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Research article

Traveling wave solutions of conformable time fractional Burgers type equations

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Abstract: In this paper, we investigate the conformable time fractional Burgers type equations. First, we construct the explicit solutions of Riccati equation by means of modified tanh function method and modified extended exp-function method respectively. In addition, based on the formulas obtained above, the traveling wave solutions of conformable time fractional Burgers equation and (2+1)-dimensional generalized conformable time fractional Burgers equations are established applying functional separation variables method. Furthermore, the three-dimensional diagrams of the obtained exact solutions are presented for the purpose of visualization.

Keywords: conformable time fractional Burgers type equation; Riccati equation; traveling wave solution; method of functional separation of variables
Mathematics Subject Classification: 26A33, 35C07

1. Introduction

Fractional calculus plays a significant role in various applied fields, such as electrochemistry, viscoelasticity, rheology, biology and physics [1–4]. The idea of generalizing integer derivative to fractional derivative has been proposed by Hospital in 1695. Since then, many types of fractional derivatives have been introduced such as Riemann-Liouville, Caputo, Grünwald-Letnikov [5] and the exact solutions of fractional partial differential equations in Riemann-Liouville or Caputo sense have been provided in [6–10]. In 2014, Khalil proposed a new fractional differential operator named as conformable fractional derivative in [11]. Researchers were taking keen interest to develop the theory of this type fractional derivative as it possesses some satisfactory properties. Abdeljawad [12] constructed the chain rule, formula of fractional integration by parts, Taylor power series representation regarding to conformable fractional derivative. In [13], Zhao pointed out that the physical interpretation of the conformable fractional derivative is a modification of classical velocity in direction and magnitude. In addition, the exact solutions of some conformable fractional partial differential equations have been constructed using various methods, such as the modified Kudryashov method [14–17], the first integral method [18, 19], the auxiliary method [20–22], the generalized $\exp(-\Phi(\xi))$-expansion method [23–25] and so on.

It was acknowledged that the method of functional separation of variables is an effective and systematic method for the construction of the exact solutions to the integer order partial differential equation [26–29]. Furthermore, for conformable fractional partial differential equations, functional separation variables method also is an effective method. In [30], some new exact solutions for the conformable space-time fractional (4+1)-dimensional Fokas equation were constructed using several methods, such as functional separation of variables, the generalized Kudryashov method and so on. Recently, we constructed the exact solutions of conformable time fractional Airy equation, conformable time fractional Telegraph equation and conformable time fractional inviscid Burgers equation with the functional variables separation method and generalized variables separation method [31].

Burgers equation is the simplest evolution equation to embody nonlinearity and dissipation and the construction of its solutions attracts much attentions. In [32], Murray mentioned simply classical Burgers equation of turbulence in the appendix. The integer-order (2+1)-dimensional generalized Burgers equation [33], $u_t + u_{xy} + uu_y + u_x\partial_x^{-1}u_y = 0$, was firstly introduced as an integral model through the Painlevé analysis [34]. In addition, if we set $x = y$, $u = v$, (2+1)-dimensional generalized Burgers equation will degenerate into Burgers equation. Furthermore, Kurt [35] studied the exact solution of the conformable time fractional Burgers equation with Hopf-Cole transform. The approximate analytical solution of the time conformable fractional Burgers equation is determined by using the homotopy analysis method. Çenesiz [36] used the first integral method to establish the exact solutions for the conformable time-fractional Burgers equation, modified Burgers equation, and Burgers-Korteweg-de-Vries equation. In [37], the rational fractional $(D_\alpha^\xi G/G)$-expansion method, the exp-function method and the extended tanh method were employed to construct the closed-form solutions of Burgers equation with conformable fractional derivative.

In this paper, we intend to utilize the method of separation of variables to construct the traveling wave solutions of the following two conformable time fractional Burgers type equations. One is the conformable time fractional Burgers equation

$$T_{\alpha l}u + \frac{\partial u}{\partial x}u - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $u(x,t)$ is the velocity of the turbulent motion and ν represents the diffusion coefficient. And the other is the (2+1)-dimensional generalized conformable time fractional Burgers equations

$$\begin{cases} T_{\alpha l}u = uu_y + \gamma v u_x + \beta u_{yy} + \gamma\beta u_{xx}, \\ u_x - v_y = 0, \end{cases} \tag{1.2}$$

where γ, β are given constants. $u(x,y,t)$ denotes the physical field and $v(x,y,t)$ denotes some potential [33]. $T_{\alpha l}$ is the (left) conformable fractional differential operator with respect to t.

The rest of this paper is organized as follows. In section 2, we recall the definition and properties of conformable fractional derivative and the steps of the separation variables method are presented. We construct the explicit solutions of certain Riccati equation by modified tanh function method and modified extended exp-function method in section 3. In section 4, functional separation variables method is applied to obtain the traveling wave solutions of the conformable time fractional Burgers type equations (1.1) and (1.2). In section 5, the three dimensional diagrams of some exact solutions are provided. In section 6, we draw a conclusion of this paper.

2. Preliminaries

In this section, we recall the definition of conformable fractional derivative and related properties. In addition, we describe the precise process of functional separation variables method.

Definition 1. [12] Let $\alpha \in (0,1], t > 0$ and $f(t) : [0,\infty) \to \mathbb{R}$. The (left) conformable fractional derivative of function $f(t)$ with order α is defined by

$$T_{\alpha l}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

In addition, if function $f(t)$ has the (left) conformable fractional derivative with order α in $(0,\infty)$, we call that $f(t)$ is (left) α-differentiable in $(0,\infty)$. If $f(t)$ is (left) α-differentiable in interval $(0,\infty)$ and $\lim_{t\to 0^+} T_{\alpha l} f(t)$ exists, then we define

$$T_{\alpha l} f(0) = \lim_{t\to 0^+} T_{\alpha l} f(t).$$

Denote $f'(s)$ to be the derivative of $f(s)$ with respect to independent variable s. Then the following properties are held to be true.

Lemma 1. *[11, 12] Let $\alpha \in (0,1]$, f be differentiable and g be (left) α-differentiable for $t > 0$. Then*
(1) $T_{\alpha l}(f)(t) = t^{1-\alpha} f'(t)$.
(2) Let $h(t) = f(g(t))$. Then $h(t)$ is (left) α-differentiable and

$$T_{\alpha l}(h)(t) = f'(g)(t) T_{\alpha l}(g)(t). \tag{2.1}$$

Next, we provide the details of the method of functional separation of variables introduced in [38]. Consider the following general conformable fractional differential equation

$$F(x,t,w,T_{\alpha l}w, w_x, w_{xx}, w_{xxx}, \cdots, w_{x^n}) = 0, \tag{2.2}$$

where w_{x^n} means the n-th order partial derivative of function $w(x,t)$ with respect to x and search for the functional separable solutions with the form as follows

$$w(x,t) = F(z),\ z = \sum_{i=1}^{k} \phi_i(x)\psi_i(t), \tag{2.3}$$

where $\phi_i(x)$, $\psi_i(t)$ $(i = 1,2,\cdots,k)$ and $F(z)$ are unknowns which will be determined later. Substitute (2.3) into original Eq (2.2) to write out the following functional differential equation

$$\sum_{i=1}^{m} \Phi_i(z)\Psi_i(t) = 0,$$

where $\Phi_i(z)$, $\Psi_i(t)$ $(i = 1, 2, \cdots, m)$ are functionals of the unknowns and use the differentiation method or the splitting method for even number m introduced in [38] to solve the determining system and find the exact formulas of functions $F(z)$, $\psi_i(t)$ and $\phi_i(x)$ $(i = 1, 2, \cdots, k)$ then insert them into (2.3) to obtain the presentation of the exact solution $w(x, t)$.

In this paper, we assume that B, C, C_1, C_2, C_3, C_4, μ are arbitrary nonzero constants and A_1 A_2, A^*, a, b, a_i $(i = -1, 0, 1)$ and b_1 are arbitrary constants. And suppose that $\varphi(x)$, $\psi(t)$, $\phi(t)$, $\phi(\frac{t^\alpha}{\alpha})$, $F(z)$, $F_1(z)$ and $F_2(z)$ are unknown functions which will be determined later.

3. Solutions of Riccati equation

Since we intend to draw support from the solutions of Riccati equation

$$F'(z) = AF(z) + BF^2(z) \tag{3.1}$$

to obtain the solutions of Burgers type equations (1.1) and (1.2), in this section we consider the construction of the following types of solutions to the Riccati equation (3.1) by means of the modified tanh function method used in [39] and the Modified extended exp-function method applied in [40].

Type 1 With the aid of the modified tanh function method and balancing the nonlinear term and the highest derivative, we obtain that the value of the balance coefficient is 1. Thus, we set

$$F(\xi) = a_0 + a_1 \tanh(\xi). \tag{3.2}$$

Insert (3.2) into Riccati equation (3.1) and equate the coefficients of all powers of $\tanh^i(\xi)$ $(i = 0, 1, 2)$ to be zero, we can obtain a system of algebraic equations for a_i $(i = 0, 1)$

$$\begin{cases} -a_1 = Ba_1^2, \\ 0 = 2Ba_1a_0 + Aa_1, \\ a_1 = Ba_0^2 + Aa_0. \end{cases}$$

Solving the above system, we can determine the values of the coefficients for $A = \pm 2$

$$a_0 = \mp\frac{1}{B}, \ a_1 = -\frac{1}{B}. \tag{3.3}$$

Plugging (3.3) into (3.2), we obtain the following exact solution of Riccati equation (3.1)

$$F_{1,2}(\xi) = \pm\frac{1}{B} - \frac{1}{B}\tanh(\xi). \tag{3.4}$$

Type 2 Suppose

$$F(\xi) = a_{-1}e^{\chi(\xi)} + a_0 + a_1e^{-\chi(\xi)} + b_1e^{\chi(\xi)}, \tag{3.5}$$

where $\chi(\xi)$ satisfies $\chi'(\xi) = e^{-\chi(\xi)} + ae^{\chi(\xi)} + b$. By means of the modified extended exp-function method [40], we can deduce the following results. When $b^2 - 4a > 0$ and $a \neq 0$,

$$\chi(\xi) = \ln\left(\frac{-\sqrt{b^2 - 4a}\tanh(\frac{\sqrt{b^2-4a}}{2}(\xi + A^*)) - b}{2a}\right). \tag{3.6}$$

When $b^2 - 4a > 0$, $a = 0$ and $b \neq 0$,

$$\chi(\xi) = -\ln(\frac{b}{e^{b(\xi + A^*)} - 1}). \tag{3.7}$$

Putting (3.5) into Riccati equation (3.1) and equating all the coefficients of the powers of e^i ($i = \chi, -\chi, 0$) to be zero, we get a system of algebraic equations for a_i and b_1 ($i = -1, 0, 1$)

$$\begin{cases} -a_1 = Ba_1^2, \\ -a_1 b = Aa_1 + 2Ba_1 a_0, \\ aa_1 + ab_1 = a_{-1}^2 B + Bb_1^2 + 2Ba_{-1}b_1, \\ a_{-1} + b_1 - aa_1 = Aa_0 + Ba_0^2 + 2Ba_1 a_{-1} + 2Ba_1 b_1, \\ a_{-1}b + bb_1 = Aa_{-1} + Ab_1 + 2Ba_0 a_{-1} + 2Ba_0 b_1. \end{cases} \tag{3.8}$$

Solving system (3.8), we deduce the solutions as follows
Case 1

$$a_{-1} = a_{-1},\ a_0 = \frac{b - A}{2B},\ a_1 = 0,\ b_1 = \frac{a}{B} - a_{-1},\ a = a,\ b^2 - 4a = A^2. \tag{3.9}$$

Case 2

$$a_{-1} = -b_1,\ a_0 = -\frac{b + A}{2B},\ a_1 = -\frac{1}{B},\ b_1 = b_1,\ a = a,\ b^2 - 4a = A^2. \tag{3.10}$$

Substituting (3.9) and (3.10) along with (3.6) and (3.7) into formula (3.5) separately, some new exact solutions, listed as follow, are obtained for Riccati equation (3.1)

$$F_3(\xi) = -\frac{A}{2B}(1 + \tanh(\frac{A(\xi + A^*)}{2})), \tag{3.11}$$

$$F_{4,5}(\xi) = \frac{\mp\sqrt{A^2 + 4a} - A}{2B} - \frac{2a}{-AB\tanh(\frac{A(\xi + A^*)}{2}) \mp B\sqrt{A^2 + 4a}}, \tag{3.12}$$

$$F_{6,7}(\xi) = \frac{\mp\sqrt{A^2 + 4a} - A}{2B} - \frac{\pm\sqrt{A^2 + 4a}}{Be^{\pm\sqrt{A^2 + 4a}(\xi + A^*)} - B}, \tag{3.13}$$

where and whereafter A is an arbitrary nonzero constant.

Remark 1. The exact solutions to Riccati equation (3.1), listed as follows, were constructed using the first integral method in [18].

$$F_8(\xi) = \frac{A}{2B}(1 + \coth_{\theta\eta}(\frac{A(\xi + A^*)}{2})), \tag{3.14}$$

$$F_{9,10}(\xi) = \frac{A}{2B}(1 + \coth_{\theta\eta}(A(\xi + A^*)) \pm \mathrm{csch}_{\theta\eta}(A(\xi + A^*))), \tag{3.15}$$

$$F_{11,12}(\xi) = \frac{A}{4B}(\pm 2 + \tanh_{\theta\eta}(\frac{A(\xi + A^*)}{4}) \pm \coth_{\theta\eta}(\frac{A(\xi + A^*)}{4})), \tag{3.16}$$

$$F_{13}(\xi) = \frac{A}{2B}(-1 + \frac{\sqrt{M^2 + N^2} - AM\cosh_{\theta\eta}(A(\xi + A^*))}{M\sinh_{\theta\eta}(A(\xi + A^*)) + N}), \tag{3.17}$$

$$F_{14}(\xi) = \frac{A}{2B}(-1 - \frac{\sqrt{N^2 - M^2} + AM\sinh_{\theta\eta}(A(\xi + A^*))}{M\cosh_{\theta\eta}(A(\xi + A^*)) + N}), \tag{3.18}$$

where M, N are two nonzero real constants satisfying $N^2 - M^2 > 0$. And $\tanh_{\theta\eta}(\xi) = \frac{\theta e^{\xi} - \eta e^{-\xi}}{\theta e^{\xi} + \eta e^{-\xi}}$, $\coth_{\theta\eta}(\xi) = \frac{\theta e^{\xi} + \eta e^{-\xi}}{\theta e^{\xi} - \eta e^{-\xi}}$, $\sinh_{\theta\eta}(\xi) = \frac{\theta e^{\xi} - \eta e^{-\xi}}{2}$, $\cosh_{\theta\eta}(\xi) = \frac{\theta e^{\xi} + \eta e^{-\xi}}{2}$, $\operatorname{csch}_{\theta\eta}(\xi) = \frac{2}{\theta e^{\xi} - \eta e^{-\xi}}$.

Remark 2. In order to distinguish different solutions of Riccati equation (3.1) introduced above, we numbered them with $F_i(\xi), i = 1, 2 \cdots 14$.

4. Application of functional separation variables method

In this section, as the application of the functional separation variables method, we construct the traveling wave solutions to conformable time fractional Burgers equation and (2+1)-dimensional generalized conformable time fractional Burgers equations with the help of the formulas of solutions to Riccati equation (3.1) provided in section 3.

4.1. Conformable time fractional Burgers equation

In this subsection, we establish the traveling wave solutions of conformable fractional Burgers equation (1.1) by means of functional separation variables method.

Case 1 Suppose

$$u(x,t) = F(z), z = x + \phi(t). \tag{4.1}$$

Insert (4.1) into (1.1) to obtain the following functional differential equation

$$F'\phi' t^{1-\alpha} + FF' - \nu F'' = 0. \tag{4.2}$$

On separating the variables in (4.2), we obtain

$$t^{1-\alpha}\phi' = \mu \tag{4.3}$$

and

$$FF' - \nu F'' + \mu F' = 0. \tag{4.4}$$

Solving (4.3), we find

$$\phi(t) = \frac{\mu t^{\alpha}}{\alpha} + A_1. \tag{4.5}$$

When (4.4) is integrated once with respect to z and the constant of integration is set to be zero, we get the following Riccati equation

$$F' = \frac{\mu}{\nu}F + \frac{1}{2\nu}F^2. \tag{4.6}$$

Due to (4.1), (4.5) and (4.6), the traveling wave solutions to Eq (1.1) can be written as follows according to the different formulas of the solutions of Riccati equation (4.6).

Type 1 In terms of (3.4), we obtain the following kink solutions with $\mu = \pm 2\nu$

$$u_{1,2}(x,t) = \pm 2\nu - 2\nu \tanh(x + \frac{\mu t^{\alpha}}{\alpha} + A_1). \tag{4.7}$$

Type 2 By virtue of (3.11)–(3.13), we deduce the following five traveling wave solutions

$$u_3(x,t) = -\mu(1 + \tanh(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{2\nu})). \tag{4.8}$$

$$u_{4,5}(x,t) = \mp \sqrt{\mu^2 + 4a\nu^2} - \mu - \frac{4a\nu^2}{-\mu \tanh(\frac{\mu(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)}{2\nu}) \mp \sqrt{\mu^2 + 4a\nu^2}}. \tag{4.9}$$

$$u_{6,7}(x,t) = \mp \sqrt{\mu^2 + 4a\nu^2} - \mu - \frac{\pm 2\sqrt{\mu^2 + 4a\nu^2}}{e^{\pm \frac{\sqrt{\mu^2+4a\nu^2}}{\nu}(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)} - 1}. \tag{4.10}$$

Type 3 Applying (3.14)–(3.18) in Remark 1, we obtain the following seven types of traveling wave solutions

$$u_8(x,t) = \mu(1 + \coth_{\theta\eta}(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{2\nu})). \tag{4.11}$$

$$u_{9,10}(x,t) = \mu(1 + \coth_{\theta\eta}(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{\nu}) \pm \mathrm{csch}_{\theta\eta}(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{\nu})). \tag{4.12}$$

$$u_{11,12}(x,t) = \frac{\mu}{2}(\pm 2 + \tanh_{\theta\eta}(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{4\nu}) \pm \coth_{\theta\eta}(\frac{\mu(x + \frac{\mu t^{\alpha}}{\alpha} + A_1 + A^*)}{4\nu})). \tag{4.13}$$

$$u_{13}(x,t) = -\mu + \frac{\mu\nu\sqrt{M^2 + N^2} - \mu^2 M \cosh_{\theta\eta}(\frac{\mu(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)}{\nu})}{M\nu \sinh_{\theta\eta}(\frac{\mu(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)}{\nu}) + N\nu}. \tag{4.14}$$

$$u_{14}(x,t) = -\mu - \frac{\mu\nu\sqrt{N^2 - M^2} + \mu^2 M \sinh_{\theta\eta}(\frac{\mu(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)}{\nu})}{M\nu \cosh_{\theta\eta}(\frac{\mu(x+\frac{\mu t^{\alpha}}{\alpha}+A_1+A^*)}{\nu}) + N\nu}. \tag{4.15}$$

Case 2 We seek an exact solution in the form

$$u(x,t) = F(z), z = \frac{t^{\alpha}}{\alpha} + \varphi(x). \tag{4.16}$$

Insert (4.16) into (1.1) to obtain the following functional differential equation

$$F' + FF'\varphi' - \nu F''(\varphi')^2 - \nu F'\varphi'' = 0. \tag{4.17}$$

Divide (4.17) by F' to yield

$$1 + F\varphi' - \frac{\nu F''(\varphi')^2}{F'} - \nu\varphi'' = 0. \tag{4.18}$$

Differentiating (4.18) with respect to z, we obtain

$$F'\varphi' - \nu(\varphi')^2 \frac{F'''F' - (F'')^2}{(F')^2} = 0,\ \varphi' \neq 0. \tag{4.19}$$

On separating the variables in (4.19), we get

$$\frac{F'''F'-(F'')^2}{(F')^2} - \mu F' = 0, \tag{4.20}$$

$$\frac{1}{\nu\varphi'} = \mu. \tag{4.21}$$

When (4.20) is integrated once with respect to z and the constant of integration is set to be A_1, we derive

$$\frac{F''}{F'} = \mu F + A_1. \tag{4.22}$$

Putting (4.22) into Eq (4.17) yields $A_1 = \mu^2\nu$. Thus (4.22) can be rewritten as

$$F'' = \mu^2\nu F' + \mu F F'. \tag{4.23}$$

When (4.23) is integrated once with respect to z and the constant of integration is set to be zero, we achieve

$$F' = \mu^2\nu F + \frac{\mu}{2}F^2. \tag{4.24}$$

In addition, Solving (4.21) yields

$$\varphi(x) = \frac{x + A_2}{\mu\nu}. \tag{4.25}$$

In terms of (4.16), (4.24) and (4.25), the traveling wave solutions for Eq (1.1) can be written as follows.
Type 1 Thanks to (3.4), we derive the following kink solutions with $\nu = \pm 2\mu^2$

$$u_{1,2}(x,t) = \pm\frac{2}{\mu} - \frac{2}{\mu}\tanh\left(\frac{t^\alpha}{\alpha} + \frac{x + A_2}{\mu\nu}\right). \tag{4.26}$$

Type 2 In view of (3.11)–(3.13), we gain the following five solutions

$$u_3(x,t) = -\mu^3\nu\left(1 + \tanh\left(\frac{\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)}{2}\right)\right). \tag{4.27}$$

$$u_{4,5}(x,t) = \mu(\mp\sqrt{\mu^4\nu^2 + 4a} - \mu^2\nu) - \frac{\pm 2\sqrt{\mu^4\nu + 4a}}{\mu(e^{\pm\sqrt{\mu^4\nu^2+4a}(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)} - 1)}. \tag{4.28}$$

$$u_{6,7}(x,t) = \mp\mu\sqrt{\mu^4\nu^2 + 4a} - \mu^3\nu + \frac{4a}{\mu(\mu^2\nu\tanh(\frac{\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)}{2}) \pm \sqrt{\mu^4\nu^2 + 4a})}. \tag{4.29}$$

Type 3 By virtue of (3.14)–(3.18) in Remark 1, we deduce the following seven types of traveling wave solutions

$$u_8(x,t) = \mu^3\nu(1 + \coth_{\theta\eta}(\frac{\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)}{2})). \tag{4.30}$$

$$u_{9,10}(x,t) = \mu^3\nu(1 + \coth_{\theta\eta}(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*) \pm \mathrm{csch}_{\theta\eta}(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)). \tag{4.31}$$

$$\begin{aligned} u_{11,12}(x,t) = \pm\mu^3\nu + \frac{\mu^3\nu}{2}\tanh_{\theta\eta}(\frac{\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)}{4}) \\ \pm\frac{\mu^3\nu}{2}\coth_{\theta\eta}(\frac{\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)}{4}). \end{aligned} \tag{4.32}$$

$$u_{13}(x,t) = \mu^3\nu(-1 + \frac{\sqrt{M^2+N^2} - M\mu^2\nu\cosh_{\theta\eta}(\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*))}{M\sinh_{\theta\eta}(\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)) + N}). \tag{4.33}$$

$$u_{14}(x,t) = \mu^3\nu(-1 - \frac{\sqrt{N^2-M^2} + M\mu^2\nu\sinh_{\theta\eta}(\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*))}{M\cosh_{\theta\eta}(\mu^2\nu(\frac{t^\alpha}{\alpha} + \frac{x+A_2}{\mu\nu} + A^*)) + N}). \tag{4.34}$$

4.2. (2+1)-dimensional generalized conformable time fractional Burgers equations

In this subsection, we consider the (2+1)-dimensional generalized conformable time fractional Burgers equations (1.2) via functional separation variables method.
Assume

$$u(x,y,t) = F_1(z),\ v(x,y,t) = F_2(z),\ z = m\varphi(x) + ny + l\frac{t^\alpha}{\alpha}, \tag{4.35}$$

where m, n and $l \in \mathbb{R}$ are non-zero constants. Inserting (4.35) into (1.2), we obtain the following functional differential equations

$$-lF_1' + nF_1F_1' + \gamma mF_2F_1'\varphi' + \beta n^2F_1'' + \gamma\beta\left(m^2F_1''(\varphi')^2 + F_1'\varphi''\right) = 0, \tag{4.36}$$

and

$$mF_1'\varphi' - nF_2' = 0. \tag{4.37}$$

In (4.37), for any constant $\mu \neq 0$, it follows

$$\varphi' = \mu, \tag{4.38}$$

$$nF_2' = \mu mF_1'. \tag{4.39}$$

When (4.38) and (4.39) are integrated once with respect to x and z respectively and the constants of integration are set to be zero, we can achieve

$$\varphi(x) = \mu x, \tag{4.40}$$

$$nF_2(z) = \mu m F_1(z). \tag{4.41}$$

Thanks to the arbitrariness of the value for μ, we might as well take $\mu = 1$. Then (4.40) and (4.41) can be written as

$$\varphi(x) = x, \tag{4.42}$$

$$nF_2(z) = mF_1(z). \tag{4.43}$$

Plugging (4.42) and (4.43) into (4.36) yields

$$-lF_1' + (n + \frac{\gamma m^2}{n})F_1F_1' + (\beta n^2 + \gamma\beta m^2)F_1'' = 0. \tag{4.44}$$

When (4.44) is integrated once with respect to z and the constants of integration are set to be zero, we get the following Riccati equation

$$F_1' = \frac{l}{\beta(n^2 + \gamma m^2)}F_1 - \frac{1}{2\beta n}F_1^2. \tag{4.45}$$

In view of (4.35), (4.41) and (4.45), the generalized traveling wave solutions to Eq (1.2) can be written as follows.

Type 1 With the help of (3.4), we derive the following kink solutions with $l = \pm 2\beta(n^2 + \gamma m^2)$

$$\begin{cases} u_{1,2}(x,y,t) = \pm 2\beta n + 2\beta n \tanh(mx + ny + l\frac{t^\alpha}{\alpha}), \\ v_{1,2}(x,y,t) = \pm 2\beta m + 2\beta m \tanh(mx + ny + l\frac{t^\alpha}{\alpha}). \end{cases} \tag{4.46}$$

Type 2 Applying (3.11)–(3.13), we get the following five solutions

$$\begin{cases} u_3(x,y,t) = \dfrac{ln}{n^2 + \gamma m^2}(1 + \tanh(\dfrac{l(mx + ny + l\frac{t^\alpha}{\alpha} + A^*)}{2\beta(n^2 + \gamma m^2)})), \\ v_3(x,y,t) = \dfrac{lm}{n^2 + \gamma m^2}(1 + \tanh(\dfrac{l(mx + ny + l\frac{t^\alpha}{\alpha} + A^*)}{2\beta(n^2 + \gamma m^2)})). \end{cases} \tag{4.47}$$

$$\begin{cases} u_{4,5}(x,y,t) = \dfrac{ln}{n^2 + \gamma m^2} \pm \beta n \sqrt{\dfrac{l^2}{\beta^2(n^2 + \gamma m^2)^2} + 4a} \\ \qquad - \dfrac{2a}{\frac{l\tanh(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{2\beta(n^2+\gamma m^2)})}{2\beta^2 n(n^2+\gamma m^2)} \pm \frac{\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}}{2\beta n}}, \\ v_{4,5}(x,y,t) = \dfrac{lm}{n^2 + \gamma m^2} \pm \beta m \sqrt{\dfrac{l^2}{\beta^2(n^2 + \gamma m^2)^2} + 4a} \\ \qquad - \dfrac{2am}{\frac{l\tanh(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{2\beta(n^2+\gamma m^2)})}{2\beta^2(n^2+\gamma m^2)} \pm \frac{\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}}{2\beta}}. \end{cases} \tag{4.48}$$

$$\begin{cases} u_{6,7}(x,y,t) = \dfrac{ln}{n^2+\gamma m^2} \pm \beta n\sqrt{\dfrac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a} \\ \qquad + \dfrac{\pm 2\beta n\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}}{e^{\pm\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}-1}, \\ v_{6,7}(x,y,t) = \dfrac{lm}{n^2+\gamma m^2} \pm \beta m\sqrt{\dfrac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a} \\ \qquad + \dfrac{\pm 2\beta m\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}}{e^{\pm\sqrt{\frac{l^2}{\beta^2(n^2+\gamma m^2)^2}+4a}(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}-1}. \end{cases} \tag{4.49}$$

Type 3 By virtue of (3.14)–(3.18) in Remark 1, we deduce the following seven types of traveling wave solutions

$$\begin{cases} u_8(x,y,t) = -\dfrac{ln}{n^2+\gamma m^2}(1+\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{2\beta(n^2+\gamma m^2)})), \\ v_8(x,y,t) = -\dfrac{lm}{n^2+\gamma m^2}(1+\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{2\beta(n^2+\gamma m^2)})). \end{cases} \tag{4.50}$$

$$\begin{cases} u_{9,10}(x,y,t) = -\dfrac{ln}{n^2+\gamma m^2}(1+\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})) \\ \qquad \mp\dfrac{ln}{n^2+\gamma m^2}(\operatorname{csch}_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})), \\ v_{9,10}(x,y,t) = -\dfrac{lm}{n^2+\gamma m^2}(1+\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})) \\ \qquad \mp\dfrac{lm}{n^2+\gamma m^2}(\operatorname{csch}_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})). \end{cases} \tag{4.51}$$

$$\begin{cases} u_{11,12}(x,y,t) = -\dfrac{ln}{2(n^2+\gamma m^2)}(\pm 2+\tanh_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{4\beta(n^2+\gamma m^2)})) \\ \qquad \pm\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{4\beta(n^2+\gamma m^2)})), \\ v_{11,12}(x,y,t) = -\dfrac{lm}{2(n^2+\gamma m^2)}(\pm 2+\tanh_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{4\beta(n^2+\gamma m^2)})) \\ \qquad \pm\coth_{\theta\eta}(\dfrac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{4\beta(n^2+\gamma m^2)})). \end{cases} \tag{4.52}$$

$$
\begin{cases}
u_{13}(x,y,t) = -\dfrac{ln}{(n^2+\gamma m^2)}(-1 + \dfrac{\sqrt{M^2+N^2} - \frac{lM}{\beta(n^2+\gamma m^2)}\cosh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})}{M\sinh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})+N}, \\
v_{13}(x,y,t) = -\dfrac{lm}{(n^2+\gamma m^2)}(-1 + \dfrac{\sqrt{M^2+N^2} - \frac{lM}{\beta(n^2+\gamma m^2)}\cosh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})}{M\sinh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})+N}.
\end{cases}
\tag{4.53}
$$

$$
\begin{cases}
u_{14}(x,y,t) = -\dfrac{ln}{(n^2+\gamma m^2)}(-1 - \dfrac{\sqrt{N^2-M^2} + \frac{lM}{\beta(n^2+\gamma m^2)}\sinh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})}{M\cosh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})+N}, \\
v_{14}(x,y,t) = -\dfrac{lm}{(n^2+\gamma m^2)}(-1 - \dfrac{\sqrt{N^2-M^2} + \frac{lM}{\beta(n^2+\gamma m^2)}\sinh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})}{M\cosh_{\theta\eta}(\frac{l(mx+ny+l\frac{t^\alpha}{\alpha}+A^*)}{\beta(n^2+\gamma m^2)})+N}.
\end{cases}
\tag{4.54}
$$

5. Graphical simulations

In this section, we present the three-dimensional graphical interpretations for certain chosen traveling wave solutions of the conformable time fractional Burgers equation (1.1) and (2+1)-dimensional generalized conformable time fractional Burgers equations (1.2). The following graphs are drawn by selecting suitable values of the involved parameters to visualize the underlying mechanisms of the considered equation for selected values of the order α.

For the conformable time fractional Burgers equation (1.1), Figures 1–6 show the 3D graphs for the solutions $u_5(x,t)$, $u_6(x,t)$, $u_8(x,t)$, $u_{10}(x,t)$, $u_{13}(x,t)$ and $u_{14}(x,t)$, respectively. Moreover, for (2+1)-dimensional generalized conformable time fractional Burgers equations (1.2), Figures 7–11 demonstrate the 3D graphs for the solutions $u_3(x,y,t)$, $u_9(x,y,t)$, $u_{10}(x,y,t)$, $u_{13}(x,y,t)$ and $u_{14}(x,y,t)$, respectively. Here we omit the graphs of $v(x,y,t)$ in (2+1)-dimensional generalized conformable fractional Burgers equations due to their similarity to $u(x,y,t)$ respectively. Moreover, concrete example of the influence of the fractional order α to the traveling wave solutions $u_{14}(x,t)$ is also presented.

From Figure 12, we observe that the velocity of the turbulent motion $u(x,t)$ tends to 0 when time $t \to \infty$ for any fixed fractional order α and $x = 1$. The time t when the velocity of the turbulent motion $u(x,t)$ arrives at the maximum value increases with the increase of the fractional order α from 0.2 to 0.999 for fixed $x = 1$. In addition, the physical interpretation of the conformable fractional derivative is a modification of classical velocity in direction and magnitude [13]. And these solutions obtained in whole paper must be helpful to explain some physical phenomena described by the conformable time fractional Burgers equation (1.1) and (2+1)-dimensional conformable time fractional Burgers equations (1.2).

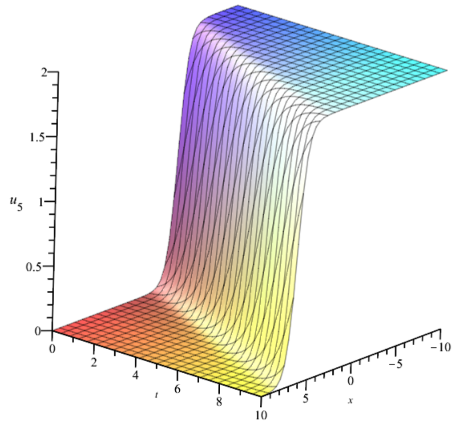

Figure 1. 3D polt of u_5 obtained in (4.9) with μ=-1, ν=0.5, a=A^*=2, A_1=1, α=0.999.

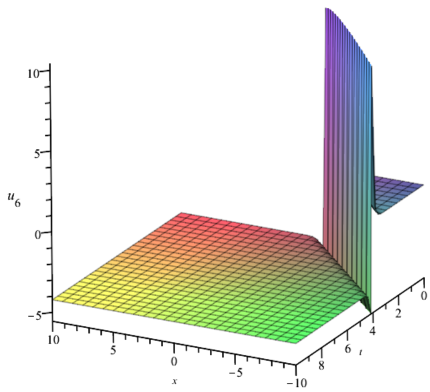

Figure 2. 3D polt of u_6 obtained in (4.10) with μ=2, ν=0.5, a=0.9, A^*=A_1=1, α=0.999.

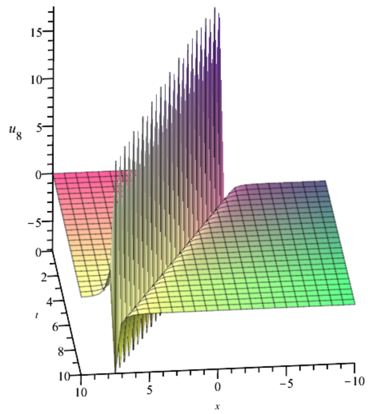

Figure 3. 3D polt of u_8 obtained in (4.11) with μ=-1,ν=0.5, η=2, θ=A^*=A_1=1, α=0.999.

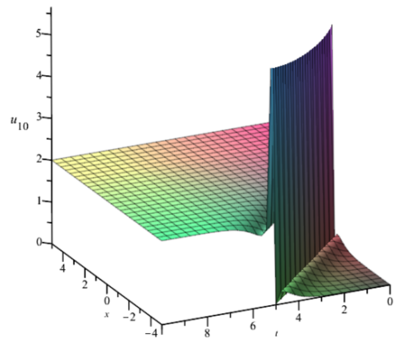

Figure 4. 3D polt of u_{10} obtained in (4.12) with μ=θ=1, ν=0.5, A^*=A_1=0, α=0.999.

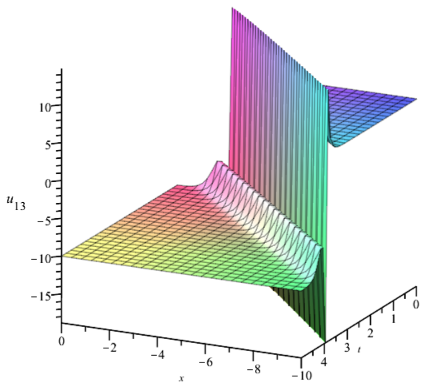

Figure 5. 3D polt of u_{13} obtained in (4.14) with μ=2, ν=0.5, M=3, N=5, θ=1, η=2, A_1=A^* = 1, α=0.999.

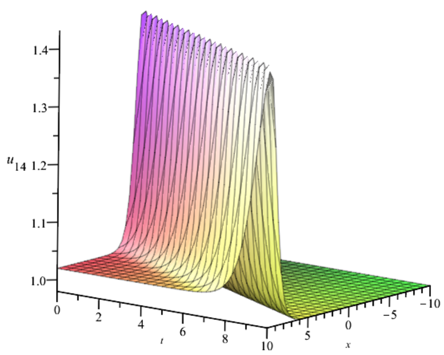

Figure 6. 3D polt of u_{14} obtained in (4.15) with μ=-1, ν=0.5, M=3, N=5, θ=1, η=2, A_1=A^* = 0, α=0.999.

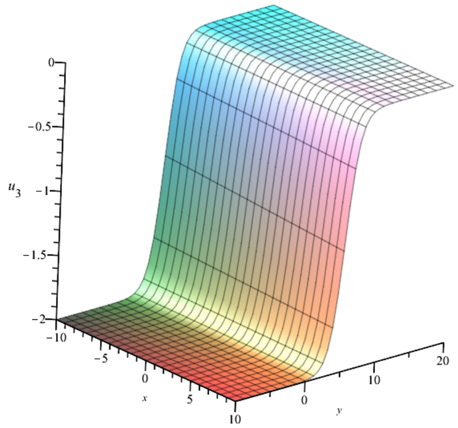

Figure 7. 3D polt of u_3 obtained in (4.47) with n=-1, m=0.01, l=1, γ=β=0.9, A^*=0, α=0.7.

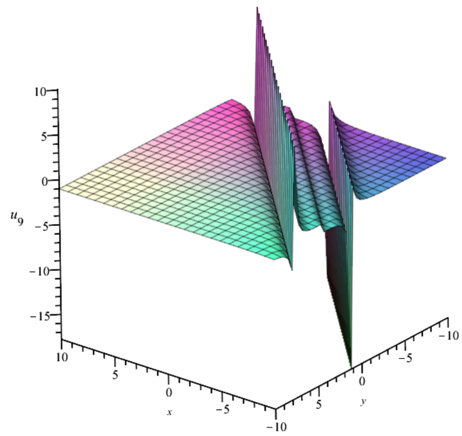

Figure 8. 3D polt of u_9 obtained in (4.51) with m=n=l=β=η=A^*=2, θ=γ=1, t=6, α=0.999.

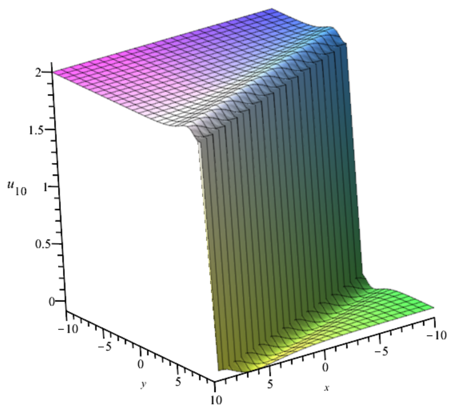

Figure 9. 3D polt of u_{10} obtained in (4.51) with m=θ=1, n=-2, β=l=η=2, A^*=0, γ=0.001, t=6, α=0.3.

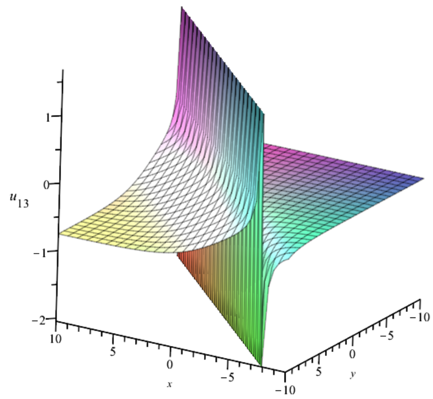

Figure 10. 3D polt of u_{13} obtained in (4.53) with m=n=β=γ=θ=A^*=1, η=2, l=-1, M=-3, N=5, t=6, α=0.999.

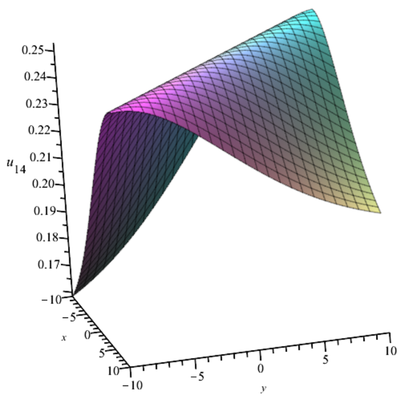

Figure 11. 3D polt of u_{14} obtained in (4.54) with m=n=η=γ=N=2, M=l=β=θ=A^*=1, t=6, α=0.999.

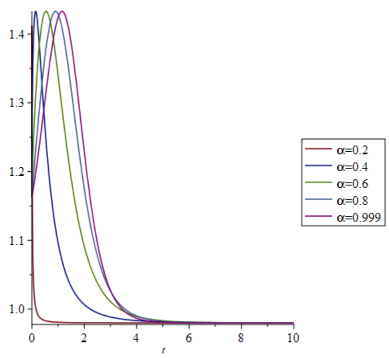

Figure 12. 2D polt of u_{14} obtained in (4.15) with μ=-1,υ=0.5, M=3, N=5, θ=1, η=2, A_1=A^* = 0, x=1.

6. Conclusions

In conclusion, for conformable time fractional Burgers equation (1.1), we achieve twenty-eight classes of solutions (4.7)–(4.15) and (4.26)–(4.34). And for (2+1)-dimensional generalized conformable time fractional Burgers equations (1.2), we obtain fourteen classes of solutions (4.46)–(4.54). Moreover, we demonstrate certain selected 3D graphs for the purpose of visualization. And all graphics are drawn with the help of Maple software. The investigation of this paper shows that functional separation of variables is an effective method to solve conformable fractional nonlinear partial differential equations.

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Conflict of interest

The authors declare that they have no competing interests.

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