min_{1\le i\le n}\{p_i\}} \right\}$$

is a family of bounded absorbing sets. From Lemma 1 and Lemma 2, we know that there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Theorem 2. Suppose that there exist three symmetric positive definite matrices P, U_1 and U_2 such that

$$\Gamma = \begin{pmatrix} \Gamma_{11} + \frac{MB_2}{1-\mu} & PA + U_1L_2 & U_2M_2 \\ * & -2U_1 & 0 \\ * & * & -2U_2 \end{pmatrix} < 0,$$

where

$$\Gamma_{11} = PB_3 - 2PC - 2U_1L_1 - 2U_2M_1.$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Proof. $\Gamma < 0$ implies there must exist a sufficient small positive constant λ such that

$$\tilde{\Gamma} = \begin{pmatrix} \Gamma_{11} + 2\lambda P + 2\lambda I + \frac{e^{\lambda\tau}}{1-\mu}MB_2 & PA + U_1L_2 & U_2M_2 \\ * & 2\lambda I - 2U_1 & 0 \\ * & * & 2\lambda I - 2U_2 \end{pmatrix} < 0.$$

For every solution $x(t)$ satisfying $\|x_{t_0}\| \le r$, we construct the following Lyapunov-Krasovskii functional

$$V(t) = e^{\lambda t}\sum_{i=1}^n p_i x_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n \frac{p_i|b_{ij}|m_j}{1-\mu}\int_{t-\tau_{ij}(t)}^t e^{\lambda(s+\tau)}x_j^2(s)ds \tag{3.11}$$

and obtain

$$\begin{aligned} V(t_0) &= e^{\lambda t_0}\sum_{i=1}^n p_i x_i^2(t_0) + \sum_{i=1}^n\sum_{j=1}^n \frac{p_i|b_{ij}|m_j}{1-\mu}\int_{t_0-\tau_{ij}(t_0)}^{t_0} e^{\lambda(s+\tau)}x_j^2(s)ds \\ &\le \max_{1\le i\le n}\{p_i\}e^{\lambda t_0}r^2 + e^{\lambda(t_0+\tau)}\tau\sum_{i=1}^n\sum_{j=1}^n \frac{p_i|b_{ij}|m_j r^2}{1-\mu} \le \alpha, \end{aligned}$$

where α is a positive constant.

Computing $\dot{V}(t)$ along the trajectories of system (1.1) and using (2.1) and (2.3), we derive

$$\begin{aligned}
\dot{V}(t) &= \lambda e^{\lambda t}\sum_{i=1}^{n} p_i x_i^2(t) + 2e^{\lambda t}\sum_{i=1}^{n} p_i x_i(t)\Big(-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) + u_i\Big) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{p_i|b_{ij}|m_j}{1-\mu}\Big(e^{\lambda(t+\tau)}x_j^2(t) - (1-\dot{\tau}_{ij}(t))e^{\lambda(t-\tau_{ij}(t)+\tau)}x_j^2(t-\tau_{ij}(t))\Big) \\
&\le e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n} a_{ij} f_j(x_j(t))x_i(t) + 2\sum_{j=1}^{n} |b_{ij}|m_j|x_j(t-\tau_{ij}(t))||x_i(t)| + 2|u_i||x_i(t)|\Big) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|m_j\Big(\frac{e^{\lambda(t+\tau)}x_j^2(t)}{1-\mu} - e^{\lambda t}x_j^2(t-\tau_{ij}(t))\Big) \\
&\le e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n} a_{ij} f_j(x_j(t))x_i(t) + \sum_{j=1}^{n} |b_{ij}|m_j(x_j^2(t-\tau_{ij}(t)) + x_i^2(t)) + \lambda^{-1}u_i^2 \\
&\quad + \lambda x_i^2(t)\Big) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|m_j\Big(\frac{e^{\lambda(t+\tau)}x_j^2(t)}{1-\mu} - e^{\lambda t}x_j^2(t-\tau_{ij}(t))\Big) \\
&= e^{\lambda t}\Big(x^T(t)(2\lambda P - 2PC + PB_1 + \frac{e^{\lambda\tau}}{1-\mu}MB_2)x(t) + 2x^T(t)PAf(x(t))\Big) + e^{\lambda t}\lambda^{-1}u^T P u.
\end{aligned}$$

The rest is similar to that of Theorem 1.

Although Theorem 1 (or Theorem 2) gives the sufficient condition of the linear matrix inequality form, it is difficult to find an executable Matlab program to solve the matrices P, U_1 and U_2 by Matlab LMI Control Toolbox because B_2 involves the elements of matrix P. That is to say, it is difficult to verify the conditions of Theorem 1 and Theorem 2. Therefore, it is necessary to give the special cases of Theorem 1 and Theorem 2 which are easy to verify by Matlab LMI Control Toolbox.

Corollary 1. Suppose that there exist three symmetric positive definite matrices $P = pI, U_1$ and U_2 such that

$$\Sigma = \begin{pmatrix} pB_1 - 2pC - 2U_1L_1 - 2U_2M_1 & pA + U_1L_2 & U_2M_2 \\ * & -2U_1 & 0 \\ * & * & -2U_2 + \frac{1}{1-\mu}B_4 \end{pmatrix} < 0.$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Corollary 2. Suppose that there exist three symmetric positive definite matrices $P = pI, U_1$ and U_2 such that

$$\Gamma = \begin{pmatrix} pB_3 - 2pC - 2U_1L_1 - 2U_2M_1 + \frac{MB_4}{1-\mu} & pA + U_1L_2 & U_2M_2 \\ * & -2U_1 & 0 \\ * & * & -2U_2 \end{pmatrix} < 0.$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Remark 1. Since system (1.2) studied in [44] is a special case of system (1.1), the above sufficient conditions of the linear matrix inequality form are valid for system (1.2). On the other hand, the results in [44] are not valid for system (1.1) because system (1.1) cannot be transformed into the vector form.

Next, we give the sufficient condition of the algebraic form.

Theorem 3. Suppose that there exist some positive constants $p_1, p_2, \cdots, p_n$ such that

$$p_i[2c_i - \sum_{j=1}^{n}|a_{ij}|l_j - \sum_{j=1}^{n}|b_{ij}|m_j] - \sum_{j=1}^{n} p_j[|a_{ji}|l_i + \frac{|b_{ji}|m_i}{1-\mu}] > 0, \forall i. \tag{3.12}$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Proof. Inequality (3.12) implies that there must exist a sufficient small positive constant λ such that

$$p_i\Big(2c_i - 2\lambda - \sum_{j=1}^{n}|a_{ij}|(l_j+\lambda) - \sum_{j=1}^{n}|b_{ij}|(m_j+\lambda)\Big) - \sum_{j=1}^{n} p_j\Big(|a_{ji}|l_i + \frac{|b_{ji}|m_i e^{\lambda\tau}}{1-\mu}\Big) > 0, \forall i. \tag{3.13}$$

For every solution $x(t)$ satisfying $\|x_{t_0}\| \le r$, we employ the Lyapunov-Krasovskii functional (3.11) and compute $\dot{V}(t)$ along the trajectories of system (1.1). From (2.1), (2.3) and (3.13), we derive

$$\begin{aligned}
\dot{V}(t) \le\ & e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n}|a_{ij}||f_j(x_j(t))||x_i(t)| + 2\sum_{j=1}^{n}|b_{ij}||g_j(x_j(t-\tau_{ij}(t)))||x_i(t)| \\
& +2|u_i||x_i(t)|\Big) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|m_j\Big(\frac{e^{\lambda(t+\tau)}x_j^2(t)}{1-\mu} - e^{\lambda t}x_j^2(t-\tau_{ij}(t))\Big) \\
\le\ & e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + 2\sum_{j=1}^{n}|a_{ij}|(l_j|x_j(t)||x_i(t)| + |f_j(0)||x_i(t)|) \\
& +2\sum_{j=1}^{n}|b_{ij}|(m_j|x_j(t-\tau_{ij}(t))||x_i(t)| + |g_j(0)||x_i(t)|) + \lambda^{-1}u_i^2 + \lambda x_i^2(t)\Big) \\
& +\sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|m_j\Big(\frac{e^{\lambda(t+\tau)}x_j^2(t)}{1-\mu} - e^{\lambda t}x_j^2(t-\tau_{ij}(t))\Big) \\
\le\ & e^{\lambda t}\sum_{i=1}^{n} p_i\Big((\lambda - 2c_i)x_i^2(t) + \sum_{j=1}^{n}|a_{ij}|(l_jx_j^2(t) + l_jx_i^2(t) + \lambda x_i^2(t) + \lambda^{-1}f_j^2(0)) \\
& +\sum_{j=1}^{n}|b_{ij}|(m_jx_j^2(t-\tau_{ij}(t)) + m_jx_i^2(t) + \lambda x_i^2(t) + \lambda^{-1}g_j^2(0)) + \lambda^{-1}u_i^2 + \lambda x_i^2(t)\Big) \\
& +\sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}|m_j\Big(\frac{e^{\lambda(t+\tau)}x_j^2(t)}{1-\mu} - e^{\lambda t}x_j^2(t-\tau_{ij}(t))\Big) \\
\le\ & e^{\lambda t}\sum_{i=1}^{n} p_i\Big(2\lambda - 2c_i + \sum_{j=1}^{n}|a_{ij}|(l_j+\lambda) + \sum_{j=1}^{n}|b_{ij}|(m_j+\lambda)\Big)x_i^2(t)
\end{aligned}$$

$$
\begin{aligned}
&+e^{\lambda t}\sum_{i=1}^{n}p_i\sum_{j=1}^{n}\Big(|a_{ij}|l_j+\frac{|b_{ij}|m_je^{\lambda\tau}}{1-\mu}\Big)x_j^2(t)+\lambda^{-1}e^{\lambda t}\sum_{i=1}^{n}p_i\Big(u_i^2+\sum_{j=1}^{n}|a_{ij}|f_j^2(0)+\sum_{j=1}^{n}|b_{ij}|g_j^2(0)\Big)\\
=\ &-e^{\lambda t}\sum_{i=1}^{n}\Big\{p_i\Big(2c_i-2\lambda-\sum_{j=1}^{n}|a_{ij}|(l_j+\lambda)-\sum_{j=1}^{n}|b_{ij}|(m_j+\lambda)\Big)-\sum_{j=1}^{n}p_j\Big(|a_{ji}|l_i+\frac{|b_{ji}|m_ie^{\lambda\tau}}{1-\mu}\Big)\Big\}x_i^2(t)\\
&+\lambda^{-1}e^{\lambda t}\beta\le\lambda^{-1}e^{\lambda t}\beta,
\end{aligned}
$$

where

$$\beta=\sum_{i=1}^{n}p_i\Big(u_i^2+\sum_{j=1}^{n}|a_{ij}|f_j^2(0)+\sum_{j=1}^{n}|b_{ij}|g_j^2(0)\Big).$$

The rest is similar to that Theorem 1.

Remark 2. As known, it is sometimes not easy to find the values of the positive constants $p_1, p_2, \cdots, p_n$ satisfying Theorem 3. It is fortune that the property of nonsingular M-matrix provides us with a way to avoid looking for these values since condition (3.12) holds is equivalent to that $W = (W_{ij})_{n\times n}$ is a nonsingular M-matrix, where

$$W_{ii}=2c_i-\sum_{j=1}^{n}|a_{ij}|l_j-\sum_{j=1}^{n}|b_{ij}|m_j-|a_{ii}|l_i-\frac{|b_{ii}|m_i}{1-\mu},\ W_{ij}=-|a_{ji}|l_i-\frac{|b_{ji}|m_i}{1-\mu},\ i\ne j.$$

Therefore, we only need to verify that all eigenvalues of the matrix W are positive [45].

Although we can obtain the eigenvalues of a matrix by calculating tool, we may prefer to see the following result without involving the constants p_i, which is a special case of Theorems 3 for $p_1 = \cdots = p_n$.

Corollary 3. Suppose that

$$2c_i-\sum_{j=1}^{n}|a_{ij}|l_j-\sum_{j=1}^{n}|b_{ij}|m_j-\sum_{j=1}^{n}[|a_{ji}|l_i+\frac{|b_{ji}|m_i}{1-\mu}]>0,\ \forall i.$$

Then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in R}$ for system (2.4).

Remark 3. Example 1 shows that our theoretical results are valid for system (1.1). Example 2 shows that the condition of the linear matrix inequality form seems better than that of the nonsingular M-matrix form. Meanwhile, the conditions of Corollary 2 seem better than those of Corollary 1.

Example 1. Consider system (1.1) involving the following matrices and functions:

$$A=\begin{pmatrix}1&-1&1&-1\\-1&1&1&-1\\1&1&1&1\\1&-1&-1&-1\end{pmatrix},B=\begin{pmatrix}1&1&-1&1\\-1&-1&1&-1\\1&1&1&1\\-1&-1&-1&1\end{pmatrix},$$

$c_1 = 7.1, c_2 = c_3 = c_4 = 7, f_i(x) = tanh(x), g_i(x) = 0.5\tanh(x), \tau_{ij}(t) = 0.5cost + 0.5, i = j; \tau_{ij}(t) = 0.5sint + 0.5, i \ne j; i, j = 1, 2, 3, 4.$

Then, we calculate $l_i = 1, m_i = 0.5, i = 1, 2, 3, 4, L_1 = M_1 = 0, L_2 = I, M_2 = 0.5I, M = 0.5I, B_1 = 4I, B_3 = 2I, B_4 = 4pI, \mu = 0.5,$

$$W = \begin{pmatrix} 6.2 & -2 & -2 & -2 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ -2 & -2 & -2 & 6 \end{pmatrix},$$

and the eigenvalues of the matrix W are 8.1509, 8, 8 and 0.0491.

From [45], we know that W is a nonsingular M-matrix, which shows that Theorem 3 holds. By using Matlab LMI Control Toolbox, we obtain

$$P = 1.315I, U_1 = diag\{6.3319, 6.3319, 6.3319, 7.0834\}, U_2 = 12.3625I$$

satisfying the condition of Corollary 1 and

$$P = 1.6739I, U_1 = diag\{6.1283, 6.1283, 6.1283, 7.0849\}, U_2 = 8.4084I$$

satisfying the condition of Corollary 2.

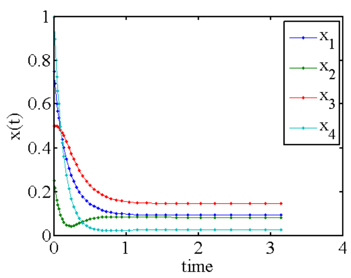

Figure 1. The solution trajectory of system (1) with initial value $(0.75, 0.25, 0.5, 1)^T$.

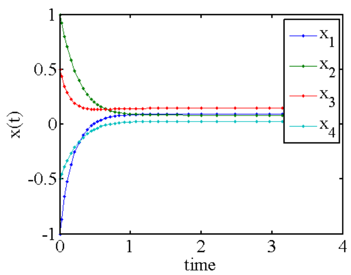

Figure 2. The solution trajectory of system (1) with initial value $(-1, 1, 0.5, -0.5)^T$.

Figures 1 and 2 show that the attractor of system (1) is an equilibrium point $(0.0923, 0.0827, 0.1457, 0.0274)^T$ and all solutions of system (1) tend to the equilibrium point.

Example 2. For the system (1.1) in Example 1, the value of c_1 is changed by 7 and the other parameters remain unchanged. We calculate

$$W = \begin{pmatrix} 6 & -2 & -2 & -2 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ -2 & -2 & -2 & 6 \end{pmatrix}.$$

It is clear that W is no longer a nonsingular M-matrix and Theorem 3 is invalid. By using Matlab LMI Control Toolbox, we obtain

$$P = 1.6812I, U_1 = diag\{6.1227, 6.1227, 6.1227, 7.0834\}, U_2 = 8.4039I$$

satisfying the condition of Corollary 2 and do not find the suitable matrices P, U_1, U_2 satisfying the condition of Corollary 1.

4. Conclusions

This paper has investigated the existence of pullback attractor of Hopfield neural networks involving multiple time-varying delays. Such neural system cannot be expressed in the vector-matrix form due to the existence of the multiple delays. So it is not easy to derive the existence conditions of the attractor by linear matrix inequality approach. By employing Lyapunov-Krasovskii functional and inequality techniques, two sets of existence conditions in linear matrix inequality form and one set of existence conditions in algebraic form are established. Two examples are given to demonstrate the effectiveness of our theoretical results and illustrate the existence conditions in linear matrix inequality form are better than those of the algebraic form.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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