AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(7): 6833–6844.
DOI:10.3934/math.2021400
Received: 21 January 2021
Accepted: 09 April 2021
Published: 22 April 2021

Research article

On differential identities of Jordan ideals of semirings

Liaqat Ali[1], **Muhammad Aslam**[2], **Ghulam Farid**[3,*] **and S. Abdel-Khalek**[4]

[1] Department of Mathematics, Govt. MAO College Lahore, Pakistan
[2] Department of Mathematics, Govt. College University Lahore, Pakistan
[3] Department of Mathematics, COMSATS Islamabad, Attock Campus, Pakistan
[4] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: faridphdsms@hotmail.com; Tel: +923334426360.

Abstract: In this article some salient characteristics of Jordan ideals of MA-semirings are discussed. We prove some results for derivations of MA-semirings satisfying different identities on their Jordan ideals and investigate commuting conditions through these ideals.

Keywords: MA-semirings; Jordan ideals; derivations
Mathematics Subject Classification: 16W10, 16Y60

1. Introduction

Semirings have significant applications in theory of automata, optimization theory, and in theoretical computer sciences (see [1–3]). A group of Russian mathematicians was able to create novel probability theory based on additive inverse semirings, called idempotent analysis (see [4, 5]) having interesting applications in quantum physics. Javed et al. [6] identified a proper subclass of semirings known as MA-Semirings. The development of commutator identities and Lie type theory of semirings [6–10] and derivations [6–8, 11, 12] make this class quite interesting for researchers. To investigate commuting conditions for rings through certain differential identities and certain ideals are still interesting problems for researchers in ring theory (see for example [13–19]) and some of them are generalized in semirings (see [6, 8–11, 20]). In this paper we investigate commuting conditions of prime MA-semirings through certain differential identities and Jordan ideals (Theorems 2.5–2.8) and also study differential identities with the help of Jordan ideals (Theorem 2.3, Theorem 2.4, Theorem 2.10). In this connection we are able to generalize a few results of Oukhtite [21] in the setting of semirings. Now we present some necessary definitions and preliminaries which will be very useful for the sequel. By a semiring S, we mean a semiring with absorbing zero '0' in which addition is commutative. A semiring S is said to be additive inverse semiring if for each $s \in S$ there is a unique $s^{'} \in S$ such that $s + s^{'} + s = s$ and $s^{'} + s + s^{'} = s^{'}$, where $s^{'}$ denotes the pseudo inverse of s. An additive inverse semiring S is said to be an MA-semiring if it satisfies $s + s^{'} \in Z(S), \forall s \in S$, where $Z(S)$ is the center of S. The class of MA-semirings properly contains the class of distributive lattices and the class of rings, we refer [6, 8, 11, 22] for examples. Throughout the paper by semiring S we mean an MA-semiring unless stated otherwise. A semiring S is prime if $aSb = \{0\}$ implies that $a = 0$ or $b = 0$ and semiprime if $aSa = \{0\}$ implies that $a = 0$. S is 2-torsion free if for $s \in S$, $2s = 0$ implies $s = 0$. An additive mapping $d : S \longrightarrow S$ is a derivation if $d(st) = d(s)t + sd(t)$. The commutator is defined as $[s, t] = st + t^{'}s$. By Jordan product, we mean $s \circ t = st + ts$ for all $s, t \in S$. The notion of Jordan ideals was introduced by Herstein [23] in rings which is further extended canonically by Sara [20] for semirings. An additive subsemigroup G of S is called the Jordan ideal if $s \circ j \in G$ for all $s \in S, j \in G$. A mapping $f : S \longrightarrow S$ is commuting if $[f(s), s] = 0, \forall s \in S$. A mapping $f : S \longrightarrow S$ is centralizing if $[[f(s), s], r] = 0, \forall s, r \in S$. Next we include some well established identities of MA-semirings which will be very useful in the sequel. If $s, t, z \in S$ and d is a derivation of S, then $[s, st] = s[s, t]$, $[st, z] = s[t, z] + [s, z]t$, $[s, tz] = [s, t]z + t[s, z]$, $[s, t] + [t, s] = t(s + s^{'}) = s(t + t^{'})$, $(st)^{'} = s^{'}t = st^{'}$, $[s, t]^{'} = [s, t^{'}] = [s^{'}, t]$, $s \circ (t + z) = s \circ t + s \circ z$, $d(s^{'}) = (d(s))^{'}$. To see more, we refer [6, 7].

From the literature we recall a few results of MA-semirings required to establish the main results.

Lemma 1. *[11] Let G be a Jordan ideal of an MA-semiring S. Then for all $j \in G$ (a). $2[S, S]G \subseteq G$ (b). $2G[S, S] \subseteq G$ (c). $4j^2S \subseteq G$ (d). $4Sj^2 \subseteq G$ (e). $4jSj \subseteq G$.*

Lemma 2. *[11] Let S be a 2-torsion free prime MA-semiring and G a Jordan ideal of S. If $aGb = \{0\}$ then $a = 0$ or $b = 0$.*

In view of Lemma 1 and Lemma 2, we give some very useful remarks.

Remark 1. *[11]*
a). If $r, s, t \in S, u \in G$, then $2[r, st]u \in G$.
b). If $aG = \{0\}$ or $Ga = \{0\}$, then $a = 0$.

Lemma 3. *[12] Let G be a nonzero Jordan ideal and d be a derivation of a 2-torsion free prime MA-semiring S such that for all $u \in G$, $d(u^2) = 0$. Then $d = 0$.*

2. Main results

Lemma 4. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S. If $a \in S$ such that for all $g \in G$, $[a, g^2] = 0$. Then $[a, s] = 0, \forall s \in S$ and hence $a \in Z(S)$.*

Proof. Define a function $d_a : S \longrightarrow S$ by $d_a(s) = [a, s]$, which is an inner derivation. As every inner derivation is derivation, therefore in view of hypothesis d_a is derivation satisfying $d_a(g^2) = [a, g^2] = 0, \forall g \in G$. By Lemma 3, $d_a = 0$, which implies that $d_a(s) = [a, s] = 0$, for all $s \in S$. Hence $a \in Z(S)$. □

Lemma 5. *Let S be a 2-torsion free prime MA-semiring and G a nonzero Jordan ideal of S. If S is noncommutative such that for all $u, v \in G$ and $r \in S$*

$$a[r, uv]b = 0, \tag{2.1}$$

then $a = 0$ *or* $b = 0$.

Proof. In (2.1) replacing r by ar and using MA-semiring identities, we obtain

$$aa[r, uv]b + a[a, uv]rb = 0 \tag{2.2}$$

Using (2.1) again, we get $a[a, uv]Sb = 0$. By the primeness of S, we have either $b = 0$ or $a[a, uv] = 0$. Suppose that

$$a[a, uv] = 0 \tag{2.3}$$

In view of Lemma 1, replacing v by $2v[s, t]$ in (2.3) and using 2-torsion freeness of S, we get $0 = a[a, uv[s, t]] = auv[a, [s, t]] + a[a, uv][s, t]$. Using (2.3) again $auv[a, [s, t]] = 0$ and therefore $auG[a, [s, t]] = \{0\}$. By the Lemma 2, we have either $aG = \{0\}$ or $[a, [s, t]] = 0$. By Remark 1, $aG = \{0\}$ implies $a = 0$. Suppose that

$$[a, [s, t]] = 0 \tag{2.4}$$

In (2.4) replacing s by sa, we get $[a, s[a, t]] + [a, [s, t]a] = 0$ and therefore $[a, s[a, t]] + [a, [s, t]]a = 0$. Using (2.4) again, we get $[a, s][a, t] = 0$. By the primeness of S, $[a, s] = 0$ and therefore $a \in Z(S)$. Hence from (2.2), we can write $aS[r, uv]b = \{0\}$. By the primeness of S, we obtain $a = 0$ or

$$[r, uv]b = 0 \tag{2.5}$$

In (2.5) replacing r by rs and using (2.5) again, we get $[r, uv]Sb = \{0\}$. By the primeness of S, we have either $b = 0$ or $[r, uv] = 0$. Suppose that

$$[r, uv] = 0 \tag{2.6}$$

In (2.6) replacing y by $2v[s, t]$ and using (2.6) again, we obtain $2[r, uv[s, t]] = 0$. As S is 2-torsion free, $[r, uv[s, t]] = 0$ which further gives $uG[r, [s, t]] = \{0\}$. As $G \neq \{0\}$, by Lemma 2 $[r, [s, t]] = 0$ which shows that S is commutative, a contradiction. Hence we conclude that $a = 0$ or $b = 0$. □

Theorem 1. *Let S be a 2-torsion free prime MA-semiring and G a nonzero Jordan ideal of S. If d_1 and d_2 are derivations of S such that for all $u \in G$,*

$$d_1 d_2(u) = 0 \tag{2.7}$$

then either $d_1 = 0$ *or* $d_2 = 0$.

Proof. Suppose that $d_2 \neq 0$. We will show that $d_1 = 0$. In view of Lemma 1, replacing u by $4u^2v, v \in G$ in (2.7), we obtain $d_1d_2(4u^2v) = 0$ and by the 2-torsion freeness of S, we have $d_1d_2(u^2v) = 0$. Using (2.7) again, we obtain

$$d_2(u^2)d_1(v) + d_1(u^2)d_2(v) = 0 \tag{2.8}$$

By lemma 1, replacing v by $2[r, jk]v$, $j, k \in G$ in (2.8), we get

$$d_2(u^2)d_1(2[r, jk]v) + d_1(u^2)d_2(2[r, jk]v) = 0$$

and

$$2d_2(u^2)[r, jk]d_1(v) + 2d_2(u^2)d_1([r, jk])v + 2d_1(u^2)[r, jk]d_2(v) + 2d_1(u^2)d_2([r, jk])v = 0$$

Using (2.8) again and hence by the 2-torsion freeness of S, we obtain

$$d_2(u^2)[r, jk]d_1(v) + d_1(u^2)[r, jk]d_2(v) = 0 \tag{2.9}$$

In (2.9), replacing v by $4v^2t, t \in S$ and using (2.9) again, we obtain

$$4d_2(u^2)[r, jk]v^2d_1(t) + 4d_1(u^2)[r, jk]v^2d_2(t) = 0$$

As S is 2-torsion free, therefore

$$d_2(u^2)[r, jk]v^2d_1(t) + d_1(u^2)[r, jk]v^2d_2(t) = 0 \tag{2.10}$$

In (2.10), taking $t = d_2(g), g \in G$ and using (2.7), we obtain

$$d_1(u^2)[r, jk]v^2d_2(d_2(g)) = 0 \tag{2.11}$$

In (2.11) writing a for $d_1(u^2)$ and b for $v^2d_2(d_2(g))$, we have $a[r, jk]b = 0, \forall r \in S, j, k \in G$.
Firstly suppose that S is not commutative. By Lemma 5, we have $a = 0$ or $b = 0$. If $d_1(u^2) = a = 0$, then by Lemma 3, $d_1 = 0$. Secondly suppose that S is commutative. In (2.7) replacing u by $2u^2$, we obtain $0 = d_1d_2(2u^2) = 2d_1d_2(u^2) = 4d_1(ud_2(u)) = 4(d_1(u)d_2(u) + ud_1d_2(u))$. Using (2.7) and the 2-torsion freeness of S, we obtain $d_1(u)d_2(u) = 0$. By our assumption S is commutative, therefore $d_1(u)S\,d_2(u) = \{0\}$. By the primeness of S, we have either $d_1(G) = \{0\}$ or $d_2(G) = \{0\}$. By Theorem 2.4 of [11], we have $d_1 = 0$ or $d_2 = 0$. But $d_2 \neq 0$. Hence $d_1 = 0$ which completes the proof. □

Theorem 2. *Let S be a 2-torsion free prime MA-semiring and G a nonzero Jordan ideal of S. If d_1 and d_2 are derivations of S such that for all $u \in G$*

$$d_1(d_2(u) + u') = 0, \tag{2.12}$$

then $d_1 = 0$.

Proof. Firstly suppose that S is commutative. Replacing u by $2u^2$ in (2.12) and using (2.12) again, we obtain $d_1(u)d_2(u) = 0$ which further implies $d_1(u)S\,d_2(u) = \{0\}$. In view of Theorem 2.4 of [11], by the primeness of S we have $d_1 = 0$ or $d_2 = 0$. If $d_2 = 0$, then from (2.12), we obtain $d_1(u) = 0, \forall u \in G$ and hence by Lemma 3, we conclude $d_1 = 0$. Secondly suppose that S is noncommutative. Further suppose that $d_2 \neq 0$. We will show that $d_1 = 0$. In (2.12) replacing u by $4u^2v, v \in G$, and using (2.12) again, we obtain $2(d_2(u^2)d_1(v) + d_1(u^2)d_2(v)) = 0$. As S is 2-torsion free, therefore

$$d_2(u^2)d_1(v) + d_1(u^2)d_2(v) = 0 \tag{2.13}$$

In (2.13) replacing v by $2[r, jk]v, r \in S, j, k, v \in G$, we obtain

$$d_2(u^2)d_1(2[r, jk])v + 2d_2(u^2)[r, jk]d_1(v) + d_1(u^2)d_2(2[r, jk])v + 2d_1(u^2)[r, jk]d_2(v) = 0$$

As by MA-semiring identities, $2[r, jk] = 2j[r, k] + 2[r, j]k$, by Lemma 1 $2[r, jk] \in G$. Therefore using (2.13) again and the 2-torsion freeness of S, we obtain

$$d_2(u^2)[r, jk]d_1(v) + d_1(u^2)[r, jk]d_2(v) = 0 \tag{2.14}$$

In (2.14) replacing v by $4v^2t, t \in S$ and using (2.14) again, we get

$$d_2(u^2)[r, jk]v^2d_1(t) + d_1(u^2)[r, jk]v^2d_2(t) = 0 \tag{2.15}$$

In (2.15) taking $t = t(d_2(w) + w^{'}), w \in G$, we get

$$d_2(u^2)[r, jk]v^2d_1(t(d_2(w) + w^{'})) + d_1(u^2)[r, jk]v^2d_2(t(d_2(w) + w^{'})) = 0$$

and therefore

$$d_2(u^2)[r, jk]v^2d_1(t)(d_2(w) + w^{'}) + d_2(u^2)[r, jk]v^2td_1((d_2(w) + w^{'}))$$
$$+d_1(u^2)[r, jk]v^2d_2(t)(d_2(w) + w^{'}) + d_1(u^2)[r, jk]v^2td_2(d_2(w) + w^{'}) = 0$$

Using (2.12) and (2.15) in the last expression, we obtain

$$(d_1(u^2))[r, jk](v^2td_2(d_2(w) + w^{'})) = 0 \tag{2.16}$$

Applying Lemma 5 on (2.15), we get either $d_1(u^2) = 0$ or $v^2td_2(d_2(w) + w^{'}) = 0$. If $d_1(u^2) = 0$ then by Lemma 3, $d_1 = 0$. If $v^2Sd_2(d_2(w) + w^{'}) = \{0\}$, the by the primeness of S, we have $v^2 = 0$ or $d_2(d_2(w) + w^{'}) = 0$. If $v^2 = 0, \forall v \in G$, then $G = \{0\}$, a contradiction. Suppose that for all $w \in G$

$$d_2(d_2(w) + w^{'}) = 0 \tag{2.17}$$

In (2.17)replacing w by $4z^2u, z, u \in G$, and using (2.17) again, we obtain

$$d_2(z^2)d_2(u) = 0 \tag{2.18}$$

In (2.18), replacing u by $4xz^2, x \in G$ and using (2.18) again, we obtain $d_2(z^2)Gd_2(z^2) = \{0\}$. By Lemma 2, $d_2(z^2) = 0$ and hence by Lemma 3, we conclude that $d_2 = 0$. Taking $d_2 = 0$ in the hypothesis to obtain $d_1(u) = 0$ and hence by Theorem 2.4 of [11], we have $d_1 = 0$. □

Theorem 3. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and d_1 and d_2 be derivations of S such that for all $u, v \in G$*

$$[d_1(u), d_2(v)] + [u, v]^{'} = 0 \tag{2.19}$$

Then S is commutative.

Proof. If $d_1 = 0$ or $d_2 = 0$, then from (2.19), we obtain $[G, G] = \{0\}$. By Theorem 2.3 of [11] S is commutative. We assume that both d_1 and d_2 are nonzero. In (2.19) replacing u by $4uw^2$ and using MA-semiring identities and 2-torsion freeness of S, we get
$d_1(u)[2w^2, d_2(v)] + ([d_1(u), d_2(v)] + [u, v]^{'})2w^2 + u([d_1(2w^2), d_2(v)]$

$$+[2w^2, v]^{'}) + [u, d_2(v)]d_1(2w^2) = 0$$

Using (2.19) again, we get

$$d_1(u)[2w^2, d_2(v)] + [u, d_2(v)]d_1(2w^2) = 0$$

and by the 2-torsion freeness of S, we have

$$d_1(u)[w^2, d_2(v)] + [u, d_2(v)]d_1(w^2) = 0 \tag{2.20}$$

Replacing u by $2u[r, jk]$ in (2.20) and using it again, we obtain

$$d_1(u)[r, jk][w^2, d_2(v)] + [u, d_2(v)][r, jk]d_1(w^2) = 0 \tag{2.21}$$

In (2.21) replacing u by $4su^2$ and using (2.21) again, we obtain

$$d_1(s)u^2[r, jk][w^2, d_2(v)] + [s, d_2(v)]u^2[r, jk]d_1(w^2) = 0 \tag{2.22}$$

In (2.22) replacing s by $d_2(v)s$ and then using commutator identities, we get

$$d_1d_2(v)su^2[r, jk][w^2, d_2(v)] = 0 \tag{2.23}$$

Therefore $d_1d_2(v)Su^2[r, jk][w^2, d_2(v)] = \{0\}$. By the primeness of S, we obtain either $d_1d_2(v) = 0$ or $u^2[r, jk][w^2, d_2(v)] = 0$. Consider the sets

$$G_1 = \{v \in G : d_1d_2(v) = 0\}$$

and

$$G_2 = \{v \in G : u^2[r, jk][w^2, d_2(v) = 0\}$$

We observe that $G = G_1 \cup G_2$. We will show that either $G = G_1$ or $G = G_2$. Suppose that $v_1 \in G_1 \setminus G_2$ and $v_2 \in G_2 \setminus G_1$. Then $v_1 + v_2 \in G_1 + G_2 \subseteq G_1 \cup G_2 = G$. We now see that $0 = d_1d_2(v_1 + v_2) = d_1d_2(v_2)$, which shows that $v_2 \in G_1$, a contradiction. On the other hand $0 = u^2[r, jk][w^2, d_2(v_1 + v_2)] = u^2[r, jk][w^2, d_2(v_1)]$, which shows that $v_1 \in G_2$, a contradiction. Therefore either $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$, which respectively show that either $G = G_1$ or $G = G_2$. Therefore we conclude that for all $v \in G$, $d_1d_2(v) = 0$ or $u^2[r, jk][w^2, d_2(v)] = 0$. Suppose that $d_1d_2(v) = 0, v \in G$. then by Lemma 2, $d_1 = 0$ or $d_2 = 0$. Secondly suppose that $u^2[r, jk][w^2, d_2(v)] = 0, u, v, w, j, k \in G, r \in S$. By Lemma 5, we have either $u^2 = 0$ or $[w^2, d_2(v)] = 0$. But $u^2 = 0$ leads to $G = \{0\}$ which is not possible. Therefore $[w^2, d_2(v)] = 0$ and employing Lemma 4, $[d_2(v), s] = 0, s \in S$. Hence by Theorem 2.2 of [22], S is commutative. □

Theorem 4. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and d_1 and d_2 be derivations of S such that for all $u, v \in G$*

$$d_1(u)d_2(v) + [u, v]' = 0 \tag{2.24}$$

Then $d_1 = 0$ or $d_2 = 0$ and thus S is commutative.

Proof. It is quite clear that if at least one of d_1 and d_2 is zero, then we obtain $[G, G] = \{0\}$. By Theorem 2.3 of [11] and Theorem 2.1 of [22], S is commutative. So we only show that at least one of the derivations is zero. Suppose that $d_2 \neq 0$. In (2.24), replacing v by $4vw^2, w \in G$, we obtain $d_1(u)d_2(4vw^2) + [u, 4vw^2]' = 0$ and therefore using MA-semirings identities, we can write

$$4d_1(u)vd_2(w^2) + 4d_1(u)d_2(v)w^2 + 4v[u, w^2]' + 4[u, v]'w^2 = 0$$

In view of Lemma 1 as $2w^2 \in G$, using (2.24) and the 2-torsion freeness of S, we obtain

$$d_1(u)vd_2(w^2) + v[u, w^2]' = 0 \tag{2.25}$$

In (2.25) replacing v by $2[s, t]v$, $s, t \in S$ and hence by the 2-torsion freeness of S, we get

$$d_1(u)[s, t]vd_2(w^2) + [s, t]v[u, w^2]' = 0 \tag{2.26}$$

Multiplying (2.25) by $[s, t]$ from the left, we get

$$[s, t]d_1(u)vd_2(w^2) + [s, t]v[u, w^2]' = 0$$

and since S is an MA-semiring, therefore

$$[s, t]d_1(u)vd_2(w^2) = [s, t]v[u, w^2] \tag{2.27}$$

Using (2.27) into (2.26), we obtain $d_1(u)[s, t]vd_2(w^2) + [s, t]'d_1(u)vd_2(w^2) = 0$. By MA-semirings identities, we further obtain $[d_1(u), [s, t]]Gd_2(w^2) = \{0\}$. By Lemma 2, we obtain either $[d_1(u), [s, t]] = 0$ or $d_2(w^2) = 0$. If $d_2(w^2) = 0$, then by Lemma 3, $d_2 = 0$. On the other hand, if

$$[d_1(u), [s, t]] = 0 \tag{2.28}$$

In (2.28) replacing t by st, we get $[d_1(u), s[s, t]] = 0$ and using (2.23) again $[d_1(u), s][s, t] = 0$ and therefore $[d_1(u), s]S[s, t] = \{0\}$ and by the primeness of S, we get $[S, S] = \{0\}$ and hence S is commutative or $[d_1(u), s] = 0$. In view of Theorem 2.2 of [22] from $[d_1(u), s] = 0$ we have $[S, S] = \{0\}$ which further implies S is commutative. Hence (2.19)becomes $d_1(u)d_2(v) = 0$. As above we have either $d_1 = 0$ or $d_2 = 0$. □

Theorem 5. *Let S be a 2-torsion free prime MA-semiring and G a nonzero Jordan ideal of S. If d_1, d_2 and d_3 be nonzero. derivations such that for all $u, v \in G$ either*
1). $d_3(v)d_1(u) + d_2(u')d_3(v) = 0$ or
2). $d_3(v)d_1(u) + d_2(u')d_3(v) + [u, v]' = 0$.
Then S is commutative and $d_1 = d_2$.

Proof. 1). By the hypothesis for the first part, we have

$$d_3(v)d_1(u) + d_2(u')d_3(v) = 0 \tag{2.29}$$

which further implies

$$d_3(v)d_1(u) = d_2(u)d_3(v) \tag{2.30}$$

In (2.29) replacing u by $4uw^2$, we obtain

$$4d_3(v)d_1(u)w^2 + 4d_3(v)ud_1(w^2) + 4d_2(u')w^2d_3(v) + 4u'd_2(w^2)d_3(v) = 0$$

and therefore by the 2-torsion freeness of S, we have

$$d_3(v)d_1(u)w^2 + d_3(v)ud_1(w^2) + d_2(u')w^2d_3(v) + u'd_2(w^2)d_3(v) = 0 \tag{2.31}$$

Using (2.30) into (2.31), we obtain

$$d_2(u)[d_3(v), w^2] + [d_3(v), u]d_1(w^2) = 0 \tag{2.32}$$

In (2.32) replacing u by $2u[r, jk], r \in S, j, k \in G$, and using (2.32) again, we get

$$d_2(u)[r, jk][d_3(v), w^2] + [d_3(v), u][r, jk]d_1(w^2) = 0 \tag{2.33}$$

In (2.33) replacing u by $4tu^2, t \in S$ and using 2-torsion freeness and (2.33) again, we get

$$d_2(t)u^2[r, jk][d_3(v), w^2] + [d_3(v), t]u^2[r, jk]d_1(w^2) = 0 \tag{2.34}$$

Taking $t = d_3(v)t$ in (2.34) and using (2.34) again we obtain

$$d_2d_3(v)tu^2[r, jk][d_3(v), w^2] = 0 \tag{2.35}$$

We see that equation (2.35) is similar as (2.23) of the previous theorem, therefore repeating the same process we obtain the required result.
2). By the hypothesis, we have

$$d_3(v)d_1(u) + d_2(u^{'})d_3(v) + [u, v]^{'} = 0 \tag{2.36}$$

For $d_3 = 0$, we obtain $[G, G] = \{0\}$ and by Theorem 2.3 of [11] this proves that S is commutative. Assume that $d_3 \neq 0$. From (2.36), using MA-semiring identities, we can write

$$d_3(v)d_1(u) = d_2(u)d_3(v) + [u, v] \tag{2.37}$$

and

$$d_3(v)d_1(u) + [u, v]^{'} = d_2(u)d_3(v) \tag{2.38}$$

In (2.36), replacing u by $4uz^2$, we obtain

$$4(d_3(v)ud_1(z^2) + d_3(v)d_1(u)z^2 + d_2(u^{'})z^2d_3(v) + u^{'}d_2(z^2)d_3(v) + u[z^2, v]^{'}) + [u, v]^{'}z^2) = 0$$

and using (2.37) and (2.38) and then 2-torsion freeness of S, we obtain

$$[d_3(v), u]d_1(z^2) + d_2(u)[d_3(v), z^2] = 0 \tag{2.39}$$

We see that (2.39) is same as (2.32) of the previous part of this result. This proves that $[S, S] = \{0\}$ and hence S is commutative. Also then by the hypothesis, since $d_3 \neq 0$, $d_1 = d_2$. □

Theorem 6. *Let G be nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and d_1 and d_2 be nonzero derivations of S such that for all $u, v \in G$*

$$[d_2(v), d_1(u)] + d_1[v, u]^{'} = 0 \tag{2.40}$$

Then S is commutative.

In (2.40) replacing u by $4uw^2, w \in G$ and using 2-torsion freeness and again using(2.40), we obtain

$$[d_2(v) + v', u]d_1(w^2) + d_1(u)[d_2(v) + v', w^2] = 0 \tag{2.41}$$

In (2.41) replacing u by $2u[r, jk], j, k \in G, r \in S$, we obtain
$u[d_2(v) + v', 2[r, jk]]d_1(w^2) + 2[d_2(v) + v', u][r, jk]d_1(w^2)$

$$+ud_1(2[r, jk])[d_2(v) + v', w^2] + 2d_1(u)[r, jk][d_2(v) + v', w^2] = 0$$

Using 2-torsion freeness and (2.41) again, we get

$$[d_2(v) + v', u][r, jk]d_1(w^2) + d_1(u)[r, jk][d_2(v) + v', w^2] = 0 \tag{2.42}$$

In(2.42) replacing u by $4tu^2, t \in S$ and using (2.42) again, we get

$$[d_2(v) + v', t]u^2[r, jk]d_1(w^2) + d_1(t)u^2[r, jk][d_2(v) + v', w^2] = 0 \tag{2.43}$$

In (2.43) taking $t = (d_2(v) + v')t$ and using MA-semirings identities, we obtain
$(d_2(v) + v')[d_2(v) + v', t]u^2[r, jk]d_1(w^2) + d_1(d_2(v) + v')tu^2[r, jk][d_2(v) + v', w^2]$

$$+(d_2(v) + v')d_1(t)u^2[r, jk][d_2(v) + v', w^2] = 0$$

and using (2.43) again, we obtain

$$d_1(d_2(v) + v')tu^2[r, jk][d_2(v) + v', w^2] = 0 \tag{2.44}$$

By the primeness we obtain either $d_1(d_2(v)+v') = 0$ or $u^2[r, jk][d_2(v)+v', w^2] = 0$. If $d_1(d_2(v)+v') = 0$, then by Theorem 2 we have $d_1 = 0$, which contradicts the hypothesis. Therefore we must suppose $u^2[r, jk][d_2(v) + v', w^2] = 0$. By Lemma 5, we have either $u^2 = 0$ or $[d_2(v) + v', w^2] = 0$. But $u^2 = 0$ implies $G = \{0\}$ which is not possible. On the other hand applying Lemma 5, we have $[d_2(v) + v', r] = 0, \forall r \in S$ and therefore taking $r = v, v \in G$ and $[d_2(v), v] + [v', v] = 0$ and using MA-semiring identities, we get

$$[d_2(v), v] + [v, v]' = 0 \tag{2.45}$$

As $[v, v]' = [v, v]$, from (2.45), we obtain $[d_2(v), v] + [v, v] = 0$ and therefore

$$[d_2(v), v] = [v, v]' \tag{2.46}$$

Using (2.46) into (2.45), we get $2[d_2(v), v] = 0$ and by the 2-torsion freeness of S, we get $[d_2(v), v] = 0$. By Theorem 2.2 of [22], we conclude that S is commutative.

Corollary 1. *Let G be nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and d be a nonzero derivation of S such that for all $u, v \in G$ $d[v,u]=0$. Then S is commutative*

Proof. In theorem (6) taking $d_2 = 0$ and $d_1 = d$, we get the required result. □

Theorem 7. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring and d_2 be derivation of S. Then there exists no nonzero derivation d_1 such that for all $u, v \in G$*

$$d_2(v) \circ d_1(u) + d_1(v' \circ u) = 0 \tag{2.47}$$

Proof. Suppose on the contrary that there is a nonzero derivation d_1 satisfying (2.47). In (2.47) replacing u by $4uw^2, w \in G$ and using (2.47) again, we obtain

$$d_1(u)[w^2, d_2(v) + v] + [u, d_2(v)]' d_1(w^2) + ud_1(v \circ w^2) + (u \circ v)d_1(w^2)' + ud_1[v, w^2]' = 0 \tag{2.48}$$

In (2.48), replacing u by $u[r, jk], r \in S, j, k \in G$ and using (2.48) again, we get

$$d_1(u)[r, jk][w^2, d_2(v) + v] + [u, d_2(v) + v]'[r, jk]d_1(w^2) = 0 \tag{2.49}$$

In (2.49) replacing u by $4tu^2, t \in S$ and using (2.49) again, we obtain
$d_1(t)u^2[r, jk][w^2, d_2(v) + v] + td_1(u^2)[r, jk][w^2, d_2(v) + v]$

$$+t[u^2, d_2(v) + v]'[r, jk]d_1(w^2) + [t, d_2(v) + v]'u^2[r, jk]d_1(w^2) = 0$$

and using2-torsion freeness and (2.49) again, we obtain

$$d_1(t)u^2[r, jk][w^2, d_2(v) + v] + [t, d_2(v) + v]'u^2[r, jk]d_1(w^2) = 0 \tag{2.50}$$

In (2.50) taking $t = (d_2(v) + v)t$ and using MA-semirings identities, we get $d_1(d_2(v) + v)tu^2[r, jk][w^2, d_2(v) + v] + (d_2(v) + v)d_1(t)u^2[r, jk][w^2, d_2(v) + v]$

$$+(d_2(v) + v)[t, d_2(v) + v]'u^2[r, jk]d_1(w^2) = 0$$

Using (2.50) again, we obtain

$$d_1(d_2(v) + v)tu^2[r, jk][w^2, d_2(v) + v] = 0 \tag{2.51}$$

that is $d_1(d_2(v)+v)S u^2[r, jk][w^2, d_2(v)+v] = 0$. Therefore by the primeness following the same process as above, we have either $d_1(d_2(v) + v) = 0$ or $u^2[r, jk][w^2, d_2(v) + v] = 0$ for all $u, v, j, k, w \in G, r \in S$. If $d_1(d_2(v) + v) = 0$. As $d_1 \neq 0$, therefore $d_2(v) + v = 0$. Secondly suppose that $u^2[r, jk][w^2, d_2(v) + v] = 0$. By Lemma 5, we have either $u^2 = 0$ or $[w^2, d_2(v) + v] = 0$. But $u^2 = 0$ implies that $G = \{0\}$, a contradiction. Therefore we consider the case when $[w^2, d_2(v) + v] = 0$, which implies, by Lemma 4, that $[d_2(v) + v, r] = 0, \forall r \in S$ and taking in particular $t = v \in G$, we have

$$[d_2(v), v] + [v, v] = 0 \tag{2.52}$$

Also by definition of MA-semirings, we have $[v, v] = [v, v]'$. Therefore $[d_2(v), v] + [v, v]' = 0$ and therefore

$$[d_2(v), v] = [v, v] \tag{2.53}$$

Using (2.53) into (2.52) and then using 2-torsion freeness of S, we obtain $[d(v), v] = 0$. By Theorem 2.2 of [22], we conclude that S is commutative. Therefore (2.47) will be rewritten as $2d_1(u)d_2(v) + 2(d_1(v')u + v'd_1(u)) = 0$ and hence by the 2-torsion freeness of S, we obtain

$$d_1(u)d_2(v) + d_1(v')u + v'd_1(u) = 0 \tag{2.54}$$

In (2.54) replacing u by $2uw$ and using 2-torsion freeness of S, we get

$$d_1(u)wd_2(v) + ud_1(w)d_2(v) + d_1(v')uw + v'd_1(u)w + v'ud_1(w) = 0$$

and therefore

$$w(d_1(u)d_2(v) + d_1(v^{'})u + v^{'}d_1(u)) + ud_1(w)d_2(v) + v^{'}ud_1(w) = 0$$

Using (2.54) again, we obtain

$$ud_1(w)d_2(v) + v^{'}ud_1(w) = 0 \tag{2.55}$$

In (2.55) replacing v by $2vz$, we get

$$ud_1(w)d_2(v)z + ud_1(w)vd_2(z) + v^{'}zud_1(w) = 0$$

and therefore

$$z(ud_1(w)d_2(v) + v^{'}ud_1(w)) + ud_1(w)vd_2(z) = 0$$

and using (2.55) again, we get $d_1(w)uGd_2(z) = \{0\}$. By the above Lemma 2,we have either $d_1(w)u = 0$ or $d_2(z) = 0$ and therefore by Remark 1, we have either $d_1(w) = 0$ or $d_2(z) = 0$. As $d_1 \neq 0$, therefore $d_2 = 0$. Therefore our hypothesis becomes $d_1(u \circ v) = 0$ and therefore $d_1(u^2) = 0$, $\forall u \in G$. By Lemma 3, $d_1 = 0$ a contraction to the assumption. Hence d_1 is zero. □

3. Open problem

We have proved the results of this paper for prime semirings and it would be interesting to generalize them for semiprime semirings, we leave it as an open problem.

Acknowledgments

Taif University Researchers Supporting Project number (TURSP-2020/154), Taif University Taif, Saudi Arabia.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. K. Glazek, *A guide to the literature on semirings and their applications in mathematics and information sciences with complete bibliography*, Springer, 2002.
2. P. Kostolányi, F. Mišún, Alternating weighted automata over commutative semirings, *Theor. Comput. Sci.*, **740** (2018), 1–27.
3. U. Hebisch, H. J. Weinert, *Semirings: Algebraic theory and applications in computer science*, World Scientific Publishing Company, 1998.
4. V. N. Kolokoltsov, V. P. Maslov, *Idempotent analysis and its applications*, Dordrecht: Kluwer Acad. Publ., 1997.
5. V. P. Maslov, S. N. Samborskii, *Idempotent analysis*, RI: American Mathematical Society, 1992.

6. M. A. Javed, M. Aslam, M. Hussain, On condition (A_2) of Bandlet and Petrich for inverse semiqrings, *Int. Math. Forum*, **7** (2012), 2903–2914.
7. S. Shafiq, M. Aslam, M. A. Javed, On centralizer of semiprime inverse semiring, *Discuss. Math. Gen. Algebra Appl.*, **36** (2016), 71–84.
8. Y. A. Khan, W. A. Dudek, Stronger Lie derivations on MA-semirings, *Afr. Mat.*, **31** (2020), 891–901.
9. L. Ali, Y. A. Khan, A. A. Mousa, S. A. Khalek, G. Farid, Some differential identities of MA-semirings with involution, *AIMS Mathematics*, **6** (2020), 2304–2314.
10. L. Ali, M. Aslam, M. I. Qureshi, Y. A. Khan, S. Ur Rehman, G. Farid, Commutativity of MA-semirings with involution through generalized derivations, *J. Math.*, **2020** (2020), 8867247.
11. L. Ali, M. Aslam, Y. A. Khan, On Jordan ideals of inverse semirings with involution, *Indian J. Sci. Technol.*, **13** (2020), 430–438.
12. L. Ali, M. Aslam, Y. A. Khan, G. Farid, On generalized derivations of semirings with involution, *J. Mech. Continua Math. Sci.*, **15** (2020), 138–152.
13. R. Awtar, Lie and Jordan structure in prime rings with derivations, *P. Am. Math. Soc.*, **41** (1973), 67–74.
14. H. E. Bell, W. S. Martindale, Centralizing mappings of semiprime rings, *Can. Math. Bull.*, **30** (1987), 92–101.
15. J. Berger, I. N. Herstein, J. W. Kerr, Lie ideals and derivations of prime rings, *J. Algebra*, **71** (1981), 259–267.
16. B. E. Johnson, Continuity of derivations on commutative Banach algebras, *Am. J. Math.*, **91** (1969), 1–10.
17. D. A. Jordan, On the ideals of a Lie algebra of derivations, *J. Lond. Math. Soc.*, **2** (1986), 33–39.
18. L. Oukhtite, A. Mamouni, Derivations satisfying certain algebraic identities on Jordan ideals, *Arab. J. Math.*, **1** (2012), 341–346.
19. E. C. Posner, Derivations in prime rings, *P. Am. Math. Soc.*, **8** (1957), 1093–1100.
20. S. Shafiq, M. Aslam, Jordan and Lie ideals of inverse semirings, *Asian-Eur J. Math.*, **2021** (2021), 2150181.
21. L. Oukhtite, A. Mamouni, C. Beddani, Derivations and Jordan ideals in prim rings, *J. Taibah Uni. Sci.*, **8** (2014), 364–369.
22. L. Ali, Y. A. Khan, M. Aslam, On Posner's second theorem for semirings with involution, *JDMSC*, **23** (2020), 1195–1202.
23. I. N. Herstein, On the Lie and Jordan rings of simple associative ring, *Am. J. Math.*, **77** (1955), 279–285.

AIMS Press

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)