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Research article

Convexity with respect to strictly monotone function and Riemann-Liouville fractional Fejér-Hadamard inequalities

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Abstract: In this paper we study the Fejér-Hadamard inequalities for convex function with respect to a strictly monotone function. We establish two inequalities for convex function with respect to a strictly monotone function via Riemann-Liouville fractional integrals. From inequalities found here many new results can be derived by selecting specific strictly monotone and weight functions. Also a variety of existing Fejér-Hadamard and Hadamard inequalities can be reproduced.

Keywords: Fejér-Hadamard inequality; convex function; monotone function; Riemann-Liouville fractional integrals
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1. Introduction

The wighted version of Hadamard inequality known as Fejér-Hadamard inequality was established by Fejér in 1906. It is stated as follows:

Theorem 1. *[1] Let $\psi : [a, b] \to \mathbb{R}$ be a convex function. Further, let $\eta : [a, b] \to \mathbb{R}$ be integrable non-negative function which is symmetric about $\frac{a+b}{2}$. Then we have*

$$\psi\left(\frac{a+b}{2}\right)\int_a^b \eta(x)dx \leq \int_a^b \psi(x)\eta(x)dx \leq \frac{\psi(a)+\psi(b)}{2}\int_a^b \eta(x)dx. \tag{1.1}$$

The Hadamard inequality is obtained if we consider $\eta(x) = 1$ in the inequality (1.1). The following definition of "convex function with respect to a strictly monotone function" is the key factor of this paper.

Definition 1. *[2] If φ is strictly monotone function, then ψ is called convex with respect to φ if $\psi o \varphi^{-1}$ is a convex function.*

Alternatively the Definition 1 can be taken as follows:
Let I, J be intervals in $\mathbb{R}$ and $\psi : I \to \mathbb{R}$ be the convex function, also let $\varphi : J \subset I \to \mathbb{R}$ be strictly monotone function. Then ψ is called convex with respect to φ if

$$\psi\left(\varphi^{-1}\left(tx + (1-t)y\right)\right) \le t\psi\left(\varphi^{-1}(x)\right) + (1-t)\psi\left(\varphi^{-1}(y)\right), \tag{1.2}$$

for $t \in [0,1],\ x, y \in \text{Range}(\varphi)$, provided Range($\varphi$) is convex set. Therefore Definition 1 is equivalently defined by inequality (1.2).
Examples: [3] 1. Let $\varphi(x) = x$. Then $\varphi^{-1}(x) = x$, the inequality (1.2) takes the form

$$\psi\left(tx + (1-t)y\right) \le t\psi(x) + (1-t)\psi(y). \tag{1.3}$$

2. Let $\varphi(x) = \ln x$. Then $\varphi^{-1}(x) = \exp x$, the inequality (1.2) takes the form

$$\psi\left(\exp\left(tx + (1-t)y\right)\right) \le t\psi\left(\exp(x)\right) + (1-t)\psi\left(\exp(y)\right). \tag{1.4}$$

By replacing x with $\ln x$ and y with $\ln y$ in (1.4), we get

$$\psi\left(x^t y^{1-t}\right) \le t\psi(x) + (1-t)\psi(y). \tag{1.5}$$

3. Let $\varphi(x) = \frac{1}{x}$. Then $\varphi^{-1}(x) = \frac{1}{x}$, the inequality (1.2) takes the form

$$\psi\left((tx + (1-t)y)^{-1}\right) \le t\psi\left(\frac{1}{x}\right) + (1-t)\psi\left(\frac{1}{y}\right). \tag{1.6}$$

By replacing x with $\frac{1}{x}$ and y with $\frac{1}{y}$ in (1.6), we get

$$\psi\left(\frac{xy}{ty + (1-t)x}\right) \le t\psi\left(x\right) + (1-t)\psi\left(y\right). \tag{1.7}$$

4. Let $\varphi(x) = x^p,\ p > 0$. Then $\varphi^{-1}(x) = x^{\frac{1}{p}}$, the inequality (1.2) takes the form

$$\psi\left((tx + (1-t)y)^{\frac{1}{p}}\right) \le t\psi\left(x^{\frac{1}{p}}\right) + (1-t)\psi\left(y^{\frac{1}{p}}\right). \tag{1.8}$$

By replacing x with x^p and y with y^p in (1.8), we get

$$\psi\left((tx^p + (1-t)y^p)^{\frac{1}{p}}\right) \le t\psi\left(x\right) + (1-t)\psi\left(y\right). \tag{1.9}$$

5. By replacing x with $\varphi(x)$, y with $\varphi(y)$, the inequality (1.2) takes the form

$$\psi\left(\varphi^{-1}\left(t\varphi(x) + (1-t)g(y)\right)\right) \le t\psi(x) + (1-t)\psi(y). \tag{1.10}$$

Inequalities (1.3), (1.5), (1.7) and (1.9) give convexity, GA-convexity, harmonic convexity and p-convexity given in [4–6]. Hence these independently defined notions are actually examples of a convex function with respect to a strictly monotone function.

Definition 2. *[7] A function ψ will be called symmetric with respect to a strictly monotone function h about $\frac{h(a)+h(b)}{2}$, $a, b \in \text{Domain}(h)$, if*

$$\psi(h^{-1}(h(a)+h(b)-x) = \psi(h^{-1}(x)) \tag{1.11}$$

holds for all $x \in \text{Rang}(h)$.

The notions of symmetric, harmonically symmetric, p-symmetric, geometrically symmetric are examples of Definition 2. These are defined explicitly in [8–10].
We have obtained the following versions of the Fejér-Hadamard inequality for convex function with respect to a strictly monotone function.

Theorem 2. *[7] Let I, J be intervals in $\mathbb{R}$ and $\psi : [a,b] \subset I \to \mathbb{R}$ be a convex function, also let $\varphi : J \supset [a,b] \to \mathbb{R}$ be a strictly monotone function. Further, let ψ be convex with respect to φ, and $\eta : [a,b] \to \mathbb{R}$ be non-negative integrable and symmetric with respect to φ about $\frac{\varphi(a)+\varphi(b)}{2}$. Then the following inequality holds:*

$$\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_{\varphi(a)}^{\varphi(b)}\eta\left(\varphi^{-1}(t)\right)d\xi \le \int_{\varphi(a)}^{\varphi(b)}\psi\left(\varphi^{-1}(t)\right)\eta\left(\varphi^{-1}(t)\right)d\xi \tag{1.12}$$
$$\le \frac{\psi(a)+\psi(b)}{2}\int_{\varphi(a)}^{\varphi(b)}\eta\left(\varphi^{-1}(t)\right)d\xi.$$

The aim of this paper is to give two Riemann-Liouville fractional versions of the Fejér-Hadamard inequality for convex function with respect to a strictly monotone function by using symmetricity with respect to strictly monotone function. These Fejér-Hadamard inequalities for specific strictly monotone functions will give results for convex, geometric convex, harmonically convex and p-convex functions published by different authors in [5, 7–16]. The following definition gives the left as well as right Riemann-Liouville fractional integral operators:

Definition 3. *[17] Let $\mu > 0$ and $\psi \in L_1[a,b]$. Then Riemann-Liouville fractional integral operators of order μ are defined by:*

$$I_{a^+}^{\mu}\psi(x) := \frac{1}{\Gamma(\mu)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\mu}}dt, \quad x > a \tag{1.13}$$

$$I_{b^-}^{\mu}\psi(x) := \frac{1}{\Gamma(\mu)}\int_x^b \frac{\psi(t)}{(t-x)^{1-\mu}}dt, \quad x < b, \tag{1.14}$$

where $\Gamma(.)$ is notation for the gamma function.

The following theorem gives first fractional version of the Hadamard inequality for Riemann-Liouville fractional integrals.

Theorem 3. *[15] Let $\psi : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $\psi \in L[a,b]$. If ψ is a convex function on $[a,b]$, then the following fractional integral inequality holds:*

$$\psi\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[I_{a^+}^{\mu}\psi(b) + I_{b^-}^{\mu}\psi(a)\right] \le \frac{\psi(a)+\psi(b)}{2}, \tag{1.15}$$

with $\mu > 0$.

Another version of the Hadamard inequality for Riemann-Liouville fractional integrals is given in the following theorem.

Theorem 4. *[16] Under the assumptions of Theorem 3, the following fractional integral inequality holds:*

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[I^{\mu}_{\left(\frac{a+b}{2}\right)^{+}}\psi(b) + I^{\mu}_{\left(\frac{a+b}{2}\right)^{-}}\psi(a)\right] \leq \frac{\psi(a)+\psi(b)}{2}, \tag{1.16}$$

with $\mu > 0$.

We have obtained the following fractional versions of the Hadamard inequality for Riemann-Liouville fractional integrals of convex function with respect to a strictly monotone function.

Theorem 5. *[7] Let* I, J *be intervals in* $\mathbb{R}$ *and* $\psi : [a,b] \subset I \to \mathbb{R}$ *be a convex function, also let* $\varphi : J \supset [a,b] \to \mathbb{R}$ *be a strictly monotone function. Further, let* ψ *be convex with respect to* φ*. Then for* $\mu > 0$ *the following inequality holds for Riemann-Liouville fractional integrals:*

$$\begin{aligned} &\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \leq \frac{\Gamma(\mu+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\mu}}\left(J^{\mu}_{\varphi(a)^{+}}\psi(b) + J^{\mu}_{\varphi(b)^{-}}\psi(a)\right) \\ &\leq \frac{\psi(a)+\psi(b)}{2}. \end{aligned} \tag{1.17}$$

Theorem 6. *[7] Under the assumptions of Theorem 5, the following inequality holds for Riemann-Liouville fractional integrals:*

$$\begin{aligned} &\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(\varphi(b)-\varphi(a))^{\mu}}\left(J^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{+}}\psi(b) + J^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{-}}\psi(a)\right) \\ &\leq \frac{\psi(a)+\psi(b)}{2}. \end{aligned} \tag{1.18}$$

In the upcoming section we establish two versions of the Fejér-Hadamard inequality for convex function with respect to a strictly monotone function by using Riemann-Liouville fractional integrals. These inequalities generate new inequalities by selecting different strictly increasing and decreasing functions of our choice. Several results published in [5, 7–16, 18, 19] are deducible from the results presented in this paper.

2. Riemann-Liouville fractional integral Fejér-Hadamard inequality for convex function with respect to a strictly monotone function

First we prove the following lemma:

Lemma 1. *Let* ψ *be symmetric with respect to strictly monotone function* φ *about* $\frac{\varphi(a)+\varphi(b)}{2}$*, and* $\varphi \in L[a,b]$*. Then the following identity holds for Riemann-Liouville fractional integrals:*

$$I^{\mu}_{\varphi(a)^{+}}\psi(b) = I^{\mu}_{\varphi(b)^{-}}\psi(a) = \frac{I^{\mu}_{\varphi(a)^{+}}\psi(b) + I^{\mu}_{\varphi(b)^{-}}\psi(a)}{2}. \tag{2.1}$$

Proof. From definition of Riemann-Liouville fractional integrals we have

$$I^{\mu}_{\varphi(a)^+}\psi(b) = I^{\mu}_{\varphi(a)^+}\psi\left(\varphi^{-1}(\varphi(b))\right) = \frac{1}{\Gamma(\mu)}\int_{\varphi(a)}^{\varphi(b)} \frac{\psi\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}. \tag{2.2}$$

By setting $\varphi(a)+\varphi(b)-u = z$ in (2.2) we get

$$I^{\mu}_{\varphi(a)^+}\psi(b) = \frac{1}{\Gamma(\mu)}\int_{\varphi(a)}^{\varphi(b)} \frac{\psi\left(\varphi^{-1}(\varphi(a)+\varphi(b)-z)\right)dz}{(z-\varphi(a))^{1-\mu}}. \tag{2.3}$$

By using symmetricity of ψ with respect to strictly monotone function φ about $\frac{\varphi(a)+\varphi(b)}{2}$, we get $I^{\mu}_{\varphi(a)^+}\psi(b) = I^{\mu}_{\varphi(b)^-}\psi\left(\varphi^{-1}(\varphi(a))\right)$ and hence (2.1) is obtained. □

Remark 1. *(i) By setting $\varphi(x) = \frac{1}{x}$ in (2.1), we get [20, Lemma 2].*
(ii) By setting $\varphi(x) = x^p$, $p \neq 0$ in (2.1), we get [21, Lemma 1].

By using Lemma 1 we prove the following Riemann-Liouville fractional Fejér-Hadamard inequality for convex function ψ with respect to a strictly monotone function φ.

Theorem 7. *Let I, J be intervals in $\mathbb{R}$ and $\psi, \eta : [a,b] \subset I \to \mathbb{R}$ be real valued functions. Let ψ be convex and w be the positive and symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$. Let $\varphi : J \supset [a,b] \to \mathbb{R}$ be a strictly monotone function. If ψ is convex with respect to φ, then the following inequality holds for Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\left(I^{\mu}_{\varphi(a)^+}\eta(b) + I^{\mu}_{\varphi(b)^-}\eta(a)\right) \\
&\leq I^{\mu}_{\varphi(a)^+}(\psi.\eta)(b) + I^{\mu}_{\varphi(b)^-}(\psi.\eta)(a) \\
&\leq \frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\varphi(a)^+}\eta(b) + I^{\mu}_{\varphi(b)^-}\eta(a)\right).
\end{aligned} \tag{2.4}$$

Proof. Let K be the interval with end points $\varphi(a)$ and $\varphi(b)$. Since ψ is convex with respect to φ, for all $x, y \in K$, the inequality

$$\psi\left(\varphi^{-1}\left(\frac{x+y}{2}\right)\right) \leq \frac{\psi(\varphi^{-1}(x)) + \psi(\varphi^{-1}(y))}{2} \tag{2.5}$$

holds. By setting $x = \xi\varphi(a) + (1-\xi)\varphi(b)$, $y = (1-\xi)\varphi(a) + \xi\varphi(b)$, $\xi \in [0,1]$, we find the following inequality:

$$\begin{aligned}
&2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\
&\leq \psi(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b))) + \psi(\varphi^{-1}((1-\xi)\varphi(a)+\xi\varphi(b))).
\end{aligned} \tag{2.6}$$

By multiplying with $\xi^{\mu-1}\eta(\varphi^{-1}(\xi\varphi(a) + (1-\xi)\varphi(b)))$ on both sides of (2.6) and then integrating over $[0,1]$, the following inequality is obtained:

$$2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_0^1 \xi^{\mu-1}\eta(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b))d\xi \tag{2.7}$$

$$\leq \int_0^1 \xi^{\mu-1}(\psi.\eta)(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))d\xi$$
$$+\int_0^1 \xi^{\mu-1}\psi(\varphi^{-1}((1-\xi)\varphi(a)+\xi\varphi(b)))\eta(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))d\xi.$$

Now setting again $u=\xi\varphi(a)+(1-\xi)\varphi(b)$ that is $\xi=\frac{\varphi(b)-u}{\varphi(b)-\varphi(a)}$ and $v=(1-\xi)\varphi(a)+\xi\varphi(b)$ that is $\xi=\frac{v-\varphi(a)}{\varphi(b)-\varphi(a)}$ in (2.7), we find the following inequality:

$$2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_{\varphi(a)}^{\varphi(b)}\frac{\eta\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}\leq\int_{\varphi(a)}^{\varphi(b)}\frac{(\psi.\eta)\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}$$
$$+\int_{\varphi(a)}^{\varphi(b)}\frac{\psi\left(\varphi^{-1}(v)\right)\eta\left(\varphi^{-1}(\varphi(a)+\varphi(b)-v)\right)dv}{(v-\varphi(a))^{1-\mu}}.$$

From which by using symmericity of w with respect to φ, one can get the first inequality of (2.4). On the other hand by using convexity of ψ with respect to φ, the following inequality can be derived:

$$\psi(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))+\psi(\varphi^{-1}((1-\xi)\varphi(a)+\xi\varphi(b)))\leq\psi(a)+\psi(b),\quad \xi\in[0,1]. \tag{2.8}$$

By multiplying with $\xi^{\mu-1}\eta(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))$ on both sides of (2.8) and then integrating over $[0,1]$, the following inequality is obtained:

$$\int_0^1 \xi^{\mu-1}(\psi.\eta)(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))d\xi \tag{2.9}$$
$$+\int_0^1 \xi^{\mu-1}\psi(\varphi^{-1}((1-\xi)\varphi(a)+\xi\varphi(b)))\eta(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))d\xi$$
$$\leq[\psi(a)+\psi(b)]\int_0^1 \xi^{\mu-1}\eta(\varphi^{-1}(\xi\varphi(a)+(1-\xi)\varphi(b)))d\xi.$$

By making substitution $u=\xi\varphi(a)+(1-\xi)\varphi(b)$ and $v=(1-\xi)\varphi(a)+\xi\varphi(b)$ in first and second integrals respectively of the left hand side of the inequality (2.9), and making substitution of $u=\xi\varphi(a)+(1-\xi)\varphi(b)$ for integral appearing on right side of this inequality we obtain

$$\int_{\varphi(a)}^{\varphi(b)}\frac{\psi\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}+\int_{\varphi(a)}^{\varphi(b)}\frac{\psi\left(\varphi^{-1}(v)\right)\eta(\varphi^{-1}(\varphi(a)+\varphi(b)-v))dv}{(v-\varphi(a))^{1-\mu}} \tag{2.10}$$
$$\leq\frac{\psi(a)+\psi(b)}{2}\int_{\varphi(a)}^{\varphi(b)}\frac{\eta\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}.$$

From which by using symmericity of w with respect to φ, one can get the second inequality of (2.4). □

In the following we give consequences the above theorem.

Corollary 1. *The following Fejér-Hadamard inequality holds for GA-convex function:*

$$\psi\left(\sqrt{ab}\right)\left(I^{\mu}_{\ln a^{+}}\eta(b)+I^{\mu}_{\ln b^{-}}\eta(a)\right)\leq I^{\mu}_{\ln a^{+}}(\psi.\eta)(b)+I^{\mu}_{\ln b^{-}}(\psi.\eta)(a) \tag{2.11}$$
$$\leq\frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\ln a^{+}}\eta(b)+I^{\mu}_{\ln b^{-}}\eta(a)\right).$$

Proof. Let $\varphi(x) = \exp x$. Then $\varphi^{-1}(x) = \ln x$, the inequality (2.4) reduces to (2.11) for GA-convex functions. □

Corollary 2. *The following Fejér-Hadamard inequality holds for $\psi \circ \ln$-convex function:*

$$\begin{aligned}
&\psi\left(\ln\left(\frac{\exp(a)+\exp(b)}{2}\right)\right)\left(I^{\mu}_{\exp(a)^+}\eta(b) + I^{\mu}_{\exp(b)^-}\eta(a)\right) \\
&\leq I^{\mu}_{\exp(a)^+}(\psi.\eta)(b) + I^{\mu}_{\exp(b)^-}(\psi.\eta)(a) \\
&\leq \frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\exp(a)^+}\eta(b) + I^{\mu}_{\exp(b)^-}\eta(a)\right).
\end{aligned} \tag{2.12}$$

Proof. Let $\varphi(x) = \ln x$. Then $\varphi^{-1}(x) = \exp x$, the inequality (2.4) reduces to (2.12) for GA-convex functions. □

Remark 2. *(i) By choosing $\eta(x) = 1$, Theorem 5 is obtained.*
(ii) By choosing $\varphi(x) = \frac{1}{x}$, [20, Theorem 5] is obtained.
(iii) By choosing $\eta(x) = 1$ and $\varphi(x) = x$, Theorem 3 is obtained.
(iv) By choosing $\eta(x) = 1$ and $\varphi(x) = \frac{1}{x}$, [12, Theorem 4] is obtained.
(v) By choosing $\eta(x) = 1$ and $\varphi(x) = x^p$, $\mu = 1$, [11, Theorem 6] is obtained.
(vi) By choosing $\eta(x) = 1$ and $\varphi(x) = \frac{1}{x}$, $\mu = 1$, [5, Theorem 2.4] is obtained.
(vii) By choosing $\varphi(x) = x^p$, $\mu = 1$, [9, Theorem 5] is obtained.
(viii) By choosing $\eta(x) = 1$ and $\varphi(x) = \ln x$, $\mu = 1$, [10, Theorem 2.2] is obtained.
(ix) By choosing $\eta(x) = 1$ and $\varphi(x) = x$, $\mu = 1$, the classical Hadamard inequality is obtained.

Lemma 2. *Let ψ be symmetric with respect to strictly monotone function φ about $\frac{\varphi(a)+\varphi(b)}{2}$, and $\varphi \in L[a,b]$. Then the following identity holds for Riemann-Liouville fractional integrals:*

$$I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi(b) = I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^-}\psi(a) = \frac{I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi(b) + I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^-}\psi(a)}{2}. \tag{2.13}$$

Proof. From definition of Riemann-Liouville fractional integrals we have

$$I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi(b) = I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi\left(\varphi^{-1}(\varphi(b))\right) = \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} \frac{\psi\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}. \tag{2.14}$$

By setting $\varphi(a) + \varphi(b) - u = z$ in (2.14) we get

$$I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi(b) = \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} \frac{\psi\left(\varphi^{-1}(\varphi(a)+\varphi(b)-z)\right)dz}{(z-\varphi(a))^{1-\mu}}. \tag{2.15}$$

By using symmetricity of ψ with respect to strictly monotone function φ about $\frac{\varphi(a)+\varphi(b)}{2}$, we get $I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^+}\psi(b) = I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^-}\psi\left(\varphi^{-1}(\varphi(a))\right)$ and hence (2.13) is obtained. □

Remark 3. *(i) By setting $\varphi(x) = \frac{1}{x}$ in (2.13), we get [14, Lemma 2].*
(ii) By setting $\varphi(x) = x^p$, $p \neq 0$ in (2.13), we get the identity for p-symmetric functions.

In the next theorem we establish another version of the Fejér-Hadamard inequality for convex function with respect to a strictly monotone function.

Theorem 8. *Under the assumptions of Theorem 7, the following inequality holds for Riemann-Liouville fractional integrals:*

$$\begin{aligned}&\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\left(I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{+}}\eta(b)+I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{-}}\eta(a)\right)\\&\le I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{+}}(\psi.\eta)(b)+I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{-}}(\psi.\eta)(a)\\&\le\frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{+}}\eta(b)+I^{\mu}_{\frac{\varphi(a)+\varphi(b)}{2}^{-}}\eta(a)\right).\end{aligned}\tag{2.16}$$

Proof. Let $x=\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)$, $y=\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)$, $\xi\in[0,1]$. Then from (2.5) we get the following inequality:

$$\begin{aligned}&2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\le\psi\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)\\&+\psi\left(\varphi^{-1}\left(\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)\right)\right).\end{aligned}\tag{2.17}$$

By multiplying with $\xi^{\mu-1}\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)$ on both sides of (2.17) and then integrating over $[0,1]$, the following inequality is obtained:

$$\begin{aligned}&2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_0^1\xi^{\mu-1}\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi\\&\le\int_0^1\xi^{\mu-1}\psi\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi\\&+\int_0^1\xi^{\mu-1}\psi\left(\varphi^{-1}\left(\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)\right)\right)\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi.\end{aligned}\tag{2.18}$$

Taking $u=\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)$ that is $\xi=\frac{2(\varphi(b)-u)}{\varphi(b)-\varphi(a)}$ and $v=\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)$ that is $\xi=\frac{2(v-\varphi(a))}{\varphi(b)-\varphi(a)}$ in (2.18), we find the following inequality:

$$\begin{aligned}&2\psi\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{\eta\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}\\&\le\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{(\psi.\eta)\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}+\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{\psi\left(\varphi^{-1}(v)\right)\eta\left(\varphi^{-1}(\varphi(a)+\varphi(b)-v)\right)dv}{(v-\varphi(a))^{1-\mu}}.\end{aligned}$$

From which by using symmericity of w with respect to φ, one can get the first inequality of (2.16). Again by using convexity of ψ with respect to φ, the following inequality is derived for $\xi\in[0,1]$:

$$\psi\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)+\psi\left(\varphi^{-1}\left(\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)\right)\right)\le\psi(a)+\psi(b).\tag{2.19}$$

By multiplying with $\xi^{\mu-1}\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)$ on both sides of (2.8) and then integrating over [0, 1], the following inequality is obtained:

$$\begin{aligned}
&\int_0^1 \xi^{\mu-1}\psi\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi \\
&+\int_0^1 \xi^{\mu-1}\psi\left(\varphi^{-1}\left(\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)\right)\right)\eta\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi. \\
&\leq [\psi(a)+\psi(b)]\int_0^1 \xi^{\mu-1}\left(\varphi^{-1}\left(\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)\right)\right)d\xi.
\end{aligned} \tag{2.20}$$

By making substitution $u=\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)$ and $v=\frac{2-\xi}{2}\varphi(a)+\frac{\xi}{2}\varphi(b)$ in first and second integrals respectively of the left hand side of the inequality (2.20), and making substitution of $u=\frac{\xi}{2}\varphi(a)+\frac{2-\xi}{2}\varphi(b)$ in the integral appearing in the right hand side of this inequality we will get

$$\begin{aligned}
&\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{(\psi.\eta)\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}+\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{\psi\left(\varphi^{-1}(v)\right)\eta\left(\varphi^{-1}(\varphi(a)+\varphi(b)-v)\right)dv}{(v-\varphi(a))^{1-\mu}} \\
&\leq \frac{\psi(a)+\psi(b)}{2}\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}\frac{\eta\left(\varphi^{-1}(u)\right)du}{(\varphi(b)-u)^{1-\mu}}.
\end{aligned} \tag{2.21}$$

From which by using symmericity of w with respect to φ, one can get the second inequality of (2.16).
□

The consequences of above theorem are given in the following corollaries and remark.

Corollary 3. *The following Fejér-Hadamard inequality holds for GA-convex function:*

$$\begin{aligned}
&\psi\left(\sqrt{ab}\right)\left(I^{\mu}_{\ln\sqrt{ab}^+}\eta(b)+I^{\mu}_{\ln\sqrt{ab}^-}\eta(a)\right)\leq I^{\mu}_{\ln\sqrt{ab}^+}(\psi.\eta)(b)+I^{\mu}_{\ln\sqrt{ab}^-}(\psi.\eta)(a) \\
&\leq \frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\ln\sqrt{ab}^+}\eta(b)+I^{\mu}_{\ln\sqrt{ab}^-}\eta(a)\right).
\end{aligned} \tag{2.22}$$

Proof. Let $\varphi(x)=\exp x$. Then $\varphi^{-1}(x)=\ln x$, the inequality (2.16) reduces to (2.22) for GA-convex functions. □

Corollary 4. *The following Fejér-Hadamard inequality holds for* $\psi\circ\ln$*-convex function:*

$$\begin{aligned}
&\psi\left(\ln\left(\frac{\exp(a)+\exp(b)}{2}\right)\right)\left(I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^+}\eta(b)+I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^-}\eta(a)\right) \\
&\leq I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^+}(\psi.\eta)(b)+I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^-}(\psi.\eta)(a) \\
&\leq \frac{\psi(a)+\psi(b)}{2}\left(I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^+}\eta(b)+I^{\mu}_{\frac{\exp(a)+\exp(b)}{2}^-}\eta(a)\right).
\end{aligned} \tag{2.23}$$

Proof. Let $\varphi(x)=\ln x$. Then $\varphi^{-1}(x)=\exp x$, the inequality (2.16) reduces to (2.23) for GA-convex functions. □

Remark 4. *(i) By choosing $\eta(x) = 1$, Theorem 6 is obtained.*
(i) By choosing $\eta(x) = 1$ and $\varphi(x) = x$, Theorem 4 is obtained.
(ii) By choosing $\eta(x) = 1$ and $\varphi(x) = \frac{1}{x}$, [14, Theorem 4] is obtained.
(iii) By choosing $\eta(x) = 1$ and $\varphi(x) = x^p$, $p \neq 0$, [13, Theorem 2.1] is obtained.
(iv) By choosing $\eta(x) = 1$ and $\varphi(x) = \frac{1}{x}$, $\mu = 1$, [5, Theorem 2.4] is obtained.
(v) By choosing $\eta(x) = 1$ and $\varphi(x) = x^p$, $p \neq= 1$, [11, Theorem 6] is obtained.

3. Conclusions

We have studied the Riemann-Liouville fractional integral versions of Fejér-Hadamard inequalities for convex function with respect to strictly monotone function. The established inequalities provide the Hadamard and Fejér-Hadamard inequalities for Riemann-Liouville fractional integrals of convex, harmonically convex, p-convex and GA-convex functions. For specific increasing/decreasing functions the reader can produce corresponding Fejér-Hadamard inequalities from results of this paper. Further, we are investigating such results for other kinds of fractional integrals for future work.

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Conflict of interest

It is declared that the author have no competing interests.

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