Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(7): 6749–6780.
DOI: 10.3934/math.2021397
Received: 13 December 2020
Accepted: 07 April 2021
Published: 21 April 2021

Research article

Langevin equation with nonlocal boundary conditions involving a ψ-Caputo fractional operators of different orders

Arjumand Seemab[1], **Mujeeb ur Rehman**[1], **Jehad Alzabut**[2,3,*], **Yassine Adjabi**[4,5] **and Mohammed S. Abdo**[6]

[1] Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan
[2] Department of Mathematics and General Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
[3] Group of Mathematics, Faculty of Engineering, Ostim Technical University, Ankara 06374, Turkey
[4] Department of Mathematics, Faculty of Sciences, University of M'hamed Bougara, U.M.B.B., Algeria
[5] Dynamic Systems Laboratory, Faculty of Mathematics, U.S.T.H.B., Algeria
[6] Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen

* **Correspondence:** Email: jalzabut@psu.edu.sa; Tel: +00966543431972; Fax: +00966114548317.

Abstract: This paper studies Langevin equation with nonlocal boundary conditions involving a ψ-Caputo fractional operators of different orders. By the aid of fixed point techniques of Krasnoselskii and Banach, we derive new results on existence and uniqueness of the problem at hand. Further, a new ψ-fractional Gronwall inequality and ψ-fractional integration by parts are employed to prove Ulam-Hyers and Ulam-Hyers-Rassias stability for the solutions. Examples are provided to demonstrate the advantage of our major results. The proposed results here are more general than the existing results in the literature which can be obtained as particular cases.

Keywords: generalized fractional operators; ψ-Caputo derivative; ψ-fractional Langevin type equation; gxistence and uniqueness; U-H stability type; Krasnoselskii fixed point theorem; ψ-fractional Gronwall inequality
Mathematics Subject Classification: 34A08, 26A33, 34A12, 34D20

1. Introduction

Lately, fractional calculus has played a very significant role in various scientific fields; see for instance [1, 2] and the references cited therein. As a result of this, fractional differential equations have caught the attention of many investigators working in different desciplines [3–11]. However, most of researchers works have been conducted by using fractional derivatives that mainly rely on Riemann-Liouville, Hadamard, Katugampola, Grunwald Letnikov and Caputo approaches.

Fractional derivatives of a function with respect to another function have been considered in the classical monographs [1, 12] as a generalization of Riemann-Liouville derivative. This fractional derivative is different from the other classical fractional derivative as the kernel appears in terms of another function ψ. Thus, this type of derivative is referred to as ψ-fractional derivative. Recently, this derivative has been reconsidered by Almeida in [14] where the Caputo-type regularization of the existing definition and some interesting properties are provided. Several properties of this operator could be found in [1, 12, 13, 15, 16]. For some particular cases of ψ, one can realize that ψ-fractional derivative can be reduced to the Caputo fractional derivative [1], the Caputo-Hadamard fractional derivative [17] and the Caputo-Erdélyi-Kober fractional derivative [18].

On the other hand, the investigation of qualitative properties of solutions for different fractional differential (and integral) equations is the key theme of applied mathematics research. Numerous interesting results concerning the existence, uniqueness, multiplicity, and stability of solutions or positive solutions by applying some fixed point techniques are obtained. However, most of the proposed problems have been handled concerning the classical fractional derivatives of the Riemann-Liouville and Caputo [19–29].

In parallel with the intensive investigation of fractional derivative, a normal generalization of the Langevin differential equation appears to be replacing the classical derivative by a fractional derivative to produce fractional Langevin equation (FLE). FLE was first introduced in [30] and then different types of FLE were the object of many scholars [31–40]. In particular, the authors studied a nonlinear FLE involving two fractional orders on different intervals with three-point boundary conditions in [40], whereas FLE involving a Hadamard derivative type was considered in [33–35].

Alternatively, the stability problem of differential equations was discussed by Ulam in [41]. Thereafter, Hyers in [42] developed the concept of Ulam stability in the case of Banach spaces. Rassias provided a fabulous generalization of the Ulam-Hyers stability of mappings by taking into account variables. His approach was refered to as Ulam-Hyers-Rassias stability [43]. Recently, the Ulam stability problem of implicit differential equations was extended into fractional implicit differential equations by some authors [44–47]. A series of papers was devoted to the investigation of existence, uniqueness and U-H stability of solutions of the FLE within different kinds of fractional derivatives.

Motivated by the recent developments on ψ-fractional calculus, in the present work, we investigate the existence, uniqueness and stability in the sense UlamHyersRassias of solutions for the following FLE within ψ-Caputo fractional derivatives of different orders involving nonlocal boundary conditions

$$\begin{cases} \left({}^c\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)\left({}^c\mathfrak{D}^{\beta,\psi}_{a+,t}+\lambda\right)[u] = \mathfrak{F}(t,u(t),{}^c\,\mathfrak{D}^{\gamma,\psi}_{a+,t}[u]),\ t\in(a,T), \\ u(a)=0,\ u(\eta)=0,\ u(T)=\mu\left(J^{\gamma,\psi}_{a+,\xi}\right)[u],\ \mu>0, \end{cases} \tag{1.1}$$

where $\left(J^{\gamma,\psi}_{a+,\xi}\right)$ and $\left({}^c\mathfrak{D}^{\theta,\psi}_{a+,t}\right)$ are ψ-fractional integral of order γ, ψ-Caputo fractional derivative of order $\theta\in\{\alpha,\beta,\gamma\}$ respectively, $0\le a<\eta<\xi<T<\infty$, $1<\alpha\le 2$, $0<\gamma<\beta\le 1$, λ is a real number and $\mathfrak{F}:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ is a continuous function. We observe that problem (1.1) is designated within a general platform in the sense that general fractional derivative is considered with respect to different fractional orders, the forcing function depends on the general fractional derivative and boundary conditions involve integral fractional operators. Furthermore, the stability analysis in the sense of Ulam is investigated by the help of new versions of ψ-fractional Gronwall inequality and ψ-fractional integration by parts. It is worth mentioning here that the proposed results in this paper which rely on ψ-fractional integrals and derivatives can generalize the existing results in the literature [31,40] and obtain them as particular cases.

The major contributions of the work are as follows: Some lemmas and definitions on ψ-fractional calculus theory are recalled in Section 2. In Section 3, we prove the existence and uniqueness of solutions for problem (1.1) via applying fixed point theorems. Section 4 devotes to discuss different types of stability results for the problem (1.1) by the help of generalized ψ-Gronwall's inequality [49] and ψ-fractional integration by parts. The proposed results are examined via Maple using several numerical examples for different values of function ψ, presented in several tables in Section 5, to check the applicability of the theoretical findings. We end the paper by a conclusion in Section 6.

2. Preliminaries and essential lemmas

The standard Riemann-Liouville fractional integral of order α, $\Re(\alpha) > 0$, has the form

$$\left(J^{\alpha}_{a+,t}\right)[u] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(\tau)\, \mathrm{d}\tau, \ t > a.$$

The left-sided factional integrals and fractional derivatives of a function u with respect to another function ψ in the sense of Riemann-Liouville are defined as follows [13,14]

$$\left(J^{\alpha,\psi}_{a+,t}\right)[u] = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)\,(\psi(t)-\psi(\tau))^{\alpha-1} u(\tau)\, \mathrm{d}\tau,$$

and

$$\left(\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)[u] = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(J^{n-\alpha,\psi}_{a+,t}\right)[u],$$

respectively, where $n = [\alpha] + 1$.

Analogous formulas can be offered for the right fractional (integral and derivative) as follows:

$$\left(J^{\alpha,\psi}_{t,b-}\right)[u] = \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(\tau)\,(\psi(\tau)-\psi(t))^{\alpha-1} u(\tau)\, \mathrm{d}\tau,$$

and

$$\left(\mathfrak{D}^{\alpha,\psi}_{t,b-}\right)[u] = \left(-\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(J^{n-\alpha,\psi}_{t,b-}\right)[u].$$

The left (right) ψ-Caputo fractional derivatives of u of order α are given by

$$\left({}^c\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)[u] = \left(J^{n-\alpha,\psi}_{a+,t}\right)\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n [u]$$

and

$$\left({}^c\mathfrak{D}^{\alpha,\psi}_{t,b-}\right)[u] = \left(J^{n-\alpha,\psi}_{t,b-}\right)\left(\frac{-1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n [u],$$

respectively. In particular, when $\alpha \in (0,1)$, we have

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)[u] = \left(J_{a+,t}^{1-\alpha,\psi}\right)\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)[u]$$

and

$$\left({}^{c}\mathfrak{D}_{t,b-}^{\alpha,\psi}\right)[u] = -\left(J_{t,b-}^{1-\alpha,\psi}\right)\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)[u],$$

where $u, \psi \in C^n[a,b]$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$.

Remark 2.1. We propose the remarkable paper [16] in which some generalizations using ψ-fractional integrals and derivatives are described. In particular, we have

$$\begin{cases} if\ \ \psi(t) \longrightarrow t, \quad \text{then } J_{a+,t}^{\alpha,\psi} \longrightarrow J_{a+,t}^{\alpha}, \\ if\ \ \psi(t) \longrightarrow \ln t, \quad \text{then } J_{a+,t}^{\alpha,\psi} \longrightarrow {}^{H}J_{a+,t}^{\alpha}, \\ if\ \ \psi(t) \longrightarrow t^{\rho}, \quad \text{then },\ J_{a+,t}^{\alpha,\psi} \longrightarrow {}^{\rho}J_{a+,t}^{\alpha},\ \rho > 0, \end{cases}$$

where $J_{a+,t}^{\alpha}, {}^{H}J_{a+,t}^{\alpha}, {}^{\rho}J_{a+,t}^{\alpha}$ are classical Riemann-Liouville, Hadamard, and Katugampola fractional operators.

Lemma 2.2. *[14] Given $u \in C([a,b])$ and $v \in C^n([a,b])$, we have that for all $\alpha > 0$*

$$\begin{aligned} \int_a^b v(\tau)\left({}^{c}\mathfrak{D}_{a+,\tau}^{\alpha,\psi}\right)[u]\,\mathrm{d}\tau &= \int_a^b u(\tau)\left({}^{c}\mathfrak{D}_{\tau,b-}^{\alpha,\psi}\right)\left[\frac{v}{\psi'}\right]\frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\tau)\ \mathrm{d}\tau \\ &\quad + \sum_{k=0}^{n-1}\left(-\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \left(J_{\tau,b-}^{n-\alpha,\psi}\right)\left[\frac{v}{\psi'}\right] u_{\psi}^{[n-k-1]}(\tau)\Bigg|_{\tau=a}^{\tau=b}, \end{aligned}$$

where

$$u_{\psi}^{[k]}(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k u(t).$$

Lemma 2.3. *[1] Let $\alpha > 0$ and $u,\ \psi \in C([a,b])$. Then*

$$\left\|J_{a+,t}^{\alpha,\psi}[u]\right\|_C \leq K_\psi \|u\|_C,\ K_\psi = \frac{1}{\Gamma(1+\alpha)}(\psi(b)-\psi(a))^{\alpha}.$$

For all $n-1 < \alpha < n$,

$$\left\|{}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}[u]\right\|_C \leq K_\psi \|u\|_{C^{[n]}\psi},\ K_\psi = \frac{1}{\Gamma(n+1-\alpha)}(\psi(b)-\psi(a))^{n-\alpha}$$

where $\|\cdot\|_C$ is the Chebyshev norm defined on $C([a,b])$.

The following results are well known and one can see [1, 14] for further details.

Lemma 2.4. *[1] Let $\alpha,\beta > 0$, consider the functions*

$$\left(J_{a+,t}^{\alpha,\psi}\right)\left[(\psi(\tau)-\psi(a))^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\psi(t)-\psi(a))^{\alpha+\beta-1},$$

$$\left(J_{a+,t}^{\alpha,\psi}\right)[1] = \frac{1}{\Gamma(1+\alpha)}\left(\psi(t)-\psi(a)\right)^{\alpha}$$

and

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)\left[\left(\psi(\tau)-\psi(a)\right)^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\psi(t)-\psi(a)\right)^{\beta-\alpha-1},$$

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)[1] = \frac{1}{\Gamma(1-\alpha)}\left(\psi(t)-\psi(a)\right)^{-\alpha},\ \alpha>0.$$

Note that

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)\left[\left(\psi(\tau)-\psi(a)\right)^{k}\right] = 0,\ k=0,..,n-1.$$

The subsequent properties are valid: If $\alpha,\beta>0$, then

$$\left(J_{a+,t}^{\alpha,\psi}\right)\left(J_{a+,t}^{\beta,\psi}\right)[u] = \left(J_{a+,t}^{\alpha+\beta,\psi}\right)[u] \text{ and } \left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)\left({}^{c}\mathfrak{D}_{a+,t}^{\beta,\psi}\right)[u] = \left({}^{c}\mathfrak{D}_{a+,t}^{\alpha+\beta,\psi}\right)[u],$$

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)\left(J_{a+,t}^{\beta,\psi}\right)[u] = \left(J_{a+,t}^{\beta-\alpha,\psi}\right)[u]. \tag{2.1}$$

Lemma 2.5. *[1] Given a function $u\in C^{n}[a,b]$ and $\alpha>0$, we have*

$$J_{a+,t}^{\alpha,\psi}\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)[u] = u(t) - \sum_{j=0}^{n-1}\left[\frac{1}{j!}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j}u(a)\right]\left(\psi(t)-\psi(a)\right)^{j}.$$

In particular, given $\alpha\in(0,1)$, we have

$$J_{a+,t}^{\alpha,\psi}\left({}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\right)[u] = u(t)-u(a).$$

Lemma 2.6. *Given a function $u\in C^{n}[a,b]$ and $1>\alpha>0$, we have*

$$\left\|\left(J_{a+,t_2}^{\alpha,\psi}\right)[u] - \left(J_{a+,t_1}^{\alpha,\psi}\right)[u]\right\| \le \frac{2\left\|u\right\|_{\infty}}{\Gamma(\alpha+1)}\left(\psi(t_2)-\psi(t_1)\right)^{\alpha}.$$

Proof. Using Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
\left|\left(J_{a+,t_2}^{\alpha,\psi}\right)[u] - \left(J_{a+,t_1}^{\alpha,\psi}\right)[u]\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_2}\psi'(\tau)\left[\left(\psi(t_2)-\psi(\tau)\right)^{\alpha-1} - \left(\psi(t_1)-\psi(\tau)\right)^{\alpha-1}\right]u(\tau)\,\mathrm{d}\tau\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\psi'(\tau)\left(\psi(t_2)-\psi(\tau)\right)^{\alpha-1}u(\tau)\,\mathrm{d}\tau\right| \\
&\le \frac{\left\|u\right\|_{\infty}}{\Gamma(\alpha+1)}\left[\left(\psi(t_2)-\psi(t_1)\right)^{\alpha} + \left(\psi(t_1)-\psi(a)\right)^{\alpha} - \left(\psi(t_2)-\psi(a)\right)^{\alpha}\right] \\
&\quad + \frac{\left\|u\right\|_{\infty}}{\Gamma(\alpha+1)}\left(\psi(t_2)-\psi(t_1)\right)^{\alpha} \\
&\le \frac{2\left\|u\right\|_{\infty}}{\Gamma(\alpha+1)}\left(\psi(t_2)-\psi(t_1)\right)^{\alpha}.
\end{aligned}$$

□

Now we state here two important fixed point theorems, namely Banach and Krasnoselskii's fixed point theorems. These will help us to develop sufficient conditions for the existence and uniqueness of solutions.

Theorem 2.7. *[48] Let $\mathcal{B}_r$ be the closed ball of radius $r > 0$, centred at zero, in a Banach space X with $\Upsilon : \mathcal{B}_r \to X$ a contraction and $\Upsilon(\partial \mathcal{B}_r) \subseteq \mathcal{B}_r$. Then, Υ has a unique fixed point in $\mathcal{B}_r$.*

Theorem 2.8. *[48] Let $\mathcal{M}$ be a closed, convex, non-empty subset of a Banach space $X \times X$. Suppose that $\mathbb{E}$ and $\mathbb{F}$ map $\mathcal{M}$ into X and that*

(i) $\mathbb{E}u + \mathbb{F}v \in \mathcal{M}$ *for all* $u, v \in \mathcal{M}$;
(ii) $\mathbb{E}$ *is compact and continuous;*
(iii) $\mathbb{F}$ *is a contraction mapping.*

Then the operator equation $\mathbb{E}w + \mathbb{F}w = w$ has at least one solution on $\mathcal{M}$.

Definition 2.9. The problem (1.1) is U-H stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\tilde{u} \in C([a, T])$ of the inequality

$$\left| \left({}^{c}\mathfrak{D}^{\alpha,\psi}_{a+,t} \right) \left({}^{c}\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda \right) [\tilde{u}] - \mathfrak{F}(t, u(t), {}^{c}\mathfrak{D}^{\gamma,\psi}_{a+,t} [\tilde{u}]) \right| \leq \epsilon, \ t \in [a, T], \tag{2.2}$$

there exists a solution $u \in C[a, T]$ of the problem (1.1) with

$$|\tilde{u}(t) - u(t)| \leq \epsilon c_f.$$

Definition 2.10. The problem (1.1) is generalized U-H stable if there exists $\Phi(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\Phi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $\tilde{u} \in C[a, T]$ of inequality (2.2), there exists a solution $u \in C[a, T]$ of problem (1.1) with

$$|\tilde{u}(t) - u(t)| \leq \Phi(\epsilon), \ t \in [a, T],$$

where $\Phi(\epsilon)$ is only dependent on ϵ.

Definition 2.11. The problem (1.1) is U-H-R stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $\tilde{u} \in C[a, T]$ of the inequality

$$\left| \left({}^{c}\mathfrak{D}^{\alpha,\psi}_{a+,t} \right) \left({}^{c}\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda \right) [\tilde{u}] - \mathfrak{F}(t, u(t), {}^{c}\mathfrak{D}^{\gamma,\psi}_{a+,t} [\tilde{u}]) \right| \leq \epsilon\Phi(t), \ t \in [a, T],$$

there exists a solution $u \in C[a, T]$ of the problem (1.1) with

$$|\tilde{u}(t) - u(t)| \leq \epsilon c_f \Phi(t).$$

Definition 2.12. The problem (1.1) is generalized U-H-R stable with respect to Φ if there exists $c_f > 0$ such that for each solution $\tilde{u} \in C[a, T]$ of the inequality

$$\left| {}^{c}\mathfrak{D}^{\alpha,\psi}_{a+,t} \left({}^{c}\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda \right) [u] - \mathfrak{F}(t, u(t), {}^{c}\mathfrak{D}^{\gamma,\psi}_{a+,t} [u]) \right| \leq \Phi(t), \ t \in [a, T],$$

there exists a solution $u \in C[a, T]$ of the problem (1.1) with

$$|\tilde{u}(t) - u(t)| \leq c_f \Phi(t).$$

We adopt the following conventions:

$$\mathfrak{F}_u(t) = \mathfrak{F}(t, u(t), {}^{c}\mathfrak{D}^{\gamma,\psi}_{a+,t}[u]) \quad \text{and} \quad \mathcal{K}(t;a) = \psi(t) - \psi(a).$$

We remark that, the following generalized ψ-Gronwall Lemma is an important tool in proving the main results of this paper.

Lemma 2.13. *[49] Let u, v be two integrable functions on $[a,b]$. Let $\psi \in C^1[a,b]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall\, t \in [a,b]$. Assume that*

(i) u and v are nonnegative;
(ii) The functions $(g_i)_{i=1\cdots n}$ are bounded and monotonic increasing functions on $[a,b]$;
(iii) The constants $\alpha_i > 0$ $(i = 1,2,\ldots,n)$. If

$$u(t) \le v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i - 1} u(\tau)\, d\tau,$$

then

$$u(t) \le v(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots,k'=1}^{n} \frac{\prod_{i=1}^{k}(g_{i'}(t)\Gamma(\alpha_{i'}))}{\Gamma(\sum_{i=1}^{k}\alpha_{i'})} \int_a^t \left[\psi'(\tau)(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\right] v(\tau)\, d\tau\right).$$

Remark 2.14. [49] For $n = 2$ in the hypotheses of Lemma 2.13. Let $v(t)$ be a nondecreasing function for $a \le t < T$. Then we have

$$u(t) \le v(t)\left[E_{\alpha_1}\left(g_1(t)\Gamma(\alpha_1)(\mathcal{K}(t;a))^{\alpha_1}\right) + E_{\alpha_2}\left(g_2(t)\Gamma(\alpha_2)(\mathcal{K}(t;a))^{\alpha_2}\right)\right],$$

where $E_{\alpha_i}(i = 1,2)$ is the Mittag-Leffler function defined below.

Definition 2.15. [50] The Mittag-Leffler function is given by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{2.3}$$

where $\Re(\alpha) > 0$ and $\Gamma(z)$ is a Gamma function. In particular, if $\alpha = 1/2$ in (2.3) we have

$$E_{1/2}(z) = \exp\left(z^2\right)[1 + erf(z)],$$

where $erf(z)$ is the error function.

In the remaining portion of the paper, we make use of the next suppositions:

(A$_1$) For each $t \in [a,T]$, there exist a constant $L_i > 0$ $(i = 1,2)$ such that

$$|\mathfrak{F}(t,u_1,v_1) - \mathfrak{F}(t,u_2,v_2)| \le L_1|u_1 - v_2| + L_2|u_1 - v_2|, \text{ for all } u_i, v_i \in \mathbb{R}.$$

(A$_2$) There exists an increasing function $\chi(t) \in (C[a,T], \mathbb{R}^+)$, for any $t \in [a,T]$,

$$|\mathfrak{F}(t,u,v)| \le \chi(t),\ u, v \in \mathbb{R}.$$

(A$_3$) There exist a constant $L > 0$ such that

$$|\mathfrak{F}(t,u,v)| \leq L, \text{ for any } t \in [a,T], \ u,v \in \mathbb{R}.$$

(A$_4$) There exists an function $\Phi(t) \in (C[a,T], \mathbb{R}^{+})$ and there exists $l_{\alpha,\psi} > 0$ such that for any $t \in [a,T]$,

$$\left(J^{\alpha,\psi}_{a+,t}\right)[\Phi] \leq l_{\alpha,\psi}\Phi(t), \ \alpha > 0.$$

Denoting

$$\sigma_{11} = \left(J^{\beta,\psi}_{a_+,\eta}\right)[1] \text{ and } \sigma_{12} = \left(J^{\beta,\psi}_{a_+,\eta}\right)[\mathcal{K}(\tau;a)],$$

and

$$\sigma_{21} = \left(J^{\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\beta+\gamma,\psi}_{a_+,\xi}\right)[1] \text{ and } \sigma_{22} = \left(\left(J^{\beta,\psi}_{a_+,T}\right)[\mathcal{K}(\tau;a)] - \mu\left(J^{\beta+\gamma,\psi}_{a_+,\xi}\right)[\mathcal{K}(\tau;a)]\right).$$

Further, we assume

$$|\sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12}| \neq 0,$$

where σ_{ij} are constants.

3. Existence and uniqueness of solution

In order to study the nonlinear FLE (1.1), we first consider the linear associated FLE and conclude the form of the solution.

3.1. Linear boundary problem

The following lemma regards a linear variant of problem

$$\begin{cases} \left({}^{c}\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)\left({}^{c}\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda\right)[u] = F(t), \ t \in (a,T), \\ u(a) = 0, \ u(\eta) = 0, \ u(T) = \mu\left(J^{\gamma,\psi}_{a+,\xi}\right)[u], \ a < \eta < \xi < T, \end{cases} \tag{3.1}$$

where $F \in C([a,T], \mathbb{R})$.

Lemma 3.1. *The unique solution of the ψ-Caputo linear problem (3.1) is given by the integral equation*

$$\begin{aligned} u(t) = & -\lambda\left(J^{\beta,\psi}_{a+,t}\right)[u] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[F] \\ & + \frac{(\mathcal{K}(t;a))^{\beta}(\mathcal{K}(t;\eta)}{\Gamma(\beta+2)\Delta}\left\{\left(J^{\alpha+\beta,\psi}_{a+,T}\right)[F] - \lambda\left(J^{\beta,\psi}_{a+,T}\right)[u] - \mu\left(J^{\alpha+\beta+\gamma,\psi}_{a+,\xi}\right)[F] + \mu\lambda\left(J^{\beta+\gamma,\psi}_{a+,\xi}\right)[u]\right\} \\ & - \frac{(\mathcal{K}(t;a))^{\beta}}{\Delta(\mathcal{K}(\eta;a))^{\beta}}\left(\frac{(\mathcal{K}(T;a))^{\beta}(\mathcal{K}(T;t))}{\Gamma(\beta+2)} - \frac{\mu(\mathcal{K}(\xi;a))^{\beta+\gamma}[(\beta+1)(\mathcal{K}(\xi;t)) - \gamma(\mathcal{K}(t;a))]}{\Gamma(\beta+\gamma+2)(\beta+1)}\right) \\ & \times\left\{\left(J^{\alpha+\beta,\psi}_{a+,\eta}\right)[F] - \lambda\left(J^{\beta,\psi}_{a+,\eta}\right)[u]\right\}, \end{aligned} \tag{3.2}$$

where

$$\Delta = \left[\frac{(\mathcal{K}(T;a))^{\beta}\mathcal{K}(T;\eta)}{\Gamma(\beta+2)} - \frac{\mu(\mathcal{K}(\xi;a))^{\beta+\gamma}[(\beta+1)\mathcal{K}(\xi;\eta) - \gamma\,\mathcal{K}(\eta;a)]}{\Gamma(\beta+\gamma+2)(\beta+1)}\right] \neq 0. \tag{3.3}$$

Proof. Applying $\left(J_{a+,t}^{\alpha,\psi}\right)$ on both sides of (3.1-a), we have

$$\left({}^{c}\mathfrak{D}_{a+,t}^{\beta,\psi}+\lambda\right)[u]=\left(J_{a+,t}^{\alpha,\psi}\right)[F]+c_1+c_2(\psi(t)-\psi(a)), \tag{3.4}$$

for $c_1,\ c_2\in\mathbb{R}$.
Now applying $\left(J_{a+,t}^{\beta,\psi}\right)$ to both sides of (3.4) , we get

$$u(t)=-\lambda\left(J_{a+,t}^{\beta,\psi}\right)[u]+\left(J_{a+,t}^{\alpha+\beta,\psi}\right)[F]+c_1\left(J_{a+,t}^{\beta,\psi}\right)[1]+c_2\left(J_{a+,t}^{\beta,\psi}\right)[\mathcal{K}(\tau;a)]+c_3,$$

where $c_3\in\mathbb{R}$.
Using the boundary conditions in (3.1-b), we obtain $c_3:=c_3(F)=0$ and

$$J_{a+,t}^{\delta,\psi}[u]=-\lambda\left(J_{a+,t}^{\beta+\delta,\psi}\right)[u]+\left(J_{a+,t}^{\alpha+\beta+\delta,\psi}\right)[F]+c_1\left(J_{a+,t}^{\beta+\delta,\psi}\right)[1]+c_2\left(J_{a+,t}^{\beta+\delta,\psi}\right)[\mathcal{K}(\tau;a)]. \tag{3.5}$$

Further, we get a system of linear equations with respect to c_1, c_2 as follows

$$\begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{21}&\sigma_{22}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\end{pmatrix},$$

where

$$b_1=\lambda\left(J_{a_+,\eta}^{\beta,\psi}\right)[u]-\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[F]$$

and

$$b_2=\lambda\left(\left(J_{a_+,T}^{\beta,\psi}\right)[u]-\mu\left(J_{a_+,\xi}^{\beta+\delta,\psi}\right)[u]\right)-\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[F]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[F]\right).$$

We note

$$\Delta\equiv\det(\sigma)=|\sigma_{11}\sigma_{22}-\sigma_{21}\sigma_{12}|.$$

Because the determinant of coefficients for $\Delta\neq 0$. Thus, we have

$$c_1:=c_1(F)=\frac{\sigma_{22}b_1-\sigma_{12}b_2}{\Delta}\text{ and }c_2:=c_2(F)=\frac{\sigma_{11}b_2-\sigma_{21}b_1}{\Delta}.$$

Substituting these values of c_1 and c_2 in (3.5), we finally obtain (3.2) as

$$u(t)=-\lambda\left(J_{a+,t}^{\beta,\psi}\right)[u]+\left(J_{a+,t}^{\alpha+\beta,\psi}\right)[F]+\tfrac{\sigma_{22}b_1-\sigma_{12}b_2}{\Delta}\left(J_{a+,t}^{\beta,\psi}\right)[1]+\tfrac{\sigma_{11}b_2-\sigma_{21}b_1}{\Delta}\left(J_{a+,t}^{\beta,\psi}\right)[\mathcal{K}(\tau;a)]. \tag{3.6}$$

That is, the integral equation (3.6) can be written as (3.2) and

$$\begin{aligned}\left(J_{a+,t}^{\delta,\psi}\right)[u]&=-\lambda\left(J_{a+,t}^{\beta+\delta,\psi}\right)[u]\\ &+\left(J_{a+,t}^{\alpha+\beta+\delta,\psi}\right)[F]+\tfrac{\sigma_{22}b_1-\sigma_{12}b_2}{\Delta}\left(J_{a+,t}^{\beta+\delta,\psi}\right)[1]+\tfrac{\sigma_{11}b_2-\sigma_{21}b_1}{\Delta}\left(J_{a+,t}^{\beta+\delta,\psi}\right)[\mathcal{K}(\tau;a)].\end{aligned}$$

Differentiating the above relations one time we obtain (3.1-a), also it is easy to get that the condition (3.1-b) is satisfied. The proof is complete. □

For convenience, we define the following functions

$$d_{11}(t)=-\tfrac{1}{\Delta}\left(\sigma_{22}\left(J_{a+,t}^{\beta,\psi}\right)[1]-\sigma_{21}\left(J_{a+,t}^{\beta,\psi}\right)[\mathcal{K}(\tau;a)]\right),\quad d_{21}(t)=-d_{11}(t) \tag{3.7}$$

and

$$d_{12}(t)=\tfrac{1}{\Delta}\left(\sigma_{12}\left(J_{a+,t}^{\beta,\psi}\right)[1]-\sigma_{11}\left(J_{a+,t}^{\beta,\psi}\right)[\mathcal{K}(\tau;a)]\right),\quad d_{22}(t)=d_{12}(t). \tag{3.8}$$

3.2. Nonlinear problem

The following result is an immediate consequence of Lemma 3.1.

Lemma 3.2. *Let* $\lambda \in \mathbb{R}$. *Then problem (1.1) is equivalent to the integral equation*

$$u(t) = -\lambda \left(J_{a_+,t}^{\beta,\psi}\right)[u] + \left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u] + \phi_u(\mathfrak{F}), \tag{3.9}$$

where

$$\begin{aligned}\phi_u(\mathfrak{F}) &= d_{11}(t)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u] + d_{12}(t)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u] - \mu\left(J_{a_+,\xi}^{\alpha+\beta+\gamma,\psi}\right)[\mathfrak{F}_u]\right) \\ &\quad +\lambda d_{21}(t)\left(J_{a_+,\eta}^{\beta,\psi}\right)[u] - \lambda d_{22}(t)\left(\left(J_{a_+,T}^{\beta,\psi}\right)[u] - \mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[u]\right)\end{aligned} \tag{3.10}$$

and d_{ij} *are defined in (3.7) and (3.8).*

From the expression of (1.1-a) and (3.9), we can see that if all the conditions in Lemmas 3.1 and 3.2 are satisfied, the solution is a $C[a,T]$ solution of the ψ-Caputo fractional boundary value problem (1.1).

In order to lighten the statement of our result, we adopt the following notation.

$$\varsigma_{11} = \sup_{t\in[a,T]} \left|\lambda\left(J_{a_+,t}^{\beta,\psi}\right)[1] + \rho_{11} + L_1\left(\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[1] + \rho_{12}\right)\right|, \tag{3.11}$$

$$\varsigma_{12} = L_2\varsigma_{13} \text{ where } \varsigma_{13} = \sup_{t\in[a,T]} \left|\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[1]\right| + \rho_{12}, \tag{3.12}$$

$$\varsigma_{21} = \sup_{t\in[a,T]} \left|\lambda\left(J_{a+,t}^{\beta-\gamma,\psi}\right)[1] + \rho_{21} + L_1\left(\left(J_{a+,t}^{\alpha+\beta-\gamma,\psi}\right)[1] + \rho_{22}\right)\right|, \tag{3.13}$$

$$\varsigma_{22} = L_2\varsigma_{23} \text{ where } \varsigma_{23} = \sup_{t\in[a,T]} \left|\left(J_{a+,t}^{\alpha+\beta-\gamma,\psi}\right)[1]\right| + \rho_{22},$$

with

$$\rho_{11} = |\lambda| \sup_{t\in[a,T]} \left(|d_{21}(t)|\left(J_{a_+,\eta}^{\beta,\psi}\right)[1] + |d_{22}(t)|\left(\left(J_{a_+,T}^{\beta,\psi}\right)[1] - \mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[1]\right)\right), \tag{3.14}$$

$$\rho_{12} = \sup_{t\in[a,T]} \left(|d_{11}(t)|\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1] + |d_{12}(t)|\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1] - \mu\left(J_{a_+,\xi}^{\alpha+\beta+\gamma,\psi}\right)[1]\right)\right), \tag{3.15}$$

$$\rho_{21} = |\lambda| \sup_{t\in[a,T]} \left(\left|\left({}^c\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{21}]\right|\left(J_{a_+,\eta}^{\beta,\psi}\right)[1] + \left|\left({}^c\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{22}]\right|\left(J_{a_+,T}^{\beta,\psi}\right)[1] - \mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[1]\right), \tag{3.16}$$

and

$$\rho_{22} = \sup_{t\in[a,T]} \left(\left|\left({}^c\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{11}]\right|\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1] + \left|\left({}^c\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{12}]\right|\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1] - \mu\left(J_{a_+,\xi}^{\alpha+\beta+\gamma,\psi}\right)[1]\right). \tag{3.17}$$

We are now in a position to establish the existence and uniqueness results. Fixed point theorems are the main tool to prove this.

Let $C = C([a,T],\mathbb{R})$ be a Banach space of all continuous functions defined on $[a,T]$ endowed with the usual supremum norm. Define the space

$$E = \{u : u \in C^3([a,T],\mathbb{R}),\ \left({}^c\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[u] \in C\}, \tag{3.18}$$

equipped with the norm

$$\|u\|_E = \max\left\{\|u\|_\infty, \left\|\left({}^c\mathfrak{D}^{\gamma,\psi}_{a+,t}\right)[u]\right\|_\infty\right\}.$$

Then, we may conclude that $(E, \|.\|_E)$ is a Banach space.

To introduce a fixed point problem associated with (3.9) we consider an integral operator $\Psi : E \to E$ defined by

$$(\Psi u)(t) = -\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[u] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_u] + \phi_u(\mathfrak{F}). \tag{3.19}$$

Theorem 3.3. *Assume that $\mathfrak{F} : [a,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous function that satisfies (A_1). If we suppose that*

$$0 < \varsigma = \max\{\varsigma_{11}, \varsigma_{12}, \varsigma_{21}, \varsigma_{22}\} < 1, \tag{3.20}$$

holds. Then the problem (1.1) has a unique solution on E.

Proof. The proof will be given in two steps.

Step 1. The operator Ψ maps bounded sets into bounded sets in E.

For our purpose, consider a function $u \in E$. It is clear that $\Psi u \in E$. Also by (2.1), (3.10) and (3.19), we have

$$\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)(\Psi u) = -\lambda\left(J^{\beta-\delta,\psi}_{a+,t}\right)[u] + \left(J^{\alpha+\beta-\delta,\psi}_{a+,t}\right)[\mathfrak{F}_u] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[\phi_u(\mathfrak{F})]. \tag{3.21}$$

Indeed, it is sufficient to prove that for any $r > 0$, for each $u \in B_r = \{u \in E : \|u\|_E \leq r\}$, we have $\|\Psi u\|_E \leq r$.

Denoting

$$L_0 = \sup_{t\in[a,T]}\{|\mathfrak{F}(t,0,0| : t \in [a,T]\} < \infty \text{ and } L_B = L_1 \sup_{t\in[a,T]}|u(t)| + L_2 \sup_{t\in[a,T]}\left|\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[u]\right| + L_0.$$

By (A_1) we have for each $t \in [a,T]$

$$|\mathfrak{F}_u(t)| = |\mathfrak{F}_u(t) - \mathfrak{F}_0(t) + \mathfrak{F}_0(t)| \leq |\mathfrak{F}_u(t) - \mathfrak{F}_0(t)| + |\mathfrak{F}_0(t)| \leq L_B.$$

Firstly, we estimate $|\phi_u(\mathfrak{F})|$ as follows

$$\begin{aligned}
&|\phi_u(\mathfrak{F})| \\
= \ &\left|\lambda d_{21}(t)\left(J^{\beta,\psi}_{a_+,\eta}\right)[u]\right| + \left|d_{12}(t)\left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[|\mathfrak{F}_u - \mathfrak{F}_0| + \mathfrak{F}_0] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[|\mathfrak{F}_u - \mathfrak{F}_0| + \mathfrak{F}_0]\right)\right| \\
&+ \left|\lambda d_{22}(t)\left(\left(J^{\beta,\psi}_{a_+,T}\right)[u] - \mu\left(J^{\beta+\delta,\psi}_{a_+,\xi}\right)[u]\right)\right| + \left|d_{11}(t)\left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[|\mathfrak{F}_u - \mathfrak{F}_0| + \mathfrak{F}_0]\right|.
\end{aligned}$$

Then

$$\begin{aligned}
|\phi_u(\mathfrak{F})| \leq \ &\left|d_{11}(t)\left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[L_B]\right| + \left|d_{12}(t)\left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[L_B] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[L_B]\right)\right| \\
&+ \left|\lambda d_{21}(t)\left(J^{\beta,\psi}_{a_+,\eta}\right)[u]\right| + \left|\lambda d_{22}(t)\left(\left(J^{\beta,\psi}_{a_+,T}\right)[u] - \mu\left(J^{\beta+\delta,\psi}_{a_+,\xi}\right)[u]\right)\right|.
\end{aligned}$$

Taking the maximum over $[a,T]$, we get

$$\sup_{t\in[a,T]}|\phi_u(\mathfrak{F})| \leq \rho_{11}\sup_{t\in[a,T]}|u(t)| + \rho_{12}\left(L_1\sup_{t\in[a,T]}|u(t)| + L_2\sup_{t\in[a,T]}\left|\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[u]\right| + L_0\right), \tag{3.22}$$

where ϕ_u, $d_{ij}(t)$ and ρ_{ij} defined by (3.10), (3.7), (3.8) and (3.14–3.17) respectively. Using (3.19) and (3.22), we obtain

$$\|(\Psi u)\|_\infty \leq \varsigma_{11} \sup_{t\in[a,T]} |u(t)| + \varsigma_{12} \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u] \right| + \varsigma_{13} L_0, \tag{3.23}$$

where ς_{ij} defined by (3.11) and (3.12). On the other hand

$$\begin{aligned} & \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[\phi_u(\mathfrak{F})] \\ = \; & \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{11}] \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right)[\mathfrak{F}_u] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{12}] \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right)[\mathfrak{F}_u] - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right)[\mathfrak{F}_u] \right) \\ & + \lambda \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{21}] \left(J^{\beta,\psi}_{a_+,\eta} \right)[u] - \lambda \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{22}] \left(\left(J^{\beta,\psi}_{a_+,T} \right)[u] - \mu \left(J^{\beta+\delta,\psi}_{a_+,\xi} \right)[u] \right). \end{aligned}$$

Taking the maximum over $[a,T]$, we get

$$\begin{aligned} & \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[\phi_u(\mathfrak{F})] \right| \\ \leq \; & \rho_{21} \sup_{t\in[a,T]} |u(t)| + \rho_{22} \left(L_1 \sup_{t\in[a,T]} |u(t)| + L_2 \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u] \right| + L_0 \right). \end{aligned} \tag{3.24}$$

Using (3.21) and (3.24), we obtain

$$\left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi u) \right\|_\infty \leq \varsigma_{21} \sup_{t\in[a,T]} |u(t)| + \varsigma_{22} \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u] \right| + \varsigma_{23} L_0. \tag{3.25}$$

Consequently, by (3.23) and (3.25), we have

$$\|(\Psi u)\|_E \leq \varsigma \|u\|_E + L_0 \max\{\varsigma_{13}, \varsigma_{23}\} \leq \varsigma r + (1-\varsigma) r = r,$$

where ς is defined by (3.20) and choose

$$r > \frac{L_0 \max\{\varsigma_{13}, \varsigma_{23}\}}{(1-\varsigma)}, \; 0 < \varsigma < 1.$$

The continuity of the functional $\mathfrak{F}_u$ would imply the continuity of (Ψu) and $\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi u)$ Hence, Ψ maps bounded sets into bounded sets in E.

Step 2. Now we show that Ψ is a contraction. By (A_1) and (3.19), for $u, v \in E$ and $t \in [a,T]$, we have

$$|(\Psi u)(t) - (\Psi v)(t)| \leq |\lambda| \left| \left(J^{\beta,\psi}_{a_+,t} \right)[u-v] \right| + \left| \left(J^{\alpha+\beta,\psi}_{a_+,t} \right)[\mathfrak{F}_u - \mathfrak{F}_v] \right| + |\phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F})|,$$

where

$$\left| \left(J^{\alpha+\beta,\psi}_{a_+,t} \right)[\mathfrak{F}_u - \mathfrak{F}_v] \right| \leq \left| \left(J^{\alpha+\beta,\psi}_{a_+,t} \right)[1] \right| \left(L_1 |u-v| + L_2 \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u-v] \right| \right)$$

and

$$\begin{aligned} & \phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F}) \\ = \; & d_{11}(t) \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right)[\mathfrak{F}_u - \mathfrak{F}_v] + d_{12}(t) \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right)[\mathfrak{F}_u - \mathfrak{F}_v] - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right)[\mathfrak{F}_u - \mathfrak{F}_v] \right) \\ & + \lambda d_{21}(t) \left(J^{\beta,\psi}_{a_+,\eta} \right)[u-v] - \lambda d_{22}(t) \left(\left(J^{\beta,\psi}_{a_+,T} \right)[u-v] - \mu \left(J^{\beta+\delta,\psi}_{a_+,\xi} \right)[u-v] \right), \end{aligned}$$

for all $t \in [a,T]$, which implies

$$|\phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F})| \leq (\rho_{11} + L_1\rho_{12}) \sup_{t\in[a,T]} |u(t) - v(t)| + \rho_{12}L_2 \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u - v] \right|.$$

Hence, we get

$$\begin{aligned} & |(\Psi u)(t) - (\Psi v)(t)| \\ \leq\ & |\lambda| \left| \sup_{t\in[a,T]} \left(J^{\beta,\psi}_{a_+,t} \right)[1] \right| \sup_{t\in[a,T]} |u(t) - v(t)| + \sup_{t\in[a,T]} \left| \left(J^{\alpha+\beta,\psi}_{a_+,t} \right)[1] \right| \\ & \left(L_1 \sup_{t\in[a,T]} |u(t) - v(t)| + L_2 \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u - v] \right| \right) + \sup_{t\in[a,T]} |\phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F})|. \end{aligned}$$

Consequently,

$$\|(\Psi u) - (\Psi v)\|_\infty \leq \varsigma_{11} \sup_{t\in[a,T]} |u(t) - v(t)| + \varsigma_{12} \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u - v] \right|. \tag{3.26}$$

A similar argument shows that

$$\begin{aligned} & \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[(\Psi u) - (\Psi v)] \right| \\ =\ & |\lambda| \left| \left(J^{\beta-\delta,\psi}_{a+,t} \right)[u - v] \right| + \left| \left(J^{\alpha+\beta-\delta,\psi}_{a+,t} \right)[\mathfrak{F}_u - \mathfrak{F}_v] \right| + \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[\phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F})] \right|, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} & \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[\phi_u(\mathfrak{F}) - \phi_v(\mathfrak{F})] \right| \\ \leq\ & (\rho_{21} + L_1\rho_{22}) \sup_{t\in[a,T]} |u(t) - v(t)| + \rho_{22}L_2 \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u - v] \right| + \rho_{22}L_0. \end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28), we obtain

$$\left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi u) - \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi v) \right\|_\infty \leq \varsigma_{21} \sup_{t\in[a,T]} |u(t) - v(t)| + \varsigma_{22} \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[u - v] \right|. \tag{3.29}$$

Consequently, by (3.26) and (3.29), we have

$$\|(\Psi u) - (\Psi v)\|_E \leq \varsigma \|u - v\|_E$$

and choose $\varsigma = \max\{\varsigma_{11}, \varsigma_{12}, \varsigma_{21}, \varsigma_{22}\} < 1$. Hence, the operator Ψ is a contraction, therefore Ψ maps bounded sets into bounded sets in E. Thus, the conclusion of the theorem follows by the contraction mapping principle. □

For simplicity of presentation, we let

$$\Lambda_{11} = \left(J^{\alpha+\beta,\psi}_{a+,T} \right)[1] + \left(J^{\alpha+\beta-\gamma,\psi}_{a+,T} \right)[1] + \left(d_{11}(T) + \left({}^c\mathfrak{D}^{\gamma,\psi}_{a+,T} \right)[d_{11}] \right) \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right)[1],$$

$$\Lambda_{12} = \left(d_{12}(T) + \left({}^c\mathfrak{D}^{\gamma,\psi}_{a+,T} \right)[d_{12}] \right) \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right)[1] - \mu \left(J^{\alpha+\beta+\gamma,\psi}_{a_+,\xi} \right)[1] \right),$$

$$\Lambda_{21}=\left(J_{a+,T}^{\beta,\psi}\right)[1]+d_{21}(T)\left(J_{a_+,\eta}^{\beta,\psi}\right)[1]+d_{22}(T)\left(\left(J_{a_+,T}^{\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[1]\right),$$
$$\Lambda_{22}=\left(J_{a+,T}^{\beta-\gamma,\psi}\right)[1]+\left({}^{c}\mathfrak{D}_{a+,T}^{\gamma,\psi}\right)[d_{21}]\left(J_{a_+,\eta}^{\beta,\psi}\right)[1]+\left({}^{c}\mathfrak{D}_{a+,T}^{\gamma,\psi}\right)[d_{22}]\left(\left(J_{a_+,T}^{\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[1]\right).$$

We consider the space defined by (3.18) equipped with the norm

$$\|u\|_E=\|u\|_\infty+\left\|\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[u]\right\|_\infty. \tag{3.30}$$

It is easy to know that $(E,\|.\|_E)$ is a Banach space with norm (3.30). On this space, by virtue of Lemma 3.2, we may define the operator $\Psi:E\longrightarrow E$ by

$$(\Psi u)(t)=(\Psi_1 u)(t)+(\Psi_2 u)(t)=\left(-\lambda\left(J_{a_+,t}^{\beta,\psi}\right)[u]+\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]+\phi_u(\mathfrak{F})\right),$$

where Ψ_1 and Ψ_2 are the two operators defined on B_r by

$$(\Psi_1 u)(t)=\left(J_{a+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]+d_{11}(t)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]+d_{12}(t)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\gamma,\psi}\right)[\mathfrak{F}_u]\right) \tag{3.31}$$

and

$$(\Psi_2 u)(t)=-\lambda\left(J_{a+,t}^{\beta,\psi}\right)[u]+\lambda d_{21}(t)\left(J_{a_+,\eta}^{\beta,\psi}\right)[u]-\lambda d_{22}(t)\left(\left(J_{a_+,T}^{\beta,\psi}\right)[u]-\mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[u]\right), \tag{3.32}$$

where $d_{ij}(t)$ are defined by (3.7) and (3.8).

Applying $\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)$ on both sides of (3.31) and (3.32), we have

$$\begin{aligned}\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[\Psi_1 u] &= \left(J_{a+,t}^{\alpha+\beta-\gamma,\psi}\right)[\mathfrak{F}_u]+\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{11}]\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]\\ &\quad+\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{12}]\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_u]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\gamma,\psi}\right)[\mathfrak{F}_u]\right)\end{aligned}$$

and

$$\begin{aligned}\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[\Psi_2 u] &= -\lambda\left(J_{a+,t}^{\beta-\gamma,\psi}\right)[u]+\lambda\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{21}]\left(J_{a_+,\eta}^{\beta,\psi}\right)[u]\\ &\quad-\lambda\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[d_{22}]\left(\left(J_{a_+,T}^{\beta,\psi}\right)[u]-\mu\left(J_{a_+,\xi}^{\beta+\gamma,\psi}\right)[u]\right).\end{aligned} \tag{3.33}$$

Thus, Ψ is well-defined because Ψ_1 and Ψ_2 are well-defined. The continuity of the functional $\mathfrak{F}_u$ confirms the continuity of $(\Psi u)(t)$ and $\left({}^{c}\mathfrak{D}_{a+,t}^{\gamma,\psi}\right)[\Psi u](t)$, for each $t\in[a,T]$. Hence the operator Ψ maps E into itself.

In what follows, we utilize fixed point techniques to demonstrate the key results of this paper. In light of Lemma 3.2, we rewrite problem (3.9) as

$$u=\Psi u,\ u\in E. \tag{3.34}$$

Notice that problem (3.9) has solutions if the operator Ψ in (3.34) has fixed points. Conversely, the fixed points of Ψ are solutions of (1.1). Consider the operator $\Psi:E\longrightarrow E$. For $u,v\in B_r$, we find that

$$\|\Psi u\|_E=\|\Psi_1 u\|_E+\|\Psi_2 u\|_E.$$

Theorem 3.4. *Assume that* $\mathfrak{F} : [a, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+}$ *is a continuous function and the assumption* (A_3) *holds. If*

$$0 < |\lambda| (\Lambda_{21} + \Lambda_{22}) < 1, \tag{3.35}$$

then, problem (1.1) has at least one fixed point on $[a, T]$.

Proof. The proof will be completed in four steps:

Step 1. Firstly, we prove that, for any $u, v \in B_r$, $\Psi_1 u + \Psi_2 v \in B_r$, it follows that

$$\begin{aligned}
&\|(\Psi_1 u)\|_E = \|(\Psi_1 u)\|_\infty + \left\|\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)(\Psi_1 u)\right\|_\infty \\
\leq\ & \left[\left(J^{\alpha+\beta,\psi}_{a+,T}\right)[1] + d_{11}(T)\left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[1] + d_{12}(T)\left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[1]\right)\right]\|\mathfrak{F}_u\|_\infty \\
& + \left[\left(J^{\alpha+\beta-\delta,\psi}_{a+,T}\right)[1] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{11}]\left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[1]\right]\|\mathfrak{F}_u\|_\infty \\
& + \left[\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{12}]\left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[1]\right)\right]\|\mathfrak{F}_u\|_\infty,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|(\Psi_1 u)\|_E \times \|\mathfrak{F}_u\|^{-1}_\infty \\
=\ & \left(J^{\alpha+\beta,\psi}_{a+,T}\right)[1] + \left(J^{\alpha+\beta-\delta,\psi}_{a+,T}\right)[1] + \left(d_{11}(T) + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{11}]\right)\left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[1] \\
& + \left(d_{12}(T) + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{12}]\right)\left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[1]\right).
\end{aligned}$$

Then, we have

$$\|(\Psi_1 u)\|_E \leq (\Lambda_{11} + \Lambda_{12}) \times \|\mathfrak{F}_u\|_\infty,\ \Lambda_{11} + \Lambda_{12} < \infty, \tag{3.36}$$

which yields that Ψ_1 is bounded. On the opposite side

$$\begin{aligned}
&\frac{1}{|\lambda|}\|(\Psi_2 v)\|_E \times \|v\|^{-1}_E \\
\leq\ & \left(J^{\beta,\psi}_{a+,T}\right) + d_{21}(T)\left(J^{\beta,\psi}_{a_+,\eta}\right)[1] + d_{22}(T)\left(\left(J^{\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\beta+\delta,\psi}_{a_+,\xi}\right)[1]\right) \\
& + \left(J^{\beta-\delta,\psi}_{a+,T}\right)[1] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{21}]\left(J^{\beta,\psi}_{a_+,\eta}\right)[1] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[d_{22}]\left(\left(J^{\beta,\psi}_{a_+,T}\right)[1] - \mu\left(J^{\beta+\delta,\psi}_{a_+,\xi}\right)[1]\right),
\end{aligned}$$

which implies

$$\|(\Psi_2 v)\|_E \leq |\lambda| (\Lambda_{21} + \Lambda_{22}) \times \|v\|_E. \tag{3.37}$$

Then, from (3.36) and (3.37), it follows that

$$\|\Psi(u, v)\|_E \leq (\Lambda_{11} + \Lambda_{12}) \times \|\mathfrak{F}_u\|_\infty + |\lambda| (\Lambda_{21} + \Lambda_{22}) \times \|v\|_E. \tag{3.38}$$

By (A_3) and (3.38), we have that

$$(\Lambda_{11} + \Lambda_{12}) \times \|\mathfrak{F}_u\|_\infty + |\lambda| (\Lambda_{21} + \Lambda_{22}) \times \|v\|_E \leq (\Lambda_{11} + \Lambda_{12}) \times L + |\lambda| (\Lambda_{21} + \Lambda_{22}) \times r \leq r.$$

Then

$$r > \frac{(\Lambda_{11} + \Lambda_{12}) \times L}{1 - |\lambda| (\Lambda_{21} + \Lambda_{22})},\ 0 < |\lambda| (\Lambda_{21} + \Lambda_{22}) < 1,$$

which concludes that $\Psi_1 u + \Psi_2 v \in B_r$ for all $u, v \in B_r$.

Step 2. Next, for $u, v \in B_r$, Ψ_2 is a contraction. From (3.32) and (3.33), we have

$$\|(\Psi_2 u) - (\Psi_2 v)\|_E = \|(\Psi_2 u) - (\Psi_2 v)\|_\infty + \left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 u) - \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 v) \right\|_\infty ,$$

where

$$\begin{aligned} &\|(\Psi_2 u) - (\Psi_2 v)\|_\infty \leq \\ &|\lambda| \left(\left(J^{\beta,\psi}_{a+,t} \right)[1] + d_{21}(T) \left(J^{\beta,\psi}_{a_+,\eta} \right)[1] + d_{22}(T) \left(\left(J^{\beta,\psi}_{a_+,T} \right)[1] - \mu \left(J^{\beta+\delta,\psi}_{a_+,\xi} \right)[1] \right) \right) \|u - v\|_\infty . \end{aligned} \tag{3.39}$$

From (3.39), we can write

$$\|(\Psi_2 u) - (\Psi_2 v)\|_\infty \leq |\lambda| \, \Lambda_{21} \times \|u - v\|_\infty . \tag{3.40}$$

On the other hand

$$\begin{aligned} &\left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 u) - \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 v) \right\|_\infty \times \|u - v\|^{-1}_\infty \leq \\ &|\lambda| \left(\left(J^{\beta-\delta,\psi}_{a+,t} \right)[1] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,T} \right)[d_{21}] \left(J^{\beta,\psi}_{a_+,\eta} \right)[1] + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,T} \right)[d_{22}] \left(\left(J^{\beta,\psi}_{a_+,T} \right)[1] - \mu \left(J^{\beta+\delta,\psi}_{a_+,\xi} \right)[1] \right) \right), \end{aligned}$$

which yields

$$\left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 u) - \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)(\Psi_2 v) \right\|_\infty \leq |\lambda| \, \Lambda_{22} \times \|u - v\|_\infty . \tag{3.41}$$

Thus, using (3.40) and (3.41), it follows that

$$\|(\Psi_2 u) + (\Psi_2 v)\|_E \leq |\lambda| \left(\Lambda_{21} + \Lambda_{22} \right) \times \|u - v\|_E ,$$

and choose $0 < |\lambda| (\Lambda_{21} + \Lambda_{22}) < 1$. Hence, the operator Ψ_2 is a contraction.

Step 3. The continuity of Ψ_1 follows from that of $\mathfrak{F}_u$. Let $\{u_n\}$ be a sequence such that $u_n \longrightarrow u$ in E. Then for each $t \in [a, T]$

$$\begin{aligned} |(\Psi_1 u_n)(t) - (\Psi_1 u)(t)| \;=\; & \left(J^{\alpha+\beta,\psi}_{a+,t} \right)[\mathfrak{F}_{u_n} - \mathfrak{F}_u] + d_{11}(t) \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right)[\mathfrak{F}_{u_n} - \mathfrak{F}_u] \\ & + d_{12}(t) \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right)[\mathfrak{F}_{u_n} - \mathfrak{F}_u] - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right)[\mathfrak{F}_{u_n} - \mathfrak{F}_u] \right). \end{aligned}$$

By last equality with Eq (3.31), we can write

$$\begin{aligned} &|(\Psi_1 u_n)(t) - (\Psi_1 u)(t)| \leq \\ &\left[\left(J^{\alpha+\beta,\psi}_{a+,t} \right)[1] + d_{11}(t) \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right)[1] + d_{12}(t) \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right)[1] - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right)[1] \right) \right] \sup_{t \in [a,T]} \left| \mathfrak{F}_{u_n} - \mathfrak{F}_u \right| . \end{aligned}$$

It follows that

$$\begin{aligned} &\|(\Psi_1 u_n) - (\Psi_1 u)\|_E = \|(\Psi_1 u_n) - (\Psi_1 u)\|_\infty + \left\| \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)((\Psi_1 u_n) - (\Psi_1 u)) \right\|_\infty \leq \\ &\left[\left(J^{\alpha+\beta,\psi}_{a+,T} \right) + d_{11}(T) \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right) + d_{12}(T) \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right) - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right) \right) \right] \left\| \mathfrak{F}_{u_n} - \mathfrak{F}_u \right\|_\infty \\ &+ \left[\left(J^{\alpha+\beta-\delta,\psi}_{a+,T} \right) + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{11}] \left(J^{\alpha+\beta,\psi}_{a_+,\eta} \right) \right. \\ &\left. + \left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t} \right)[d_{12}] \left(\left(J^{\alpha+\beta,\psi}_{a_+,T} \right) - \mu \left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi} \right) \right) \right] \left\| \mathfrak{F}_{u_n} - \mathfrak{F}_u \right\|_\infty . \end{aligned} \tag{3.42}$$

By (3.42), we have

$$\|(\Psi_1 u_n) - (\Psi_1 u)\|_E \left\| \mathfrak{F}_{u_n} - \mathfrak{F}_u \right\|^{-1}_\infty \leq$$

$$\left(J_{a+,T}^{\alpha+\beta,\psi}[1]\right)+\left(J_{a+,T}^{\alpha+\beta-\delta,\psi}[1]\right)+\left(d_{11}(T)+\left({}^{c}\mathfrak{D}_{a+,t}^{\delta,\psi}\right)[d_{11}]\right)\left(J_{a_{+},\eta}^{\alpha+\beta,\psi}[1]\right)$$
$$+\left(d_{12}(T)+\left({}^{c}\mathfrak{D}_{a+,t}^{\delta,\psi}\right)[d_{12}]\right)\left(\left(J_{a_{+},T}^{\alpha+\beta,\psi}[1]\right)-\mu\left(J_{a_{+},\xi}^{\alpha+\beta+\delta,\psi}[1]\right)\right). \tag{3.43}$$

Consequently, by (3.43), we have

$$\|(\Psi_1 u_n)-(\Psi_1 u)\|_\infty \leq (\Lambda_{11}+\Lambda_{12})\times\left\|\mathfrak{F}_{u_n}-\mathfrak{F}_u\right\|_\infty,\ \Lambda_{11}+\Lambda_{12}<\infty.$$

Since $\mathfrak{F}_u$ is a continuous function, then by Lebesgue's dominated convergence theorem it follows that

$$\|(\Psi_1 u_n)-(\Psi_1 u)\|_\infty \longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Furthermore, Ψ_1 is uniformly bounded on B_r as $\|(\Psi_1 u)\|_E \leq (\Lambda_{11}+\Lambda_{12})\times\|\mathfrak{F}_u\|_\infty$, due to (3.36).

Step 4. Finally, we establish the compactness of Ψ_1. Let $u, v\in B_r$, for $t_1,t_2\in[a,T],\ t_1<t_2$, we have

$$\begin{aligned}
&\|(\Psi_1 u)(t_2)-(\Psi_1 u)(t_1)\|_\infty\\
\leq\ &\left[\left(J_{a+,t_2}^{\alpha+\beta,\psi}\right)[1]+d_{11}(t_2)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1]+d_{12}(t_2)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[1]\right)\right]\|\mathfrak{F}_u\|_\infty\\
&-\left[\left(J_{a+,t_1}^{\alpha+\beta,\psi}\right)[1]+d_{11}(t_1)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1]+d_{12}(t_1)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)\right)[1]\right]\|\mathfrak{F}_u\|_\infty\\
\leq\ &\left[\left(\left(J_{a+,t_2}^{\alpha+\beta,\psi}\right)[1]-\left(J_{a+,t_1}^{\alpha+\beta,\psi}\right)[1]\right)+\Lambda_{41}\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1]\right.\\
&\left.+\Lambda_{42}\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[1]\right)\right]\|\mathfrak{F}_u\|_\infty.
\end{aligned}\tag{3.44}$$

On the other hand

$$\begin{aligned}
&\left\|\left({}^{c}\mathfrak{D}_{a+,t}^{\delta,\psi}\right)(\Psi_1 u)(t_2)-\left({}^{c}\mathfrak{D}_{a+,t}^{\delta,\psi}\right)(\Psi_1 u)(t_1)\right\|_\infty\\
\leq\ &\left[\left(J_{a+,t_2}^{\alpha+\beta-\delta,\psi}\right)+\left({}^{c}\mathfrak{D}_{a+,t_2}^{\delta,\psi}\right)[d_{11}]\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)+\left({}^{c}\mathfrak{D}_{a+,t_2}^{\delta,\psi}\right)[d_{12}]\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)\right)\right]\|\mathfrak{F}_u\|_\infty\\
&-\left[\left(J_{a+,t_1}^{\alpha+\beta-\delta,\psi}\right)+\left({}^{c}\mathfrak{D}_{a+,t_1}^{\delta,\psi}\right)[d_{11}]\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)+\left({}^{c}\mathfrak{D}_{a+,t_1}^{\delta,\psi}\right)[d_{12}]\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)\right)\right]\|\mathfrak{F}_u\|_\infty\\
&\left[\left(\left(J_{a+,t_2}^{\alpha+\beta-\delta,\psi}\right)-\left(J_{a+,t_1}^{\alpha+\beta-\delta,\psi}\right)\right)+\Lambda_{43}\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)+\Lambda_{44}\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)\right)\right]\|\mathfrak{F}_u\|_\infty.
\end{aligned}\tag{3.45}$$

Using (3.44) and (3.45), we get

$$\begin{aligned}
&\|(\Psi_1 u)(t_2)-(\Psi_1 u)(t_1)\|_E\\
\leq\ &\left[\left(\left(J_{a+,t_2}^{\alpha+\beta,\psi}\right)[1]-\left(J_{a+,t_1}^{\alpha+\beta,\psi}\right)[1]\right)+\left(\left(J_{a+,t_2}^{\alpha+\beta-\delta,\psi}\right)-\left(J_{a+,t_1}^{\alpha+\beta-\delta,\psi}\right)\right)\right.\\
&\left.+(\Lambda_{41}+\Lambda_{43})\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[1]+(\Lambda_{42}+\Lambda_{44})\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[1]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[1]\right)\right]\|\mathfrak{F}_u\|_\infty,
\end{aligned}$$

where

$$\Lambda_{41}=d_{11}(t_2)-d_{11}(t_1),\Lambda_{42}=d_{12}(t_2)-d_{12}(t_1)$$

and

$$\Lambda_{43}=\left({}^{c}\mathfrak{D}_{a+,t_2}^{\delta,\psi}\right)[d_{11}]-\left({}^{c}\mathfrak{D}_{a+,t_1}^{\delta,\psi}\right)[d_{11}],\Lambda_{44}=\left({}^{c}\mathfrak{D}_{a+,t_2}^{\delta,\psi}\right)[d_{12}]-\left({}^{c}\mathfrak{D}_{a+,t_1}^{\delta,\psi}\right)[d_{11}].$$

Consequently, we have

$$\|(\Psi_1 u)(t_2)-(\Psi_1 u)(t_1)\|\times\|\mathfrak{F}_u\|^{-1}\longrightarrow 0 \text{ as } t_1\to t_2.$$

Thus, Ψ_1 is relatively compact on B_r. Hence, by the Arzela-Ascoli Theorem, Ψ_1 is completely continuous on B_r. Therefore, according to Theorem 2.8, the Problem (1.1) has at least one solution on B_r. This completes the proof. □

4. Stability of solutions

Hereafter, we discuss the UlamHyers and UlamHyersRassias stability of solutions of the FLE (1.1). In the proofs of Theorems 4.4 and 4.9, we use integration by parts in the settings of ψ-fractional operators. Denoting

$$\varphi_1(t) = \left(J^{\beta,\psi}_{a_+,t}\right)[1] \text{ and } \varphi_2(t) = \left(J^{\beta,\psi}_{a_+,t}\right)[\mathcal{K}(\tau;a)]. \tag{4.1}$$

Remark 4.1. For every $\epsilon > 0$, a function $\tilde{u} \in C$ is a solution of of the inequality

$$\left|\left({}^c\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)\left({}^c\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda\right)[\tilde{u}] - \mathfrak{F}(t,\tilde{u}(t),{}^c\,\mathfrak{D}^{\gamma,\psi}_{a+,t}[\tilde{u}])\right| \le \epsilon\Phi(t),\ t \in [a,T], \tag{4.2}$$

where $\Phi(t) \ge 0$ if and only if there exists a function $g \in C$, (which depends on $\tilde{u}$) such that

(i) $|g(t)| \le \epsilon\Phi(t),\ \forall t \in [a,T];$
(ii) $\left({}^c\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)\left({}^c\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda\right)[\tilde{u}] = \mathfrak{F}(t,\tilde{u}(t),{}^c\,\mathfrak{D}^{\gamma,\psi}_{a+,t}[\tilde{u}]) + g(t).$

Lemma 4.2. *If $\tilde{u} \in C$ is a solution of the inequality (4.2) then $\tilde{u}$ is a solution of the following integral inequality*

$$\left|\tilde{u}(t) - \left(-\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}}] + \phi_{\tilde{u}}(\mathfrak{F})\right)\right| \le C_\Phi(t), \tag{4.3}$$

where

$$C_\Phi(t) = \epsilon\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\Phi] + c_1(\epsilon\Phi)\varphi_1(t) + c_2(\epsilon\Phi)\varphi_2(t) + c_3(\epsilon\Phi),$$

where $c_1(\epsilon\Phi) - c_3(\epsilon\Phi)$ are real constants with $\mathfrak{F}_{\tilde{u}} = \Phi$ and C_Φ is independent of $\tilde{u}(t)$ and $\mathfrak{F}_{\tilde{u}}$.

Proof. Let $\tilde{u} \in C$ be a solution of the inequality (4.2). Then by Remark 4.1-(*ii*), we have that

$$\tilde{u}(t) = -\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}} + g] + \phi_{\tilde{u}}(\mathfrak{F}_{\tilde{u}} + g), \tag{4.4}$$

where

$$\phi_{\tilde{u}}(\mathfrak{F}_{\tilde{u}} + g) = c_1(\mathfrak{F}_{\tilde{u}} + g)\varphi_1(t) + c_2(\mathfrak{F}_{\tilde{u}} + g)\varphi_2(t) + c_3(\mathfrak{F}_{\tilde{u}} + g),$$

with

$$c_j(\mathfrak{F}_{\tilde{u}} + g) = c_j(\mathfrak{F}_{\tilde{u}}) + c_j(g),\ j = 1,2,3.$$

In view of (A_1) and (4.3), we obtain

$$\begin{aligned}
&\left|\tilde{u}(t) - \left(-\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}}] + \phi_{\tilde{u}}(\mathfrak{F})\right)\right| \\
= &\left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[g] + c_1(g)\varphi_1(t) + c_2(g)\varphi_2(t) + c_3(g)\right| \\
\le &\left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\epsilon\Phi] + c_1(\epsilon\Phi)\varphi_1(t) + c_2(\epsilon\Phi)\varphi_2(t) + c_3(\epsilon\Phi)\right| = C_\Phi(t).
\end{aligned}$$

□

As an outcome of Lemma 4.2, we have the following result:

Corollary 4.3. *Assume that $\mathfrak{F}_{\tilde{u}}$ is a continuous function that satisfies (A_1). If $\tilde{u} \in C$ is a solution of the inequality*

$$\left|{}^{c}\mathfrak{D}_{a+,t}^{\alpha,\psi}\left({}^{c}\mathfrak{D}_{a+,t}^{\beta,\psi}+\lambda\right)u(t)-\mathfrak{F}(t,u(t),{}^{c}\,\mathfrak{D}_{a+,t}^{\gamma,\psi}\,[u]\,(t))\right|\leq\epsilon,\ t\in[a,T]\,, \tag{4.5}$$

then $\tilde{u}$ is a solution of the following integral inequality

$$\left|\tilde{u}(t)-\left(-\lambda\left(J_{a_+,t}^{\beta,\psi}\right)[\tilde{u}]+\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]+\phi_{\tilde{u}}\left(\mathfrak{F}\right)\right)\right|\leq C_{\epsilon}, \tag{4.6}$$

with

$$\tilde{u}(a)=0,\ \tilde{u}(\eta)=0,\ \tilde{u}(T)=\mu\left(J_{a+,\xi}^{\gamma,\psi}\right)[\tilde{u}]\,,\ a<\eta<\xi<T,\ 0<\mu, \tag{4.7}$$

where

$$C_{\epsilon}=\epsilon\varsigma_{13}, \tag{4.8}$$

where ς_{13} is given by (3.12).

Proof. By Remark 4.1-(*ii*), (4.4), and by using (3.10) with the conditions (4.7), we have

$$\begin{aligned}\phi_{\tilde{u}}\left(\mathfrak{F}+g\right) &= d_{11}\left(t\right)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]+d_{12}\left(t\right)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]\right)\\ &\quad+\lambda d_{21}\left(t\right)\left(J_{a_+,\eta}^{\beta,\psi}\right)[\tilde{u}]-\lambda d_{22}\left(t\right)\left(\left(J_{a_+,T}^{\beta,\psi}\right)[\tilde{u}]-\mu\left(J_{a_+,\xi}^{\beta+\delta,\psi}\right)[\tilde{u}]\right)\\ &\quad+d_{11}\left(t\right)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[g]+d_{12}\left(t\right)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[g]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[g]\right).\end{aligned}$$

The solution of the problem (4.4) is given by

$$\begin{aligned}&\left|\tilde{u}\left(t\right)-\left(-\lambda\left(J_{a_+,t}^{\beta,\psi}\right)[\tilde{u}]+\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]+\phi_{\tilde{u}}\left(\mathfrak{F}\right)\right)\right|\\ \leq\ &\left|\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[g]+d_{11}\left(t\right)\left(J_{a_+,\eta}^{\alpha+\beta,\psi}\right)[g]+d_{12}\left(t\right)\left(\left(J_{a_+,T}^{\alpha+\beta,\psi}\right)[g]-\mu\left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi}\right)[g]\right)\right|,\end{aligned}$$

which implies that

$$\left|\tilde{u}\left(t\right)-\left(-\lambda\left(J_{a_+,t}^{\beta,\psi}\right)[\tilde{u}]+\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}]+\phi_{\tilde{u}}\left(\mathfrak{F}\right)\right)\right|\leq\Phi_{\epsilon}\left(t\right),$$

where

$$\Phi_{\epsilon}\left(t\right)=\left(J_{a_+,t}^{\alpha+\beta,\psi}\right)[\epsilon]+c_1\left(\epsilon\right)\varphi_1\left(t\right)+c_2\left(\epsilon\right)\varphi_2\left(t\right)+c_3\left(\epsilon\right)$$

with

$$C_{\epsilon}\equiv\sup_{t\in[a,T]}|\Phi_{\epsilon}\left(t\right)|=\epsilon\varsigma_{13},$$

which is the desired inequality (4.6). □

This corollary is obtained from Lemma 4.2 by setting $\Phi\left(t\right)=1$, for all $t\in[a,T]$, with (4.7).

Theorem 4.4. *Assume that $\mathfrak{F}_{\tilde{u}}$ is a continuous function that satisfies (A_1) and (A_4). The Eq (1.1-a) is H-U-R stable with respect to Φ if there exists a real number $l_{\alpha,\psi}>0$ such that for each $\epsilon>0$ and for each solution $\tilde{u}\in C^3\left([a,T],\mathbb{R}\right)$ of the inequality (4.2), there exists a solution $u^*\in C^3\left([a,T],\mathbb{R}\right)$ of (1.1-a) with*

$$|\tilde{u}(t)-u^*(t)|\leq\epsilon l_{\alpha,\psi}\Phi\left(t\right). \tag{4.9}$$

Proof. Using (4.2) and (1.1), we obtain

$$\begin{aligned}\tilde{u}(t) &= -\lambda\left(J^{\beta,\psi}_{a+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{\tilde{u}}+g] + c_1(\mathfrak{F}_{\tilde{u}}+g)\varphi_1(t) + c_2(\mathfrak{F}_{\tilde{u}}+g)\varphi_2(t) + c_3(\mathfrak{F}_{\tilde{u}}+g)\\ &= \theta_{\tilde{u}}(t,\mathfrak{F}_{\tilde{u}}+g) + c_1(\mathfrak{F}+g)\varphi_1(t) + c_2(\mathfrak{F}+g)\varphi_2(t) + c_3(\mathfrak{F}+g) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}u^*(t) &= -\lambda\left(J^{\beta,\psi}_{a+,t}\right)[u^*] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{u^*}] + c_1'(\mathfrak{F}_{u^*})\varphi_1(t) + c_2'(\mathfrak{F}_{u^*})\varphi_2(t) + c_3'(\mathfrak{F}_{u^*})\\ &= \theta_{u^*}(t,\mathfrak{F}) + c_1'(\mathfrak{F}_{u^*})\varphi_1(t) + c_1'(\mathfrak{F}_{u^*})\varphi_1(t) + c_2'(\mathfrak{F}_{u^*})\varphi_2(t) + c_3'(\mathfrak{F}_{u^*}),\end{aligned}$$

where

$$\theta_{\tilde{u}}(t,\mathfrak{F}) = -\lambda\left(J^{\beta,\psi}_{a+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{\tilde{u}}]. \tag{4.11}$$

By using (4.9) and (4.10), we have the following inequalities

$$\begin{aligned}|\tilde{u}(t)-u^*(t)| &\le \left|\tilde{u}(t) - \left(\theta_{u^*}(t,\mathfrak{F}) + c_1'(\mathfrak{F}_{u^*})\varphi_1(t) + c_2'(\mathfrak{F}_{u^*})\varphi_2(t) + c_3'(\mathfrak{F}_{u^*})\right)\right|\\ &\le |\tilde{u}(t) - (\theta_{\tilde{u}}(t,\mathfrak{F}) + c_1(\mathfrak{F}_{\tilde{u}})\varphi_1(t) + c_1(\mathfrak{F}_{\tilde{u}})\varphi_1(t) + c_2(\mathfrak{F}_{\tilde{u}})\varphi_2(t) + c_3(\mathfrak{F}_{\tilde{u}}))|\\ &\quad + |\theta_{\tilde{u}}(t,\mathfrak{F}) - \theta_{u^*}(t,\mathfrak{F})| + \left|(c_1(\mathfrak{F}_{\tilde{u}}) - c_1'(\mathfrak{F}_{u^*}))\varphi_1(t)\right|\\ &\quad + \left|(c_2(\mathfrak{F}_{\tilde{u}}) - c_2'(\mathfrak{F}_{u^*}))\right|\varphi_2(t) + \left|c_3(\mathfrak{F}_{\tilde{u}}) - c_3'(\mathfrak{F}_{u^*})\right|.\end{aligned}$$

By setting

$$c_{33} = c_3(\mathfrak{F}_{\tilde{u}}) - c_3'(\mathfrak{F}_{u^*}) = \tilde{u}(a) - u^*(a),\ c_{11} = c_1(\mathfrak{F}_{\tilde{u}}) - c_1'(\mathfrak{F}_{u^*}) \text{ and } c_{22} = c_2(\mathfrak{F}_{\tilde{u}}) - c_2'(\mathfrak{F}_{u^*}),$$

and

$$w(\eta) = \tilde{u}(\eta) - u^*(\eta) + \theta_{\tilde{u}}(\eta,\mathfrak{F}) - \theta_{u^*}(\eta,\mathfrak{F}) \text{ and } w(T) = \tilde{u}(T) - u^*(T) + \theta_{\tilde{u}}(T,\mathfrak{F}) - \theta_{u^*}(T,\mathfrak{F}).$$

It follows from (4.9) and (4.10), that

$$\begin{pmatrix}\varphi_1(\eta) & \varphi_2(\eta)\\ \varphi_1(T) & \varphi_2(T)\end{pmatrix}\begin{pmatrix}c_{11}\\ c_{22}\end{pmatrix} = \begin{pmatrix}w(\eta)\\ w(T)\end{pmatrix},$$

Applying Lemma 4.2 and from estimation (4.11), it follows

$$|\tilde{u}(t)-u^*(t)| \le C_\Phi(t) + |\theta_{\tilde{u}}(t,\mathfrak{F}) - \theta_{u^*}(t,\mathfrak{F})| + c_{11}\varphi_1(t) + c_{22}\varphi_2(t) + c_{33},$$

where

$$\begin{aligned}|\theta_{\tilde{u}}(t,\mathfrak{F}) - \theta_{u^*}(t,\mathfrak{F})| &= \left|-\lambda\left(J^{\beta,\psi}_{a+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{\tilde{u}}] - \left(-\lambda\left(J^{\beta,\psi}_{a+,t}\right)[u^*] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{u^*}]\right)\right|\\ &\le \left|-\lambda\left(J^{\beta,\psi}_{a+,t}\right)[\tilde{u}-u^*] + \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[\mathfrak{F}_{\tilde{u}} - \mathfrak{F}_{u^*}]\right|.\end{aligned}$$

Using Lemma 2.4 and (A_1), we have

$$\left(J^{\alpha+\beta,\psi}_{a+,t}\right)\left[\left|\left({}^c\mathfrak{D}^{\delta,\psi}_{a+,t}\right)[\tilde{u}-u^*]\right|\right] = z_0(t) - \left(J^{\alpha+\beta-\delta,\psi}_{a+,t}\right)[\tilde{u}-u^*]$$

and

$$\left|\left(J_{a_{+},t}^{\alpha+\beta,\psi}\right)[\mathfrak{F}_{\tilde{u}}-\mathfrak{F}_{u^{*}}]\right| \leq \left|L_{1}\left(J_{a_{+},t}^{\alpha+\beta,\psi}\right)[\tilde{u}-u^{*}]\right| + L_{2}\left|z_{0}(t)-\left(J_{a_{+},t}^{\alpha+\beta-\delta,\psi}\right)[\tilde{u}-u^{*}]\right|,$$

where

$$z_{0}(t)=\frac{|\tilde{u}(a)-u^{*}(a)|}{\Gamma(\alpha+\beta)}\times\left(J_{a,t-}^{1-\delta,\psi}\right)\left[(\mathcal{K}(t;a))^{\alpha+\beta-1}\right].$$

Set

$$q(t)=\mathcal{G}(t)+L_{2}\frac{|\tilde{u}(a)-u^{*}(a)|}{\Gamma(\alpha+\beta)}\times\left(J_{a,t-}^{1-\delta,\psi}\right)\left[(\mathcal{K}(t;a))^{\alpha+\beta-1}\right],$$

where

$$\mathcal{G}(t)=C_{\Phi}(t)+c_{11}\varphi_{1}(t)+c_{22}\varphi_{2}(t)+c_{33},$$

with

$$C_{\Phi}(t)=\epsilon\left(J_{a_{+},t}^{\alpha+\beta,\psi}\right)[\Phi]+c_{1}(\epsilon\Phi)\varphi_{1}(t)+c_{2}(\epsilon\Phi)\varphi_{2}(t)+c_{3}(\epsilon\Phi).$$

This means that

$$p(t)\leq q(t)+\lambda\left(J_{a_{+},t}^{\beta,\psi}\right)[\tilde{u}-u^{*}]+L_{1}\left(J_{a_{+},t}^{\alpha+\beta,\psi}\right)[\tilde{u}-u^{*}]+L_{2}\left(J_{a_{+},t}^{\alpha+\beta-\delta,\psi}\right)[\tilde{u}-u^{*}]. \tag{4.12}$$

Using Lemma 2.13, the above inequality implies the estimation for $p(t)$ such as

$$p(t)\leq q(t)+\sum_{k=1}^{\infty}\left(\begin{array}{l}\frac{(\lambda\Gamma(\beta))^{k}}{\Gamma(k\beta)}\int_{a}^{t}\left[\psi'(\tau)(\mathcal{K}(t;\tau))^{k\beta-1}\right]q(\tau)\,\mathrm{d}\tau\\ +\\ \frac{(L_{1}\Gamma(\alpha+\beta))^{k}}{\Gamma(k(\alpha+\beta))}\int_{a}^{t}\left[\psi'(\tau)(\mathcal{K}(t;\tau))^{k(\alpha+\beta)-1}\right]q(\tau)\,\mathrm{d}\tau\\ +\\ \frac{(L_{2}\Gamma(\alpha+\beta-\delta))^{k}}{\Gamma(k(\alpha+\beta-\delta))}\int_{a}^{t}\left[\psi'(\tau)(\mathcal{K}(t;\tau))^{k(\alpha+\beta-\delta)-1}\right]q(\tau)\,\mathrm{d}\tau\end{array}\right).$$

Therefore, with (A_4), the inequality (4.12) can be rewritten as

$$p(t)=|\tilde{u}(t)-u^{*}(t)|\leq\epsilon l_{\alpha,\psi}\Phi(t).$$

By Remark 2.14, one can obtain

$$\begin{aligned}p(t) \quad\leq\quad & q(t)\left[E_{\beta}\left(\lambda\Gamma(\beta)(\mathcal{K}(t;a))^{\beta}\right)+E_{\alpha+\beta}\left(\lambda\Gamma(\alpha+\beta)(\mathcal{K}(t;a))^{\alpha+\beta}\right)\right.\\ &\left.+\,E_{\alpha+\beta+\delta}\left(\lambda\Gamma(\alpha+\beta+\delta)(\mathcal{K}(t;a))^{\alpha+\beta+\delta}\right)\right].\end{aligned}$$

Thus, we complete the proof. □

Theorem 4.5. *Assume that the assumptions (A_1) and (A_4). If a continuously differentiable function $\tilde{u}:[a,T]\longrightarrow\mathbb{R}$ satisfies (4.2), where $\Phi:[a,T]\longrightarrow\mathbb{R}^{+}$ is a continuous function with (A_3), then there exists a unique continuous function $u^{*}:[a,T]\longrightarrow\mathbb{R}$ of problem (1.1) such that*

$$|\tilde{u}(t)-u^{*}(t)|\leq\epsilon l_{\alpha,\psi}\Phi(t), \tag{4.13}$$

with

$$|\tilde{u}(a)-u^{*}(a)|=|\tilde{u}(\eta)-u^{*}(\eta)|=|\tilde{u}(T)-u^{*}(T)|=0. \tag{4.14}$$

Proof. Assume that $\tilde{u} \in C^3([a,T],\mathbb{R})$ is a solution of the (4.2). In view of proof of Theorem 4.4, we get

$$\mathcal{G}(t) = C_\Phi(t) + c_{11}\varphi_1(t) + c_{22}\varphi_2(t) + c_{33} = C_\Phi(t),$$

with the conditions (4.14), we have

$$C_\Phi(t) = \epsilon\left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\Phi] + \left(J^{\alpha+\beta,\psi}_{a_+,\eta}\right)[\Phi]\, d_{11}(t) + \left(\left(J^{\alpha+\beta,\psi}_{a_+,T}\right)[\Phi] - \mu\left(J^{\alpha+\beta+\delta,\psi}_{a_+,\xi}\right)[\Phi]\right) d_{12}(t)\right|.$$

Set $q(t) = C_\Phi(t)$. Using Theorem 4.4 and (A_4), we conclude that, the estimation for $p(t) = |u(t) - \tilde{u}(t)|$ such as (4.12). So the inequality (4.12) can be rewritten as

$$p(t) = |u(t) - \tilde{u}(t)| \le \epsilon l_{\alpha,\psi}\Phi(t).$$

By Remark 2.14, one can obtain

$$\begin{aligned} p(t) \quad &\le \quad q(t)\left[E_\beta\left(\lambda\Gamma(\beta)(\mathcal{K}(t;a))^\beta\right) + E_{\alpha+\beta}\left(\lambda\Gamma(\alpha+\beta)(\mathcal{K}(t;a))^{\alpha+\beta}\right)\right. \\ &\quad \left. + E_{\alpha+\beta+\delta}\left(\lambda\Gamma(\alpha+\beta+\delta)(\mathcal{K}(t;a))^{\alpha+\beta+\delta}\right)\right]. \end{aligned}$$

This proves that the problem (1.1) is, UlamHyersRassias stable. □

Theorem 4.6. *Assume that the assumptions (A_2), (A_4) and (4.2) hold. Then Eq (1.1-a) is H-U-R stable.*

Proof. By (A_2) and (4.11), we have

$$\left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}}] - \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_u]\right| \le \left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\tilde{\chi}] - \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\chi]\right|$$

and

$$|\tilde{u}(t) - u^*(t)| \le C_\Phi(t) + |\theta_{\tilde{u}}(t,\mathfrak{F}+g) - \theta_{u^*}(t,\mathfrak{F})| + c_{11}|\varphi_1(t)| + c_{22}|\varphi_2(t)| + c_{33},$$

where

$$\begin{aligned} &|\theta_{\tilde{u}}(t,\mathfrak{F}+g) - \theta_{u^*}(t,\mathfrak{F})| \\ = \;& \left|-\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u}] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}} + g] - \left(-\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[u^*] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{u^*}]\right)\right| \\ \le \;& \left|-\lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u} - u^*] + \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\mathfrak{F}_{\tilde{u}} - \mathfrak{F}_{u^*}]\right| + \left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[g]\right|. \end{aligned}$$

Using Lemma 4.2, we have

$$p(t) \le q(t) + \lambda\left(J^{\beta,\psi}_{a_+,t}\right)[\tilde{u} - u^*],$$

where

$$q(t) = \mathcal{G}(t) + \left|\left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\tilde{\chi}] - \left(J^{\alpha+\beta,\psi}_{a_+,t}\right)[\chi]\right|,$$

with

$$\mathcal{G}(t) = C_\Phi(t) + c_{11}\varphi_1(t) + c_{22}\varphi_2(t) + c_{33}.$$

From the above, it follows

$$p(t) \le q(t) + \sum_{k=1}^{\infty} \frac{(\lambda\Gamma(\beta))^k}{\Gamma(k\beta)} \int_a^t \left[\psi'(\tau)(\mathcal{K}(t;\tau))^{k\beta-1}\right] q(\tau)\,\mathrm{d}\tau .$$

By Remark 2.14, one can obtain

$$p(t) \le q(t)\, E_\beta\left(\lambda\Gamma(\beta)(\mathcal{K}(t;a))^\beta\right).$$

□

Remark 4.7. If $\Phi(t)$ is a constant function in the inequalities (4.2), then we say that (1.1-a) is UlamHyers stable.

Corollary 4.8. *Assume that the assumptions (A_2), (A_4) and (4.2) hold. Then Eq (1.1-a) with (4.13) is UlamHyersRassias stable.*

Proof. Using Theorem 4.6, we have

$$p(t) \leq q(t) + \lambda \left(J_{a_+,t}^{\beta,\psi} \right) [\tilde{u} - u^*],$$

where

$$p(t) = |\tilde{u}(t) - u(t)| \text{ and } q(t) = C_\Phi(t) + \left| \left(J_{a_+,t}^{\alpha+\beta,\psi} \right) [\tilde{\chi}] - \left(J_{a_+,t}^{\alpha+\beta,\psi} \right) [\chi] \right|.$$

We conclude that

$$p(t) \leq q(t) + \sum_{k=1}^{\infty} \frac{(\lambda \Gamma(\beta))^k}{\Gamma(k\beta)} \int_a^t \left[\psi'(\tau) (\mathcal{K}(t;\tau))^{k\beta-1} \right] q(\tau) \, d\tau \ .$$

By Remark 2.14, one can obtain

$$p(t) \leq q(t) E_\beta \left(\lambda \Gamma(\beta) (\mathcal{K}(t;a))^\beta \right).$$

□

Theorem 4.9. *Assume that the assumptions (A_1) and (4.2) with (4.14) hold. Then problem (1.1) is UlamHyers stable and consequently generalized UlamHyers stable.*

Proof. Let u^* be a unique solution of the fractional Langevin type problem (1.1), that is, $u^*(t) = (\Psi u^*)(t)$. Assume that $\tilde{u} \in C([a,T], \mathbb{R})$ is a solution of the (4.2). By using the estimation

$$|(\Psi\tilde{u})(t) - (\Psi u^*)(t)| \leq \lambda \left| \left(J_{a_+,t}^{\beta,\psi} \right) [\tilde{u} - u^*] \right| + \left| \left(J_{a_+,t}^{\alpha+\beta,\psi} \right) [\mathfrak{F}_{u^*} - (\mathfrak{F}_{\tilde{u}} + g)] \right| + |\phi_{\tilde{u}}(\mathfrak{F} + g) - \phi_{u^*}(t, \mathfrak{F})|,$$

where

$$\begin{aligned} & |\phi_{\tilde{u}}(\mathfrak{F} + g) - \phi_{u^*}(t, \mathfrak{F})| \\ = \; & d_{11}(t) \left(J_{a_+,\eta}^{\alpha+\beta,\psi} \right) [\mathfrak{F}_{\tilde{u}} - \mathfrak{F}_{u^*}] + d_{12}(t) \left(\left(J_{a_+,T}^{\alpha+\beta,\psi} \right) [\mathfrak{F}_{\tilde{u}} - \mathfrak{F}_{u^*}] - \mu \left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi} \right) [\mathfrak{F}_{\tilde{u}} - \mathfrak{F}_{u^*}] \right) \\ & + \lambda d_{21}(t) \left(J_{a_+,\eta}^{\beta,\psi} \right) [\tilde{u} - u^*] - \lambda d_{22}(t) \left(\left(J_{a_+,T}^{\beta,\psi} \right) [\tilde{u} - u^*] - \mu \left(J_{a_+,\xi}^{\beta+\delta,\psi} \right) [\tilde{u} - u^*] \right) \\ & + d_{11}(t) \left(J_{a_+,\eta}^{\alpha+\beta,\psi} \right) [g] + d_{12}(t) \left(\left(J_{a_+,T}^{\alpha+\beta,\psi} \right) [g] - \mu \left(J_{a_+,\xi}^{\alpha+\beta+\delta,\psi} \right) [g] \right). \end{aligned} \tag{4.15}$$

Taking the maximum over $[a,T]$, we get

$$\sup_{t\in[a,T]} |(\Psi\tilde{u})(t) - (\Psi u^*)(t)| \leq \varsigma_{11} \sup_{t\in[a,T]} |\tilde{u}(t) - u^*(t)| + \varsigma_{12} \sup_{t\in[a,T]} \left| \left({}^c\mathfrak{D}_{a_+,t}^{\delta,\psi} \right) [\tilde{u} - u^*] \right| + \epsilon\varsigma_{13}.$$

Using Lemma 2.3 and (4.15), we obtain

$$\sup_{t\in[a,T]} |(\Psi\tilde{u})(t) - (\Psi u^*)(t)| \leq (\varsigma_{11} + \varsigma_{12}\kappa_0) \sup_{t\in[a,T]} |\tilde{u}(t) - u^*(t)| + \epsilon\varsigma_{13},$$

where

$$\kappa_0 = \frac{1}{\Gamma(2-\delta)} (\psi(T) - \psi(a))^{1-\delta}.$$

We conclude that

$$\|\tilde{u} - u^*\|_\infty \le \frac{\epsilon \varsigma_{13}}{(1 - \varsigma_{11} - \varsigma_{12}\kappa_0)}, \; 0 < 1 - \varsigma_{11} - \varsigma_{12}\kappa_0 < 1.$$

Thus problem (1.1) is UlamHyers stable. Further, using Theorem 4.5 implies that solution of (1.1) is generalized UlamHyers stable. This completes the proof. □

Corollary 4.10. *Let the conditions of Theorem 4.9 hold. Then Problem (1.1) is generalized UlamHyersRassias stable.*

Proof. Set $\epsilon = 1$ in the proof of Theorem 4.9, we get

$$\|\tilde{u} - u^*\|_\infty \le \frac{\varsigma_{13}}{(1 - \varsigma_{11} - \varsigma_{12}\kappa_0)}, \;\; 0 < 1 - \varsigma_{11} - \varsigma_{12}\kappa_0 < 1. \tag{4.16}$$

□

Remark 4.11. (**i**) Considering (1.1) and inequality (4.2), then under the assumptions of Theorem 4.5, one can follow the same procedure to confirm that (1.1) is UlamHyers stable.
(ii) Other stability results for the Eq (1.1) can be discussed in a similar manner.

5. Applications

In this section, we provide some test problems to illustrate the applicability of the established results.

Example 5.1. Without loss of generality, we only consider the following ψ-Caputo Langevin equations

$$\left({}^c\mathfrak{D}^{\alpha,\psi}_{a+,t}\right)\left({}^c\mathfrak{D}^{\beta,\psi}_{a+,t} + \lambda\right)[u] = \mathfrak{F}(t, u(t), {}^c\mathfrak{D}^{\gamma,\psi}_{a+,t}[u]).$$

By taking

$$\mathfrak{F}(t, u, v) = \frac{\kappa}{20} E_{1/2}\left(t^{1/2}\right) u + \frac{\kappa}{10} v, \; \kappa \in [0, +\infty) \tag{5.1}$$

we have

$$L_1 = \frac{\kappa}{20} \sup\left\{E_{1/2}\left(t^{1/2}\right) : t \in [a, T]\right\} \;\text{ and }\; L_2 = \frac{\kappa}{10}.$$

For $\psi(t) = t$, we shall show that condition (3.3) holds with

$$\alpha = 3/2, \; \beta = 6/7, \; \gamma = 5/7, \; \lambda = 1/6, \; \mu = 2, \; a = 0, \eta = 4/7, \; \xi = 4/5, \; T = 1. \tag{5.2}$$

A simple computation shows that

$$\sigma_{11} = 0.653, \; \sigma_{12} = 0.201, \; \sigma_{21} = 0.0483 \text{ and } \sigma_{22} = 0.255.$$

$$\Delta \equiv |\sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12}| = 0.157.$$

(**i**) Thus, the hypotheses (A_1) and (3.20) are satisfied with

$$d_{11}(T) = 1.54, \; d_{12}(T) = 1.02, \; d_{21}(T) = 1.54 \text{ and } d_{22}(T) = 1.02,$$

$$\rho_{11} = 0.217,\ \rho_{12} = 0.345,\ \rho_{21} = 0.280 \text{ and } \rho_{22} = 0.635$$

and

$$\varsigma_{11} = 0.393 + 0.174\kappa,\ \varsigma_{12} = 0.0696\kappa,\ \varsigma_{21} = 0.458 + 0.328\kappa \text{ and } \varsigma_{22} = 0.131\kappa,$$

where $\rho_{11},\ \rho_{12},\ \rho_{21}, \rho_{22},\ \varsigma_{11},\ \varsigma_{12}$ and ς_{21} are given by (3.14)–(3.17) and (3.11)–(3.13) respectively.

Thus condition (3.20), with

$$0 < \kappa \leq 1.648$$

is

$$\varsigma \equiv \max\{\varsigma_{11}, \varsigma_{12}, \varsigma_{21}, \varsigma_{22}\} = 0.998 < 1, \text{ with } \kappa = 1.648$$

and

$$L_0 = 0,\ \varsigma_{13} = 0.696,\ \varsigma_{23} = 1.31 \text{ and } \max\{\varsigma_{13}, \varsigma_{23}\} = 1.31.$$

Hence, by Theorem 3.3, the Problem (1.1) with (5.1) and (5.2) has a unique solution.

(ii) On the other hand, using (3.35), the condition

$$\Lambda_{21} = 2.11,\ \Lambda_{22} = 2.14,\ 0 < |\lambda| (\Lambda_{21} + \Lambda_{22}) = 0.708 < 1,$$

is satisfied and

$$r > \frac{(\Lambda_{11} + \Lambda_{12}) \times L}{1 - |\lambda| (\Lambda_{21} + \Lambda_{22})} = 6.88L \text{ with } \Lambda_{11} = 1.31, \Lambda_{12} = 0.700.$$

So, by Theorem 3.4, the Problem (1.1) with (5.1) and (5.2) has at least one fixed point on $[0, 1]$.

(iii) It is easy to check that the condition (4.8) is satisfied. Indeed,

$$C_\epsilon = \epsilon\varsigma_{13} = .696\epsilon.$$

Then by Corollary 4.3, we have, if $\tilde{u} \in C$ is a solution of the inequality (4.5), then $\tilde{u}$ is a solution of the integral inequality (4.6).

(iv) Let $\Phi(t) = \psi(t) - \psi(a)$ in Remark 2.14 satisfy (A_4).

From (4.2) and the condition (A_4), we get

$$\left(J^{\alpha,\psi}_{a+,t}\right)[\tau] = \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} \leq \frac{\Gamma(2)\,T^\alpha}{\Gamma(\alpha+2)} t,\ l_{\alpha,t} = \frac{\Gamma(2)\,T^\alpha}{\Gamma(\alpha+2)} \tag{5.3}$$

and

$$\sup_{t\in[a,T]} |C_\epsilon(t)| \leq \epsilon \left| \tfrac{\Gamma(2)T^{\alpha+\beta}}{\Gamma(\alpha+\beta+2)} + \tfrac{\Gamma(2)\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+2)} d_{11}(T) + \left(\tfrac{\Gamma(2)T^{\alpha+\beta}}{\Gamma(\alpha+\beta+2)} - \mu \tfrac{\Gamma(2)\xi^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+2)} \right) d_{12}(T) \right|.$$

With (5.2), we obtain

$$\epsilon l_{\alpha,\psi} \equiv \sup_{t\in[a,T]} C_\epsilon(t) = 0.216\epsilon.$$

By Lemma 2.13 and Remark 2.14, there exists $l_{\alpha,\psi} \equiv 0.216 > 0$ such that for each $\varepsilon > 0$, we have

$$p(t) \leq .216\epsilon t \left[E_{6/7}\left(1/6\Gamma(6/7)\, t^{6/7}\right) + E_{33/14}\left(1/6\Gamma(33/14)\, t^{33/14}\right) + E_{43/14}\left(1/6\Gamma(43/14)\, t^{43/14}\right) \right].$$

Therefore, by Theorem 4.5, the Problem (1.1) with (5.1) and (5.2) is generalized UlamHyersRassias-Mittag-Leffler stable.

(v) The condition (4.16) is satisfied with

$$\kappa_0 = 1.11,\ \varsigma_{11} + \varsigma_{12}\kappa_0 = 0.807 < 1.$$

By Theorem 4.9, this implies that Problem (1.1) with (5.1) and (5.2) has UlamHyersRassias stability.

Further in the below tables (as Tables 1 and 2), we list the consequences of proposed theorems for different values of functions ψ.

Table 1. Numerical values for different parameters.

		Theorem 3.3		Theorem 3.4		Corollary 4.3	Theorem 4.9
$\psi(t)$	κ	ς	$r >$	$\lvert\lambda\rvert(\Lambda_{21} + \Lambda_{22})$	$r >$	C_ϵ	$\varsigma_{11} + \varsigma_{12}\kappa_0$
t	1.647	0.998	0	0.708	$6.88L$	0.696ε	0.807
$t^{1/3}$	1.825	0.997	0	0.708	$6.86L$	0.693ε	0.723
$\ln(t+1)$	1.262	0.999	0	0.594	$2.49L$	0.292ε	0.511
$\exp(t)$	0.749	0.998	0	0.950	$111L$	2.52ε	0.919
$\sin t,\ [0, \frac{\pi}{2}]$	0.406	0.999	0	0.708	$6.87L$	0.696ε	0.679

and

Table 2. Numerical values for different parameters.

				Theorem 4.5	Corollary 4.8
$\psi(t)$	$[a, T]$	κ	(d_{11}, d_{12})	$l_{\alpha,\psi}$	$l_{\alpha,\psi}$
t	$[0, 1]$	1.647	$(1.54, 1.020)$	0.216	0.105
$t^{1/3}$	$[0, 1]$	1.825	$(1.29, 0.470)$	0.212	0.105
$\ln(t+1)$	$[0, 1]$	1.262	$(1.17, 0.914)$	0.0901	0.00442
$\exp(t)$	$[0, 1]$	0.749	$(2.39, 1.16)$	0.789	0.378
$\sin t,$	$[0, \frac{\pi}{2}]$	0.406	$(1.37, 0.962)$	0.213	0.105

Example 5.2. Let

$$\mathfrak{F}(t, u, v) = E_{1/2}\left(t^{1/2}\right) + \frac{\kappa}{20}E_{1/2}\left(t^{1/2}\right)u + \frac{\kappa}{10}v,\ \kappa \in [0, +\infty) \tag{5.4}$$

and

$$\alpha = 5/3,\ \beta = 3/4,\ \gamma = 1/2,\ \lambda = 1/8,\ \mu = 2,\ \eta = 5/4,\ \xi = 5/3. \tag{5.5}$$

Then, we have

$$L_0 = \sup E\left\{_{1/2}\left(t^{1/2}\right) : t \in [a, T]\right\},\ L_1 = \frac{\kappa}{20}\sup E\left\{_{1/2}\left(t^{1/2}\right) : t \in [a, T]\right\},\ L_2 = \frac{\kappa}{10}.$$

In the below tables (as Tables 3 and 4), we list the consequences of proposed theorems for different values of functions ψ.

If $\Phi(t) = \psi(t) - \psi(a)$ then

Table 3. Numerical values for different parameters.

			Theorem 3.3		Theorem 3.4		Corollary 4.3	Theorem 4.9
$\psi(t)$	$[a,T]$	κ	ς	$r>$	$\lvert\lambda\rvert(\Lambda_{21}+\Lambda_{22})$	$r>$	C_ϵ	$\varsigma_{11}+\varsigma_{12}\kappa_0$
$\ln t$	$[1,e]$	0.131	0.999	31600	0.548	$3.82L$	0.674ε	0.750
$\sqrt{t}$	$[1,2]$	2.439	0.988	213	0.144	$0.315L$	0.0781ε	0.455
t^2	$[1/2,1]$	4.911	0.996	746	0.476	$5.27L$	0.213ε	0.885
$\sin t$	$[0,\pi/2]$	0.406	0.999	12100	0.708	$6.87L$	0.696ε	0.679
2^t	$[1,2]$	0.1512	0.998	30200	0.785	$37.4L$	3.82ε	0.967
$\exp(t^2)$	$[0,1]$	0.686	0.998	6330	0.723	$89.5L$	3.77ε	1.22

Table 4. Numerical values for different parameters.

				Theorem 4.5	Corollary 4.8
$\psi(t)$	$[a,T]$	κ	(d_{11},d_{12})	$l_{\alpha,\psi}$	$l_{\alpha,\psi}$
$\ln t$	$[1,e]$	0.131	(2.00, 1.19)	0.199	0.096
$\sqrt{t}$	$[1,2]$	2.439	(1.46, 1.34)	0.0233	0.0111
t^2	$[1/2,1]$	4.911	(1.07, 0.116)	0.0865	0.0479
$\sin t$	$[0,\pi/2]$	0.406	(1.37, 0.962)	0.213	0.105
2^t	$[1,2]$	0.1525	(4.58, 1.66)	1.15	0.514
$\exp(t^2)$	$[0,1]$	0.686	(0.688, 0.00777)	0.009	0.357

Example 5.3. (**i**) If we set $\Phi(t)=\exp(\theta(\psi(t)-\psi(a)))$ for every $\theta\neq 0$, then by the changing of variables $\theta(\psi(t)-\psi(a))=u$, we obtain

$$\left(J_{a+,t}^{\alpha,\psi}\right)[\Phi(t)]=\frac{\gamma(\alpha,\theta(\psi(t)-\psi(a)))}{\theta^\alpha\Gamma(\alpha)}\exp(\theta(\psi(t)-\psi(a))), \tag{5.6}$$

where $\gamma(\alpha,t)$ is the incomplete Gamma function defined by

$$\gamma(\alpha,t)=\int_0^t\tau^{\alpha-1}e^{-\tau}\mathrm{d}\tau,\ \Re(t)>0,\ \left|\arg(t)\right|<\pi. \tag{5.7}$$

Thus

$$\frac{t^\alpha}{\alpha}e^{-t}\le\gamma(\alpha,t)\le\frac{t^\alpha}{\alpha}. \tag{5.8}$$

Then by the above inequality, we obtain

$$\left(J_{a+,t}^{\alpha,\psi}\right)[\Phi(t)]\le\frac{(\theta(\psi(t)-\psi(a)))^\alpha}{\theta^\alpha\Gamma(\alpha+1)}\exp(\theta(\psi(t)-\psi(a))),\ \text{for all } t\in[a,T].$$

Hence function $\Phi(t)$ satisfies the condition (A_4) with

$$l_{\alpha,\psi}=\frac{(\theta(\psi(t)-\psi(a)))^\alpha}{\theta^\alpha\Gamma(\alpha+1)}.$$

(**ii**) If we set $\Phi(t)=E_\alpha(\theta(\psi(t)-\psi(a))^\alpha)$ for every $\theta\neq 0$, then

$$\left(J_{a+,t}^{\alpha,\psi}\right)[\Phi(t)]=\frac{1}{\theta}(E_\alpha(\theta(\psi(t)-\psi(a))^\alpha)-1)\le\frac{1}{\theta}E_\alpha(\theta(\psi(t)-\psi(a))^\alpha).$$

Thus function $\Phi(t)$ satisfies condition (A_4) with

$$l_{\alpha,\psi} = \frac{1}{\theta}.$$

(**iii**) The function $\Phi(t)$ is positive and there exists a constant $l_{\alpha,\psi}$ such that the condition (A_4) is satisfied.

Indeed, for each $t \leq [a, T]$, we get

$$(\psi(t) - \psi(\tau))^{\alpha-1} \leq (\psi(t) - \psi(a))^{\alpha-1}.$$

Where as $\tau \in [a, t]$, $\alpha \geq 1$ and $\psi'(\tau) \geq 0$

$$\begin{aligned} \left(J_{a+,t}^{\alpha,\psi}\right)[\Phi] &= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \Phi(\tau)\,\mathrm{d}\tau \\ &\leq \frac{(\psi(t) - \psi(a))^{\alpha-1}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)\Phi(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{5.9}$$

then

$$\begin{aligned} \int_a^t \psi'(\tau)\Phi(\tau)\,\mathrm{d}\tau &= (\psi(t)\Phi(t) - \psi(a)\Phi(a)) - \int_a^t \psi(\tau)\Phi'(\tau)\,\mathrm{d}\tau \\ &\leq (\psi(t)\Phi(t) - \psi(a)\Phi(a)) - \psi(a)\int_a^t \Phi'(\tau)\,\mathrm{d}\tau \\ &= (\psi(t) - \psi(a))\Phi(t). \end{aligned}$$

$$\left(J_{a+,t}^{\alpha,\psi}\right)[\Phi] \leq \frac{(\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha)}\Phi(t).$$

Thus function $\Phi(t)$ satisfies the condition (A_4) with

$$l_{\alpha,\psi} = \frac{(\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha)}.$$

Example 5.4. Let $h \in C^2\left(\mathbb{R}^2\right)$ be bounded and let $g \in L_1[0, T]$. Then the functions

$$\mathfrak{F}(t, u, v) = g(t)h(u, v) \quad \text{or} \quad \mathfrak{F}(t, u, v) = g(t) + h(u, v),$$

satisfies (5.4). In view of condition (A_2), we consider the different values of function $\mathfrak{F}$ (as Table 5),

Table 5. Upper bounds.

$\mathfrak{F}(t, u, v)$	$\lvert\mathfrak{F}(t, u, v)\rvert \leq$	$\mathfrak{F}(t, u, v)$	$\lvert\mathfrak{F}(t, u, v)\rvert \leq$
$g(t)\left(\frac{\lvert u\rvert+\lvert v\rvert}{\lvert u\rvert+\lvert v\rvert+1}\right)$	$\lvert g(t)\rvert$	$g(t) + \frac{\lvert u\rvert}{\lvert u\rvert+1} + \frac{\lvert v\rvert}{\lvert v\rvert+1}$	$\lvert g(t)\rvert + 2$
$g(t)\sin u + \frac{2}{\pi}\arctan v$	$2\lvert g(t)\rvert$	$g(t)\tanh u + sgn(v)$	$\lvert g(t)\rvert + 1$

where

$$sgn(v) = \begin{cases} \frac{v}{|v|} & \text{if} \quad v \neq 0, \\ 0 & \text{if} \quad v = 0. \end{cases}$$

6. Conclusions

The Langevin equation has been introduced to characterize dynamical processes in a fractal medium in which the fractal and memory features with a dissipative memory kernel are incorporated. Therefore, the consideration of Langevin equation in frame of fractional derivatives settings would be providing better interpretation for real phenomena. Consequently, scholars have considered different versions of Langevin equation and thus many interesting papers have been reported in this regard. However, one can notice that most of existing results have been carried out with respect to the classical fractional derivatives.

In this paper, we have tried to promote the current results and considered the FLE in a general platform. The boundary value problem of nonlinear FLE involving ψ- fractional operators of different orders was investigated. One of the major differences in the problem considered in this work and relevant work already published in literature, is that, we are dealing with general fractional operator. Secondly, the forcing function depends on fractional derivative of unknown function. We employ the newly accommodated ψ- fractional calculus to prove the following for the considered problem:

(i) The existence and uniqueness of solutions: Techniques of fixed point theorems are used to prove the results. Prior to the main theorems, the forms of solutions are derived for both linear and nonlinear problems.

(ii) Stability in sense of Ulam: We adopt the required definitions of UlamHyers stability with respect to ψ- fractional derivative. The UlamHyersRassias and generalized U-H-R stability of the solution are discussed. Gronwall inequality and integration by parts in frame of ψ- fractional derivative are also employed to complete the proofs.

(iii) Applications: Particular examples are addressed at the end of the paper to show the consistency of the theoretical results.

We claim that the results of this paper are new and generalize some earlier results. Moreover, by fixing the parameters involved in the given problem, we can obtain some new results as special cases of the ones presented in this paper. For example, letting $\psi = t, \mu = 0, a = 0$ and $T = 1$ in the results of Section 3, we get the ones derived in [40]. Besides, the existence results for the initial value problem of nonlinear classical Langevin equation of the form:

$$\dddot{u} + \lambda \ddot{u} = \mathfrak{F}(t, u, \dot{u}),\ u(0) = 0,\ u(\eta) = 0,\ u(1) = 0,\ 0 < \eta < 1,$$

can be addressed by fixing a $\alpha = 2$ and $\beta = 1$ in the results of this paper.

For further investigation, one can propose to study the properties of the solution of the considered problem via some numerical computations and simulations. We leave this as promising future work. Results obtained in the present paper can be considered as a contribution to the developing field of fractional calculus via generalized fractional derivative operators.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgement

J. Alzabut would like to thank Prince Sultan University for funding and supporting this work.

References

1. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier, 2006.
2. R. Herrmann, *Fractional Calculus for Physicist*, World Scientific Public, 2014.
3. M. Yavuz, European option pricing models described by fractional operators with classical and generalized MittagLeffler kernels, *Numer. Methods Partial Differ. Equations*, 2020. Available from: `https://doi.org/10.1002/num.22645`.
4. M. R. Jena, S. Chakraverty, M. Yavuz, Two-hybrid techniques coupled with an integral transform for caputo time-fractional Navier-Stokes equations, *Prog. Fract. Differ. Appl.*, **6** (2020), 201–213.
5. I. Podlubny, *Fractional Differential Equations*, San Diego: Academic Press, 1999.
6. K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics, Berlin: Springer-Verlag, 2010.
7. N. Sene, Mathematical views of the fractional Chuas electricalcircuit described by the Caputo-Liouville derivative, *Rev. Mex. Fis.*, **67** (2020), 91–99.
8. N. Sene, Cascade of fractional differential equations and eeneralized Mittag-Leffler stability, *Int. J. Math. Modell. Comput.*, **10** (2020), 25–35.
9. H. M. Fahad, M. Rehman, Generalized substantial fractional operators and well-posedness of Cauchy problem, *Bull. Malays. Math. Sci. Soc.*, 2020. Available from: `https://doi.org/10.1007/s40840-020-01008-4`.
10. A. Keten, M. Yavuz, D. Baleanu, Nonlocal Cauchy problem via a fractional operator involving power kernel in Banach spaces, *Fractal Fractional*, **3** (2019), 27.
11. M. Yavuz, Novel solution methods for initial boundary value problems of fractional order with conformable differentiation, *Int. J. Optim. Control: Theor. Appl.*, **8** (2018), 1–7.
12. S. G. Samko, A. A. Kilbas, O. I. Mariche, *Fractional Integrals and Derivatives*, translated from the 1987 Russian original, Yverdon: Gordon and Breach, 1993.
13. T. J. Osler, Fractional derivatives of a composite function, *SIAM J. Math. Anal.*, **1** (1970), 288–293.
14. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44** (2017), 460–481.
15. V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, New York: John Wiley & Sons Inc., 1994.
16. O. P. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Fractional Calculus Anal. Appl.*, **15** (2012), 700–711.
17. M. D. Kassim, N. E. Tatar, Stability of logarithmic type for a Hadamard fractional differential problem, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 447–466.

18. Y. Luchko, J. J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative, *Fractional Calculus Anal. Appl.*, **10** (2007), 249–276.

19. B. Ahmad, M. M. Matar, O. M. EL-Salmy, Existence of solutions and ulam stability for Caputo type sequential fractional differential equations of order $\alpha \in (2, 3)$, *Int. J. Anal. Appl.*, **15** (2017), 86–101.

20. M. A. Darwich, S. K. Ntouyas, Existence results for a fractional functional differential equation of mixed type, *Commun. Appl. Nonlinear Anal.*, **15** (2008), 47–55.

21. V. Obukhovskii, P. Zecca, M. Afanasova, On some boundary value problems for fractional feedback control systems, *Differ. Equations Dyn. Sys.*, 2018. Available from: `https://doi.org/10.1007/s12591-018-0435-5`.

22. M. El-Shahed, Positive solutions for boundary value problem of nonlinear fractional differential equation, *Abstr. Appl. Anal.*, **2007** (2007), 010368.

23. E. M. Elsayed, On the existence and stability of solution of boundary value problem for fractional integro-differential equations with complex order, *Filomat*, **32** (2018).

24. R. A. Yan, S. R. Sun, Z. L. Han, Existence of solutions of boundary value problems for Caputo fractional differential equations on time scales, *Bull. Iran. Math. Soc.*, **42** (2016), 247–262.

25. Y. Zhao, S. Sun, Z. Han, Q. Li, The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simu.*, **16** (2011), 2086–2097.

26. H. Qin, X. Zuo, J. Liu, Existence and controllability results for fractional impulsive integrodifferential systems in Banach spaces, *Abstr. Appl. Anal.*, **2013** (2013), 295837.

27. R. Sakthivel, Y. Ren, N. I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Comput. Math. Appl.*, **62** (2011), 1451–1459.

28. G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut, C. Huang, Hybrid control scheme for projective lag synchronization of Riemann-Liouville sense fractional order memristive BAM neural networks with mixed delays, *Mathematics*, **7** (2019), 759. Available from: `https://doi.org/10.3390/math7080759`.

29. M. Ismail, U. Saeed, J. Alzabut, M. ur Rehman, Approximate solutions for fractional boundary value problems via Green-CAS method, *Mathematics*, **7** (2019), 1164. Available from: `https://doi.org/10.3390/math7121164`.

30. F. Mainradi, P. Pironi, The fractional Langevin equation: Brownian motion revisted, *Extracta Math.*, **10** (1996), 140–154.

31. B. Ahmad, A. Alsaedi, S. Salem, On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders, *Adv. Differ. Equations*, **2019** (2019), 1–44.

32. H. Baghani, J. Alzabut, Juan Nieto, A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions, *Math. Meth. Appl. Sci.*, 2020. Available from: `https://doi.org/10.1002/mma.6639`.

33. C. Kiataramkul, K. N. Sotiris, J. Tariboon, A. Kijjathanakorn, Generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions, *Boundary Value Prob.*, **2016** (2016), 217.

34. W. Sudsutad, S. K. Ntouyas, J. Tariboon, Systems of fractional Langevin equations of Riemann-Liouville and Hadamard types, *Adv. Differ. Equations*, **2015** (2015), 1–24.
35. A. Berhail, N. Bouache, M. M. Matar, J. Alzabut, On nonlocal integral and derivative boundary value problem of nonlinear Hadamard Langevin equation with three different fractional orders, *Bol. Soc. Mat. Mex.*, **26** (2020), 303–318.
36. X. Li, S. Sun, Y. Sun, Existence of solutions for fractional Langevin equation with infinite-point boundary conditions, *Appl. Math. Comput.*, **53** (2016), 1–10.
37. W. Yukunthorn, S. K. Ntouyas, J. Tariboon, Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions, *Adv. Differ. Equations*, **2014** (2014), 315.
38. Z. Zhou, Y. Qiao, Solutions for a class of fractional Langevin equations with integral and anti-periodic boundary conditions, *Boundary Value Prob.*, **2018** (2018), 152.
39. H. Zhou, J. Alzabut, L. Yang, On fractional Langevin differential equations with anti-periodic boundary conditions, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3577–3590.
40. B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal.*, **13** (2012), 599–606.
41. S. M. Ulam, A collection of mathematical problems, *Bull. Am. Math. Soc.*, **60** (1960), 361–363.
42. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, **27** (1941), 222–224.
43. T. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.
44. J. R. Wang, Z. Lin, Ulam's type stability of Hadamard type fractional integral equations, *Filomat*, **28** (2014), 1323–1331.
45. S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Palm Harbor: Hadronic Press, 2001.
46. M. Ahmad, A. Zada, J. Alzabut, Hyres-Ulam stability of coupled system of fractional differential equations of Hilfer-Hadamard type, *Demonstr. Math.*, **52** (2019), 283–295.
47. M. Ahmad, A. Zada, J. Alzabut, Stability analysis for a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian, *Adv. Differ. Equations*, **2019** (2019), 436.
48. D. R. Smart, *Fixed Point Theorems*, Cambridge: University Press, 1980.
49. J. Alzabut, Y. Adjabi, W. Sudsutad, M. ur Rehman, New generalizations for Gronwall type inequalities involving a ψ-fractional operator and their applications, *AIMS Math.*, **6** (2021), 5053–5077.
50. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. Rogosin, *Mittag-Leffler Functions*, Related Topics and Applications, Berlin: Springer, 2014.