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Research article

Periodic bouncing solutions for sublinear impact oscillator

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Abstract: The existence of periodic bouncing solutions for sublinear impact oscillator is proved by using Poincaré-Birkhoff twist theorem. The approach of this paper is based on a well defined successor map and the phase-plane analysis of the spiral properties.

Keywords: impact oscillator; sublinear; periodic solution; successor map; Poincaré-Birkhoff twist theorem
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1. Introduction

In this paper, we consider the impact oscillator of the form

$$\begin{cases} x'' + f(t,x) = 0, \quad \text{for } x(t) > 0; \\ x(t) \geq 0; \\ \text{For any } t_0, \text{ if } x(t_0) = 0 \text{ then } x'(t_0+) = -x'(t_0-). \end{cases} \tag{1.1}$$

We assume that $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and 2π-periodic in the first variable.

From the perspective of mechanics, the system (1.1) simulates the motion of a particle attached to a nonlinear spring and bouncing elastically against the barrier described by $x = 0$. Thus, it serves as a model of dynamical system with discontinuities [16]. Systems of this type are special cases of vibro-impact systems(see, e.g., [2, 3, 8, 24]). There are also interesting relations with the Fermi accelerator [17], dual billiards [5] and certain models used in celestial mechanics [9]. Although it's important and some results are known (see, e.g., [4, 17, 18, 23, 25–29]), comparing with the second order equation without impact, even for one-degree-of-freedom linear oscillators or asymptotically linear oscillators with impacts, the dynamics of system (1.1) is far from being understood.

Our purpose in this paper is to investigate the existence of nontrivial subharmonic bouncing solutions with prescribed number of impacts for sublinear impact oscillators. As is known to us, the existence of subharmonics of arbitrary order is usually a hint of a complex dynamics, although their connection needs further study. The following definition clarifies the concept of bouncing solution we mention here.

Definition 1.1. *A continuous function $x : \mathbb{R} \to \mathbb{R}$ is a bouncing solution to problem (1.1) if the following conditions hold.*

(i) $x(t) \geq 0$ *for all* $t \in \mathbb{R}$.

(ii) *the impact set* $W = \{t : x(t) = 0\}$ *is discrete and not empty.*

(iii) *elastic impact condition* $x'(t_0+) = -x'(t_0-)$ *holds for any* $t_0 \in W$.

(iv) *given an interval* I, *if* $I \cap W = \emptyset$, *then* $x \in C^2(I, \mathbb{R}^+)$ *and it is a classical solution of*

$$x'' + f(t, x) = 0. \tag{1.2}$$

We note that a grazing orbit also satisfies the above definition of bouncing solution, that is, an "impact" with zero velocity. A grazing impact indicates a bifurcation or pass to a different type of dynamical behaviour, see, e.g., [19, 20, 31] and the references therein. Also, another interesting physical phenomenon is the chattering, when there is an accumulation of impacts, that is, an infinite number of impacts occurring in a finite time, see, e.g., [6, 7]. In our next setting grazing and chattering phenomenons are not present, but they play an important role in models coming from mechanical engineering. The researches of grazing and chattering phenomenons need different approaches, we will not mention here. In the following, we will focus on the existence and multiplicity of "regular" periodic bouncing solutions, that is the bouncing solution without zero velocity at impact.

There are several interesting mathematical researches on the existence and multiplicity of periodic bouncing solutions for impact oscillators. We recall that Bonheure and Fabry proved in [4] the existence of 2π-periodic bouncing solutions for a linear impact oscillator ($f(t, x) = \lambda x - p(t)$) by means of an approximation approach. They considered the linear impact oscillator as the limitation of some second order equations without impact. Later, Qian and Torres proved in [26] and [27] the existence of 2π-periodic bouncing solutions for some nonlinear equation which has an attractive singularity at the origin and for Hill's equation ($f(t, x) = a(t)x - p(t)$), respectively. The approach in [26] and [27] is based on the successor map. In addition, Lazer and McKenna proved [18] the existence of large amplitude periodic bouncing solutions with one impact during one period for impact oscillators with damping. Their result was extended to the existence of periodic bouncing solutions with multiple impacts during one period by Qian [25].

In this paper, we consider the existence of infinitely many large subharmonic bouncing solutions for sublinear impact oscillator (1.1). We recall that in [10], Ding and Zanolin proved the existence and multiplicity of subharmonic solutions for sublinear oscillator (1.2) without impact by using the Poincaré-Birkhoff theorem.

In order to apply Poincaré-Birkhoff theorem for sublinear oscillator (1.2), one needs to consider the iterates of the Poincaré map of Eq (1.2) to show that the twist condition is fulfilled at the boundary of a given annulus in the phase-plane. Thus, the trajectories of the equivalent system staring from the boundary of the annulus may pass the origin, which leads to a bad evaluation of the rotations. Hence in [10], some careful phase-plane analysis of the dynamics of the solutions is performed.

Let $x_0(t)$ be a 2π-periodic solution of Eq (1.2) (the existence of 2π-periodic solution to sublinear oscillators can be easily realized by topological degree arguments). By the transformation $z = x - x_0(t)$, Eq (1.2) can be written as the form

$$z'' + f(t, z + x_0(t)) - f(t, x_0(t)) = 0.$$

Moreover, we consider the above equation as the following equivalent system

$$z' = w, \quad w' = -f(t, z + x_0(t)) + f(t, x_0(t)). \tag{1.3}$$

Then $x(t)$ is a solution of Eq (1.2) if and only if $(z(t), z'(t))$ is a solution of system (1.3). Furthermore, $(0, 0)$ is the unique solution of system (1.3) passing the origin. Hence, any trajectory of system (1.3) staring from the boundary of the suitable annulus doesn't pass the origin, which will help to construct some annulus with the twist condition at it's boundary.

But the approach in [10] is not valid for impact oscillator since we don't know if there is such 2π-periodic bouncing solution $x_0(t)$ of impact oscillator (1.1) (topological degree arguments is not suitable for impact oscillator). In addition, even if one can find such $x_0(t)$, yet after transformation $z = x - x_0(t)$, the impact condition

"For any t_0, if $z(t_0) = 0$ then $z'(t_0+) = -z'(t_0-)$"

will not be satisfied.

Therefore, in this paper, we use the "successor map" approach as in [23], [26] and [27]. There is a difference here. In [26], oscillatory properties of the Hill's equation are used to ensure that the successor map is well defined. In this paper, we will analyse the spiral property of the solutions for sublinear impact oscillator to prove that the successor map is well defined (see Lemma 2.1).

In the following, we assume, without loss of generality, that $f(t, x)$ is locally Lipschitzian in the x-variable, in order that uniqueness for the associated Cauchy problems is guaranteed. Otherwise we can use the approximation approach as in [10].

Then, for any τ and $v > 0$, there exists a unique solution $x(t; \tau, v)$ of impact oscillator (1.1) with initial conditions

$$x(\tau; \tau, v) = 0, \quad x'(\tau; \tau, v) = v > 0.$$

Furthermore, $x(t; \tau, v)$ is continuous with respect to (τ, v) (see, e.g., [14]). Moreover, if $x(t; \tau, v)$ vanishes at some time $\hat{\tau} > \tau$, then $\hat{\tau}$ is the time of the next impact. As the bouncing is elastic, the velocity after this impact is

$$\hat{v} = -x'(\hat{\tau}; \tau, v).$$

If $\hat{v}$ is finite and positive, the map

$$\mathcal{S} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+, \quad \mathcal{S}(\tau, v) = (\hat{\tau}, \hat{v})$$

is well defined, continuous, and one to one. According to [1, 21–23], this function is called successor map. The iteration of the successor map is denoted by $\mathcal{S}^n(\tau, v) = (\hat{\tau}^n(\tau, v),\ \hat{v}^n(\tau, v))$ and we will use $\hat{\tau}^n = \hat{\tau}^n(\tau, v)$, $\hat{v}^n = \hat{v}^n(\tau, v)$ for short.

We assume that $f(t, x)$ satisfies the following condition

(H_0) $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, 2π-periodic in the first variable and locally Lipschitzian in the second variable. Moreover, it satisfies the Landesman-Lazer condition

$$\int_0^{2\pi} \liminf_{x\to+\infty} f(t,x)dt > 0.$$

In this paper, we will prove, by using hypothesis (H_0), that for any $n \in \mathbb{N}$ the iteration of the successor map $\mathcal{S}^n(\tau, v)$ is well defined for v sufficiently large, that is the next n-th impact exists and has non-zero velocity. Moreover, let

$$T_*^n(v) = \inf\{\ \hat{\tau}^n(\tau, v) - \tau\ :\ \tau \in \mathbb{R}\ \}.$$

Our main result can be stated as the following theorem.

Theorem 1.1. *Assume (H_0) and there exists a sequence $\{v_k\}$ with $v_k \to +\infty$ as $k \to +\infty$, such that*

$$(T_0) \qquad \lim_{k\to+\infty} T_*^n(v_k) = +\infty,$$

then for any $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that for $m \geq m_n$, there exists at least one $2m\pi$-periodic bouncing solution $x_{n,m}(t)$ of (1.1) with exactly n impacts in $[0, 2m\pi)$. Moreover, $\max\{|x'_{n,m}(\hat{\tau}_w-)| : \hat{\tau}_w \in [0, 2m\pi),\ w = 1, \ldots, n$ are impact times of $x_{n,m}(t)\} \to +\infty$ as $m \to \infty$, and if m and n are relatively prime, then $x_{n,m}(t)$ is a $2m\pi$-periodic solution which is not $2l\pi$-periodic, for every $l \in \{1, 2, \cdots, m-1\}$.

Remark 1.1. *Hypothesis (T_0) means that there exists a sequence $\{v_k\}$ with $v_k \to +\infty$ as $k \to +\infty$, such that the time for next n-th impact of $x(t; \tau, v_k)$ tends to infinity with the increase of initial velocity v_k. This leads to infinitely many twists in the phase plane if the successor map $\mathcal{S}$ is well defined. But hypothesis (T_0) is not easy to check. Later, we will give some explicit sufficient conditions of assumption (T_0) as the corollaries of Theorem 1.1 for application. Our theorem can be applied to the following interesting classes of impact models (The details see Remarks 1.3 and 1.5).*
1. Sublinear impact equation, that is $f(t, x)$ satisfies sublinear growth condition

$$\lim_{x\to+\infty} \frac{f(t,x)}{x} = 0, \quad \textit{uniformly for} \quad t \in [0, 2\pi]. \tag{1.4}$$

As a typical case of sublinear impact osillctor, we can apply our theorem to periodically forced pendulum with impact

$$x'' + \sin x = p(t), \quad \textit{where} \quad \int_0^{2\pi} p(t)dt < -1. \tag{1.5}$$

2. "Sublinearity" in a more general sense. For example, let $f(t, x) = g(x) + p(t)$, where $\int_0^{2\pi} p(t)dt < 0$ and g is of the form

$$g(x) = \begin{cases} x^3 \sin^2 x, & \textit{for } x \in [x_{2k-1}, x_{2k}], \\ 0, & \textit{for } x \in [x_{2k}, x_{2k+1}], \end{cases}$$

where $x_k \to +\infty$ as $k \to +\infty$, such that for $G(x) = \int_0^x g(s)ds$ we have

$$\frac{G(x_{2k+1})}{x_{2k+1}^2} \leq \frac{1}{4k}, \quad \frac{G(x_{2k})}{x_{2k}^2} \geq k, \quad \textit{for} \quad k = 1, 2, \cdots.$$

Obviously, we have

$$\liminf_{x\to+\infty} \frac{g(x)}{x} = 0 \quad \textit{and} \quad \limsup_{x\to+\infty} \frac{g(x)}{x} = +\infty.$$

Remark 1.2. *In Theorem 1.1, when $f(t,x)$ is 2π-least periodic in t and m and n are relatively prime, then $x_m(t)$ is a $2m\pi$-least periodic solution, that is $x_m(t)$ is a m-subharmonic solution. When m and n have a common factor $d \in \mathbb{N},\ d > 1$, then $x_m(t)$ may be a m/d-subharmonic solution.*

Recall that the Landesman-Lazer condition in assumption (H_0) can be written as the following equivalent statment (see, [12], Lemma 1): there exist a constant $d_0 > 0$ and a L^1-function $\psi : [0,2\pi] \to \mathbb{R}$, such that for $x \geq d_0$, $f(t,x) \geq \psi(t)$ and $\bar{\psi} = \frac{1}{2\pi}\int_0^{2\pi}\psi(t)dt > 0$. Then, denoting $\widetilde{f}(t,x) = f(t,x) - \psi(t) + \bar{\psi}$ and $\Psi(t) = -\int_0^t \psi(s)ds + t\bar{\psi}$, the equation (1.2) has a equivalent 2π-periodic system of the form

$$x' = y + \Psi(t), \quad y' = -\widetilde{f}(t,x). \tag{1.6}$$

Moreover, $\widetilde{f}(t,x) \geq \bar{\psi}$ for $x \geq d_0$.

As the first corollary of the above theorem, we can prove the existence and multiplicity of subharmonic bouncing solutions for sublinear impact oscillator which is a generalization of the main theorem in [10] to impact oscillator.

Corollary 1.1. *Assume (H_0) and there is a continuous function g, such that*

$$\widetilde{f}(t,x) \leq g(x) \ \text{ for } t \in [0,2\pi] \text{ and } x \geq d_0.$$

Moreover, for $G(x) = \int_0^x g(s)ds$, we have

$$(G_0) \qquad \lim_{c\to+\infty}\int_0^c \frac{1}{\sqrt{G(c)-G(s)}}ds = +\infty.$$

Then for any $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that for $m \geq m_n$, there exists at least one $2m\pi$-periodic bouncing solution of (1.1) with exactly n impacts in $[0,2m\pi)$.

Remark 1.3. *If $f(t,x)$ satisfies sublinear growth condition (1.4), we can choose $g(x) = \max\{\widetilde{f}(t,x) : t \in [0,2\pi]\}$. Then $\widetilde{f}(t,x) \leq g(x)$ for $t \in [0,2\pi]$, $x > 0$ and*

$$\lim_{x\to+\infty}\frac{g(x)}{x} = 0,$$

which implies that

$$\int_0^c \frac{1}{\sqrt{G(c)-G(s)}}ds \geq \int_{c/2}^c \frac{1}{\sqrt{G(c)-G(s)}}ds \geq \frac{\sqrt{2\xi}}{\sqrt{g(\xi)}} \to +\infty \quad as \quad c \to +\infty.$$

Hence, Corollary 1.1 can be applied to sublinear impact equation. Now we consider following periodically forced pendulum with impact

$$(E_\lambda) \qquad x'' + \sin x = \lambda p(t), \quad where \quad \int_0^{2\pi} p(t)dt < -1.$$

When $\lambda \geq 1$, we can use Corollary 1.1 to prove the existence of m-subharmonic solution $x_{n,m}(t)$ for m being sufficiently large. Moreover, we know $x_{n,m}(t)$ has exactly n impacts in $[0,2m\pi)$ and $\max\{|x'_{n,m}(\hat{\tau}_w-)| : \hat{\tau}_w \in [0,2m\pi),\ w = 1,\dots,n$ are impact times of $x_{n,m}(t)\} \to +\infty$ as $m \to \infty$. When

$\lambda = 0$, then the solution of (E_0) satisfies that $(x'(t))^2/2 = \cos x(t) + C$ for $x(t) \geq 0$, where C is a constant. Thus, for $v > 2$, the solution $x(t;\tau,v)$ of (E_0) satisfies that $x'(t;\tau,v)$ does not vanish at any time $t > \tau$. Moreover, $x(t;\tau,v)$ is not a periodic solution. If m is sufficiently large, we have $\varepsilon_0 > 0$ such that (E_λ) has no $2m\pi$-periodic solution $x(t;\tau,v)$ with $v > 2$ for $\lambda \in [0,\varepsilon_0]$ by using the continuous dependence theorem of solutions for parameters.

When (1.1) is a forced impact oscillator, i.e. $f(t,x) = g(x) + p(t)$, with g, p being continuous functions and p being 2π-periodic, we have the following corollary of Theorem 1.1 in which hypothesis (G_0) is substituted by a more explicit and general hypothesis

$$(G_1) \qquad \liminf_{x\to+\infty} G(x)/x^2 = 0.$$

Corollary 1.2. *Assume (H_0) and (G_1). Then for any $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that for $m \geq m_n$, there exists at least one $2m\pi$-periodic bouncing solution of (1.1) with exactly n impacts in $[0, 2m\pi)$.*

Remark 1.4. *The condition similar to (G_1) is firstly introduced in [11] to guarantee the existence of 2π-periodic solution.*

Remark 1.5. *Assumption (G_0) implies (G_1). Actually, if (G_0) holds but*

$$\liminf_{x\to+\infty} G(x)/x^2 = 2\alpha > 0,$$

then

$$\lim_{x\to+\infty} (G(x) - \alpha x^2) = +\infty,$$

which implies that there is a sequence $\{x_k\}$ with $x_k \to +\infty$ as $k \to +\infty$, such that

$$G(x_k) - G(s) \geq \alpha(x_k^2 - s^2) \quad for \quad s \in (0, x_k], \quad for \quad k = 1, 2, \cdots.$$

Thus,

$$\int_0^{x_k} (G(x_k) - G(s))^{-1/2} ds \leq (\sqrt{\alpha})^{-1} \int_0^{x_k} (x_k^2 - s^2)^{-1/2} ds = (2\sqrt{\alpha})^{-1}\pi,$$

for $k = 1, 2, \cdots$, which contradicts to (G_0). It is easy to see that Corollary 1.2 can be applied to the models in Remark 1.1.

The rest of the paper is organized as follows. In Section 2, the proof of Theorem 1.1 is given. Sections 3 and 4 are used to prove Corollaries 1.1 and 1.2, respectively. Section 5 is an appendix, which contains the proof of a key lemma in Section 2.

2. Existence of periodic bouncing solutions

In order to prove the existence of periodic bouncing solutions of impact oscillators (1.1), we will apply the Poincaré-Birkhoff twist theorem to the successor map $\mathcal{S}$. In this paper, we adopt the following version of the Poincaré-Birkhoff twist theorem.

Theorem 2.1. *Let $\mathcal{A}$ be an annulus in the plane, whose boundaries $\Gamma_i,\ i = 1, 2$ are starlike closed curves around the origin $O \in \mathcal{D}_1$, where $\mathcal{D}_1$ is the bounded region confined by the inner boundary Γ_1. Suppose*

$$f : \mathcal{A} \to \mathcal{B}$$

is an area-preserving homeomorphism homotopic to the inclusion such that $\mathcal{A} \subset \mathcal{B}$ and $f(\mathcal{A}) \subset \mathcal{B}$, where $\mathcal{B}$ is an annulus and $f(\mathcal{A}) \cap \partial\mathcal{B} = \emptyset$. Moreover, f possesses a continuous lift

$$\tilde{f} : (r, \theta) \to (h(r, \theta),\ \phi(r, \theta))$$

satisfying the boundary twist condition, that is

$$\begin{cases} \phi(r, \theta) - \theta - 2j_0\pi > 0 \ (or\ < 0), & for\ (r, \theta) \in \tilde{\Gamma}_1; \\ \phi(r, \theta) - \theta - 2j_0\pi < 0 \ (or\ > 0), & for\ (r, \theta) \in \tilde{\Gamma}_2 \end{cases}$$

for some $j_0 \in \mathbb{Z}$, where $\tilde{\Gamma}_i$ is the lift of $\Gamma_i,\ i = 1, 2$, respectively. In addition, the areas of the two connected components of the complement of $f(\mathcal{A})$ in $\mathcal{B}$ are the same as the areas of the corresponding connected components of the complement of $\mathcal{A}$ in $\mathcal{B}$, respectively.

Then, f has at least two geometrically distinct fixed points (r_i, θ_i) for $i = 1, 2$, which satisfy

$$\phi(r_i, \theta_i) - \theta_i = 2j_0\pi,\ i = 1, 2.$$

Remark 2.1. *The proof of this theorem can be obtained in [27], one can also refer to [30]. Its proof basically combines the proofs of the classic form of the Poincaré-Birkhoff twist theorem and a geometric lemma mentioned in [13] by Franks.*

Next, we will discuss the properties of the successor map in order to apply the above Poincaré-Birkhoff twist theorem.

First, we will prove that the successor map $\mathcal{S}$ is well defined where v is sufficiently large.

Lemma 2.1. *Assume (H_0) holds. Then for any $n \in \mathbb{N}$, there exists $v(n) > 0$, such that $\mathcal{S}^n(\tau, v)$ is well defined for $v \geq v(n)$ and $\tau \in \mathbb{R}$.*

The proof of Lemma 2.1 will be given in Section 5.

From the uniqueness of the solution of (1.1) with respect to the initial condition, we know that the successor map $\mathcal{S}$

$$\mathcal{S} : (\tau,\ v) \mapsto (\hat{\tau},\ \hat{v})$$

is well defined, one to one, and continuous in its domain $\mathbb{R} \times [v_0, +\infty)$, where v_0 is sufficiently large. Moreover, it satisfies

$$\mathcal{S}(\tau + 2\pi, v) = \mathcal{S}(\tau, v) + (2\pi, 0).$$

Thus, we can interpret τ and v as polar coordinates and $\mathcal{S}$ is an embedding homeomorphism on $\mathbf{S}^1 \times [v_0, +\infty)$.

Lemma 2.2. *$\mathcal{S}$ is an area-preserving map with the area element $v\mathrm{d}v\mathrm{d}\tau$. Moreover, $\mathcal{S}$ is area-preserving homotopic to the inclusion, and for any annuli $\mathcal{A} \subset \mathcal{B} \subset \mathbf{S}^1 \times \mathbb{R}^+$ with $\mathcal{S}(\mathcal{A}) \subset \mathrm{int}(\mathcal{B})$, the areas of the two connected components of the complement of $\mathcal{S}(\mathcal{A})$ in $\mathcal{B}$ are the same as the areas of the corresponding components of the complement of $\mathcal{A}$ in $\mathcal{B}$, respectively.*

The proof of this lemma is similar to the proof of Lemma 1 in [15].

It is easy to show that for any $n, m \in \mathbb{N}$, a fixed point of the $\mathcal{S}^n - (2m\pi, 0)$ corresponds to a $2m\pi$-periodic bouncing solution of the equation (1.1) with n impacts in each period.

Now, we apply Theorem 2.1 to the successor map $\mathcal{S}$.

First, denote $T_n^*(v) = \sup\{\ \hat{\tau}^n(\tau, v) - \tau\ :\ \tau \in \mathbb{R}\ \}$, then $T_*^n(v) \leq T_n^*(v)$. From Lemma 2.1, $\mathcal{S}^n$ is well defined and continuous in $\tau \in [0, 2\pi]$ and $v \geq v(n)$, same as $\hat{\tau}^n$. Then, for fixed n and v, $\hat{\tau}^n(\tau, v) - \tau$ is well defined. Thus, $T_n^*(v)$ must be limited by the finite covering theorem. Thus, there exists $m_n \in \mathbb{N}$, such that $T_n^*(v(n)) < 2m_n\pi$. For $m > m_n,\ m \in \mathbb{N}$, we have $T_n^*(v(n)) < 2m\pi$, that is

$$\hat{\tau}^n(\tau, v(n)) - \tau < 2m\pi, \qquad \tau \in [0, 2\pi]. \tag{2.1}$$

On the other hand, for the above m, there exits k_m under (T_0), such that $T_*^n(v_{k_m}) > 2m\pi$, that is

$$\hat{\tau}^n(\tau, v_{k_m}) - \tau > 2m\pi, \qquad \tau \in [0, 2\pi]. \tag{2.2}$$

Let $\mathcal{A}$ denote the annulus bounded by $\mathbf{S}^1 \times \{v(n)\}$ and $\mathbf{S}^1 \times \{v_{k_m}\}$ and $\mathcal{B}$ denote the annulus bounded by $\mathbf{S}^1 \times \{v_*\}$ and $\mathbf{S}^1 \times \{v^*\}$. Then we can prove, when $v_* > 0$ is sufficiently small and v^* is sufficiently large, $\mathcal{P}(\mathcal{A}) \subset \mathrm{int}(\mathcal{B})$, where $\mathcal{P} :\ \mathcal{A} \to \mathcal{B}$ is defined as follows

$$\mathcal{P}(\tau, v) = \mathcal{S}^n(\tau, v) - (2m\pi, 0).$$

(2.1) and (2.2) show that the lift $\widetilde{\mathcal{P}}$ possesses the boundary twist condition on $\mathcal{A}$. Moreover, Lemma 2.2 implies that $\mathcal{P}$ meets all the other conditions of Theorem 2.1. Thus the conclusion of Theorem 1.1 is generated by a direct application of Theorem 2.1. Specifically, if the number of bouncing n is greater than or equal to 2, the two fixed points provided by Theorem 2.1 may correspond to the same bouncing solution $x_{n,m}(t)$, so we can only assure the existence of the two different $2m\pi$- periodic bouncing solutions when there is only one impact in each period. Moreover, $T_n^*(v)$ is continuous with respect to v. So $T_n^*(v)$ is bounded for $v \in D$, where D is compact. This implies that $\max\{|x'_{n,m}(\hat{\tau}_w-)|\ :\ \hat{\tau}_w \in [0, 2m\pi),\ w = 1, \ldots, n$ are impact times of $x_{n,m}(t)\} \to +\infty$ as $m \to \infty$.

Theorem 1.1 is thus proved. □

3. Subharmonic bouncing solutions of sublinear impact oscillator

In this section, we prove Corollary 1.1 by Theorem 1.1. Let $x(t; \tau, v)$ be the solution of Eq (1.1) with the initial conditions

$$x(\tau; \tau, v) = 0, \quad x'(\tau; \tau, v) = v > 0.$$

For $v > 0$, we know that there exists $\tau_1 > 0$ such that $x(t; \tau, v) > 0$ for $t \in (\tau, \tau_1)$. Moreover, $x(t; \tau, v)$ is a classical solution of Eq (1.2) for $t \in (\tau, \tau_1)$. Let $y(t; \tau, v) = x'(t; \tau, v) - \Psi(t)$, then for $t \in (\tau, \tau_1)$, $(x(t; \tau, v), y(t; \tau, v))$ satisfies the system (1.6). Without loss of generality, we suppose that

$$d_0 > 2\Psi_0 = 2 \max_{t \in [0, 2\pi]} \{|\Psi(t)|\} > 0.$$

From assumption (H_0) and Lemma 2.1, we know that the successor map $\mathcal{S}(\tau, v) = (\hat{\tau}, \hat{v})$ is well defined when v is sufficiently large. Moreover, there exist $\alpha(\tau, v), \beta(\tau, v) \in (\tau, \hat{\tau})$ such that $x(\alpha(\tau, v)) = d_0, y(\beta(\tau, v)) = d_0$ and

$$x(t; \tau, v) \geq d_0, \quad y(t; \tau, v) \geq d_0, \quad \text{for} \quad t \in [\alpha(\tau, v), \beta(\tau, v)].$$

Next we will prove

Lemma 3.1. *Assume* (H_0) *and* (G_0)*, then*

$$\lim_{v\to+\infty}(\beta(\tau,v)-\alpha(\tau,v))=+\infty,\quad \textit{uniformly for}\quad \tau\in\mathbb{R}.$$

Proof. In the proof, we will use $\alpha=\alpha(\tau,v)$, $\beta=\beta(\tau,v)$ for short. Define

$$H_+(x,y)=\frac{y^2}{4}+G(x).$$

Then

$$\begin{aligned}H'_+(x(t),y(t))&=\frac{1}{2}yy'+g(x)x'=-\frac{y}{2}\widetilde{f}(t,x)+g(x)(y+\Psi)\\&=\frac{y}{2}[g(x)-\widetilde{f}(t,x)]+\left(\frac{y}{2}+\Psi\right)g(x).\end{aligned}$$

Hence,

$$H'_+(x(t),y(t))>0,\quad \text{for}\quad t\in(\alpha,\beta).$$

It implies

$$G(x(t))+y^2(t)/4<H_+(x(\beta),y(\beta)),\quad \text{for}\quad t\in(\alpha,\beta).$$

Then

$$x'(t)=y(t)+\Psi(t)<d_0+2\sqrt{d_0^2/4+G(x(\beta))-G(x(t))},\quad \text{for}\quad t\in(\alpha,\beta).$$

Through simple calculation and integrating on $[\alpha,\beta]$, we have

$$\beta-\alpha>\int_{d_0}^{x(\beta)}\frac{ds}{d_0+2\sqrt{d_0^2/4+G(x(\beta))-G(s)}}.\tag{3.1}$$

On the other hand, from the condition $g(x)\geq\widetilde{f}(t,x)\geq\bar{\psi}$ for $x\geq d_0$, we have a $L>0$ such that

$$d_0+2\sqrt{d_0^2/4+G(c)-G(s)}\leq 4\sqrt{G(c)-G(s)}\quad \text{for}\quad s\in[d_0,c-L],$$

where $c\gg 1$. Moreover

$$\int_{c-L}^{c}\frac{ds}{\sqrt{G(c)-G(s)}}=\int_{c-L}^{c}\frac{ds}{\sqrt{g(\xi)(c-s)}}\leq 2\sqrt{\frac{L}{d_0}}.$$

Then

$$\begin{aligned}\int_{d_0}^{c-L}\frac{ds}{d_0+2\sqrt{d_0^2/4+G(c)-G(s)}}&\geq\int_{d_0}^{c-L}\frac{ds}{4\sqrt{G(c)-G(s)}}\\&\geq\int_{d_0}^{c}\frac{ds}{4\sqrt{G(c)-G(s)}}-2\sqrt{\frac{L}{d_0}}\to+\infty\end{aligned}$$

as $c\to+\infty$ by (G_0).

Finally, from the estimation (5.7) in Section 5, we have

$$\lim_{v\to+\infty}x(\beta(\tau,v))=+\infty,\quad \text{uniformly for}\quad \tau\in\mathbb{R}.$$

Lemma 3.1 is thus proved with (3.1). □

Lemma 3.1 implies that (T_0) holds, so we obtain the conclusion of Corollary 1.1 by Theorem 1.1.

4. Subharmonic bouncing solutions of the forced impact oscillator with weak sub-quadratic potential

In this section, we prove Corollary 1.2. Consider the forced impact oscillator of the form

$$\begin{cases} x'' + g(x) = p(t), \quad \text{for } x(t) > 0; \\ x(t) \geq 0; \\ \text{If there exists } t_0 \text{ such that } x(t_0) = 0 \text{ then } x'(t_0+) = -x'(t_0-). \end{cases} \tag{4.1}$$

In this case, (H_0) can be expressed as the following

(H_0') $g : \mathbb{R}^+ \to \mathbb{R}$ is continuous, and $p : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic. Moreover, there is $d_0 > 0$ such that

$$g(x) > \bar{p} = \frac{1}{2\pi} \int_0^{2\pi} p(s)ds, \quad \text{for} \quad x \geq d_0.$$

Let $x(t; \tau, v)$ be the solution of (4.1) with initial conditions

$$x(\tau; \tau, v) = 0, \quad x'(\tau; \tau, v) = v > 0.$$

For $v > 0$, we know that there exists $\tau_1 > 0$ such that $x(t; \tau, v) > 0$ for $t \in (\tau, \tau_1)$. Let $y(t; \tau, v) = x'(t; \tau, v) - P(t)$, where $P(t) = \int_0^t (p(s) - \bar{p})ds$. Then for $t \in (\tau, \tau_1)$, $(x(t; \tau, v), y(t; \tau, v))$ satisfies the following system

$$x' = y + P(t), \quad y' = -g(x) + \bar{p}. \tag{4.2}$$

Furthermore, there exists $\delta_0 > 0$ such that

$$g(x) - \bar{p} \geq \delta_0, \quad \text{for} \quad x \geq d_0.$$

Without loss of generality, we suppose that

$$d_0 > 2P_0 = 2 \max_{t \in [0, 2\pi]} \{|P(t)|\} > 0.$$

By assumption (H_0') and the method similar to that used in Lemma 2.1, we can prove that the successor map $\mathcal{S}(\tau, v) = (\hat{\tau}, \hat{v})$ is well defined for v being sufficiently large. Moreover, there exists $\alpha(\tau, v)$ and $\beta(\tau, v) \in (\tau, \hat{\tau})$ such that

$$x(\alpha(\tau, v)) = d_0, \quad y(\beta(\tau, v)) = d_0$$

and $(x(t; \tau, v), y(t; \tau, v))$ moves from $(d_0, y(\alpha(\tau, v)))$ to $(x(\beta(\tau, v)), d_0)$.

Next we will prove

Lemma 4.1. *Assume (H_0') and (G_1) hold, then there exists a sequence $\{v_k\}$ with $v_k \to +\infty$ as $k \to +\infty$, such that*

$$\lim_{k \to +\infty} (\beta(\tau, v_k) - \alpha(\tau, v_k)) = +\infty, \quad \textit{uniformly for} \quad \tau \in \mathbb{R}. \tag{4.3}$$

Proof. In the proof, we will use $\alpha = \alpha(\tau, v), \beta = \beta(\tau, v)$ for short. Define

$$I_{\pm}(x, y) = \widetilde{G}(x) + (y \pm d_0)^2/2,$$

where $\tilde{G}(x) = G(x) - \bar{p}x$. It is easy to verify that

$$I'_{+}(x(t), y(t)) < 0, \quad I'_{-}(x(t), y(t)) > 0, \quad \text{for} \quad t \in (\alpha, \beta),$$

which imply that

$$\widetilde{G}(x(t)) + (y(t) - d_0)^2/2 < \widetilde{G}(x(\beta)) < \widetilde{G}(x(t)) + (y(t) + d_0)^2/2, \quad \text{for} \quad t \in (\alpha, \beta). \tag{4.4}$$

Then

$$x'(t) = y(t) + P(t) < 2d_0 + \sqrt{2(\widetilde{G}(x(\beta)) - \widetilde{G}(x(t)))}, \quad \text{for} \quad t \in (\alpha, \beta),$$

and therefore, through simple computations and an integration on $[\alpha, \beta]$, we obtain

$$\beta - \alpha > \int_{d_0}^{x(\beta)} \frac{ds}{2d_0 + \sqrt{2(\widetilde{G}(x(\beta)) - \widetilde{G}(s))}} := \Lambda(x(\beta)). \tag{4.5}$$

On the other hand, in view of (4.4), we have two positive constants K_1 and K_2 such that

$$y^2(\alpha)/2 - K_1 y(\alpha) - K_2 \leq \widetilde{G}(x(\beta)) \leq y^2(\alpha)/2 + K_1 y(\alpha) + K_2.$$

Moreover, as the argument in Lemma 2.1, we have a positive constant K_3 such that

$$v - K_3 \leq y(\alpha) \leq v + K_3.$$

Then, for v being sufficiently large, we have a positive constant K_4 such that

$$v^2/2 - K_4 v \leq \widetilde{G}(x(\beta)) \leq v^2/2 + K_4 v. \tag{4.6}$$

From the assumption (G_1), we have a sequence $\{x_k\}$ with $x_k \to +\infty$ as $k \to \infty$, such that

$$\lim_{k \to +\infty} \frac{G(x_k)}{x_k^2} = 0,$$

which implies

$$\lim_{k \to +\infty} \frac{\widetilde{G}(x_k)}{x_k^2} = \lim_{k \to +\infty} \frac{G(x_k) - \bar{p}x_k}{x_k^2} = 0.$$

$\forall \varepsilon > 0$, choose k_0, such that

$$\frac{\widetilde{G}(x_k)}{x_k^2} < \varepsilon, \quad \text{for} \quad k \geq k_0.$$

Let $v_k = K_4 + \sqrt{K_4^2 + 2\widetilde{G}(x_k)}$, that is $v_k^2/2 - K_4 v_k = \widetilde{G}(x_k)$, then from (4.6), we have $x(\beta) \geq x_k$. Moreover, through (4.5) and (4.6), we obtain

$$\begin{aligned}
\Lambda(x(\beta)) &= \int_{d_0}^{x(\beta)} \frac{ds}{2d_0 + \sqrt{2(\widetilde{G}(x(\beta)) - \widetilde{G}(s))}} \\
&\geq \frac{x(\beta) - d_0}{2d_0 + \sqrt{2\widetilde{G}(x(\beta))}} \cdot \frac{x_k}{\sqrt{\widetilde{G}(x_k)}} \cdot \frac{\sqrt{\widetilde{G}(x_k)}}{x(\beta)} \\
&\geq \frac{x_k}{\sqrt{\widetilde{G}(x_k)}} \cdot \frac{\sqrt{v_k^2/2 - K_4 v_k}}{2d_0 + \sqrt{v_k^2 + 2K_4 v_k}} \cdot \frac{x(\beta) - d_0}{x(\beta)}.
\end{aligned}$$

Thus, for $k \geq k_0$, we have a positive constant K_5 such that

$$\Lambda(x(\beta)) \geq K_5/\sqrt{\varepsilon}.$$

Therefore, by (4.5), we obtain

$$\lim_{k\to+\infty} (\beta(\tau, v_k) - \alpha(\tau, v_k)) = +\infty, \quad \text{uniformly for} \quad \tau \in \mathbb{R}. \qquad \square$$

Lemma 4.1 implies that (T_0) holds, so we can prove Corollary 1.2 by Theorem 1.1.

5. The proof of Lemma 2.1

The Landesman-Lazer condition in Assumption (H_0) can be written in a different form. For example: without loss of generality, we suppose that

$$d_0 > 2\Psi_0 = 2 \max_{t\in[0,2\pi]} \{|\Psi(t)|\} > 0,$$

where $\Psi(t) = -\int_0^t \psi(s)ds + t\bar{\psi}$ is 2π-periodic. Let

$$g_-(x) := \min_{t\in[0,2\pi]} \{\widetilde{f}(t,x)\}, \quad g_+(x) := \max_{t\in[0,2\pi]} \{\widetilde{f}(t,x)\},$$

then $g_\pm(x) \geq \bar{\psi} > 0$ for $x \geq d_0$, which implies that

$$G_\pm(x) = \int_0^x g_\pm(s)ds \to +\infty, \quad \text{as} \quad x \to +\infty.$$

When $v > 0$, there exists $\tau_1 > 0$ such that $x(t;\tau,v) > 0$ for $t \in (\tau,\tau_1)$. Moreover, $x(t;\tau,v)$ is a solution of Eq (1.2). Let

$$y(t;\tau,v) = x'(t;\tau,v) - \Psi(t).$$

Then for $t \in (\tau,\tau_1)$, $(x(t;\tau,v), y(t;\tau,v))$ satisfies the system of the form

$$x' = y + \Psi(t), \quad y' = -\widetilde{f}(t,x). \tag{5.1}$$

Denote by $(x(t), y(t)) = (x(t;\tau, v), y(t;\tau, v))$ and let

$$D_1 = \{(x,y) : 0 \le x \le d_0,\ y \ge d_0\}; \quad D_2 = \{(x,y) : x \ge d_0,\ y \ge d_0\};$$
$$D_3 = \{(x,y) : x \ge d_0,\ |y| \le d_0\}; \quad D_4 = \{(x,y) : x \ge d_0,\ y \le -d_0\};$$
$$D_5 = \{(x,y) : 0 \le x \le d_0,\ y \le -d_0\}.$$

Our proof will be divided into five steps.

Step 1. Let $M_0 = \max_{t\in[0,2\pi],x\in[0,d_0]}\{|\widetilde{f}(t,x)|\}$ and $v \ge M_0 + 3d_0$. For $t > \tau$, if $0 < x(t) \le d_0$, then $y(t) > 2d_0$. Otherwise, we have $\tau' > \tau$ such that $y(\tau') = 2d_0$ and

$$0 < x(t) \le d_0, \quad y(t) > 2d_0 \quad \text{for} \quad t \in (\tau, \tau').$$

Then

$$|v - \Psi(\tau) - y(\tau')| = \left|\int_\tau^{\tau'} \left(-\widetilde{f}(s, x(s))\right) ds\right| \le (\tau' - \tau)M_0,$$

which implies

$$\tau' - \tau \ge (v - 3d_0)/M_0.$$

Thus

$$|x(\tau') - x(\tau)| = \left|\int_\tau^{\tau'} (y(s) + \Psi(s))ds\right| > (\tau' - \tau)d_0 \ge d_0(v - 3d_0)/M_0,$$

that is

$$v < M_0 + 3d_0,$$

which leads to a contradiction.

Moreover, for $t > \tau$ and $0 < x(t) \le d_0$, we have

$$|y(t) - v - \Psi(\tau)| = \left|\int_{x(\tau)}^{x(t)} \frac{-\widetilde{f}(t,x)}{y + \Psi} dx\right| \le \frac{2M_0 d_0}{3d_0} = \frac{2M_0}{3}. \tag{5.2}$$

According to the continuation theorem (e.g., [14]), there exits $t_1 > \tau$, such that $x(t_1) = d_0$. Thus, (5.2) implies

$$v - d_0 - 2M_0/3 \le y(t_1) \le v + d_0 + 2M_0/3. \tag{5.3}$$

Then

$$x'(t_1) > 0.$$

Step 2. From Step 1, we have $(x(t), y(t)) \in D_2$ for $t > t_1$ and t being near t_1. In order to describe the motion of $(x(t), y(t))$ in D_2, we define

$$H_-(x,y) = y^2 + G_-(x), \quad H_+(x,y) = y^2/4 + G_+(x).$$

Then

$$r = \sqrt{x^2 + y^2} \to +\infty \Longleftrightarrow H_\pm(x,y) \to +\infty. \tag{5.4}$$

Moreover, it is easy to verify that

$$H'_-(x(t), y(t)) < 0, \quad H'_+(x(t), y(t)) > 0, \quad \text{for} \quad (x(t), y(t)) \in D_2,$$

which implies that

$$H_-(x(t), y(t)) < H_-(x(t_1), y(t_1)), \quad \text{for} \quad (x(t), y(t)) \in D_2, \tag{5.5}$$

and

$$H_+(x(t), y(t)) > H_+(x(t_1), y(t_1)), \quad \text{for} \quad (x(t), y(t)) \in D_2. \tag{5.6}$$

From (5.4)-(5.6) and the continuation theorem, we know that $(x(t), y(t))$ will intersect with $y = d_0$ at certain point $t_2 \in (t_1, \tau_1)$. Moreover, by (5.3), (5.5) and (5.6), we have

$$M_1^-(v) < x(t_2) < M_1^+(v), \tag{5.7}$$

where

$$M_1^-(v) = G_+^{-1}(G_+(d_0) + (v - d_0 - 2M_0/3)^2/4 - (d_0)^2/4)$$

and

$$M_1^+(v) = G_-^{-1}[G_-(d_0) + (v + d_0 + 2M_0/3)^2 - (d_0)^2].$$

Step 3. (5.7) implies that $y'(t_2) = -\widetilde{f}(t_2, x(t_2)) \leq -\bar{\psi} < 0$. Hence, for $t > t_2$ and t being near t_2, we have $(x(t), y(t)) \in D_3$. Moreover, there exists $t_3 \in (t_2, \tau_1)$ such that $y(t_3) = -d_0$ and

$$|x(t) - x(t_2)| < M_2, \quad |y(t)| < d_0, \quad \text{for} \quad t \in (t_2, t_3), \tag{5.8}$$

where $M_2 = 5(d_0)^2/\bar{\psi}$. Otherwise, according to the continuation theorem, there exists $t_2' \in (t_2, \tau_1)$ such that $|x(t_2') - x(t_2)| = M_2$ and

$$|x(t) - x(t_2)| \leq M_2, \quad |y(t)| < d_0, \quad \text{for} \quad t \in (t_2, t_2'].$$

Thus if v is sufficiently large, from (5.7), we have $x(t) \geq d_0$ for $t \in (t_2, t_2']$. Then

$$M_2 = |x(t_2') - x(t_2)| = \left| \int_{y(t_2)}^{y(t_2')} \frac{y + \Psi}{-\widetilde{f}(t, x)} dy \right| \leq 4d_0^2/\bar{\psi},$$

which leads to a contradiction.

Hence, by (5.8), we have

$$M_1^-(v) - M_2 < x(t_3) < M_1^+(v) + M_2. \tag{5.9}$$

Then

$$y'(t_3) = -\widetilde{f}(t_3, x(t_3)) < 0.$$

Step 4. From Step 3, we have $(x(t), y(t)) \in D_4$ for $t > t_3$ and

$$H_-'(x(t), y(t)) > 0, \quad H_+'(x(t), y(t)) < 0, \quad \text{for} \quad (x(t), y(t)) \in D_4.$$

From similar argument as in Step 2, we know that $(x(t), y(t))$ will intersect with $x = d_0$ at certain point $t_4 \in (t_3, \tau_1)$. Furthermore, there exist $M_3^{\pm}(v)$ with

$$M_3^{\pm}(v) \to -\infty \Longleftrightarrow v \to +\infty,$$

such that

$$M_3^-(v) < y(t_4) < M_3^+(v). \tag{5.10}$$

Step 5. From Step 4, we have $(x(t), y(t)) \in D_5$ for $t > t_4$. Moreover,

$$y(t) < -2d_0, \quad \text{when } v \text{ being sufficiently large.} \tag{5.11}$$

Otherwise, suppose there exists t_5, such that $y(t_5) = -2d_0$ and

$$0 < x(t) \le d_0,\ y(t) > -2d_0, \qquad t \in (t_4, t_5).$$

Then,

$$|y(t_5) - y(t_4)| = \left|\int_{t_4}^{t_5} -\widetilde{f}(s, x(s))ds\right| \le M_0(t_5 - t_4),$$

that is

$$t_5 - t_4 \ge \frac{1}{M_0}|y(t_5) - y(t_4)| \ge \frac{1}{M_0}|M_3^+(v) + 2d_0|.$$

Hence,

$$\begin{aligned} d_0 > |x(t_5) - x(t_4)| &= \left|\int_{t_4}^{t_5} x'(s)ds\right| = \left|\int_{t_4}^{t_5} (y(s) + \Psi(s))ds\right| \\ &> (t_5 - t_4)d_0 \ge \frac{d_0}{M_0}|M_3^+(v) + 2d_0|, \end{aligned}$$

which leads to a contradiction.

The inequality (5.11) implies that $x'(t) = y(t) + \Psi(t) < -3d_0/2$ and

$$\begin{aligned} |y(t) - y(t_4)| &= \left|\int_{t_4}^{t} y'(s)ds\right| = \left|\int_{x(t_4)}^{x(t)} \frac{y'}{x'}dx\right| \\ &= \left|\int_{x(t_4)}^{x(t)} \frac{-\widetilde{f}(t, x)}{y + \Psi}dx\right| \le \frac{2M_0d_0}{3d_0} = \frac{2M_0}{3}. \end{aligned}$$

According to the continuation theorem, there exists $\hat{\tau} > t_4 > \tau$ such that $x(\hat{\tau}) = 0$.

Let $\hat{v} = -(y(\hat{\tau}-) + \Psi(\hat{\tau}))$. Then $\mathcal{S} :\ (\tau, v) \mapsto (\hat{\tau}, \hat{v})$ is well defined. Moreover,

$$|y(\hat{\tau}-) - y(t_4)| = \left|\int_{t_4}^{\hat{\tau}} y'(s)ds\right| = \frac{2M_0}{3}.$$

With (5.10), we have

$$M_3^-(v) - \frac{2M_0}{3} < y(\hat{\tau}-) < M_3^+(v) + \frac{2M_0}{3}.$$

Hence

$$\hat{v} \to +\infty \Longleftrightarrow v \to +\infty.$$

For any $n \in \mathbb{N}$, we can define $\mathcal{S}^n$ by applying the above discussion recursively. Lemma 2.1 is thus proved.

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Conflict of interest

All authors declare no conflicts of interest.

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