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Research article

Existence results for Riemann-Liouville fractional integro-differential inclusions with fractional nonlocal integral boundary conditions

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Abstract: We introduce a new class of problems consisting of Riemann-Liouville fractional integro-differential inclusions supplemented with fractional nonlocal multi-point boundary conditions. The existence results for the given problem are derived in the weighted space with the aid of appropriate fixed point theorems for multi-valued maps. Numerical examples are constructed for the illustration of the obtained results.

Keywords: Riemann-Liouville fractional derivative; integro-differential inclusions; nonlocal multi-point boundary conditions; existence; fixed point
Mathematics Subject Classification: 34A08, 34A60, 34B15

1. Introduction

In this paper, we introduce and investigate the existence of solutions for a Riemann-Liouville nonlinear fractional integro-differential inclusion

$$D^{\alpha}u(t) \in F(t, u(t), (\phi u)(t), (\psi u)(t)), \ 1 < \alpha \leq 2, \quad t \in [0, T], \tag{1.1}$$

supplemented with nonlocal integral multi-point boundary conditions of fractional order

$$\begin{cases} D^{\alpha-2}u(0^+) + a_0 D^{\alpha-2}u(T^-) &= \varphi(u), \\ D^{\alpha-1}u(0^+) + a_1 D^{\alpha-1}u(T^-) &= \nu I^{\alpha-1}u(\eta) + \sum_{i=1}^{m} \mu_i u(\xi_i), \end{cases} \tag{1.2}$$

where $0 < \eta < \xi_1 < \xi_2 < \ldots < \xi_m < T$, D^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha > 0$, I^{α} denotes the Riemann-Liouville fractional integral of order $\alpha > 0$, $\nu, a_1, a_0, \ \mu_i, i \in \{1, 2, \ldots, m\}$

are real constants with $a_0 \neq -1$, $F : [0,T] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ ($\mathcal{P}(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$), $\varphi : C([0,T],\mathbb{R}) \to \mathbb{R}$ is given appropriate continuous function and

$$(\phi u)(t) = \int_0^t \gamma(t,s)u(s)ds, \quad (\psi u)(t) = \int_0^t \delta(t,s)u(s)ds,$$

with γ and δ being continuous functions on $[0,T] \times [0,T]$. In (1.2), one can notice that $D^{\alpha-2}$ is a fractional integral operator as $\alpha - 2 \in (-1,0]$. For the details of such notation, we refer the reader to Section 2.3.2 in [6].

The subject of differential equations of fractional order has evolved as an interesting and important field of research in view of its role describing some real world problems in physics, mechanics and other fields, for example, see [1–3] and the references cited therein. The investigation of theoretical aspects of fractional calculus received a lot of attention and now it is regarded as a significant area of research. Numerous research papers and monographs indeed reflect the popularity of the topic of fractional differential equations, for instance, see [4–10].

The study of fractional differential inclusions also gained much attraction and momentum as these tools of mathematical analysis are found to be of great utility in modeling stochastic and optimal controls problems [11]. Existence results for boundary value problems with different kinds of boundary conditions were established by many researchers with the aid of techniques of nonlinear analysis. Nonlocal boundary value problems have become a rapidly growing area of research, because mathematical models for several phenomena in engineering, physics and life sciences involve such problems. Some recent contributions to the subject of fractional differential inclusions can be seen in [12–18] and references cited therein.

Riemann-Liouville fractional derivative operators naturally arise in real world phenomena in several diverse disciplines such as viscoelastic materials, electrochemistry, control theory, neurons in biology, etc. For application details, we refer the reader to the works [19–22]. Riemann-Liouville fractional differential equations are used to study slow relaxation processes in complex systems like polymers, biological tissue and self-similar protein dynamics [23]. The behavior of real materials was discussed with the aid of Riemann-Liouville fractional derivatives in [24]. Riemann-Liouville fractional operators are employed to investigate projectile motion and electrical circuits in [25] and [26] respectively. In a recent paper [27], the authors investigated a backward problem for fractional diffusion equations involving Riemann-Liouville derivative.

In contrast to classical boundary conditions, nonlocal conditions help to formulate the changes happening on certain positions and segments of the given domain [28]. On the other hand, boundary conditions involving derivatives and integrals provide a decent approach to describe non-uniformities on segments of curved boundary structures. Examples include fluid problems [29], biomedical applications [30], bacterial self-organization [31], engineering applications [32, 33], etc.

The present work is motivated by the recent interest in Riemann-Liouville fractional derivative operators. We plan to study the problem (1.1)–(1.2) for convex valued (upper semicontinuous case) as well as non-convex valued (Lipschitz case) multifunctions. In case when the multivalued map F has convex values, we apply Bohnenblust-Karlin fixed point theorem and nonlinear alternative for multivalued maps to obtain the existence results. We apply a fixed point theorem for contractive multivalued maps due to Covitz and Nadler to discuss the existence of solutions for the problem at hand when F is non-convex multivalued map. The outline of the paper is as follows. Section 2 contains some preliminary concepts related to our problem, while the main results are presented in Section 3. Examples are constructed to illustrate the main results in Section 4. Some interesting observations are presented in Section 5.

2. Preliminaries

Let us begin this section with some basic definitions [5].

Definition 2.1. *The (left) Riemann-Liouville fractional integral for a locally integrable real-valued function v of order $\alpha \in \mathbb{R}$ ($\alpha > 0$), denoted by $I_a^\alpha v$, is defined by*

$$I_a^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} v(\tau) d\tau, -\infty \leq a < t < b \leq +\infty$$

where Γ denotes the Euler gamma function.

Definition 2.2. *Let $v, v^{(m)} \in L^1[a,b]$ for $-\infty \leq a < t < b \leq +\infty$. The (left) Riemann-Liouville fractional derivative $D_a^\alpha v$ of order $\alpha > 0$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as*

$$D_a^\alpha v(t) = \frac{d^m}{dt^m} I_a^{1-\alpha} v(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-1-\alpha} v(\tau) d\tau.$$

In this paper, we write Riemann-Liouville fractional integral and derivative operators as I^α and D^α instead of I_{a+}^α and D_{a+}^α respectively.

Related to a linear variant of the problem (1.1)–(1.2), we need the following lemma. This result will be used to define the solution operator (introduced in Section 3) for the problem at hand.

Lemma 2.3. *Let $\sigma \in C([0,T],\mathbb{R})$, $T > 0$ $a_0 \neq -1$ and $\nu_1 \neq 0$. Then the solution of the fractional differential equation*

$$D^\alpha u(t) = \sigma(t),\ 1 < \alpha \leq 2,\ \ t \in [0,T], \tag{2.1}$$

subject to the fractional boundary conditions

$$\begin{cases} D^{\alpha-2}u(0^+) + a_0 D^{\alpha-2}u(T^-) & = \lambda, \\ D^{\alpha-1}u(0^+) + a_1 D^{\alpha-1}u(T^-) & = \nu I^{\alpha-1}u(\eta) + \sum_{i=1}^{m} \mu_i u(\xi_i), \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} u(t) \quad = \quad & \lambda b_1(t) + b_2(t) \int_0^T (T-s)\sigma(s)ds + b_3(t) \int_0^T \sigma(s)ds \\ & + b_4(t) \sum_{i=1}^{m} \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s)ds \\ & + b_5(t) \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1))} \sigma(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s)ds, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
b_1(t) &= (1 - a_0 T\Gamma(\alpha)\nu_2)\frac{t^{\alpha-2}}{(1+a_0)\Gamma(\alpha-1)} + \nu_2 t^{\alpha-1},\\
b_2(t) &= a_0\Big[(a_0 T\Gamma(\alpha)\nu_2 - 1)\frac{t^{\alpha-2}}{(1+a_0)\Gamma(\alpha-1)} - \nu_2 t^{\alpha-1}\Big],\\
b_3(t) &= \frac{a_1}{\nu_1}\Big(\frac{a_0 T(\alpha-1)t^{\alpha-2}}{(1+a_0)} - t^{\alpha-1}\Big),\ b_4(t) = \frac{1}{\nu_1}\Big(-\frac{a_0 T(\alpha-1)t^{\alpha-2}}{(1+a_0)} + t^{\alpha-1}\Big),\\
b_5(t) &= \nu_3\Big(-\frac{a_0 T(\alpha-1)t^{\alpha-2}}{(1+a_0)} + t^{\alpha-1}\Big),
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\nu_1 &= (a_1+1)\Gamma(\alpha) - \nu\eta^{2\alpha-2}\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} - \sum_{i=1}^{m}\mu_i\xi_i^{\alpha-1}\\
&\quad + \frac{\Gamma(\alpha)Ta_0}{(1+a_0)}\Big(\frac{\sum_{i=1}^{m}\mu_i\xi_i^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\nu\eta^{2\alpha-3}}{\Gamma(2\alpha-2)}\Big),\\
\nu_2 &= \frac{1}{\nu_1}\Big(\frac{\nu\eta^{2\alpha-3}}{(1+a_0)\Gamma(2\alpha-2)} + \frac{\sum_{i=1}^{m}\mu_i\xi_i^{\alpha-2}}{(1+a_0)\Gamma(\alpha-1)}\Big),\ \nu_3 = \frac{\nu}{\nu_1}.
\end{aligned} \tag{2.5}$$

Proof. Applying the Riemann-Liouville fractional integral operator I^α to both sides of (2.1), we get

$$u(t) = c_1 t^{\alpha-1} + c_0 t^{\alpha-2} + I^\alpha\sigma(t), \tag{2.6}$$

where c_0 and c_1 are unknown arbitrary constants. From (2.6), we have

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) + I^1\sigma(t), \tag{2.7}$$

$$D^{\alpha-2}u(t) = c_1\Gamma(\alpha)t + c_0\Gamma(\alpha-1) + I^2\sigma(t). \tag{2.8}$$

Using (2.8) in the first equation in (2.2), we find that

$$c_0 = \frac{1}{(1+a_0)\Gamma(\alpha-1)}[\lambda - a_0 T\Gamma(\alpha)c_1 - a_0 I^2\sigma(T)]. \tag{2.9}$$

Making use of (2.7) in the second equation in (2.2), together with notation (2.5), we get

$$c_1 = \lambda\nu_2 - a_0\nu_2 I^2\sigma(T) - \frac{a_1}{\nu_1}I^1\sigma(T) + \nu_3 I^{2\alpha-1}\sigma(\eta) + \sum_{i=1}^{m}\frac{\mu_i}{\nu_1}I^\alpha\sigma(\xi_i). \tag{2.10}$$

Using (2.10) in (2.9) together with the notation (2.5), we have

$$\begin{aligned}
c_0 &= \frac{1}{(1+a_0)\Gamma(\alpha-1)}\Big[\lambda - a_0\lambda T\nu_2\Gamma(\alpha) + a_0(a_0 T\nu_2\Gamma(\alpha) - 1)I^2\sigma(T)\\
&\quad + \frac{a_0 a_1 T\Gamma(\alpha)}{\nu_1}I^1\sigma(T) - a_0 T\Gamma(\alpha)\nu_3 I^{2\alpha-1}\sigma(\eta) - \frac{a_0 T\Gamma(\alpha)}{\nu_1}\sum_{i=1}^{m}\mu_i I^\alpha\sigma(\xi_i)\Big].
\end{aligned}$$

Inserting the value of c_0 and c_1 in (2.6) and using the notation (2.4), we get the solution (2.3). The converse can be proved by direct computation. The proof is completed. □

Let us define $\mathcal{P}_q(X) = \{Y \in \mathcal{P}(X) : Y \text{ has the property q}\}$, where $(X, \|\cdot\|)$ is a normed space. For example, $\mathcal{P}_{b,cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded and closed }\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$, etc.

Define the set of selections of F for each $u \in C([0, T], \mathbb{R})$ as

$$S_{F,u} := \{v \in L^1([0, T], \mathbb{R}) : v(t) \in F(t, u(t), (\phi u)(t), (\psi u)(t)) \text{ on } [0, T]\}.$$

Definition 2.4. *$F : [0, T] \times \mathbb{R}^3 \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory function, if (i) $t \to F(t, u, v, w)$ is measurable for each $(u, v, w) \in \mathbb{R}^3$; (ii) $(u, v, w) \to F(t, u, v, w)$ is upper semicontinuous for almost all $t \in [0, T]$ and (iii) for each $r > 0$, there exists a function $m_r \in L^1([0, T], \mathbb{R}^+)$ such that*

$$\|F(t, u, v, w)\| = \sup\{|z| : z \in F(t, u, v, w\} < m_r(t),$$

for each $u, v, w \in \mathbb{R}$ with $\max\{|u|, |v|, |w|\} \le r$ and for almost every $t \in [0, T]$.

Definition 2.5. *A function $u \in AC^2([0, T], \mathbb{R})$ satisfying the boundary conditions (1.2) is a solution of the problem (1.1)-(1.2) if there exists a function $v \in L^1([0, T], \mathbb{R})$ with $v(t) \in F(t, u(t), (\phi u)(t), (\psi u)(t))$ for almost every $t \in [0, T]$ such that*

$$\begin{aligned} u(t) \;=\; & \varphi(u)b_1(t) + b_2(t)\int_0^T (T-s)v(s)ds + b_3(t)\int_0^T v(s)ds \\ & + b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ & + b_5(t)\int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha - 1))} v(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds, \end{aligned}$$

where $b_i, 1 = 1, 2, 3, 4, 5$ are defined in (2.4).

3. Main results

3.1. The upper semicontinuous case

Our first existence result for problem (1.1)–(1.2) with convex-valued multivalued map F is based on a fixed point theorem due to Bohnenblust-Karlin [34].

Theorem 3.1. *Assume that:*

(A_1) *$F : [0, T] \times \mathbb{R}^3 \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory function.*
(A_2) *For the function m_r appeared in Definition 2.4 in the definition of L^1-Carathéodory function,*

$$\lim_{r\to+\infty} \inf \left(\frac{\int_0^T m_r(s)ds}{r} \right) = \omega < \infty. \tag{3.1}$$

(A_3) *There exists a constant $K > 0$ such that*

$$|\varphi(u)| \le K, \textit{ for all } u \in C([0, T], \mathbb{R}).$$

Then the Riemann-Liouville nonlinear fractional integro-differential inclusion boundary value problem (1.1)–(1.2) has at least one solution on $[0, T]$, *provided that*

$$\omega\Lambda < 1, \tag{3.2}$$

where

$$\Lambda := \delta_2 T + \delta_3 + \delta_4 \sum_{i=1}^{m} |\mu_i| \frac{\xi_i^{\alpha-1}}{\Gamma(\alpha)} + \delta_5 \frac{\eta^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{T}{\Gamma(\alpha)} \tag{3.3}$$

and

$$\delta_m = \sup_{t\in[0,T]} \{t^{2-\alpha}|b_m(t)|\}, \quad m \in \{1, 2, 3, 4, 5\}.$$

Proof. To transform the problem (1.1)–(1.2) into a fixed point problem, by using Lemma 2.3, we define a multi-valued map

$$\Omega : C_{2-\alpha}([0, T], \mathbb{R}) \to \mathcal{P}(C_{2-\alpha}([0, T], \mathbb{R}))$$

as

$$\begin{aligned} \Omega(u) \;=\; & \Big\{ h \in C_{2-\alpha}([0, T], \mathbb{R}) : h(t) = \varphi(u) b_1(t) + b_2(t) \int_0^T (T-s) f(s) ds \\ & + b_3(t) \int_0^T f(s) ds + b_4(t) \sum_{i=1}^{m} \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds \\ & + b_5(t) \int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad f \in S_{F,u} \Big\}, \end{aligned}$$

where $C_{2-\alpha}([0, T], \mathbb{R}) = \{f : (0, T] \to \mathbb{R} : t^{2-\alpha} x(t) \in C([0, T], \mathbb{R}\}$ with norm $\|x\|_{2-\alpha} = \sup_{t\in[0,T]} |t^{2-\alpha} x(t)|$.

We will show that the operator Ω satisfies the hypotheses of Bohnenblust-Karlin fixed point theorem, and hence it has a fixed point which will be a solution of the problem (1.1)–(1.2). We split the proof is several steps. We show that $\Omega(u)$ is convex for each $u \in C_{2-\alpha}([0, T], \mathbb{R})$, in the first step. Let us take $w_1, w_2 \in \Omega(u)$. Then there exist $f_1, f_2 \in S_{F,u}$ such that, for $t \in [0, T]$, we have

$$\begin{aligned} w_i(t) \;=\; & \varphi(u) b_1(t) + b_2(t) \int_0^T (T-s) f_i(s) ds + b_3(t) \int_0^T f_i(s) ds \\ & + b_4(t) \sum_{i=1}^{m} \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s) ds + b_5(t) \int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f_i(s) ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s) ds, \quad i = 1, 2. \end{aligned}$$

Let $0 \le \theta \le 1$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned} [\theta w_1 + (1-\theta) w_2](t) \;=\; & \varphi(u) b_1(t) + b_2(t) \int_0^T (T-s)(\theta f_1(s) + (1-\theta) f_2(s)) ds \\ & + b_3(t) \int_0^T (\theta f_1(s) + (1-\theta) f_2(s)) ds \end{aligned}$$

$$
\begin{aligned}
&+b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i}\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}(\theta f_1(s)+(1-\theta)f_2(s))ds\\
&+b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}(\theta f_1(s)+(1-\theta)f_2(s))ds\\
&+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\theta f_1(s)+(1-\theta)f_2(s))ds.
\end{aligned}
$$

This show that Ω has convex values since $S_{F,u}$ is convex. Consequently $\theta w_1+(1-\theta)w_2\in\Omega(u)$.

Let $\mathbb{B}_r=\{u\in C_{2-\alpha}([0,T],\mathbb{R}):\|u\|_{2-\alpha}\leq r\}$, $r>0$, be a bounded ball in $C_{2-\alpha}([0,T],\mathbb{R})$. It will be shown that there exists a positive number r' such that $\Omega(\mathbb{B}_{r'})\subseteq\mathbb{B}_{r'}$. The proof is by the method of contradiction. Then, for each $r>0$, there exists a function $u_r(\cdot)\in\mathbb{B}_r$, such that $\|\Omega(u_r)\|_{2-\alpha}>r$ and

$$
\begin{aligned}
h_r(t) &= \varphi(u)b_1(t)+b_2(t)\int_0^T(T-s)f_r(s)ds+b_3(t)\int_0^T f_r(s)ds\\
&\quad+b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i}\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f_r(s)ds+b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f_r(s)ds\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_r(s)ds, \qquad (3.4)
\end{aligned}
$$

for some $f_r\in S_{F,u_r}$. In consequence, we have

$$
\begin{aligned}
r &< \|\Omega(u_r)\|_{2-\alpha}\\
&\leq |\varphi(u_r)|\delta_1+\delta_2\int_0^T|T-s||f_r(s)|ds+\delta_2\int_0^T|T-s||f_r(s)|ds+\delta_3\int_0^T|f_r(s)|ds\\
&\quad+\delta_4\sum_{i=1}^{m}\int_0^{\xi_i}|\mu_i|\frac{|\xi_i-s|^{\alpha-1}}{\Gamma(\alpha)}|f_r(s)|ds+\delta_5\int_0^{\eta}\frac{|\eta-s|^{2\alpha-2}}{\Gamma(2\alpha-1)}|f_r(s)|ds\\
&\quad+t^{2-\alpha}\int_0^t\frac{|t-s|^{\alpha-1}}{\Gamma(\alpha)}|f_r(s)|ds\\
&\leq K\delta_1+\Big[\delta_2T+\delta_3+\delta_4\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\alpha-1}}{\Gamma(\alpha)}+\delta_5\frac{\eta^{2\alpha-2}}{\Gamma(2\alpha-1)}+\frac{T}{\Gamma(\alpha)}\Big]\int_0^T m_r(s)ds.
\end{aligned}
$$

It then follows by dividing both sides of the above inequality by r and taking the lower limit as $r\to\infty$ that $1\leq\omega\Lambda$, which contradicts (3.2). Therefore there exist a positive number r' such that $\Omega(\mathbb{B}_{r'})\subseteq\mathbb{B}_{r'}$.

Next it will be established that $\Omega(B_{r'})$ is equicontinuous. For $t_1,t_2\in[0,T]$ with $t_1<t_2$, $u\in\mathbb{B}_{r'}$ and $h\in\Omega(u)$, we can find $f\in S_{F,u}$ such that

$$
\begin{aligned}
h(t) &= \varphi(u)b_1(t)+b_2(t)\int_0^T(T-s)f(s)ds+b_3(t)\int_0^T f(s)ds\\
&\quad+b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i}\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f(s)ds\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds. \qquad (3.5)
\end{aligned}
$$

Using (3.5), we have

$$
\begin{aligned}
& |t_2^{2-\alpha}h(t_2) - t_1^{2-\alpha}h(t_1)| \\
\leq\ & |\varphi(u)||t_2^{2-\alpha}b_1(t_2) - t_1^{2-\alpha}b_1(t_1)| + |t_2^{2-\alpha}b_2(t_2) - t_1^{2-\alpha}b_2(t_1)| \int_0^T |T-s| m_{r'}(s)ds \\
& + |t_2^{2-\alpha}b_3(t_2) - t_1^{2-\alpha}b_3(t_1)| \int_0^T m_{r'}(s)ds \\
& + |t_2^{2-\alpha}b_4(t_2) - t_1^{2-\alpha}b_4(t_1)| \sum_{i=1}^m \int_0^{\xi_i} |\mu_i| \frac{|\xi_i - s|^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \\
& + |t_2^{2-\alpha}b_5(t_2) - t_1^{2-\alpha}b_5(t_1)| \int_0^{\eta} \frac{|\eta - s|^{2\alpha-2}}{\Gamma(2\alpha-1)} m_{r'}(s)ds \\
& + \Big| t_2^{2-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds - t_1^{2-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \Big| \\
\leq\ & |\varphi u||\nu_2||t_2 - t_1| + |t_2 - t_1||a_0\nu_2| \int_0^T |T-s| m_{r'}(s)ds + |t_2 - t_1| \Big| \frac{a_1}{\nu_1} \Big| \int_0^T m_{r'}(s)ds \\
& + |t_2 - t_1| \frac{1}{|\nu_1|} \sum_{i=1}^m \int_0^{\xi_i} |\mu_i| \frac{|\xi_i - s|^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds + |t_2 - t_1||\nu_3| \int_0^{\eta} \frac{|\eta - s|^{2\alpha-2}}{\Gamma(2\alpha-1)} m_{r'}(s)ds \\
& + \Big| t_2^{2-\alpha} \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds + t_2^{2-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \\
& - t_1^{2-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \Big| \\
\leq\ & |t_2 - t_1| \Big[K|\nu_2| + |a_0\nu_2| \int_0^T |T-s| m_{r'}(s)ds + \Big| \frac{a_1}{\nu_1} \Big| \int_0^T m_{r'}(s)ds \\
& + \frac{1}{|\nu_1|} \sum_{i=1}^m \int_0^{\xi_i} |\mu_i| \frac{|\xi_i - s|^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds + |\nu_3| \int_0^{\eta} \frac{|\eta - s|^{2\alpha-2}}{\Gamma(2\alpha-1))} m_{r'}(s)ds \Big] \\
& + \Big| \int_0^{t_1} t_2^{2-\alpha} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds - t_1^{2-\alpha} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \Big| \\
& + \Big| t_2^{2-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_{r'}(s)ds \Big| \\
\leq\ & |t_2 - t_1| \Big[K|\nu_2| + \Big(|a_0\nu_2|T + \frac{|a_1|}{\nu_1} + \frac{T^{\alpha-1|}}{|\nu_1|\Gamma(\alpha)} \sum_{i=1}^m |\mu_i| + |\nu_3| \frac{T^{2\alpha-2}}{\Gamma(2\alpha-1)} \Big) \int_0^T m_{r'}(s)ds \Big] \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t_1} [t_2^{2-\alpha}(t_2-s)^{\alpha-1} - t_1^{2-\alpha}(t_1-s)^{\alpha-1}] m_{r'}(s)ds \Big| \\
& + \Big| \frac{t_2^{2-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} m_{r'}(s)ds \Big| \to 0,
\end{aligned}
$$

as $t_2 \to t_1$ independently of $u \in \mathbb{B}_{r'}$. Thus, Ω is equi-continuous. Therefore we deduce by Arzelá-Ascoli theorem that Ω is a compact multi-valued map.

Now we will show that $\Omega(u)$ is closed for each $u \in C_{2-\alpha}([0,T],\mathbb{R})$. Let $\{x_n\}_{n\geq 0} \in \Omega(u)$ be such that $x_n \to x$ $(n \to \infty)$ in $C_{2-\alpha}([0,T],\mathbb{R})$. Then $x \in C_{2-\alpha}([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} x_n(t) &= \varphi(u_n)b_1(t) + b_2(t)\int_0^T ((T-s)v_n(s)ds + b_3(t)\int_0^T v_n(s)ds \\ &+ b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + b_5(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} v_n(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds, \end{aligned}$$

which implies that v_n converges to v in $L^1([0,T],\mathbb{R})$ as F has compact values. Thus $v \in S_{F,u}$ and for each $t \in [0,T]$, we have

$$\begin{aligned} x_n(t) \to x(t) &= \varphi(u)b_1(t) + b_2(t)\int_0^T (T-s)v(s)ds + b_3(t)\int_0^T v(s)ds \\ &+ b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + b_5(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds. \end{aligned}$$

Hence $x \in \Omega(u)$.

Finally, we show that Ω has a closed graph. Consider $u_n \to u_*$, $h_n \in \Omega(u_n)$ and $h_n \to h_*$. We will show that $h_* \in \Omega(u_*)$. By the relation $h_n \in \Omega(u_n)$, there exists $f_n \in S_{F,u_n}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= \varphi(u_n)b_1(t) + b_2(t)\int_0^T (T-s)f_n(s)ds + b_3(t)\int_0^T f_n(s)ds \\ &+ b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds + b_5(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f_n(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds. \end{aligned}$$

We need to prove that there exists $f_* \in S_{F,u_*}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \varphi(u_*)b_1(t) + b_2(t)\int_0^T (T-s)f_*(s)ds + b_3(t)\int_0^T f_*(s)ds \\ &+ b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} f_*(s)ds + b_5(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f_*(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_*(s)ds. \end{aligned}$$

Introduce a linear continuous operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ as

$$f \to \Theta(f)(t) = \varphi(u)b_1(t) + b_2(t)\int_0^T (T-s)f(s)ds + b_3(t)\int_0^T f(s)ds$$

$$+b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i}\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f(s)ds$$
$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds.$$

Observe that

$$\begin{aligned}
&\|h_n-h_*\|_{2-\alpha}\\
=\ &\Big\|b_1(t)(\varphi(u_n)-\varphi(u_*))+b_2(t)\int_0^T(T-s)(f_n(s)-f_*(s))ds+b_3(t)\int_0^T(f_n(s)-f_*(s))ds\\
&+b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i}\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}(f_n(s)-f_*(s))ds+b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}(f_n(s)-f_*(s))ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f_n(s)-f_*(s))ds\Big\|_{2-\alpha}\longrightarrow 0,\ \text{ as } n\to\infty.
\end{aligned}$$

Consequently, by using a result from [35] on closed graph operators, we deduce that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n\in S_{F,u_n}$. Since $u_n\to u_*$, therefore, the result from [35] yields

$$\begin{aligned}
h_*(t)\ =\ &\varphi(u_*)b_1(t)+b_2(t)\int_0^T(T-s)f_*(s)ds+b_3(t)\int_0^T f_*(s)ds\\
&+\sum_{i=1}^{m}\int_0^{\xi_i}b_4(t)\mu_i\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)ds+\int_0^{\eta}b_5(t)\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f_*(s)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)ds. \qquad (3.6)
\end{aligned}$$

Using the fact that an operator is upper semicontinuous if it has a closed graph ([36, Proposition 1.2]), we deduce that Ω is upper semicontinuous. Hence, we conclude that Ω is a compact and upper semicontinuous multivalued map with convex closed values. As all the assumptions of Bohnenblust-Karlin fixed point theorem [34] are satisfied, so its conclusion implies that the operator Ω has a fixed point u, which is a solution of the problem (1.1)–(1.2). The proof is finished. □

Our next existence result is based on Leray-Schauder nonlinear alternative for Kakutani maps [37].

Theorem 3.2. *Assume that* (A_1) *and* (A_3) *hold. In addition we assume that:*

(A_4) *There exist a continuous nondecreasing function* $\chi:[0,\infty)\to(0,\infty)$ *and a function* $p\in C([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,u,v,w)\|=\sup\{|y|:y\in F(t,u,v,w)\}\le p(t)\chi(|u|+|v|+|w|),$$

for each $(t,u,v,w)\in[0,T]\times\mathbb{R}^3$.

(A_5) *There exists a positive real number* M *satisfying the inequality:*

$$\frac{M}{K\delta_1+\Lambda_1\|p\|\chi((1+\gamma_0+\delta_0)M)}>1,$$

where

$$\begin{aligned}\Lambda_1 : &= \delta_2 \frac{T^{\alpha+1}\Gamma(\alpha-1)}{\Gamma(\alpha+2)} + \delta_3 \frac{T^{\alpha}}{\alpha(\alpha-1)} + \delta_4 \sum_{i=1}^{m} |\mu_i| \frac{\xi_i^{2\alpha-1}\Gamma(\alpha-1)}{\Gamma(2\alpha)} \\ &+ \delta_5 \frac{\eta^{3\alpha-2}\Gamma(\alpha-1)}{\Gamma(3\alpha-1)} + \frac{T^{\alpha+1}\Gamma(\alpha-1)}{\Gamma(2\alpha)} \end{aligned} \tag{3.7}$$

and

$$\gamma_0 = \max|\gamma(t,s)|, \quad \delta_0 = \max|\delta(t,s)|, \ t,s \in [0,T]. \tag{3.8}$$

Then there exists at least one solution for the Riemann-Liouville fractional integro-differential inclusion problem (1.1)–(1.2) on $[0,T]$.

Proof. Let $u \in \mu\Omega(x)$ for some $\mu \in (0,1)$, where the operator Ω defined in the proof of Theorem 3.1. We show that there exists $U \subseteq C([0,T],\mathbb{R})$, U an open set, with $u \notin \Omega(u)$ for any $\mu \in (0,1)$ and all $u \in \partial U$. Assume that $\mu \in (0,1)$ and $u \in \mu\Omega(u)$. Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,u}$ such that, for $t \in [0,T]$, we have

$$\begin{aligned} u(t) &= \mu\Bigg[\varphi(u)b_1(t) + b_2(t)\int_0^T (T-s)f(s)ds + b_3(t)\int_0^T f(s)ds \\ &+ b_4(t)\sum_{i=1}^{m}\int_0^{\xi_i} \mu_i \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + b_5(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds\Bigg]. \end{aligned} \tag{3.9}$$

In view of (A_4), we have for each $t \in [0,T]$,

$$\begin{aligned} \|u\|_{2-\alpha} &\leq |\varphi(u)|\delta_1 + \delta_2\int_0^T |T-s||f(s)|ds + \delta_3\int_0^T |f(s)|ds \\ &+ \delta_4\sum_{i=1}^{m}\int_0^{\xi_i} |\mu_i|\frac{|\xi_i-s|^{\alpha-1}}{\Gamma(\alpha)}|f(s)|ds + \delta_5\int_0^{\eta}\frac{|\eta-s|^{2\alpha-2}}{\Gamma(2\alpha-1)}|f(s)|ds \\ &+ t^{2-\alpha}\int_0^t \frac{|t-s|^{\alpha-1}}{\Gamma(\alpha)}|f(s)|ds \\ &\leq K\delta_1 + \Big[\delta_2\frac{T^{\alpha+1}\Gamma(\alpha-1)}{\Gamma(\alpha+2)} + \delta_3\frac{T^{\alpha}}{\alpha(\alpha-1)} + \delta_4\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{2\alpha-1}\Gamma(\alpha-1)}{\Gamma(2\alpha)} \\ &+ \delta_5\frac{\eta^{3\alpha-2}\Gamma(\alpha-1)}{\Gamma(3\alpha-1)} + \frac{T^{\alpha+1}\Gamma(\alpha-1)}{\Gamma(2\alpha)}\Big] \times \|p\|\chi((1+\gamma_0+\delta_0)\|u\|_{2-\alpha}) \\ &= K\delta_1 + \Lambda_1\|p\|\chi((1+\gamma_0+\delta_0)\|u\|_{2-\alpha}). \end{aligned}$$

Consequently, we have

$$\frac{\|u\|_{2-\alpha}}{K\delta_1 + \Lambda_1\|p\|\chi((1+\gamma_0+\delta_0)\|u\|_{2-\alpha})} \leq 1.$$

In view of (A_5), there exists M such that $\|u\|_{2-\alpha} \neq M$. Let us set

$$U = \{u \in C_{2-\alpha}([0,T],\mathbb{R}) : \|u\|_{2-\alpha} < M\}.$$

As in the proof of Theorem 3.1, it is easy to verify that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T], \mathbb{R}))$ is a compact, upper semicontinuous multi-valued map with convex closed values. Moreover, by the choice of U, there is no $u \in \partial U$ such that $u \in \mu\Omega(u)$ for some $\mu \in (0,1)$. In consequence, we conclude by the nonlinear alternative of Leray-Schauder type ([37]) that Ω has a fixed point $u \in \overline{U}$, which is indeed a solution of the problem (1.1)–(1.2). This finishes the proof. □

3.2. The Lipschitz case

This section deals with the case when the right hand side of the inclusion (1.1) is nonconvex valued. Here the main tool of our study is a fixed point theorem for multivalued maps due to Covitz and Nadler [38]. We need the following assumptions to prove our next existence result.

(S_1) $F : [0,T] \times \mathbb{R}^3 \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, u, v, w) : [0,T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $(u,v,w) \in \mathbb{R}^3$.

(S_2) $H_d(F(t,u,v,w), F(t,\overline{u},\overline{v},\overline{w})) \leq m(t)(|u-\overline{u}| + |v-\overline{v}| + |w-\overline{w}|)$ for almost all $t \in [0,T]$ and $u,v,w,\overline{u},\overline{v},\overline{w} \in \mathbb{R}$ with $m \in C([0,T],\mathbb{R}^+)$ and $d(0, F(t,0,0,0)) \leq m(t)$ for almost all $t \in [0,T]$.

(S_3) There exists a constant $L > 0$ such that $|(\varphi u) - (\varphi v)| < L\|u - v\|_{2-\alpha}$ for all $u, v \in C_{2-\alpha}([0,T],\mathbb{R})$.

Theorem 3.3. *If the hypothesis* (S_1)*-*(S_3) *are satisfied, then the Riemann-Liouville inclusion problem (1.1)-(1.2) has at least one solution on* $[0,T]$, *provided that*

$$[\delta_1 L + \Lambda_1\|m\|](1 + \gamma_0 + \delta_0) < 1,$$

where Λ_1 *is given by (3.7) and* γ_0, δ_0 *are defined by (3.8).*

Proof. Consider the operator Ω defined at the beginning of the proof of Theorem 3.1. By assumption (S_1) the set $S_{F,u}$ is non empty for each $t \in [0,T]$, and hence F has a measurable selection. Moreover $\Omega(u) \in \mathcal{P}_{cl}(C([0,T],\mathbb{R}))$ for each $u \in C([0,T],\mathbb{R})$, as proved in Theorem 3.1.

Now we will prove that here exists $0 < \widehat{\lambda} < 1$ $\left(\widehat{\lambda} = [\delta_1 L + \Lambda_1\|m\|](1 + \gamma_0 + \delta_0)\right)$ such that

$$H_d(\Omega(u), \Omega(\overline{u})) \leq \widehat{\lambda}\|u - \overline{u}\|_{2-\alpha} \quad \text{for each } u \in C_{2-\alpha}([0,T],\mathbb{R}).$$

Let $u, \overline{u} \in C_{2-\alpha}([0,T],\mathbb{R})$ and $h_1 \in \Omega(u)$. Then there exists $v_1(t) \in F(t, u(t), (\phi u)(t), (\psi u)(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) &= \varphi(u)b_1(t) + b_2(t)\int_0^T (T-s)v_1(s)ds + b_3(t)\int_0^T v_1(s)ds \\ &\quad + b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds + b_5(t)\int_0^\eta \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha-1)} v_1(s)ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds. \end{aligned}$$

By (S_2), we have

$$\begin{aligned} & H_d(F(t, u(t), (\phi u)(t), (\psi u)(t)), F(t, \overline{u}(t), (\phi\overline{u})(t), (\psi\overline{u})(t))) \\ \leq\ & m(t)\big(|u(t) - \overline{u}(t)| + |(\phi u)(t) - (\phi\overline{u})(t)| + |(\psi u)(t) - (\psi\overline{u})(t)|\big). \end{aligned}$$

Thus, we have that there exists $w \in F(t, \overline{u}(t), (\phi\overline{u})(t), (\psi\overline{u})(t))$ such that

$$|v_1(t) - w| \leq m(t)\big(|u(t) - \overline{u}(t)| + |(\phi u)(t) - (\phi\overline{u})(t)| + |(\psi u)(t) - (\psi\overline{u})(t)|\big), \quad t \in [0, T].$$

Define $U : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)\big(|u(t) - \overline{u}(t)| + |(\phi u)(t) - (\phi\overline{u})(t)| + |(\psi u)(t) - (\psi\overline{u})(t)|\big).$$

Since the multivalued operator $U(t) \cap F(t, \overline{u}(t), (\phi\overline{u})(t), (\psi\overline{u})(t))$ is measurable ([39]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \overline{u}(t), (\phi\overline{u})(t), (\psi\overline{u})(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \leq m(t)\big(|u(t) - \overline{u}(t)| + |(\phi u)(t) - (\phi\overline{u})(t)| + |(\psi u)(t) - (\psi\overline{u})(t)|\big)$. For each $t \in [0, T]$, let us define

$$\begin{aligned} h_2(t) &= \varphi(u)b_1(t) + b_2(t)\int_0^T (T - s)v_2(s)ds + b_3(t)\int_0^T v_2(s)ds \\ &\quad + b_4(t)\sum_{i=1}^m \int_0^{\xi_i} \mu_i \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds + b_5(t)\int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha - 1)} v_2(s)ds \\ &\quad + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds. \end{aligned}$$

In consequence, we get

$$\begin{aligned} &\|h_1 - h_2\|_{2-\alpha} \\ \leq\ & \sup_{t\in[0,T]} \Big\{ t^{2-\alpha}|b_1(t)||\varphi(u) - \varphi(\overline{u})| + t^{2-\alpha}|b_2(t)| \int_0^T (T - s)|v_1(s) - v_2(s)|ds \\ & + t^{2-\alpha}|b_3(t)| \int_0^T |v_1(s) - v_2(s)|ds + t^{2-\alpha}|b_4(t)| \sum_{i=1}^m |\mu_i| \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} |v_1(s) - v_2(s)|ds \\ & + |b_5(t)| \int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha - 1)} |v_1(s) - v_2(s)|ds + t^{2-\alpha} \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |v_1(s) - v_2(s)|ds \Big\} \\ \leq\ & \delta_1 L\|u - \overline{u}\|_{2-\alpha} \\ & + \Big\{ \Big[\delta_2 \int_0^T (T - s)\frac{s^{\alpha-1}}{\alpha - 1}ds + \delta_3 \int_0^T \frac{s^{\alpha-1}}{\alpha - 1}ds + \delta_4 \sum_{i=1}^m |\mu_i| \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{s^{\alpha-1}}{\alpha - 1}ds \\ & + \delta_5 \int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha - 1)} \frac{s^{\alpha-1}}{\alpha - 1}ds + t^{2-\alpha} \int_0^T \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{s^{\alpha-1}}{\alpha - 1}ds \Big] \Big\} \|m\|(1 + \gamma_0 + \delta_0)\|u - \overline{u}\|_{2-\alpha} \\ \leq\ & \delta_1 L(1 + \gamma_0 + \delta_0)\|u - \overline{u}\|_{2-\alpha} \\ & + \Big\{ \Big[\delta_2 \int_0^T (T - s)\frac{s^{\alpha-1}}{\alpha - 1}ds + \delta_3 \int_0^T \frac{s^{\alpha-1}}{\alpha - 1}ds + \delta_4 \sum_{i=1}^m |\mu_i| \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{s^{\alpha-1}}{\alpha - 1}ds \\ & + \delta_5 \int_0^{\eta} \frac{(\eta - s)^{2\alpha-2}}{\Gamma(2\alpha - 1)} \frac{s^{\alpha-1}}{\alpha - 1}ds + t^{2-\alpha} \int_0^T \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{s^{\alpha-1}}{\alpha - 1}ds \Big] \Big\} \|m\|(1 + \gamma_0 + \delta_0)\|u - \overline{u}\|_{2-\alpha} \\ =\ & (\delta_1 L + \Lambda_1\|m\|)(1 + \gamma_0 + \delta_0)\|u - \overline{u}\|_{2-\alpha} \end{aligned}$$

Analogously, interchanging the roles of u and $\overline{u}$ leads to

$$H_d(\Omega(u), \Omega(\overline{u})) \leq (\delta_1 L + \Lambda_1\|m\|)(1 + \gamma_0 + \delta_0)\|u - \overline{u}\|_{2-\alpha}.$$

In consequence, it follows by Covitz and Nadler's theorem ([38]) that Ω is a contraction and has a fixed point u, which corresponds to a solution of (1.1)–(1.2). This completes the proof. □

4. Examples

Example 4.1. *Consider the Riemann-Liouville nonlinear fractional integro-differential inclusion boundary value problem, consisting by the fractional differential inclusion*

$$D^{\alpha}u(t) \in F(t, u(t), (\phi u)(t), (\psi u)(t)),\ 1 < \alpha \leq 2,\ \ t \in [0, 1], \tag{4.1}$$

supplemented with the fractional order boundary conditions

$$\begin{cases} D^{\alpha-2}u(0^+) + a_0 D^{\alpha-2}u(T^-) &= \sin(u), \\ D^{\alpha-1}u(0^+) + a_1 D^{\alpha-1}u(T^-) &= \nu I^{\alpha-1}u(\eta) + \sum_{i=1}^{3} \mu_i u(\xi_i), \end{cases} \tag{4.2}$$

where $\alpha = 5/4, a_0 = 1/4, a_1 = -3/2, \nu = 2, \mu_1 = 4, \mu_2 = 2, \mu_3 = 6, \eta = 1/4, \xi_1 = 1/2, \xi_2 = 2/3, \xi_3 = 3/4, \varphi(u) = \sin u$.

Using the given value, we find that $L = 1, \delta_1 \simeq 0.0335887688,\ \delta_2 \simeq 0.00839719221,\ \delta_3 \simeq 0.1461565467,\ \delta_4 \simeq 0.09743769782,\ \delta_5 \simeq 0.1948753956.$

Let us take

$$\begin{aligned} & F(t, u(t), (\phi u)(t), (\psi u)(t)) \\ = \ & \left[\frac{1}{120}\left(\frac{1}{\sqrt{40+t}} + (3t + t^2)u + \int_0^t \frac{(t-s)^{\frac{7}{2}}}{15} u(s)ds + \int_0^t (t-s)^{\frac{5}{3}} u(s)ds\right),\right. \\ & \left. \frac{1}{150}\left(\frac{e^t}{80} + \frac{t^2}{15}u + \int_0^t \frac{(t-s)}{4} u(s)ds\right)\right] \end{aligned} \tag{4.3}$$

Then we have

$$\begin{aligned} & \|F(t, u, \phi u, \psi u)\| \\ \leq \ & \frac{1}{120}\left(\frac{1}{\sqrt{40+t}} + (3t + t^2)|u| + \int_0^t \frac{(t-s)^{\frac{7}{2}}}{15} |u(s)|ds + \int_0^t (t-s)^{\frac{5}{3}} |u(s)|ds\right) \\ \leq \ & \frac{1}{120\sqrt{40+t}} + \frac{\|u\|_{2-\alpha}}{120}\left((3t + t^2)t^{\alpha-2} + \int_0^t \frac{(t-s)^{\frac{7}{2}}}{15} s^{\alpha-2}ds + \int_0^t (t-s)^{\frac{5}{3}} s^{\alpha-2}ds\right), \end{aligned}$$

and

$$\omega \simeq 0.03240519123,\ \Lambda \simeq 2.523856335 \ \textit{and}\ \omega\Lambda \simeq 0.08178604717 < 1.$$

Thus, by Theorem 3.1, the problem (4.1)–(4.2) has at least one solution on $[0, 1]$.

Example 4.2. *Consider the problem that is given in Example 4.1 with*

$$\begin{aligned} & F(t, u(t), (\phi u)(t), (\psi u)(t)) \\ = \ & \left[\frac{1}{(150+t)^2}\left(cosu + \int_0^t \frac{(t-s)}{25} u(s)ds + \int_0^t (s-t)^2 u(s)ds\right), \frac{e^t}{180} + \frac{t^2}{15}u\right]. \end{aligned} \tag{4.4}$$

It is easy to see that

$$\|F(t, u, \phi u, \psi u))\| \leq p(t)\chi(|u| + |\phi u| + |\psi u|)$$

where

$$p(t)=\frac{1}{(150+t)^2},\ \chi(|u|+|\phi u|+|\psi u|)=cos|u|+|\int_0^t\frac{(t-s)}{25}u(s)ds|+|\int_0^t(s-t)^2u(s)ds|.$$

Also we have $\gamma_0=\frac{1}{25}$, $\delta_0=1$, $K=1$, $\|p\|=\frac{1}{150^2}$, $\Lambda_1\approx 4.94663$. *By direct computation, there exists a constant* $M=0.5$ *such that* $\frac{M}{\delta_1 K+\Lambda_1\|p\|\chi((1+\gamma_0+\delta_0)M)}\simeq 14.75099>1$. *Hence all assumption of Theorem 3.2 are satisfied and hence the problem (4.1)–(4.2), with F given by (4.4) has at least one solution on* $[0,1]$. *In addition,* $d(0,F(t,0,0,0)\leq\frac{1}{(150+t)^2}=m(t)$,

$$H_d(F(t,u,\phi u,\psi u),F(t,\overline{u},\phi\overline{u},\psi\overline{u}))\leq m(t)(|u-\overline{u}|+|\phi u-\phi\overline{u}|+|\psi u-\psi\overline{u}|),$$

$|\varphi u-\varphi\overline{u}|\leq|\ u-\overline{u}|$ *with* $L=1$, *and*

$$[\delta_1 L+\Lambda_1\|m\|](1+\gamma_0+\delta_0)\simeq 0.06897<1.$$

Hence all assumption of Theorem 3.3 are satisfied and hence the problem (4.1)–(4.2), with F given by (4.4) has at least one solution on $[0,1]$.

5. Conclusions

In the present work, the existence of solutions for Riemann-Liouville fractional integro-differential inclusions, supplemented with nonlocal integral boundary conditions is discussed for the cases of convex and non-convex multivalued maps involved in the problem at hand. The modern tools of the fixed point theory for multivalued maps are employed to obtain the desired results. Theorems 3.1 and 3.2 provide different criteria ensuring the existence of solutions for the problem at hand when the multivalued map is convex-valued, while Theorem 3.3 provides the existence criteria for the solutions of the given problem for nonconvex valued map. It is imperative to note that the study carried out for the problem (1.1)–(1.2) is new and enrich the literature on the multivalued problems involving Riemann-Liouville fractional derivative operators in the inclusions as well as in the boundary conditions. Furthermore, our results specialize to the new ones associated with nonlocal multi-point and nonlocal Riemann-Liouville integral boundary conditions for $\nu=0$ and $\mu_i=0$ for all $i=1,\dots,m$, respectively.

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Conflict of interest

The authors declare no conflict of interest.

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